Penalty Methods for the Hyperbolic System Modelling the Wall-Plasma Interaction in a Tokamak

Philippe Angot, Thomas Auphan and Olivier Guès

Aix-Marseille Université (Provence), LATP-CMI, UMR CNRS 6632, Marseille

[angot, tauphan, gues]@cmi.univ-mrs.fr

Objectives and motivations

Wall-plasma interactions in a tokamak [Tamain, PhD thesis 2007]

- A challenge for ITER project : control wall-plasma interactions. ⇒ development of a fast solver for numerical simulations.

A first approach [Isoardi *et al*, JCP 2010]

- Show that this approach is possible with encouraging results.
- But their penalization cuts the flux term at the plasma-limiter interface which seems to be "hazardous".

The original hyperbolic system

N : plasma density ; Γ : particle flux ; $M = \frac{\Gamma}{N}$: plasma velocity

$(t,x) \in \mathbb{R}^+ \times [-L, L]$
$\partial_t N + \partial_x \Gamma = S_N$
$$\partial_t \Gamma + \partial_x \left(\frac{\Gamma^2}{N} + N \right) = S_\Gamma$$

Boundary conditions : $M(-L) = -1$ and $M(L) = 1$ (Bohm condition)
Initial conditions : $N(0,.) = N_0$ and $\Gamma(0,.) = \Gamma_0$

Numerical tests : 2[nd] order finite volume scheme (VF Roe ncv [Gallouët *et al*, Computers & fluids 2003] with entropy correction, MUSCL (with slope limiter) and RK2 (Heun) time discretization.

Well-posedness issue

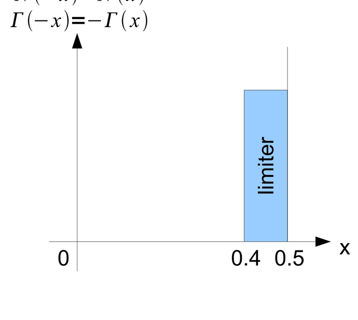
The waves speeds for the hyperbolic system are the eigenvalues : $M-1$ and $M+1$. From the boundary conditions $|M| = 1$, we infer that there is no ingoing wave (one characteristic wave and one outgoing wave at $x = \pm L$). So, we can't impose any boundary condition.
Thus, we change the boundary conditions to get one ingoing wave :

$(t,x) \in \mathbb{R}^+ \times [-L, L]$
$\partial_t N + \partial_x \Gamma = S_N$
$$\partial_t \Gamma + \partial_x \left(\frac{\Gamma^2}{N} + N \right) = S_\Gamma$$

Boundary conditions : $M(-L) = -1 + \epsilon$ and $M(L) = 1 - \epsilon$, $\epsilon > 0$
Initial conditions : $N(0,.) = N_0$ and $\Gamma(0,.) = \Gamma_0$

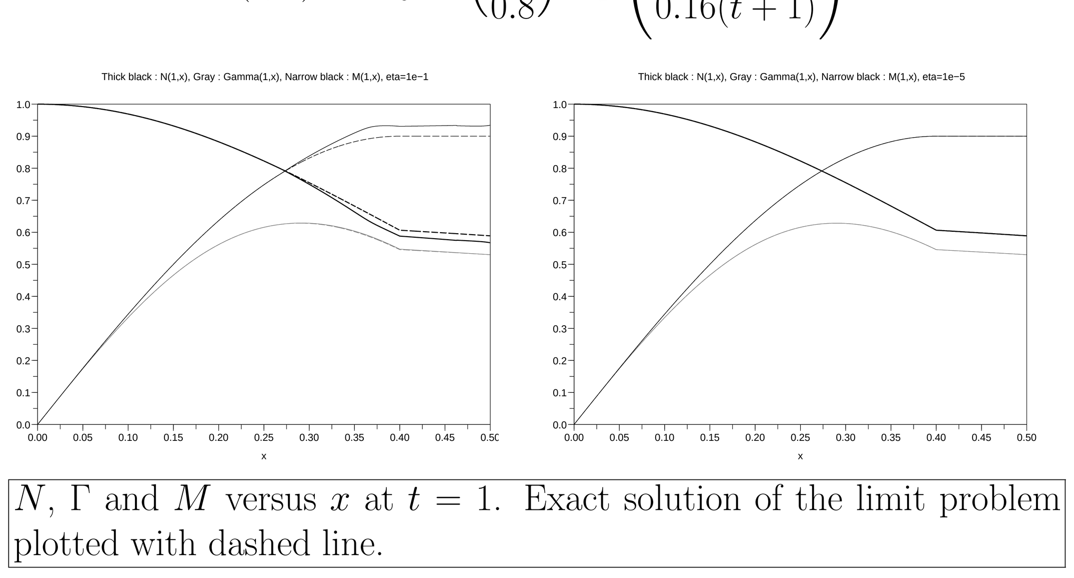
On each boundary we have only one condition to impose (this was not the case in the previous penalization).
Further numerical results obtained with $\epsilon = 0.1$ and $M_0 = 1 - \epsilon = 0.9$.

