Defaultable bond pricing using regime switching intensity model

Stéphane Goutte, Armand Ngoupeyou

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Abstract

In this paper, we are interested in finding explicit numerical formulas for the defaultable bonds prices of firms which fit well with real financial data. For this purpose, we use a default intensity whose values depend on the credit rating of these firms. Each credit rating corresponds to a regime of the default intensity. Then, this regime switches as soon as the credit rating of the firms also changes. This regime switching default intensity model allows us to capture well some market features or economics behaviors. We obtain two explicit different formulas to evaluate the conditional Laplace transform of a regime switching Cox Ingersoll Ross model. One using the property of semi-affine of this model and the other one using analytic approximation. We conclude by giving some numerical illustrations of these formulas and real data estimation results.

Keywords: Defaultable bond; Regime switching; Conditional Laplace Transform; Credit rating; Markov copula.

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Introduction

In an economic crisis situation where the credit ratings of countries or firms have a big impact in the general financial market, we need to understand and capture the change of these ratings in the dynamic of a the firm bond price. Moreover, we also have to model the contagion risk due to a bad rating of a firm on other one. For example, the Bond of countries in the Euro zone are affected by the Greek bad rating. In the literature, models for pricing defaultable securities have been introduced by Merton [23]. It consists of explicitly linking the risk of firm default and the value of the firm. Although this model is a good issue to understand the default risk, it is less useful in practical applications since it is too difficult to capture all the macroeconomics factors which appear in the dynamics of the value of the firm. Hence, Duffie and Singleton [9] introduced

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the reduced form modeling, followed by Madan and Unal [22], Jeanblanc and Rutkowski [20] and others. The main tool of this approach is the "default intensity process" which describes in short terms the instantaneous probability of default. To deal with contagion risk, the most popular approach is copula. The credit rating of each firm is modeled by a Markov chain on which we will construct our copula. In this regard, we use a continuous time Markov chain called credit migration process studied by Bielecki and Rutkowski in [4]. Hence, our copula which depends on the credit ratings will affect the dynamic of the default intensity. In fact, we define default intensity process by a Cox-Ingersoll-Ross (CIR) model whose parameters values depend on this copula.

The Cox-Ingersoll-Ross model was first considered to model the term structure of interest rate by Cox and al. in [7]. The study of this class of processes was caution by the fact that it allows us a closed form expression of Laplace transform (see Duffie and al. [8]) and model well the default intensity (Alfonsi and Brigo [1]). Moreover, Choi in [5] shows that regime switching CIR process captures more short term interest rate than standard models. In a econometric point of view, regime switching model were introduced by Hamilton in [16].

In this framework, we obtain explicit formulas to evaluate defaultable bond prices. More precisely, we obtain two different formulas to evaluate the Laplace transform of defaultable intensity. In a first time, we use the semi affine property of the regime switching Cox Ingersoll Ross model and then solve a system of Riccati's equations. In a second time, we extend the analytic approximation found in Choi and Wirjanto [6]. Indeed Choi and Wirjanto in [6] give an analytic approximation of the value of bond price with constant CIR parameter and with constant time step model discretization. We extend this result in three ways: firstly to evaluate conditional Laplace transform of a regime switching Cox Ingersoll Ross, secondly to evaluate defaultable regime switching bond price and thirdly in the case of non uniform deterministic time step model discretization (in our case, the time step model discretization depends on the regime switching stopping time). We apply these two formulas to price defaultable bond. We illustrate the efficiency of our new modelization of regime switching intensity firstly by comparing the computing time of each formulas, secondly by showing (using real historical data based on the Greece spread CDS) that our model estimates well data and that each regime captures well some market features or economics behaviors.

In Section 1, we introduce the Markov copula, the credit migration process and the regime switching Cox-Ingersoll-Ross model. In Section 2, we give the two formulas to evaluate the conditional Laplace transform in this framework. Finally, in Section 3, we show some simulations to compare the formula results, illustrate the model and then we give some estimation on real data.

1 The defaultable model

1.1 Credit migration model

Let $T > 0$ be a fixed maturity time and denote by $(\Omega, \overline{\mathbb{F}} := (\overline{\mathcal{F}}_t)_{[0,T]}, \mathbb{P})$ an underlying probability space.

