Non-diffusive large time behaviour for a degenerate viscous Hamilton-Jacobi equation

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Abstract

The convergence to non-diffusive self-similar solutions is investigated for non-negative solutions to the Cauchy problem $\partial_t u = \Delta_p u + |\nabla u|^q$ when the initial data converge to zero at infinity. Sufficient conditions on the exponents $p > 2$ and $q > 1$ are given that guarantee that the diffusion becomes negligible for large times and the L^∞ -norm of $u(t)$ converges to a positive value as $t \rightarrow \infty$.

1 Introduction

The quasilinear degenerate parabolic equation

$$(1.1) \quad \partial_t u = \Delta_p u + |\nabla u|^q, \quad (t, x) \in Q_\infty := (0, \infty) \times \mathbb{R}^N,$$

includes two competing mechanisms acting on the space variable x , a degenerate diffusion $\Delta_p u$ involving the p -Laplacian operator defined by

$$\Delta_p u := \operatorname{div} (|\nabla u|^{p-2} \nabla u), \quad p > 2,$$

and a source term $|\nabla u|^q$, $q > 1$, depending solely on the gradient of u . The aim of this work is to identify a range of the parameters p and q for which the large time behaviour of non-negative solutions to (1.1) is dominated by the source term. More precisely, we consider the Cauchy problem and supplement (1.1) with the initial condition

$$(1.2) \quad u(0) = u_0 \geq 0, \quad x \in \mathbb{R}^N.$$

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Throughout the paper, the initial condition u_0 is assumed to fulfill

$$(1.3) \quad u_0 \in \mathcal{C}_0(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N), \quad u_0 \geq 0, \quad u_0 \not\equiv 0,$$

where

$$\mathcal{C}_0(\mathbb{R}^N) := \left\{ w \in \mathcal{BC}(\mathbb{R}^N) : \lim_{R \rightarrow \infty} \sup_{\{|x| \geq R\}} \{|w(x)|\} = 0 \right\},$$

and $\mathcal{BC}(\mathbb{R}^N) := \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

For such an initial condition, the Cauchy problem (1.1), (1.2) has a unique non-negative (viscosity) solution $u \in \mathcal{BC}([0, \infty) \times \mathbb{R}^N)$ (see Proposition 2.1 below). Moreover, $t \mapsto \|u(t)\|_\infty$ is a non-increasing function and has a limit $M_\infty \in [0, \|u_0\|_\infty]$ as $t \rightarrow \infty$. Our main result is then the following:

Theorem 1.1 *Assume that $p > 2$ and $q \in (1, p)$. Consider a non-negative function u_0 satisfying (1.3) and let u be the corresponding (viscosity) solution to (1.1), (1.2). Assume further that*

$$(1.4) \quad M_\infty := \lim_{t \rightarrow \infty} \|u(t)\|_\infty > 0.$$

Then

$$(1.5) \quad \lim_{t \rightarrow \infty} \|u(t) - h_\infty(t)\|_\infty = 0,$$

where h_∞ is given by

$$(1.6) \quad h_\infty(t, x) := H_\infty\left(\frac{x}{t^{1/q}}\right) \quad \text{and} \quad H_\infty(x) := (M_\infty - \gamma_q |x|^{q/(q-1)})_+$$

for $(t, x) \in Q_\infty$ and $\gamma_q := (q-1)^{-q/(q-1)}$.

Here and below, $r_+ := \max\{r, 0\}$ denotes the positive part of the real number r .

The convergence (1.5) clearly indicates that the large time behaviour of non-negative solutions to (1.1), (1.2) fulfilling the condition (1.4) is governed by the gradient source term. Indeed, h_∞ is actually a self-similar solution to the Hamilton-Jacobi equation

$$(1.7) \quad \partial_t h = |\nabla h|^q, \quad (t, x) \in Q_\infty,$$

and an alternative formula for h_∞ reads

$$(1.8) \quad h_\infty(t, x) = \sup_{y \in \mathbb{R}^N} \left\{ M_\infty \mathbf{1}_{\{0\}}(y) - \gamma_q \frac{|x - y|^{q/(q-1)}}{t^{1/(q-1)}} \right\}$$

for $(t, x) \in [0, \infty) \times \mathbb{R}^N$, $\mathbf{1}_{\{0\}}$ denoting the indicator function of the singleton set $\{0\}$. The formula (1.8) is the well-known Hopf-Lax-Oleinik representation formula for viscosity solutions to (1.7) (see, e.g., [10, Chapter 3]) and h_∞ turns out to be the unique viscosity solution in $\mathcal{BUC}(Q_\infty)$ to (1.7) with the bounded and upper semicontinuous initial condition $h_\infty(0, x) = \mathbf{1}_{\{0\}}(x)$ for $x \in \mathbb{R}^N$ [24].

Remark 1.2 *The convergence (1.5) also holds true for the viscosity solution to the Hamilton-Jacobi equation (1.7) with a non-negative initial condition $u_0 \in \mathcal{C}_0(\mathbb{R}^N)$ but with $\|u_0\|_\infty$ instead of M_∞ in the formula (1.6) giving H_∞ . For (1.1), (1.2), the constant M_∞ takes into account that, though negligible for large times, the diffusion erodes the supremum of u during the time evolution.*

For $p = 2$, Theorem 1.1 is also valid and is proved in [7], the proof relying on a rescaling technique: The crucial step is then to identify the possible limits of the rescaled sequence and this is done by an extensive use of the Hopf-Lax-Oleinik representation formula. The proof we perform here is of a completely different nature and relies on the relaxed half-limits method introduced in [3]. A similar approach has been used in [22] and [23] to investigate the large time behaviour of solutions to first-order Hamilton-Jacobi equations $\partial_t w + H(x, \nabla w) = 0$ in Q_∞ . It has also been used in [20] to study the convergence to non-diffusive localized self-similar patterns for non-negative and compactly supported solutions to $\partial_t w - \Delta_p w + |\nabla w|^q = 0$ in Q_∞ when $p > 2$ and $q \in (1, p - 1)$.

In order to apply Theorem 1.1, one should check whether there are non-negative solutions to (1.1), (1.2) for which (1.4) holds true. The next result provides sufficient conditions for (1.4) to be fulfilled.

Theorem 1.3 *Assume that $p > 2$ and $q > 1$. Consider a non-negative function u_0 satisfying (1.3) and let u be the corresponding solution to (1.1), (1.2). Introducing*

$$(1.9) \quad q_\star := p - \frac{N}{N+1},$$

then u fulfills (1.4) if

- (a) either $q \in (1, q_\star]$,*
- (b) or $q \in (q_\star, p)$, $u_0 \in W^{2,\infty}(\mathbb{R}^N)$, and*

$$(1.10) \quad \|u_0\|_\infty > \kappa_0 \left| \inf_{y \in \mathbb{R}^N} \{\Delta_p u_0(y)\} \right|^{(p-q)/q}.$$

for some $\kappa_0 > 0$ which depends only on N , p , and q .

A similar result is already available for $p = 2$ and has been established in [7, 12]. It is also shown in [7, 12, 18] that, still for $p = 2$, $\|u(t)\|_\infty \rightarrow 0$ as $t \rightarrow \infty$ if either $q \geq p$ or $q \in (q_\star, p)$ and a suitable norm of u_0 is small enough. A similar result is likely to be true also for $p > 2$, so that different large time behaviours are only to be found in the intermediate range $q \in (q_\star, p)$.

The proof of Theorem 1.3 for $q \in (p - 1, p)$ and $p > 2$ borrows some steps from the case $p = 2$. However, it relies on semiconvexity estimates for solutions to (1.1), (1.2) which seem to be new for $p > 2$ and $q \in (1, p)$ and are stated now.

Proposition 1.4 *Assume that $p > 2$ and $q \in (1, p]$. Let u be the viscosity solution to (1.1), (1.2) with initial condition $u_0 \in \mathcal{BUC}(\mathbb{R}^N)$ (that is, $u_0 \in \mathcal{BC}(\mathbb{R}^N)$ and is uniformly continuous in $\mathbb{R}^N$). Then $\nabla u(t)$ belongs to $L^\infty(\mathbb{R}^N)$ for each $t > 0$ and there is $\kappa_1 > 0$ depending only on N , p , and q such that*

$$(1.11) \quad \Delta_p u(t, x) \geq -\kappa_1 \|u(s)\|_\infty^{(p-q)/q} (t-s)^{-p/q}, \quad t > s \geq 0,$$

in the sense of distributions. In addition, if $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, there holds

$$(1.12) \quad \Delta_p u(t, x) \geq -\frac{N(p-1)}{q(q-1)} \frac{\|\nabla u_0\|_\infty^{p-q}}{t}$$

for $t > 0$ in the sense of distributions.

The proof of Proposition 1.4 relies on the comparison principle combined with a gradient estimate established in [6].

Similar semiconvexity estimates for solutions to (1.1), (1.2) have already been obtained in [14] and [21, Lemma 5.1] for $p = q = 2$, in [7, Proposition 3.2] for $p = 2$ and $q \in (1, 2]$, and in [9, Theorem 1] for $p = q > 2$. We extend these results to the range $p > 2$ and $q \in (1, p]$. As we shall see below, the estimate (1.11) plays an important role in the proof of Theorem 1.3 and is also helpful to construct a subsolution in the proof of Theorem 1.1.

Let us finally emphasize that the validity of Proposition 1.4 is not restricted to non-negative solutions and that the solutions to the Hamilton-Jacobi equation (1.7) also enjoy the semiconvexity estimates (1.11) and (1.12). These two estimates thus stem from the reaction term $|\nabla u|^q$ and not from the diffusion.

In the next section, we recall the well-posedness of (1.1), (1.2) in $\mathcal{BUC}(\mathbb{R}^N)$, as well as some properties of the solutions established in [6]. We also show the finite speed of propagation of the support for non-negative compactly supported initial data. Section 3 is devoted to the proof of the semiconvexity estimates (Proposition 1.4) and Section 4 to that of Theorem 1.1. Theorem 1.3 is shown in the last section, its proof combining arguments of [7, 12, 19] used to established analogous results when $p = 2$.