Numerical tests



Test solutions :

$$N(t,x) = \exp\left(\frac{-x^2}{0.16(t+1)} \right)$$
$$\Gamma(t,x) = M_0 \sin\left(\frac{\pi x}{0.8}\right) \exp\left(\frac{-x^2}{0.16(t+1)} \right)$$



N, Γ and M versus x at $t = 1$. Exact solution of the limit problem plotted with dashed line.

A first approach [Isoardi *et al*, JCP 2010]

χ : characteristic function of the limiter ; η : penalization parameter.

$$\partial_t N + \partial_x \Gamma + \frac{\chi}{\eta} N = (1-\chi) S$$
$$\partial_t \Gamma + (1-\chi)\partial_x \left(\frac{\Gamma^2}{N} + N \right) + \frac{\chi}{\eta}(\Gamma - M_0 N) = 0$$

Initial conditions : $N(0,.) = N_0$ and $\Gamma(0,.) = \Gamma_0$

Problem of meaning for the term $(1-\chi)\partial_x \left(\frac{\Gamma^2}{N} + N \right)$:

For example, there is no solution piecewise $\mathcal{C}^1$ such that $\frac{\Gamma^2}{N} + N$ is continuous on $x = L$.

So this first approach was not completely satisfactory. This is confirmed by numerical tests (see next page).

A new penalty method

For the semi-linear case, [Fornet and Guès, DCDS, 2009] proposed a penalization method without any boundary layer.
Though our system is quasi-linear, we apply this method to our case and we obtain interesting results. The penalized system is :

$$\partial_t N + \partial_x \Gamma = (1-\chi) S_N$$
$$\partial_t \Gamma + \partial_x \left(\frac{\Gamma^2}{N} + N \right) + \frac{\chi}{\eta} \left(\frac{\Gamma}{M_0} - N \right) = (1-\chi) S_\Gamma$$