Definition 1.1. *A notation is a label given by an entity which measures the viability of a firm. This graduate notation goes from 1 to K. 1 for the best economic and financial situation and K for the worst. We will call an indicator of notation a continuous time homogeneous Markov chain on the finite space $\mathcal{S} = \{1, \ldots, K\}$.*

Let A and B be two firms with their own indicator of notation $(X^A)_{t\in[0,T]}$ and $(X^B)_{t\in[0,T]}$. Hence X^A and X^B are Markov chains with generator matrix Π^A and Π^B. We recall that the generator matrix of $C \in \{A, B\}$ is given by $\Pi^C_{ij} \geq 0$ if $i \neq j$ for all $i, j \in \mathcal{S}$ and $\Pi^C_{ii} = -\sum_{j\neq i} \Pi_{ij}$ otherwise. We can remark that Π^C_{ij} represents the intensity of the jump from state i to state j. Moreover, we denote by $\mathcal{F}^A_t := \{\sigma(X^A_s); 0 \leq s \leq t\}$ and $\mathcal{F}^B_t := \{\sigma(X^B_s); 0 \leq s \leq t\}$ the natural filtrations generated by X^A and X^B.

1.1.1 Markov Copula

Let denote by X the bivariate process $X = (X^A, X^B)$, which is a finite continuous time Markov chain with respect to its natural filtration $\mathcal{F}^X = \mathcal{F}^{A,B}$. We recall now the Corollary 5.1 of Bielecki and al. [2], applied to our case, which gives the condition that the components of the bivariate processes X are themselves Markov chain with respect to their own natural filtration.

Corollary 1.1. *Consider two Markov chains X^A and X^B, with respect to their own filtrations $\mathcal{F}^A$ and $\mathcal{F}^B$, and with values in $\mathcal{S}$. Suppose that their respective generators are Π^A_{ij} and Π^B_{hk} with i, j, h and k are in $\mathcal{S}$. Consider the system of equations in the unknown $\Pi^X_{ij,hk}$ where $i, j, h, k \in \mathcal{S}$ and $(i, h) \neq (j, k)$:*

$$\sum_{k\in\mathcal{S}} \Pi^X_{ij,hk} = \Pi^A_{ij} \quad \forall h, i, j \in \mathcal{S}, i \neq j \quad \textit{and} \quad \sum_{j\in\mathcal{S}} \Pi^X_{ij,hk} = \Pi^B_{hk} \quad \forall i, h, k \in \mathcal{S}, h \neq k \tag{1.1}$$

Suppose that the above system admits a solution such that the matrix $\Pi^Z := \left(\Pi^Z_{ij,hk}\right)_{i,j,h,k\in\mathcal{S}}$ with

$$\Pi^X_{ii,hh} = -\sum_{(j,k)\in\mathcal{S}\times\mathcal{S},(j,k)\neq(i,h)} \Pi^X_{ij,hk} \tag{1.2}$$

properly defines an infinitesimal generator of a Markov chain with values in $\mathcal{S} \times \mathcal{S}$. Consider, the bivariate Markov chain $X = (X^A, X^B)$ on $\mathcal{S} \times \mathcal{S}$ with generator matrix Π^X. Then, the components X^A and X^B are Markov chains with respect to their own filtrations, their generators are Π^A and Π^B.

Hence we can now formulate the Definition of a Markov copula.

Definition 1.2. *A Markov copula between the Markov chains X^A and X^B is any solution to system (1.1) such that the matrix Π^X, with $\Pi^X_{ii,hh}$ given in (1.2), properly defines an infinitesimal generator of a Markov chain with values in $\mathcal{S} \times \mathcal{S}$.*

Moreover, the infinitesimal generator process of X which is a matrix with $N := K^2$ rows and columns, since the cardinal of the state of notation is K, can be written as

$$\Pi^X = \begin{pmatrix} \pi_{(1,1)} & \cdots & \pi_{(1,K)} \\ \pi_{(2,1} & \cdots & \pi_{(2,K)} \\ \vdots & & \vdots \\ \pi_{(K,1)} & \cdots & \pi_{(K,K)} \end{pmatrix}$$

Then the possible states are N couples which are given by

$$\mathcal{E} := \{(1, 1), (1, 2), \ldots , (1, K), (2, 1), (2, 2), \ldots , (2, K), \ldots (K, 1), (K, 2), \ldots , (K, K)\}$$

1.1.2 Markov copula in the hazard rate framework

We denote by $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ the filtration such that $\mathcal{F}_t = \overline{\mathcal{F}}_t \vee \mathcal{F}_t^X$. Let τ^A and τ^B be the two default times of firms A and B. Let define for all $t \in [0,T]$:

$$H_t^A = 1_{\{\tau^A \leq t\}} \quad \text{and} \quad H_t^B = 1_{\{\tau^B \leq t\}} \tag{1.3}$$