Throughout the paper, C and C_i , $i \geq 1$, denote positive constants depending only on p , q , and N . Dependence upon additional parameters will be indicated explicitly. Also, $\mathcal{M}_N(\mathbb{R})$ denotes the space of real-valued $N \times N$ matrices and $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, $1 \leq i, j \leq N$. Given a matrix $A = (a_{ij}) \in \mathcal{M}_N(\mathbb{R})$, $tr(A)$ denotes its trace and is given by $tr(A) := \sum a_{ii}$.

2 Preliminary results

Let us first recall the well-posedness (in the framework of viscosity solutions) of (1.1), (1.2), together with some properties of the solutions established in [6].

Proposition 2.1 *Consider a non-negative initial condition $u_0 \in \mathcal{BUC}(\mathbb{R}^N)$. There is a unique non-negative viscosity solution $u \in \mathcal{BC}([0, \infty) \times \mathbb{R}^N)$ to (1.1), (1.2) such that*

$$(2.1) \quad 0 \leq u(t, x) \leq \|u_0\|_\infty, \quad (t, x) \in Q_\infty,$$

$$(2.2) \quad \|\nabla u(t)\|_\infty \leq \min \{C_1 \|u(s)\|_\infty^{1/q} (t-s)^{-1/q}, \|\nabla u(s)\|_\infty\},$$

and

$$(2.3) \quad \int_{\mathbb{R}^N} (u(t, x) - u(s, x)) \vartheta(x) dx + \int_s^t \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \cdot \nabla \vartheta - |\nabla u|^q \vartheta) dx d\tau = 0$$

for $t > s \geq 0$ and $\vartheta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. In addition, $t \mapsto \|u(t)\|_\infty$ is a non-increasing function.

Proof. We put $\tilde{u}_0 := \|u_0\|_\infty - u_0$. As $\tilde{u}_0$ is a non-negative function in $\mathcal{BUC}(\mathbb{R}^N)$, it follows from [6, Theorem 1.1] that there is a unique non-negative viscosity solution $\tilde{u}$ to

$$(2.4) \quad \partial_t \tilde{u} - \Delta_p \tilde{u} + |\nabla \tilde{u}|^q = 0, \quad (t, x) \in Q_\infty,$$

with initial condition $\tilde{u}(0, x) = \tilde{u}_0(x)$ for $x \in \mathbb{R}^N$. It also satisfies $0 \leq \tilde{u}(t, x) \leq \|u_0\|_\infty$ and

$$\int_{\mathbb{R}^N} (\tilde{u}(t, x) - \tilde{u}(s, x)) \vartheta(x) dx + \int_s^t \int_{\mathbb{R}^N} (|\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \vartheta + |\nabla \tilde{u}|^q \vartheta) dx d\tau = 0$$

for $t > s \geq 0$, $x \in \mathbb{R}^N$, and $\vartheta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. In addition, $\nabla \tilde{u}(t)$ belongs to $L^\infty(\mathbb{R}^N)$ for each $t > 0$ and

$$\|\nabla \tilde{u}(t)\|_\infty \leq C_1 \|\tilde{u}_0\|_\infty^{1/q} t^{-1/q}$$

by [6, Lemma 4.1]. Setting $u := \|u_0\|_\infty - \tilde{u}$, we readily deduce from the properties of $\tilde{u}$ that u is a non-negative viscosity solution to (1.1), (1.2) satisfying (2.1) and (2.3). Also, $\nabla u(t)$ belongs to $L^\infty(\mathbb{R}^N)$ for each $t > 0$. The uniqueness and the time monotonicity of $\|u\|_\infty$ then both follow from the comparison principle, see [8] or [11, Theorem 2.1]. Finally, given $s \geq 0$, $(t, x) \mapsto \|u(s)\|_\infty - u(t + s, x)$ is the unique non-negative viscosity solution to the Cauchy problem (2.4) with initial condition $x \mapsto \|u(s)\|_\infty - u(s, x)$ and we infer from [6, Lemma 4.1] that

$$\|\nabla u(t + s)\|_\infty \leq C_1 \|\|u(s)\|_\infty - u(s)\|_\infty^{1/q} t^{-1/q} \leq C_1 \|u(s)\|_\infty^{1/q} t^{-1/q}$$

for $t > 0$, whence (2.2). $\square$

We next turn to the propagation of the support of non-negative solutions to (1.1), (1.2) with non-negative compactly supported initial data.

Proposition 2.2 *Consider a non-negative solution u to (1.1), (1.2) with an initial condition u_0 satisfying (1.3). Assume further that u_0 is compactly supported in a ball $B(0, R_0)$ of $\mathbb{R}^N$ for some $R_0 > 0$. Then $u(t)$ is compactly supported for each $t \geq 0$.*

Proof. We argue by comparison with travelling wave solutions. By [13, Application 9.4], there is a travelling wave solution w to the convection-diffusion equation

$$(2.5) \quad \partial_t w - \partial_1^2 (w^{p-1}) + \partial_1 (w^q) = 0, \quad (t, x_1) \in (0, \infty) \times \mathbb{R},$$

with wave speed unity. It is given by $w(t, x_1) = f(x_1 - t)$ for $(t, x_1) \in (0, \infty) \times \mathbb{R}$, the function f being implicitly defined by

$$(p-1) \int_0^{f(y)} \frac{z^{p-3}}{1-z^{q-1}} dz = (-y)_+, \quad y \in \mathbb{R}.$$

In particular, f satisfies $f(y) = 0$ if $y > 0$ and $f(y) \rightarrow 1$ as $y \rightarrow -\infty$. Introducing

$$F(y) := \int_y^\infty f(z) dz, \quad y \in \mathbb{R},$$

the properties of f ensure that F is a decreasing function on $(-\infty, 0)$ with $F(y) = 0$ if $y > 0$, $F(y) \leq |y|$ if $y < 0$, and $F(y) \rightarrow \infty$ as $y \rightarrow -\infty$. There is therefore a unique $\mu \in (-\infty, 0)$ such that $F(R_0 + \mu) = \|u_0\|_\infty$. In addition, it readily follows from (2.5) and the invariance by translation of (1.1) that $W_\mu(t, x) := F(x_1 + \mu - t)$ is a travelling wave solution to (1.1). Now, u and W_μ are both solutions to (1.1) in $(0, \infty) \times H_+$, the half-space H_+ being defined by $H_+ := \{x \in \mathbb{R}^N : x_1 > R_0\}$. Owing to the monotonicity of F , the bound $0 \leq f \leq 1$, and (2.1), we have also

$$u_0(x) - W_\mu(0, y) = 0 - W_\mu(0, y) \leq W_\mu(0, x) - W_\mu(0, y) \leq |x - y|$$

for $x \in H_+$ and $y \in H_+$,

$$\begin{aligned} u(t, x) - W_\mu(t, y) &\leq \|u_0\|_\infty - W_\mu(t, y) \\ &\leq F(R_0 + \mu - t) - W_\mu(t, y) = W_\mu(t, x) - W_\mu(t, y) \leq |x - y| \end{aligned}$$

for $t > 0$, $x \in \partial H_+$, $y \in H_+$, and

$$u(t, x) - W_\mu(t, y) \leq \|u_0\|_\infty - F(R_0 + \mu - t) \leq 0$$

for $t > 0$, $x \in H_+$, $y \in \partial H_+$. We are then in a position to use the comparison principle stated in [11, Theorem 2.1] to conclude that $u(t, x) \leq W_\mu(t, x)$ for $(t, x) \in (0, \infty) \times H_+$. Consequently, $u(t, x) \leq F(x_1 + \mu - t) = 0$ if $t \geq 0$ and $x_1 > \max\{R_0, t - \mu\}$, and the rotational invariance of (1.1) allows us to conclude that $u(t, x) = 0$ for $t \geq 0$ and $|x| > \max\{R_0, t - \mu\}$. $\square$

We finally recall the convergence to self-similar solutions for non-negative and compactly supported solutions to the p -Laplacian equation [17]

$$(2.6) \quad \partial_t \varphi = \Delta_p \varphi, \quad (t, x) \in Q_\infty.$$

Proposition 2.3 *Let φ_0 be a non-negative and compactly supported function in $L^1(\mathbb{R}^N)$ and φ denote the unique weak solution to (2.6) with initial condition φ_0 . Then*

$$(2.7) \quad \lim_{t \rightarrow \infty} t^{(N(r-1))/(r(N(p-2)+p))} \|\varphi(t) - \mathcal{B}_{\|\varphi_0\|_1}(t)\|_r = 0 \quad \text{for } r \in [1, \infty],$$

where $\mathcal{B}_L$ denotes the Barenblatt solution to (2.6) given by

$$\begin{aligned} \mathcal{B}_L(t, x) &:= t^{-N/(N(p-2)+p)} b_L(xt^{-1/(N(p-2)+p)}), \\ b_L(x) &:= (C_2 L^{(p(p-2))/((p-1)(N(p-2)+p))} - C_3 |x|^{p/(p-1)})_+^{(p-1)/(p-2)} \end{aligned}$$

for $(t, x) \in Q_\infty$ and $L > 0$.

The convergence (2.7) is proved in [17, Theorem 2] for $r = \infty$. As φ_0 is compactly supported, so is $\varphi(t)$ for each $t > 0$ and the support of $\varphi(t)$ is included in $B(0, C_4(\varphi_0) t^{1/(N(p-2)+p)})$ for $t \geq 1$ [17, Proposition 2.2]. Combining this property with [17, Theorem 2] readily provide the convergence (2.7) for all $r \in [1, \infty)$.