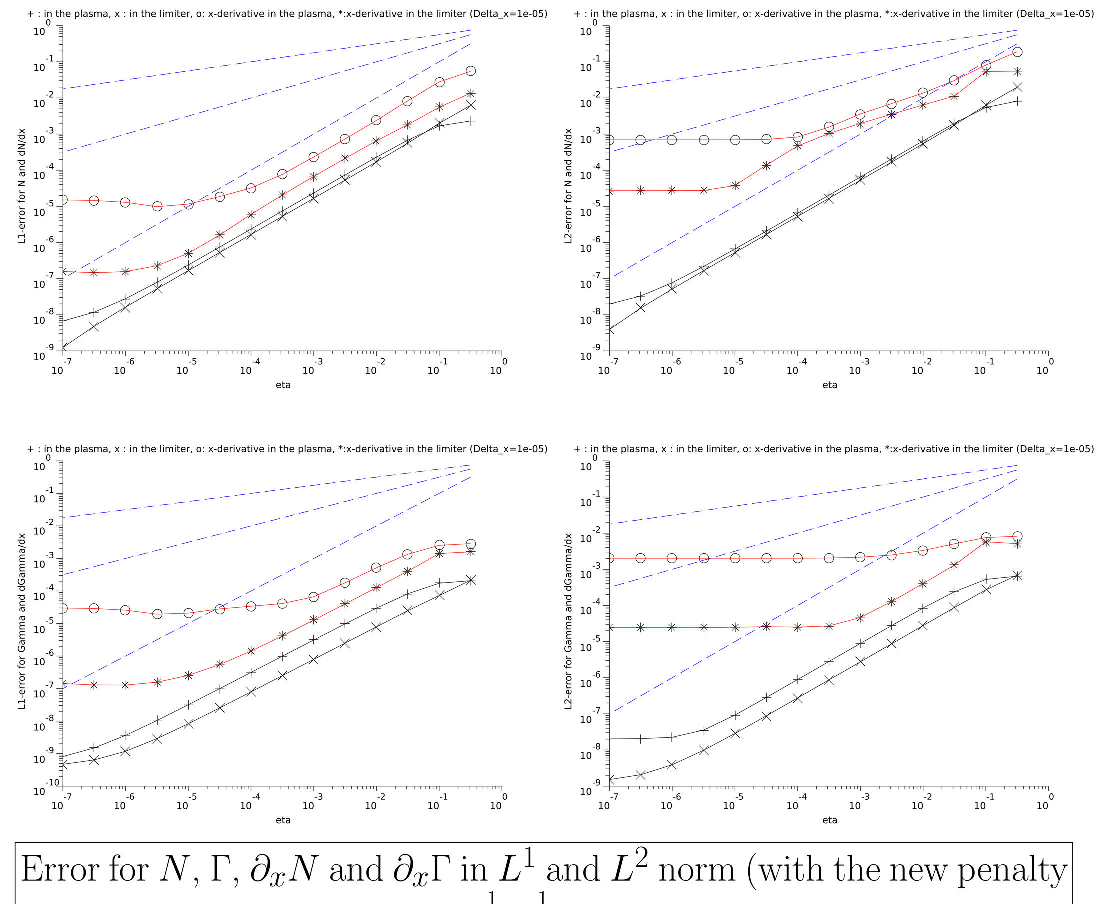
Initial conditions : $N(0,.) = N_0$ and $\Gamma(0,.) = \Gamma_0$

"Theorem" (WKB analysis)**.** *For any $k \in \mathbb{N}$, there exists an approximate solution of the penalized system*

$$N_{\eta,app}(t,x) = \sum_{n=0}^{k} \eta^n \mathbf{N}^n(t,x), \qquad \Gamma_{\eta,app}(t,x) = \sum_{n=0}^{k} \eta^n \mathbf{\Gamma}^n(t,x)$$

satisfying the equations up to an error in η^k for arbitrarily large k.

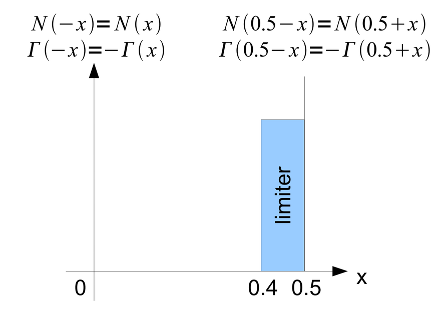
Numerical study



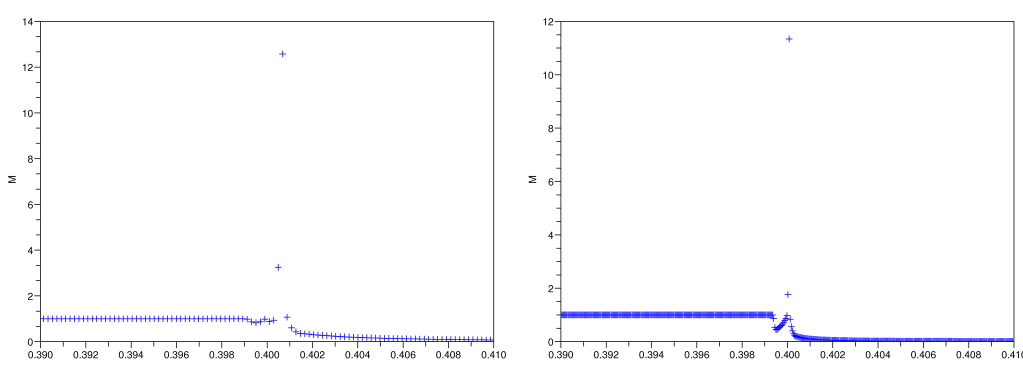
Error for N, Γ, $\partial_x N$ and $\partial_x \Gamma$ in L^1 and L^2 norm (with the new penalty method). Blue dashed lines : $\eta^{\frac{1}{4}}, \eta^{\frac{1}{2}}$ and η