We define now some others filtrations

$$\mathcal{G}_t^A = \mathcal{F}_t \vee \mathcal{H}_t^B, \quad , \quad \mathcal{G}_t^B = \mathcal{F}_t \vee \mathcal{H}_t^A \quad \textit{and} \quad \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^A \vee \mathcal{H}_t^B$$

where $\mathcal{H}^A$ (resp. $\mathcal{H}^B$) is the natural filtration generated by H^A (resp. H^B) and we will denote $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,T]}$, $\mathbb{G}^A := \left(\mathcal{G}_t^A\right)_{t\in[0,T]}$ and $\mathbb{G}^B := \left(\mathcal{G}_t^B\right)_{t\in[0,T]}$. Let now consider $\lambda^i := \lambda^i(X)$, for $i \in \{A,B\}$ two $\mathbb{F}$-progressively non-negative processes defined on $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with the filtration $\mathbb{F}$. We assume that $\int_0^\infty \lambda^i(X_s)ds = +\infty$ and we set:

$$\tau^i = \inf\left\{t \in \mathbb{R}^+, \int_0^t \lambda^i(X_s)ds \geq -\ln(U^i)\right\}, \quad i \in \{A,B\}.$$

where U^i are mutually independent uniform random variables defined on $(\Omega, \mathcal{G}, \mathbb{P})$ which are independent of λ^i. The stopping times τ^A and τ^B are totally inaccessible and conditionally independent given the filtration $\mathbb{F}$, moreover the $(\mathcal{H})$-hypothesis is satisfied (i.e. that every local $\mathbb{F}$-martingale is a local $\mathbb{G}$-martingale too). The process λ^i is called the $\mathbb{F}$-intensity of the firm i and we have that

$$M_t^i = H_t^i - \int_0^{t\wedge\tau^i} \lambda^i(X_s)ds$$

are $\mathbb{G}$-martingales. In general case, processes λ^i are $\mathbb{F} \vee \mathbb{G}^{(i)}$-adapted which jump when any default occurs. This jump impacts the default of the firm and makes some correlation between the firms. In our case, the correlation is constructed using the $\mathbb{F}$-Markov chain $X = (X^A, X^B)$. Since from the explicit formula of the intensity given the survey probability for each $i \in \{A,B\}$:

$$\lambda_t^i = -\frac{1}{\mathbb{P}(\tau^i \geq t|\mathcal{G}_t^i)} \frac{d\mathbb{P}(\tau^i \geq \theta|\mathcal{G}_t^i)}{d\theta}\Big|_{\theta=t}$$

we can find, from Bielecki and al. [3] (Example 4.5.1 p 94), that the formula of the conditional survey probability is given by:

$$\mathbb{P}(\tau^i \geq \theta|\mathcal{G}_t) = 1_{\{\tau^i \geq t\}} \mathbb{E}\left[e^{-\int_t^\theta \lambda^i(X_s)ds}|\mathcal{F}_t\right] \tag{1.4}$$

for $i \in \{A,B\}$. The Markov chain X will explain how the curve of the default bond moves with different states (regimes) of the financial market.

1.1.3 Construction of the Markov chain

We are now going to present the canonical construction of a conditional Markov chain X, based on a given filtration $\mathbb{F}$ and a stochastic infinitesimal generator Π^X. This construction can be found in Bielecki and Rukowski [4] or Eberlein and Ozkan [10], which we follow closely in the exposition. Each component $\Pi^X_{ij} : \Omega \times [0, T] \to \mathbb{R}^+$ are bounded, $\mathbb{F}$-progressively measurable stochastic processes. We recall that for every $i, j \in \mathcal{S}, i \neq j$, processes Π^X_{ij} are non-negative and $\Pi^X_{ii}(t) = -\sum_{j\neq i} \Pi^X_{ij(t)}$. The process X is constructed from an initial distribution μ and the $\overline{\mathbb{F}}$-conditional adapted infinitesimal generator Π^X by enlarging the underlying probability space $(\Omega, \overline{\mathcal{F}}, \mathbb{P}_T)$ to a probability space denoted in the sequel by $(\Omega, \mathcal{F}, \mathbb{Q}_T)$. The new probability space is obtained as a product space of the underlying one with a probability space supporting the initial distribution μ of X and a probability space supporting a sequence of uniformly distributed random variables, which control, together with the entries of the infinitesimal generator Π^X, the laws of jump times $(\tau_k)_{k\in\mathbb{N}}$ of X and jump heights. We denote by $\mathbb{F}$ its trivial extension from the original probability space $(\Omega, \overline{\mathcal{F}}, \mathbb{P}_T)$ to $(\Omega, \mathcal{F}, \mathbb{Q}_T)$. We refer to [4] or [13] for details of this construction. However an important step of this construction is that they construct a discrete time process $(\overline{X}_k)_{k\in\mathbb{N}}$ which allows us to construct the credit migration process X as

$$X_t := \overline{X}_{k-1} \quad \text{for all} \quad t \in [\tau_{k-1}, \tau_k[, \quad k \geq 1 \tag{1.5}$$

where τ_k are the jump times. An important result is that the progressive enlargement of filtration $\mathcal{F}_t := \overline{\mathcal{F}}_t \vee \mathcal{F}^X_t$, $t \in [0, T]$ satisfies the $(\mathcal{H})$-hypothesis. In the sequel, we will work under the enlarging probability space $(\Omega, \mathcal{F}, \mathbb{Q}_T)$. The expectations will be taken under the probability measure $\mathbb{Q}_T$ but for simplicity of notation, we will write $\mathbb{E}$ for $\mathbb{E}^{\mathbb{Q}_T}$.