3 Semiconvexity

In this section, we prove the semiconvexity estimates (1.11) and (1.12). To this end, we would like to derive an equation for $\Delta_p u$ to which we could apply the comparison principle. The poor regularity of u however does not allow to perform directly such a computation and an approximation procedure is needed. As a first step, we report the following result:

Lemma 3.1 *Let a and b be two non-negative function in $C^\infty([0, \infty))$ satisfying*

$$(3.1) \quad a(r) > 0, \quad a'(r) > 0, \quad a'(r) b'(r) - a(r) b''(r) > 0,$$

$$(3.2) \quad c(r) := 2 \left(\frac{b'}{a} \right) (r) + \frac{4r (a b'' - a' b')(r)}{a^2(r) + 2r a(r) a'(r)} \geq 0.$$

Consider a classical solution v to

$$(3.3) \quad \partial_t v - \operatorname{div} (a(|\nabla v|^2) \nabla v) = b(|\nabla v|^2), \quad (t, x) \in Q_\infty,$$

and put

$$w := \operatorname{div} (a(|\nabla v|^2) \nabla v) \quad \text{and} \quad z_i := a(|\nabla v|^2) \partial_i v$$

for $i \in \{1, \dots, N\}$. Then

$$(3.4) \quad \partial_t w - \mathcal{L}w - \mathcal{V} \cdot \nabla w - \frac{c(|\nabla v|^2)}{N} w^2 \geq 0 \quad \text{in } Q_\infty,$$

where

$$\begin{aligned} \mathcal{L}w &:= \sum_{i,j} \partial_i (a(|\nabla v|^2) E_{ij} \partial_j w), \quad \mathcal{V} := 2 b'(|\nabla v|^2) \nabla v, \\ E_{ij} &:= \delta_{ij} + 2 \frac{a'}{a} (|\nabla v|^2) \partial_i v \partial_j v, \quad 1 \leq i, j \leq N. \end{aligned}$$

The proof of Lemma 3.1 borrows some steps from the proof of [9, Theorem 1] for $p = q > 2$ but requires additional arguments to handle the term coming from the fact that $q \neq p$. In particular, we recall the following elementary result which will be helpful to estimate this term.

Lemma 3.2 *Let A and B be two symmetric matrices in $\mathcal{M}_N(\mathbb{R})$ and put $M := ABA$. Then M is a symmetric matrix in $\mathcal{M}_N(\mathbb{R})$ and*

$$(3.5) \quad |MX|^2 \leq \text{tr}(M^2) |X|^2 \quad \text{for } X \in \mathbb{R}^N.$$

Proof of Lemma 3.1. We first note that

$$(3.6) \quad \partial_j z_i = a(|\nabla v|^2) \sum_k E_{ik} \partial_k \partial_j v,$$

$$(3.7) \quad \partial_t z_i = a(|\nabla v|^2) \sum_k E_{ik} \partial_k \partial_t v,$$

for $1 \leq i, j \leq N$. According to the definition of w , we infer from (3.3), (3.6), and (3.7) that

$$\begin{aligned} \partial_t w &= \sum_{i,k} \partial_i (a(|\nabla v|^2) E_{ik} \partial_k \partial_t v) \\ &= \sum_{i,k} \partial_i (a(|\nabla v|^2) E_{ik} \partial_k (w + b(|\nabla v|^2))) \\ &= \mathcal{L}w + 2 \sum_{i,k} \partial_i \left((ab') (|\nabla v|^2) E_{ik} \sum_j \partial_j v \partial_j \partial_k v \right) \\ &= \mathcal{L}w + 2 \sum_{i,j} \partial_i \left((ab') (|\nabla v|^2) \partial_j v \sum_k E_{ik} \partial_k \partial_j v \right) \\ &= \mathcal{L}w + 2 \sum_{i,j} \partial_i \left(\left(\frac{b'}{a} \right) (|\nabla v|^2) z_j \partial_j z_i \right) \\ &= \mathcal{L}w + 4 \sum_{i,j} \left(\frac{b'}{a} \right)' (|\nabla v|^2) \sum_k \partial_k v \partial_k \partial_i v z_j \partial_j z_i \\ &\quad + 2 \sum_{i,j} \left(\frac{b'}{a} \right) (|\nabla v|^2) \partial_i z_j \partial_j z_i + 2 \sum_{i,j} \left(\frac{b'}{a} \right) (|\nabla v|^2) z_j \partial_j \partial_i z_i. \end{aligned}$$

Since $w = \sum \partial_i z_i$, the last term of the right-hand side of the above inequality is equal to $\mathcal{V} \cdot \nabla w$ and

$$(3.8) \quad \begin{aligned} \partial_t w &= \mathcal{L}w + \mathcal{V} \cdot \nabla w + 4 \left[a \left(\frac{b'}{a} \right)' \right] (|\nabla v|^2) \sum_{i,j,k} \partial_j v \partial_k v \partial_k \partial_i v \partial_j z_i \\ &\quad + 2 \left(\frac{b'}{a} \right) (|\nabla v|^2) \sum_{i,j} \partial_i z_j \partial_j z_i. \end{aligned}$$

On the one hand, introducing the matrix $\mathcal{E} := (E_{ij})$ and the Hessian matrix $D^2v = (\partial_i \partial_j v)$ of v , we infer from (3.6) that

$$\begin{aligned}
\sum_{i,j} \partial_i z_j \partial_j z_i &= a^2 (|\nabla v|^2) \sum_{i,j,k,l} E_{ik} \partial_k \partial_j v E_{jl} \partial_l \partial_i v \\
&= a^2 (|\nabla v|^2) \sum_{i,j} (\mathcal{E} D^2 v)_{ij} (\mathcal{E} D^2 v)_{ji} \\
(3.9) \quad \sum_{i,j} \partial_i z_j \partial_j z_i &= a^2 (|\nabla v|^2) \operatorname{tr} \left((\mathcal{E} D^2 v)^2 \right).
\end{aligned}$$

On the other hand, using once more (3.6), we obtain

$$\begin{aligned}
\sum_{i,j,k} \partial_j v \partial_k v \partial_k \partial_i v \partial_j z_i &= a (|\nabla v|^2) \sum_{i,j,k,l} \partial_j v \partial_k v \partial_k \partial_i v E_{il} \partial_l \partial_j v \\
&= a (|\nabla v|^2) \sum_{i,l} \left(\sum_k \partial_i \partial_k v \partial_k v \right) E_{il} \left(\sum_j \partial_l \partial_j v \partial_j v \right) \\
(3.10) \quad \sum_{i,j,k} \partial_j v \partial_k v \partial_k \partial_i v \partial_j z_i &= a (|\nabla v|^2) \langle D^2 v \nabla v, (\mathcal{E} D^2 v) \nabla v \rangle.
\end{aligned}$$

Inserting (3.9) and (3.10) in (3.8), we end up with

$$\begin{aligned}
(3.11) \quad \partial_t w &= \mathcal{L}w + \mathcal{V} \cdot \nabla w + 2 (ab') (|\nabla v|^2) \operatorname{tr} \left((\mathcal{E} D^2 v)^2 \right) \\
&\quad + 4 (a b'' - a' b') (|\nabla v|^2) \langle D^2 v \nabla v, (\mathcal{E} D^2 v) \nabla v \rangle.
\end{aligned}$$

We next observe that

$$(3.12) \quad \mathcal{E} \nabla v = \left(1 + 2 |\nabla v|^2 \left(\frac{a'}{a} \right) (|\nabla v|^2) \right) \nabla v$$

and that, for $X \in \mathbb{R}^N$,

$$\langle \mathcal{E} X, X \rangle = |X|^2 + 2 \left(\frac{a'}{a} \right) (|\nabla v|^2) \langle X, \nabla v \rangle^2 \geq |X|^2$$

as a and a' are both positive by (3.1). Consequently, $\mathcal{E}$ is a positive definite symmetric matrix in $\mathcal{M}_N(\mathbb{R})$ and there exists a positive definite matrix $\mathcal{E}_{1/2}$ such that $\mathcal{E}_{1/2}^2 = \mathcal{E}$. We then infer from the definition of $\mathcal{E}_{1/2}$, (3.12), and Lemma 3.2 (with $A = \mathcal{E}_{1/2}$, $B = D^2 v$ and

$X = \mathcal{E}_{1/2}^{-1} \nabla v$) that

$$\begin{aligned}
\langle D^2 v \nabla v, (\mathcal{E} D^2 v) \nabla v \rangle &= |(\mathcal{E}_{1/2} D^2 v) \nabla v|^2 \\
&= \left| \left((\mathcal{E}_{1/2} D^2 v \mathcal{E}_{1/2}) \mathcal{E}_{1/2}^{-1} \right) \nabla v \right|^2 \\
&\leq \operatorname{tr} (\mathcal{E}_{1/2} D^2 v \mathcal{E}_{1/2} \mathcal{E}_{1/2} D^2 v \mathcal{E}_{1/2}) \langle \mathcal{E}_{1/2}^{-1} \nabla v, \mathcal{E}_{1/2}^{-1} \nabla v \rangle \\
&\leq \operatorname{tr} \left((\mathcal{E} D^2 v)^2 \right) \langle \nabla v, \mathcal{E}^{-1} \nabla v \rangle \\
&\leq \operatorname{tr} \left((\mathcal{E} D^2 v)^2 \right) |\nabla v|^2 \left(1 + 2 |\nabla v|^2 \left(\frac{a'}{a} \right) (|\nabla v|^2) \right)^{-1}.
\end{aligned}$$