Optimal convergence rate for N and Γ : $\mathcal{O}(\eta)$
Non optimal rate for the x-derivatives of N in the L^2 error ⇒ boundary layer or artefact ?

Some numerical results



Flux is cut in the limiter ⇒ we can think that a Dirac measure may appear on the plasma-limiter interface. Computations stopped when $\max_{i \in \{1,\dots,J\}}(|M_i^n|) > 10$ (normally $M \leq 1$) with $\eta = 10^{-3}$.
Mesh convergence study :



M versus x with $\Delta x = 1.952 \cdot 10^{-4}$ (left graph at $t = 0.004107$) and with $\Delta x = 4.88 \cdot 10^{-5}$ (right graph, at $t = 0.0015834$).

When the resolution increases, the peak is nearer and nearer to the interface. This phenomenon is observed when $M_0 = 10\,(0.5 - x)$ and when $M_0 = 1$ but not when $M_0 = 0$.

Asymptotic analysis

Change of unknown from both sides of the interface :
$$\mathbf{U}_\eta^\pm(t,x) = \begin{pmatrix} ln(N(t,x)) \\ M(t,x) - M_0 \end{pmatrix}$$
$$\partial_t \mathbf{U}_\eta^\pm + \mathbf{f}'(\mathbf{U}_\eta^\pm)\partial_x \mathbf{U}_\eta^\pm + \frac{\chi}{M_0 \eta} P \mathbf{U}_\eta^\pm = \mathbf{S}$$

P = projection matrix.

Formal asymptotic expansion of a continuous solution of the form : $\mathbf{U}_\eta^\pm(t,x) = \sum_{n=0}^{+\infty} \eta^n \mathbf{U}^{n,\pm}(t,x)$

Substituting the expansion and classifying gives :

- Inside the plasma : $\sum_{n=0}^{\infty} \eta^n \left(\partial_t \mathbf{U}^{n,-} + \dots \right) = \mathbf{S}$
- In the limiter set : $\frac{\eta^{-1}}{M_0} P \mathbf{U}^{0,+} + \sum_{n=0}^{\infty} \eta^n \left(\partial_t \mathbf{U}^{n,+} + \dots + \frac{1}{M_0} P \mathbf{U}^{n+1,+} \right) = \mathbf{S}$

Construction by induction with continuous connection at the interface $(\mathbf{U}^{n,-}(t,0) = \mathbf{U}^{n,+}(t,0))$:

- $P\mathbf{U}^{0,+} = \mathbf{0}$
- Then, construction of $\mathbf{U}^{0,-}$, $(Id - P)\mathbf{U}^{0,+}$ and $P\mathbf{U}^{1,+}$.
- ... $\mathbf{U}^{n,-}$, $(Id - P)\mathbf{U}^{n,+}$ and $P\mathbf{U}^{n+1,+}$

If there is a boundary layer : presence of terms in $\underline{\mathbf{U}}^{n,\pm}(t, x, \frac{x}{\eta^\alpha})$.

⇒ Probably no boundary layer (error analysis needs to be done).

Conclusion and perspectives

- The first penalization does not work correctly. Besides the system considered is ill-posed.
- After a modification of the system, we experiment two penalty methods which give similar results.
- Penalizing both $N = 0$ and $M = M_0$ creates a boundary layer.
- These results need to be extended to a more complex model for edge simulation of ITER (3-space dimensions, energy equation...).

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