Owing to the non-positivity (3.1) of $a b'' - a' b'$, we deduce from (3.11) and the above inequality that

$$\partial_t w \geq \mathcal{L}w + \mathcal{V} \cdot \nabla w + (a^2 c) (|\nabla v|^2) \operatorname{tr} \left((\mathcal{E} D^2 v)^2 \right),$$

the function c being defined in (3.2). We finally use the inequality

$$\operatorname{tr} (A^2) \geq \frac{1}{N} \operatorname{tr} (A)^2, \quad A \in \mathcal{M}_N(\mathbb{R}),$$

the identity

$$w = \sum_i \partial_i z_i = a (|\nabla v|^2) \operatorname{tr} (\mathcal{E} D^2 v),$$

and the non-negativity (3.2) of c to conclude that

$$\begin{aligned}
\partial_t w &\geq \mathcal{L}w + \mathcal{V} \cdot \nabla w + \frac{1}{N} (a^2 c) (|\nabla v|^2) \operatorname{tr} (\mathcal{E} D^2 v)^2 \\
&\geq \mathcal{L}w + \mathcal{V} \cdot \nabla w + \frac{c (|\nabla v|^2)}{N} w^2,
\end{aligned}$$

and complete the proof. $\square$

Proof of Proposition 1.4. To be able to use Lemma 3.1, we shall first construct a suitable approximation of (1.1), (1.2). Such a construction has already been performed in [6] for similar purposes and we recall it now. Given u_0 satisfying (1.3), there is a sequence of functions $(u_{0,k})_{k \geq 1}$ such that, for each integer $k \geq 1$, $u_{0,k} \in \mathcal{BC}^\infty(\mathbb{R}^N)$, $u_0 \leq u_{0,k+1} \leq u_{0,k}$, and $(u_{0,k}, \nabla u_{0,k})_k$ converge towards $(u_0, \nabla u_0)$ uniformly on every compact subset of $\mathbb{R}^N$ as $k \rightarrow \infty$. Next, for $\varepsilon \in (0, 1)$ and $r \geq 0$, we set

$$a_\varepsilon(r) := (r + \varepsilon^2)^{(p-2)/2} \quad \text{and} \quad b_\varepsilon(r) := (r + \varepsilon^2)^{q/2} - \varepsilon^q.$$

Then the Cauchy problem

$$(3.13) \quad \partial_t u_{k,\varepsilon} = \operatorname{div} (a_\varepsilon (|\nabla u_{k,\varepsilon}|^2) \nabla u_{k,\varepsilon}) + b_\varepsilon (|\nabla u_{k,\varepsilon}|^2), \quad (t, x) \in Q_\infty,$$

$$(3.14) \quad u_{k,\varepsilon}(0) = u_{0,k} + \varepsilon^\nu, \quad x \in \mathbb{R}^N,$$

has a unique classical solution $u_{k,\varepsilon}$, the parameter $\nu > 0$ depending p , q , and N and being appropriately chosen. Furthermore,

$$(3.15) \quad \|\nabla u_{k,\varepsilon}(t)\|_\infty \leq \|\nabla u_{0,k}\|_\infty, \quad t \geq 0,$$

$$(3.16) \quad \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} u_{k,\varepsilon}(t, x) = u(t, x),$$

the latter convergence being uniform on every compact subset of $[0, \infty) \times \mathbb{R}^N$, see [6, Section 3] (after performing the same change of unknown function as in the proof of Proposition 2.1).

Introducing

$$c_\varepsilon(r) = 2 \left(\frac{b'_\varepsilon}{a_\varepsilon} \right) (r) + \frac{4r (a_\varepsilon b''_\varepsilon - a'_\varepsilon b'_\varepsilon)(r)}{a_\varepsilon^2(r) + 2r a_\varepsilon(r) a'_\varepsilon(r)}, \quad r \geq 0,$$

let us check that a_ε and b_ε fulfill the conditions (3.1) and (3.2). Clearly, $a_\varepsilon > 0$ and $a'_\varepsilon > 0$ as $p > 2$. Next, since $1 < q \leq p$,

$$\begin{aligned} (a'_\varepsilon b'_\varepsilon - a_\varepsilon b''_\varepsilon)(r) &= \frac{q(p-q)}{4} (r + \varepsilon^2)^{(p+q-6)/2} \geq 0, \\ c_\varepsilon(r) &= q \frac{r(q-1) + \varepsilon^2}{r(p-1) + \varepsilon^2} (r + \varepsilon^2)^{(q-p)/2} \geq 0. \end{aligned}$$

We may then apply Lemma 3.1 to deduce that $w_{k,\varepsilon} := \operatorname{div} (a_\varepsilon (|\nabla u_{k,\varepsilon}|^2) \nabla u_{k,\varepsilon})$ satisfies

$$\partial_t w_{k,\varepsilon} - \mathcal{L}_{k,\varepsilon} w_{k,\varepsilon} - \mathcal{V}_{k,\varepsilon} \cdot \nabla w_{k,\varepsilon} - \frac{c_\varepsilon (|\nabla u_{k,\varepsilon}|^2)}{N} w_{k,\varepsilon}^2 \geq 0$$

in Q_∞ . Observe next that the condition $1 < q \leq p$ implies that c_ε is a non-increasing function. It then follows from (3.15) that $c_\varepsilon (|\nabla u_{k,\varepsilon}|^2) \geq c_\varepsilon (\|\nabla u_{0,k}\|_\infty^2)$ and we end up with

$$(3.17) \quad \partial_t w_{k,\varepsilon} - \mathcal{L}_{k,\varepsilon} w_{k,\varepsilon} - \mathcal{V}_{k,\varepsilon} \cdot \nabla w_{k,\varepsilon} - \frac{c_\varepsilon (\|\nabla u_{0,k}\|_\infty^2)}{N} w_{k,\varepsilon}^2 \geq 0$$

in Q_∞ . Clearly, $t \mapsto -N / (c_\varepsilon (\|\nabla u_{0,k}\|_\infty^2) t)$ is a subsolution to (3.17) and the comparison principle warrants that

$$(3.18) \quad w_{k,\varepsilon}(t, x) \geq -\frac{N}{c_\varepsilon (\|\nabla u_{0,k}\|_\infty^2) t}, \quad (t, x) \in Q_\infty.$$

Letting $\varepsilon \rightarrow 0$ and $k \rightarrow \infty$ in the previous inequality with the help of (3.16) gives (1.12).

Next, since (1.1) is autonomous, we infer from (2.2) (with $s = 0$) and (1.12) that

$$\begin{aligned} \Delta_p u(t, x) &\geq -\frac{2N(p-1)}{q(q-1)} \frac{\|\nabla u(t/2)\|_\infty^{p-q}}{t} \\ &\geq -\frac{2^{p/q} N(p-1)}{q(q-1)} C_1^{p-q} \|u_0\|_\infty^{(p-q)/q} t^{-p/q}, \end{aligned}$$

whence (1.11) for $s = 0$. To prove the general case $s \in (0, t)$, we use again the fact that (1.1) is autonomous. $\square$

We have a similar result when u_0 is more regular.

Corollary 3.3 *Assume that $p > 2$ and $q \in (1, p]$. Let u be the solution to (1.1), (1.2) with an initial condition u_0 satisfying $u_0 \in W^{2,\infty}(\mathbb{R}^N)$ in addition to (1.3). Then*

$$(3.19) \quad \Delta_p u(t, x) \geq - \left| \inf_{y \in \mathbb{R}^N} \Delta_p u_0(y) \right|$$

in the sense of distributions.

Proof. Keeping the notations introduced in the proof of Proposition 1.4, we readily infer from (3.17) and the comparison principle that

$$(3.20) \quad w_{k,\varepsilon}(t, x) \geq - \left| \inf_{y \in \mathbb{R}^N} \Delta_p u_{0,k}(y) \right|, \quad (t, x) \in Q_\infty.$$

Owing to the regularity of u_0 , it is possible to construct the sequence $(u_{0,k})_k$ such that it satisfies

$$\lim_{k \rightarrow \infty} \inf_{y \in \mathbb{R}^N} \Delta_p u_{0,k}(y) = \inf_{y \in \mathbb{R}^N} \Delta_p u_0(y).$$

We may then pass to the limit first as $\varepsilon \rightarrow 0$ and then as $k \rightarrow \infty$ in (3.20) and use (3.16) and the above convergence to complete the proof. $\square$

Another useful consequence of the semiconvexity estimates derived in Proposition 1.4 is that the solution u to (1.1), (1.2) is a supersolution to a first-order Hamilton-Jacobi equation.

Corollary 3.4 *Consider an initial condition u_0 satisfying (1.3). Setting $F(t, \xi_0, \xi) := \xi_0 - |\xi|^q + \kappa_1 \|u_0\|_\infty^{(p-q)/q} t^{-p/q}$ for $t \in (0, \infty)$, $\xi_0 \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$ (recall that κ_1 is defined in (1.11)), the solution u to (1.1), (1.2) is a supersolution to $F(t, \partial_t w, \nabla w) = 0$ in Q_∞ .*

Proof. We still use the notations introduced in the proof of Proposition 1.4. As $w_{k,\varepsilon} = \operatorname{div} (a_\varepsilon (|\nabla u_{k,\varepsilon}|^2) \nabla u_{k,\varepsilon})$, we infer from (3.13) and (3.18) that

$$\partial_t u_{k,\varepsilon} - b_\varepsilon (|\nabla u_{k,\varepsilon}|^2) \geq - \frac{N}{c_\varepsilon (\|\nabla u_{0,k}\|_\infty^2) t}$$

in Q_∞ . We then use (3.16) and the stability of viscosity solutions [1, 2, 8] to pass to the limit as $\varepsilon \rightarrow 0$ and $k \rightarrow \infty$ in the previous inequality and conclude that u is a supersolution to

$$\partial_t w - |\nabla w|^q + \frac{N(p-1)}{q(q-1)} \frac{\|\nabla u_0\|_\infty^{p-q}}{t} = 0 \quad \text{in } Q_\infty.$$

Now, fix $T \geq 0$. As (1.1) is an autonomous equation, the function $(t, x) \mapsto u(t+T, x)$ is the solution to (1.1) with initial condition $u(T)$ and the above analysis allows us to conclude that u is a supersolution to

$$\partial_t w - |\nabla w|^q + \frac{N(p-1)}{q(q-1)} \frac{\|\nabla u(T)\|_\infty^{p-q}}{t-T} = 0 \quad \text{in } (T, \infty) \times \mathbb{R}^N.$$

We then use (2.2) (with $T = t/2$) to complete the proof. $\square$

4 Convergence to self-similarity

We change the variables and the unknown function so that the convergence (1.5) is transformed to the convergence towards a steady state. More precisely, we introduce the self-similar (or scaling) variables

$$\tau = \frac{1}{q} \log(1+t), \quad y = \frac{x}{(1+t)^{1/q}},$$

and the new unknown function v defined by

$$(4.1) \quad u(t, x) = v\left(\frac{\log(1+t)}{q}, \frac{x}{(1+t)^{1/q}}\right), \quad (t, x) \in [0, \infty) \times \mathbb{R}^N.$$

Equivalently, $v(\tau, y) = u(e^{q\tau} - 1, ye^\tau)$ for $(\tau, y) \in [0, \infty) \times \mathbb{R}^N$ and it follows from (1.1), (1.2) that v solves

$$(4.2) \quad \partial_\tau v = y \cdot \nabla v + q |\nabla v|^q + q e^{-(p-q)\tau} \Delta_p v, \quad (\tau, y) \in Q_\infty,$$

$$(4.3) \quad v(0) = u_0, \quad y \in \mathbb{R}^N.$$

We also infer from (2.1) and (2.2) that there is a positive constant $C_5(u_0)$ depending only on N, p, q , and u_0 such that

$$(4.4) \quad \|v(\tau)\|_\infty + \|\nabla v(\tau)\|_\infty \leq C_5(u_0), \quad \tau \geq 0,$$

while (1.4) reads

$$(4.5) \quad \lim_{\tau \rightarrow \infty} \|v(\tau)\|_\infty = M_\infty > 0.$$

Formally, since $p > q$, the diffusion term vanishes in the large time limit and we expect the large time behaviour of the solution v to (4.2), (4.3) to look like that of the solutions to the first-order Hamilton-Jacobi equation

$$(4.6) \quad \partial_\tau w - y \cdot \nabla w - q |\nabla w|^q = 0 \quad \text{in } Q_\infty.$$

Now, to investigate the large time behaviour of first-order Hamilton-Jacobi equations, an efficient approach has been developed in [22, 23] which relies on the relaxed half-limits method introduced in [3]. More precisely, for $(\tau, y) \in Q_\infty$, we define the relaxed half-limits v_* and v^* by

$$(4.7) \quad v_*(y) := \liminf_{(\sigma, z, \lambda) \rightarrow (\tau, y, \infty)} v(\sigma + \lambda, z) \quad \text{and} \quad v^*(y) := \limsup_{(\sigma, z, \lambda) \rightarrow (\tau, y, \infty)} v(\sigma + \lambda, z).$$

These relaxed half-limits are well-defined thanks to (4.4) and we first note that the right-hand sides of the above definitions indeed do not depend on $\tau > 0$. In addition,

$$(4.8) \quad 0 \leq v_*(x) \leq v^*(x) \leq M_\infty \quad \text{for } y \in \mathbb{R}^N$$

by (4.5), while (4.4) and the Rademacher theorem ensure that v_* and v^* both belong to $W^{1,\infty}(\mathbb{R}^N)$. Finally, by [2, Théorème 4.1] applied to equation (4.2), v^* and v_* are viscosity subsolution and supersolution, respectively, to the Hamilton-Jacobi equation

$$(4.9) \quad \mathcal{H}(y, \nabla w) := -y \cdot \nabla w - q |\nabla w|^q = 0 \quad \text{in } \mathbb{R}^N.$$

We now aim at showing that v^* and v_* coincide. However, the equation (4.9) has infinitely many solutions as $y \mapsto (c - \gamma_q |y|^{q/(q-1)})_+$ solves (4.9) for any $c > 0$. The information obtained so far on v_* and v^* are thus not sufficient and are supplemented by the next two results.

Lemma 4.1 *Given $\varepsilon \in (0, 1)$, there is $R_\varepsilon > 1/\varepsilon$ such that*

$$(4.10) \quad v(\tau, y) \leq \varepsilon \quad \text{for } \tau \geq 0 \quad \text{and } y \in \mathbb{R}^N \setminus B(0, R_\varepsilon),$$

and $0 \leq v_*(y) \leq v^*(y) \leq \varepsilon$ for $y \in \mathbb{R}^N \setminus B(0, R_\varepsilon)$.

In other words, $v(\tau)$ belongs to $\mathcal{C}_0(\mathbb{R}^N)$ for each $\tau \geq 0$ in a way which is uniform with respect to $\tau \geq 0$.

Proof. We first construct a supersolution to (4.2) in $(0, \infty) \times (\mathbb{R}^N \setminus B(0, R))$ for R large enough. To this end, consider $R \geq R_c := 1 + (q(2\|u_0\|_\infty)^{q-1} + 3pq(2\|u_0\|_\infty)^{p-2})^{1/q}$ and put $\Sigma_R(y) = \|u_0\|_\infty R^2 |y|^{-2}$ for $y \in \mathbb{R}^N \setminus B(0, R)$. Let $\mathcal{L}$ be the parabolic operator defined by

$$\mathcal{L}w(\tau, y) := \partial_\tau w(\tau, y) - y \cdot \nabla w(\tau, y) - q |\nabla w(\tau, y)|^q - q e^{-(p-q)\tau} \Delta_p w(\tau, y)$$

for $(\tau, y) \in Q_\infty$ (so that $\mathcal{L}v = 0$ by (4.2)). Then, if $y \in \mathbb{R}^N \setminus B(0, R)$, we have

$$\begin{aligned} \mathcal{L}\Sigma_R(y) &= 2\Sigma_R(y) - q \frac{2^q}{|y|^q} \Sigma_R(y)^q + q 2^{p-1} \frac{N+2-3p}{|y|^p} \Sigma_R(y)^{p-1} e^{-(p-q)\tau} \\ &\geq 2\Sigma_R(y) \left\{ 1 - q(2\|u_0\|_\infty)^{q-1} \frac{R^{2(q-1)}}{|y|^{3q-2}} - 3pq e^{-(p-q)\tau} (2\|u_0\|_\infty)^{p-2} \frac{R^{2(p-2)}}{|y|^{3p-4}} \right\} \\ &\geq 2\Sigma_R(y) \left\{ 1 - q(2\|u_0\|_\infty)^{q-1} R^{-q} - 3pq e^{-(p-q)\tau} (2\|u_0\|_\infty)^{p-2} R^{-p} \right\} \\ &\geq 0 \end{aligned}$$

by the choice of R . Consequently, Σ_R is a supersolution to (4.2) in $(0, \infty) \times (\mathbb{R}^N \setminus B(0, R))$ for $R \geq R_c$.

Now, fix $\varepsilon \in (0, 1)$. Since $u_0 \in \mathcal{C}_0(\mathbb{R}^N)$, there is $\rho_\varepsilon \geq \max\{R_c, \varepsilon^{-1}\}$ such that $u_0(y) \leq \varepsilon/2$ if $|y| \geq \rho_\varepsilon$. We then infer from the monotonicity of Σ_R and (2.1) that

$$u_0(y) - \frac{\varepsilon}{2} - \Sigma_{\rho_\varepsilon}(z) \leq -\Sigma_{\rho_\varepsilon}(z) \leq 0$$

if $|y| \geq \rho_\varepsilon$ and $|z| \geq \rho_\varepsilon$,

$$v(\tau, y) - \frac{\varepsilon}{2} - \Sigma_{\rho_\varepsilon}(z) \leq \|u_0\|_\infty - \Sigma_{\rho_\varepsilon}(z) = \Sigma_{\rho_\varepsilon}(y) - \Sigma_{\rho_\varepsilon}(z) \leq \frac{2\|u_0\|_\infty}{\rho_\varepsilon} |y - z|$$

if $|y| = \rho_\varepsilon$, $|z| \geq \rho_\varepsilon$, and $\tau \geq 0$, and

$$v(\tau, y) - \frac{\varepsilon}{2} - \Sigma_{\rho_\varepsilon}(z) \leq \|u_0\|_\infty - \|u_0\|_\infty \leq 0$$

if $|y| \geq \rho_\varepsilon$, $|z| = \rho_\varepsilon$, and $\tau \geq 0$. As $v - \varepsilon/2$ and $\Sigma_{\rho_\varepsilon}$ are subsolution and supersolution, respectively, to (4.2), the comparison principle [11, Theorem 4.1] warrants that $v(\tau, y) - \varepsilon/2 \leq \Sigma_{\rho_\varepsilon}(y)$ for $\tau \geq 0$ and $|y| \geq \rho_\varepsilon$. It remains to choose $R_\varepsilon \geq \rho_\varepsilon$ such that $\Sigma_{\rho_\varepsilon}(y) \leq \varepsilon/2$ for $|y| \geq R_\varepsilon$ to complete the proof of (4.10). The last assertion of Lemma 4.1 is then a straightforward consequence of the definition (4.7) and (4.10). $\square$

We next use the semiconvexity estimate (1.11) (and more precisely its consequence stated in Corollary 3.4) to show that v_* lies above the profile H_∞ defined in (1.6).

Lemma 4.2 *For $y \in \mathbb{R}^N$, we have*

$$(4.11) \quad H_\infty(y) \leq v_*(y) \leq v^*(y).$$

Proof. For $\tau \geq 0$, $y \in \mathbb{R}^N$, $\xi_0 \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, we set $\mathcal{F}(\tau, y, \xi_0, \xi) := \xi_0 - y \cdot \xi - q |\xi|^q + \kappa_2 e^{-(p-q)\tau}$ with $\kappa_2 := q \kappa_1 \|u_0\|_\infty^{(p-q)/q} (e^q/(e^q - 1))^{p/q}$, the constant κ_1 being defined in (1.11). It then readily follows from Corollary 3.4 that

$$(4.12) \quad v \text{ is a supersolution to } \mathcal{F}(\tau, y, \partial_\tau w, \nabla w) = 0 \text{ in } (1, \infty) \times \mathbb{R}^N.$$

We next fix $\tau_0 > 1$ and denote by V the (viscosity) solution to

$$\begin{aligned} \partial_\tau V - y \cdot \nabla V - q |\nabla V|^q &= 0, & (\tau, y) \in (\tau_0, \infty) \times \mathbb{R}^N, \\ V(\tau_0) &= v(\tau_0), & y \in \mathbb{R}^N. \end{aligned}$$

On the one hand, a straightforward computation shows that the function $\tilde{V}$ defined by

$$\tilde{V}(\tau, y) := V(\tau, y) - \kappa_2 \int_{\tau_0}^{\tau} e^{-(p-q)s} ds, \quad (\tau, y) \in (\tau_0, \infty) \times \mathbb{R}^N,$$

is the (viscosity) solution to $\mathcal{F}(\tau, y, \partial_\tau \tilde{V}, \nabla \tilde{V}) = 0$ in $(\tau_0, \infty) \times \mathbb{R}^N$ with initial condition $\tilde{V}(\tau_0) = v(\tau_0)$. Recalling (4.12), we infer from the comparison principle that

$$(4.13) \quad \tilde{V}(\tau, y) \leq v(\tau, y) \text{ for } (\tau, y) \in (\tau_0, \infty) \times \mathbb{R}^N.$$

On the other hand, it follows from Proposition A.1 that

$$\lim_{\tau \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \left| V(\tau, y) - (\|v(\tau_0)\|_\infty - \gamma_q |y|^{q/(q-1)})_+ \right| = 0.$$

We may then pass to the limit as $\tau \rightarrow \infty$ in (4.13) and use the definition (4.7) to conclude that

$$(\|v(\tau_0)\|_\infty - \gamma_q |y|^{q/(q-1)})_+ - \kappa_2 \int_{\tau_0}^{\infty} e^{-(p-q)s} ds \leq v_*(y) \leq v^*(y)$$

for $y \in \mathbb{R}^N$. Letting $\tau_0 \rightarrow \infty$ in the above inequality with the help of (4.5) completes the proof of the lemma. $\square$

We are now in a position to complete the proof of Theorem 1.1. To this end, fix $\varepsilon \in (0, 1)$. Lemma 4.1 ensures that $v^*(y) \leq \varepsilon$ for $|y| \geq R_\varepsilon \geq 1/\varepsilon$ while the continuity of H_∞ implies that there is $r_\varepsilon \in (0, \varepsilon)$ such that $H_\infty(y) \geq M_\infty - \varepsilon$ for $|y| \leq r_\varepsilon$. Recalling (4.8), we realize that

$$(4.14) \quad \begin{cases} v^*(y) - \varepsilon \leq 0 \leq H_\infty(y) & \text{if } |y| = R_\varepsilon, \\ v^*(y) - \varepsilon \leq M_\infty - \varepsilon \leq H_\infty(y) & \text{if } |y| = r_\varepsilon. \end{cases}$$

Moreover, introducing $\psi(y) = -\gamma_q |y|^{q/(q-1)}/2$, we have

$$(4.15) \quad \mathcal{H}(y, \nabla \psi(y)) = \frac{q\gamma_q}{2(q-1)} |y|^{q/(q-1)} \left(1 - \frac{1}{2^{q-1}}\right) > 0 \quad \text{if } r_\varepsilon < |y| < R_\varepsilon,$$

the Hamiltonian $\mathcal{H}$ being defined in (4.9). Summarizing, we have shown that H_∞ and $v^* - \varepsilon$ are supersolution and subsolution, respectively, to (4.9) in $\Omega_\varepsilon := \{y \in \mathbb{R}^N : r_\varepsilon < |y| < R_\varepsilon\}$ with $v^* - \varepsilon \leq H_\infty$ on $\partial\Omega_\varepsilon$ by (4.14). Owing to (4.15) and the concavity of $\mathcal{H}$ with respect to its second variable, we may apply [15, Theorem 1] to conclude that $v^* - \varepsilon \leq H_\infty$ in Ω_ε . This property being valid for each $\varepsilon \in (0, 1)$, we actually have $v^* \leq H_\infty$ in $\mathbb{R}^N$ by passing to the limit as $\varepsilon \rightarrow 0$ thanks to the properties of r_ε and R_ε . Recalling (4.11), we have thus established that $v^* = v_* = H_\infty$ in $\mathbb{R}^N$. In particular, the property $v^* = v_*$ and the definition (4.7) provide the uniform convergence of $\{v(\tau)\}_{\tau \geq 0}$ towards $v^* = H_\infty$ on every compact subset of $\mathbb{R}^N$ as $\tau \rightarrow \infty$, see [2, Lemme 4.1] or [1, Lemma V.1.9]. Combining this local convergence with Lemma 4.1 actually gives

$$(4.16) \quad \lim_{\tau \rightarrow \infty} \|v(\tau) - H_\infty\|_\infty = 0.$$

Theorem 1.1 then readily follows after writing the convergence (4.16) in the original variables (t, x) for the function u and noticing that $\|h_\infty(1+t) - h_\infty(t)\|_\infty \rightarrow 0$ as $t \rightarrow \infty$. $\square$

5 Limit value of $\|u(t)\|_\infty$

This section is devoted to the proof of Proposition 1.4, for which three cases are to be distinguished and handled differently: $q \in (1, p-1]$, $q \in (p-1, q_\star]$, and $q \in (q_\star, p)$.

Proof of Proposition 1.4: $q \in (1, p-1]$. We proceed as in [19, Proposition 1] (where a similar result is proved for $p = 2$ and $q = 1$). For $\alpha > N/2$, $\delta > 0$, and $x \in \mathbb{R}^N$, we set $\varrho_\delta(x) := (1 + \delta |x|^2)^{-\alpha}$. Clearly, $\varrho_\delta \in L^1(\mathbb{R}^N)$ and it follows from (2.3) that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} \varrho_\delta(x) u(t, x) dx &= \int_{\mathbb{R}^N} \left\{ \varrho_\delta(x) |\nabla u(t, x)|^q - |\nabla u(t, x)|^{p-2} \nabla u(t, x) \cdot \nabla \varrho_\delta(x) \right\} dx \\ &\geq \int_{\mathbb{R}^N} \varrho_\delta(x) |\nabla u(t, x)|^q \left(1 - |\nabla u(t, x)|^{p-1-q} \frac{|\nabla \varrho_\delta(x)|}{\varrho_\delta(x)} \right) dx. \end{aligned}$$

Recalling that $\|\nabla u(t)\|_\infty \leq \|\nabla u_0\|_\infty$ by (1.3) and (2.2) and noticing that $|\nabla \varrho_\delta| \leq \alpha \delta^{1/2} \varrho_\delta$, we further obtain

$$\frac{d}{dt} \int_{\mathbb{R}^N} \varrho_\delta(x) u(t, x) dx \geq \int_{\mathbb{R}^N} \varrho_\delta(x) |\nabla u(t, x)|^q (1 - \alpha \delta^{1/2} \|\nabla u_0\|_\infty^{p-1-q}) dx.$$

Choosing $\delta = \|\nabla u_0\|_\infty^{2(q+1-p)}/\alpha^2$ and integrating with respect to time give

$$\|u(t)\|_\infty \|\varrho_\delta\|_1 \geq \int_{\mathbb{R}^N} \varrho_\delta(x) u(t, x) dx \geq \int_{\mathbb{R}^N} \varrho_\delta(x) u_0(x) dx > 0.$$

We then pass to the limit as $t \rightarrow \infty$ to conclude that $M_\infty > 0$. $\square$

We next turn to the case $q \in (p-1, q_\star]$ which turns out to be more complicated and requires two preparatory results.

Lemma 5.1 *Assume that $q \in (1, q_\star]$ and let u be a non-negative solution to (1.1), (1.2) with a compactly supported initial condition u_0 satisfying (1.3). Then $u(t) \in L^1(\mathbb{R}^N)$ for each $t \geq 0$, the function $t \mapsto \|u(t)\|_1$ is non-decreasing and*

$$(5.1) \quad \lim_{t \rightarrow \infty} \|u(t)\|_1 = \infty.$$

Proof. For every $t \geq 0$, $u(t)$ is bounded and compactly supported by (2.1) and Proposition 2.2, and is thus in $L^1(\mathbb{R}^N)$. The time monotonicity of the L^1 -norm of u then readily follows from (2.3) with $\vartheta = 1$, a valid choice in this particular case as $u(t)$ is compactly supported. It further follows from (2.3) with $\vartheta = 1$ that

$$(5.2) \quad \|u(t)\|_1 \geq \|u(T)\|_1 + \int_T^t \|\nabla u(s)\|_q^q ds \quad \text{for } t > T \geq 0.$$

Consider next $T > 0$ and $t > T$. Recalling the Gagliardo-Nirenberg inequality

$$(5.3) \quad \|w\|_q \leq C_6 \|\nabla w\|_q^{N(q-1)/(N(q-1)+q)} \|w\|_1^{q/(N(q-1)+q)}, \quad w \in W^{1,q}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N),$$

we infer from (5.2), (5.3), and the time monotonicity of the L^1 -norm of u that

$$\begin{aligned} \|u(t)\|_1^{1+(q^2/N(q-1))} &\geq \|u(t)\|_1^{q^2/N(q-1)} \left(\|u(T)\|_1 + \int_T^t \|\nabla u(s)\|_q^q ds \right) \\ &\geq \int_T^t \|u(s)\|_1^{q^2/N(q-1)} \|\nabla u(s)\|_q^q ds \\ &\geq C_7 \int_T^t (\|u(s)\|_q^q)^{(N(q-1)+q)/N(q-1)} ds. \end{aligned}$$

If φ denotes the solution to the p -Laplacian equation $\partial_t \varphi - \Delta_p \varphi = 0$ in Q_∞ with initial condition $\varphi(0) = u_0$, the comparison principle readily implies that

$$(5.4) \quad \varphi(t, x) \leq u(t, x), \quad (t, x) \in Q_\infty.$$

Inserting this estimate in the previous lower bound for $\|u(t)\|_1$, we end up with

$$(5.5) \quad \|u(t)\|_1^{1+(q^2/N(q-1))} \geq C_7 \int_T^t (\|\varphi(s)\|_q^q)^{(N(q-1)+q)/N(q-1)} ds.$$

Now, by Proposition 2.3 we have

$$\lim_{s \rightarrow \infty} s^{N(q-1)/(N(p-2)+p)} \|\varphi(s) - \mathcal{B}_{\|u_0\|_1}(s)\|_q^q = 0$$

and

$$\|\mathcal{B}_{\|u_0\|_1}(s)\|_q^q = C_8 s^{-N(q-1)/(N(p-2)+p)},$$

so that

$$\begin{aligned} \|\varphi(s)\|_q^q &\geq \left(\|\mathcal{B}_{\|u_0\|_1}(s)\|_q - \|\varphi(s) - \mathcal{B}_{\|u_0\|_1}(s)\|_q \right)^q \\ &\geq s^{-N(q-1)/(N(p-2)+p)} \left(C_8 - \|\varphi(s) - \mathcal{B}_{\|u_0\|_1}(s)\|_q \right)^q \\ &\geq \left(\frac{C_8}{2} \right)^q s^{-N(q-1)/(N(p-2)+p)} \end{aligned}$$

for $s \geq T$, provided T is chosen sufficiently large. Inserting this estimate in (5.5) gives

$$\begin{aligned} \|u(t)\|_1^{1+(q^2/N(q-1))} &\geq C_9 \int_T^t s^{-(N(q-1)+q)/(N(p-2)+p)} ds \\ &\geq C_{10} \begin{cases} t^{(N+1)(q_*-q)/(N(p-2)+p)} - T^{(N+1)(q_*-q)/(N(p-2)+p)} & \text{if } q \in (1, q_*) , \\ \log(t/T) & \text{if } q = q_* . \end{cases} \end{aligned}$$

We then let $t \rightarrow \infty$ to obtain the claimed result. $\square$

We next argue as in [12, Lemma 14] (for $p = 2$) to show that, if $q \in (p-1, p)$ and $M_\infty = 0$, then the L^∞ -norm of $u(t)$ decays faster than an explicit rate.

Lemma 5.2 *Assume that $q \in (p-1, p)$ and let u be a non-negative solution to (1.1), (1.2) with an initial condition u_0 satisfying (1.3). If $M_\infty = 0$ in (1.4), then*

$$(5.6) \quad \|u(t)\|_\infty \leq C_{11} t^{-(p-q)/(2q-p)} \quad \text{for } t > 0.$$

Observe that the assumptions $p > 2$ and $q \in (p-1, p)$ imply that $2q > p$ and $(p-q)/(2q-p) > 0$.

Proof. Consider a non-negative function $\eta \in \mathcal{C}^\infty(\mathbb{R}^N)$ with compact support in $B(0, 1)$ and $\|\eta\|_1 = 1$. We then define a sequence of mollifiers $(\eta_\delta)_\delta$ by $\eta_\delta(x) := \eta(x/\delta)/\delta^N$ for $x \in \mathbb{R}^N$

and $\delta \in (0, 1)$. For $(t, x_0) \in Q_\infty$ and $T > t$, we take $\vartheta(x) = \eta_\delta(x - x_0)$ in (2.3) and infer from (1.11) (with $s = t/2$) that

$$\begin{aligned}
\|u(T)\|_\infty &\geq \int_{\mathbb{R}^N} u(T, x) \eta_\delta(x - x_0) dx \\
&\geq \int_{\mathbb{R}^N} u(t, x) \eta_\delta(x - x_0) dx - \int_t^T \int_{\mathbb{R}^N} |\nabla u(s, x)|^{p-2} \nabla u(s, x) \cdot \nabla \eta_\delta(t, x - x_0) dx ds \\
&\geq \int_{\mathbb{R}^N} u(t, x) \eta_\delta(x - x_0) dx - 2^{p/q} \kappa_1 \left\| u\left(\frac{t}{2}\right) \right\|_\infty^{(p-q)/q} \int_t^T (2s - t)^{-p/q} ds \\
&\geq \int_{\mathbb{R}^N} u(t, x) \eta_\delta(x - x_0) dx - C_{12} \left\| u\left(\frac{t}{2}\right) \right\|_\infty^{(p-q)/q} (t^{(q-p)/q} - T^{(q-p)/q}).
\end{aligned}$$

Owing to the continuity of u , we may pass to the limit as $\delta \rightarrow 0$ in the above inequality and deduce that

$$\|u(T)\|_\infty \geq u(t, x_0) - C_{12} \left\| u\left(\frac{t}{2}\right) \right\|_\infty^{(p-q)/q} (t^{(q-p)/q} - T^{(q-p)/q}).$$

But the above inequality is valid for all $x_0 \in \mathbb{R}^N$ and we thus end up with

$$\|u(T)\|_\infty \geq \|u(t)\|_\infty - C_{12} \left\| u\left(\frac{t}{2}\right) \right\|_\infty^{(p-q)/q} (t^{(q-p)/q} - T^{(q-p)/q}).$$

Finally, as $q < p$, we may let $T \rightarrow \infty$ in the previous inequality and use the assumption $M_\infty = 0$ to conclude that

$$\|u(t)\|_\infty \leq C_{12} \left\| u\left(\frac{t}{2}\right) \right\|_\infty^{(p-q)/q} t^{(q-p)/q},$$

or, equivalently, as $2q > p$,

$$t^{(p-q)/(2q-p)} \|u(t)\|_\infty \leq C_{13} \left\{ \left(\frac{t}{2}\right)^{(p-q)/(2q-p)} \left\| u\left(\frac{t}{2}\right) \right\|_\infty \right\}^{(p-q)/q}$$

for $t \geq 0$. Introducing

$$A(t) := \sup_{s \in (0, t)} \{s^{(p-q)/(2q-p)} \|u(s)\|_\infty\} \in [0, \infty), \quad t \geq 0,$$

we deduce from the previous inequality that $A(t) \leq C_{13} A(t)^{(p-q)/q}$, whence $A(t) \leq C_{13}^{q/(2q-p)}$ for $t \geq 0$. This bound being valid for each $t > 0$, the proof of (5.6) is complete. $\square$

Proof of Proposition 1.4: $q \in (p - 1, q_\star]$.

Step 1: We first consider a compactly supported initial condition u_0 satisfying (1.3) and assume for contradiction that $M_\infty = 0$. On the one hand, according to Lemma 5.2 and the assumption $q \leq q_\star$, there holds

$$(5.7) \quad \limsup_{t \rightarrow \infty} t^{N/(N(p-2)+p)} \|u(t)\|_\infty \leq C_{11} t^{(p(N+1)(q-q_\star))/((2q-p)(N(p-2)+p))} \leq C_{11}.$$

On the other hand, fix $t_0 > 0$ and let φ be the solution to the p -Laplacian equation $\partial_t \varphi - \Delta_p \varphi = 0$ in Q_∞ with initial condition $\varphi(0) = u(t_0)$. As u_0 is compactly supported, so is $u(t_0)$ by Proposition 2.2 and $u(t_0)$ thus belongs to $L^1(\mathbb{R}^N)$. Moreover, the comparison principle warrants that $u(t, x) \geq \varphi(t - t_0, x)$ for $(t, x) \in [t_0, \infty) \times \mathbb{R}^N$. We then infer from the above properties and Proposition 2.3 that, for $t > t_0$,

$$\begin{aligned} t^{N/(N(p-2)+p)} \|u(t)\|_\infty &\geq (t - t_0)^{N/(N(p-2)+p)} \|u(t)\|_\infty \\ &\geq (t - t_0)^{N/(N(p-2)+p)} \|\varphi(t - t_0)\|_\infty \\ &\geq (t - t_0)^{N/(N(p-2)+p)} \|\mathcal{B}_{\|u(t_0)\|_1}(t - t_0)\|_\infty \\ &\quad - (t - t_0)^{N/(N(p-2)+p)} \|\mathcal{B}_{\|u(t_0)\|_1}(t - t_0) - \varphi(t - t_0)\|_\infty \\ &\geq C_{14} \|u(t_0)\|_1^{p/(N(p-2)+p)} \\ &\quad - (t - t_0)^{N/(N(p-2)+p)} \|\mathcal{B}_{\|u(t_0)\|_1}(t - t_0) - \varphi(t - t_0)\|_\infty. \end{aligned}$$

Using once more Proposition 2.3, we may pass to the limit as $t \rightarrow \infty$ in the previous inequality to obtain

$$(5.8) \quad \liminf_{t \rightarrow \infty} t^{N/(N(p-2)+p)} \|u(t)\|_\infty \geq C_{14} \|u(t_0)\|_1^{p/(N(p-2)+p)}.$$

Combining (5.7) and (5.8) yields $\|u(t_0)\|_1 \leq C_{15}$ for all $t_0 > 0$ which contradicts Lemma 5.1. Therefore, $M_\infty > 0$.

Step 2: Now, if u_0 is an arbitrary initial condition satisfying (1.3), there clearly exists a compactly supported initial condition $\tilde{u}_0$ satisfying (1.3) and such that $u_0 \geq \tilde{u}_0$ in $\mathbb{R}^N$. Introducing the solution $\tilde{u}$ to (1.1) with initial condition $\tilde{u}_0$, the comparison principle entails that $u \geq \tilde{u}$ in Q_∞ , hence

$$M_\infty \geq \lim_{t \rightarrow \infty} \|\tilde{u}(t)\|_\infty.$$

The first step of the proof ensures that the right-hand side of the above inequality is positive which completes the proof. $\square$

It remains to investigate the case $q \in (q_\star, p)$, for which we adapt the proof of [7, Theorem 2.4(b)].

Proof of Proposition 1.4: $q \in (q_\star, p)$. We put

$$m_0 := \left| \inf_{y \in \mathbb{R}^N} \Delta_p u_0(y) \right|.$$

As in the proof of Lemma 5.1, let $\eta \in \mathcal{C}^\infty(\mathbb{R}^N)$ be a non-negative function with compact support in $B(0, 1)$ and $\|\eta\|_1 = 1$, and define a sequence of mollifiers $(\eta_\delta)_\delta$ by $\eta_\delta(x) := \eta(x/\delta)/\delta^N$ for $x \in \mathbb{R}^N$ and $\delta \in (0, 1)$. For $(t, x_0) \in Q_\infty$ and $T \in (0, t)$, we take $\vartheta(x) = \eta_\delta(x - x_0)$ in (2.3) and infer from (1.11) (with $s = 0$) and Corollary 3.3 that

$$\begin{aligned} \|u(t)\|_\infty &\geq \int_{\mathbb{R}^N} u(t, x) \eta_\delta(x - x_0) dx \\ &\geq \int_{\mathbb{R}^N} u_0(x) \eta_\delta(x - x_0) dx - \int_0^t \int_{\mathbb{R}^N} |\nabla u(s, x)|^{p-2} \nabla u(s, x) \cdot \nabla \eta_\delta(t, x - x_0) dx ds \\ &\geq \int_{\mathbb{R}^N} u_0(x) \eta_\delta(x - x_0) dx - \int_0^T m_0 ds - \kappa_1 \|u_0\|_\infty^{(p-q)/q} \int_T^t s^{-p/q} ds \\ &\geq \int_{\mathbb{R}^N} u_0(x) \eta_\delta(x - x_0) dx - T m_0 - C_{16} \|u_0\|_\infty^{(p-q)/q} (T^{(q-p)/q} - t^{(q-p)/q}). \end{aligned}$$

Owing to the continuity of u_0 , we may pass to the limit as $\delta \rightarrow 0$ in the above inequality and deduce that

$$\|u(t)\|_\infty \geq u_0(x_0) - T m_0 - C_{16} \|u_0\|_\infty^{(p-q)/q} (T^{(q-p)/q} - t^{(q-p)/q}).$$

Since $q < p$, we may let $t \rightarrow \infty$ in the above inequality and take the supremum with respect to x_0 to conclude that

$$M_\infty \geq \|u_0\|_\infty - T m_0 - C_{16} \|u_0\|_\infty^{(p-q)/q} T^{(q-p)/q}.$$

Next, for $\beta \in (0, 1)$, the choice $T = \|u_0\|_\infty^{(p-q)/p} (\beta + m_0)^{-q/p}$ in the previous inequality yields

$$M_\infty \geq \|u_0\|_\infty^{(p-q)/p} (\|u_0\|_\infty^{q/p} - (1 + C_{16}) (\beta + m_0)^{(p-q)/p}).$$

This inequality being valid for every $\beta \in (0, 1)$, we conclude that

$$M_\infty \geq \|u_0\|_\infty^{(p-q)/p} \left(\|u_0\|_\infty^{q/p} - (1 + C_{16}) m_0^{(p-q)/p} \right) > 0$$

as soon as (1.10) is fulfilled with $\kappa_0 = (1 + C_{16})^{p/q}$. $\square$

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A Convergence for the Hamilton-Jacobi equation (4.6)

In this section, we study the large behaviour of non-negative solutions to the Hamilton-Jacobi equation (4.6) with initial data in $\mathcal{C}_0(\mathbb{R}^N)$ and show their convergence to a steady state uniquely determined by the L^∞ -norm of the initial data. Though the large time behaviour of solutions to first-order Hamilton-Jacobi equations has received considerable attention in recent years (see [4, 5, 16, 22, 23] and the references therein), the particular case of (4.6) does not seem to have been investigated in the literature. We thus provide a simple proof relying on the Hopf-Lax-Oleinik formula.

Proposition A.1 *Let $q > 1$ and consider a non-negative function $h_0 \in \mathcal{C}_0(\mathbb{R}^N)$. Let h be the unique viscosity solution to the Cauchy problem*

$$(A.1) \quad \partial_\tau h - y \cdot \nabla h - q |\nabla h|^q = 0, \quad (\tau, y) \in (0, \infty) \times \mathbb{R}^N,$$

$$(A.2) \quad h(0) = h_0, \quad y \in \mathbb{R}^N.$$

Then

$$(A.3) \quad \lim_{\tau \rightarrow \infty} \|h(\tau) - h_s\|_\infty = 0$$

with

$$h_s(y) := \left(\|h_0\|_\infty - \gamma_q |y|^{q/(q-1)} \right)_+, \quad y \in \mathbb{R}^N,$$

the constant $\gamma_q = (q-1) q^{-q/(q-1)}$ being defined in Theorem 1.1.

Thanks to the concavity of the Hamiltonian $\mathcal{H}(y, \xi) = -y \cdot \xi - q |\xi|^q$, $(y, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$, with respect to its second variable, the Hopf-Lax-Oleinik formula provides a representation formula for the solution h to (A.1), (A.2) which can be used to prove (A.3).

Proof. We first recall that h is given by the Hopf-Lax-Oleinik formula

$$h(\tau, y) = \sup_{z \in \mathbb{R}^N} \left\{ h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \right\}$$

for $(\tau, y) \in [0, \infty) \times \mathbb{R}^N$, see, e.g., [10, Chapter 3]. Since $h(\tau, y) \geq h_0(ye^\tau) \geq 0$, we have in fact

$$h(\tau, y) = \sup_{z \in \mathbb{R}^N} \left\{ \left(h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \right)_+ \right\}$$

for $(\tau, y) \in [0, \infty) \times \mathbb{R}^N$.

Consider now $\beta \in (0, 1)$. As $h_0 \in \mathcal{C}_0(\mathbb{R}^N)$, there is $R_\beta > (\|h_0\|_\infty / \gamma_q)^{(q-1)/q}$ such that

$$(A.4) \quad h_0(z) \leq \beta \quad \text{for} \quad |z| \geq R_\beta.$$

On the one hand, if $(\tau, y) \in [\log R_\beta, \infty) \times \mathbb{R}^N$ and $z \in \mathbb{R}^N$, we have either $|z| \geq R_\beta$ and

$$\begin{aligned} & \left| \left(h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \right)_+ - \left(h_0(z) - \gamma_q |y|^{q/(q-1)} \right)_+ \right| \\ & \leq \left(h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \right)_+ + \left(h_0(z) - \gamma_q |y|^{q/(q-1)} \right)_+ \\ & \leq 2\beta \end{aligned}$$

by (A.4) or $z \in B(0, R_\beta)$ and

$$\begin{aligned}
& \left| \left(h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \right)_+ - \left(h_0(z) - \gamma_q |y|^{q/(q-1)} \right)_+ \right| \\
& \leq \gamma_q |y - z e^{-\tau}|^{q/(q-1)} \left\{ (1 - e^{-q\tau})^{-1/(q-1)} - 1 \right\} + \gamma_q \left| |y - z e^{-\tau}|^{q/(q-1)} - |y|^{q/(q-1)} \right| \\
& \leq \gamma_q (|y| + R_\beta e^{-\tau})^{q/(q-1)} \left\{ (1 - e^{-q\tau})^{-1/(q-1)} - 1 \right\} + \frac{q \gamma_q}{q-1} (|y| + |z| e^{-\tau})^{1/(q-1)} |z| e^{-\tau} \\
& \leq \gamma_q (|y| + 1)^{1/(q-1)} \left\{ \frac{q}{q-1} + |y| + 1 \right\} \left\{ (1 - e^{-q\tau})^{-1/(q-1)} - 1 + R_\beta e^{-\tau} \right\}
\end{aligned}$$

as $\tau \geq \log R_\beta$. Combining the above two estimates gives

$$\begin{aligned}
& \left| h(\tau, y) - \sup_{z \in \mathbb{R}^N} \left\{ \left(h_0(z) - \gamma_q |y|^{q/(q-1)} \right)_+ \right\} \right| \\
& \leq C(q) (|y| + 1)^{q/(q-1)} \left\{ (1 - e^{-q\tau})^{-1/(q-1)} - 1 + R_\beta e^{-\tau} \right\} + 2\beta,
\end{aligned}$$

whence

$$(A.5) \quad |h(\tau, y) - h_s(y)| \leq C(q) (|y| + 1)^{q/(q-1)} \left\{ (1 - e^{-q\tau})^{-1/(q-1)} - 1 + R_\beta e^{-\tau} \right\} + 2\beta$$

for $(\tau, y) \in [\log R_\beta, \infty) \times \mathbb{R}^N$. On the other hand, if $\tau \geq \log(R_\beta)$, $|y| \geq Y := 1 + (\|h_0\|_\infty / \gamma_q)^{(q-1)/q}$ and $z \in \mathbb{R}^N$, we have either $|y - z e^{-\tau}| \geq Y - 1$ and

$$\begin{aligned}
& h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \\
& \leq (1 - e^{-q\tau})^{-1/(q-1)} \left\{ \|h_0\|_\infty (1 - e^{-q\tau})^{1/(q-1)} - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} \right\} \\
& \leq (1 - e^{-q\tau})^{-1/(q-1)} \left\{ \|h_0\|_\infty - \gamma_q (Y - 1)^{q/(q-1)} \right\} \\
& \leq 0,
\end{aligned}$$

or $|y - z e^{-\tau}| < Y - 1$ and

$$|z| \geq |y e^\tau| - |z - y e^\tau| \geq Y e^\tau - (Y - 1) e^\tau = e^\tau \geq R_\beta,$$

so that

$$h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \leq \beta$$

by (A.4). Therefore,

$$(A.6) \quad h(\tau, y) \leq \beta \quad \text{for} \quad (\tau, y) \in [\log R_\beta, \infty) \times (\mathbb{R}^N \setminus B(0, Y)).$$

The claim (A.3) then easily follows from (A.5) and (A.6). $\square$

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