1.2 Pricing defaultable bond with Markov copula

1.2.1 Defaultable Model

Let W be a standard real Brownian motion with filtration $\overline{\mathcal{F}_t} = \sigma\{W_s; 0 \leq s \leq t\}$.

We recall that a Cox Ingersoll Ross (CIR) process is the solution of the stochastic differential equation given by

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t, \quad t \in [0, T] \tag{1.6}$$

where κ, θ and σ are constants which satisfy the condition $\sigma > 0$ and $\kappa\theta > 0$. We will assume that $\lambda_0 \in \mathbb{R}^+$ and that $2\kappa\theta \geq \sigma^2$. This is to ensure that the process (λ_t) is positive. We will now define the notion of CIR process with each parameters values depend on the value of a Markov chain.

Definition 1.3. *Let $(X)_t$ be a two-dimensional continuous time Markov chain on finite space $\mathcal{S}^2 := \{1, \ldots, K\}^2$ for all $t \in [0, T]$. We will call a Regime switching CIR the process (λ_t) which is the solution of the stochastic differential equation given by*

$$d\lambda_t = \kappa(X_t)(\theta(X_t) - \lambda_t)dt + \sigma(X_t)\sqrt{\lambda_t}dW_t, \quad t \in [0, T]. \tag{1.7}$$

For all $j \in \{1, \ldots, K\}^2$, we have that $\kappa(j)\theta(j) > 0$ and $2\kappa(j)\theta(j) \geq \sigma(j)^2$

For simplicity, we will denote the values $\kappa(X_t)$, $\theta(X_t)$ and $\sigma(X_t)$ by κ_t, θ_t and σ_t.

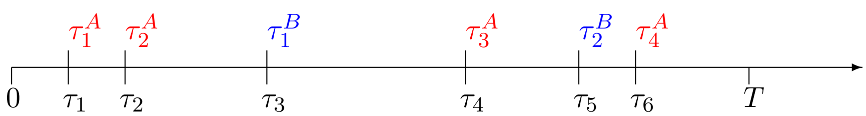
Assumption 1.1. *We assume that both intensities processes λ^A and λ^B follow a regime switching CIR given for $i = \{A, B\}$ by*

$$d\lambda^i_t = \kappa(X_t)(\theta(X_t) - \lambda^i_t)dt + \sigma(X_t)\sqrt{\lambda^i_t}dW_t. \tag{1.8}$$

Remark 1.1. *We have that the intensity process* (λ_t^i) *depends on the value of the credit migration process* $X = (X^A, X^B)$*. Hence each firm* A *and* B *has an increasing sequence of* $\mathbb{F}^X$*-stopping times given by:*

- *for the firm* A *it is* $0 \leq \tau_1^A < \tau_2^A < \cdots < \tau_n^A \leq T$.
- *for the firm* B *it is* $0 \leq \tau_1^B < \tau_2^B < \cdots < \tau_m^B \leq T$.

Hence with these two sequences, we construct another sequence by a rearrangement of these two sequences in one where we put every stopping time τ_i^A, $i \in \{1, \ldots n\}$ *and* τ_j^B, $j \in \{1, \ldots, m\}$ *in an increasing order. We obtain a new increasing sequence of stopping times of size* $M \in \mathbb{N}$ *given by* $0 \leq \tau_1 < \tau_2 < \cdots < \tau_M \leq T$*. As an example of this construction*



Remark 1.2. *By this construction, we have that on each interval* $t \in [\tau_k, \tau_{k+1}[$ *that the regime switching CIR process* λ^i *defined in* (1.8) *is a classical CIR with constant parameters.*

1.2.2 Zero coupon bond price

We can now define the defaultable Zero coupon bond price.

Definition 1.4. *We will denote by* $\left(D_{t,T}^i\right)_{t\in[0,T]}$, $i = \{A, B\}$ *the price of a defaultable discounted bond price which pays \$1 at the maturity* T.

Using the partitioning time, the notation defined in the previous subsection and the general asset pricing theory in Harrison and Pliska [17] and [18], the conditional defaultable discounted bond price $D_{t,T}$ is given by

Proposition 1.1. *For* $i = \{A, B\}$*, we have for all* $t \in [0, T]$ *that*

$$D_{t,T}^i = (1 - H_t^i)\mathbb{E}\left[\exp\left(-\int_t^T (r_s + \lambda_s^i)ds\right) | \mathcal{F}_t^X, \lambda_0\right]. \tag{1.9}$$

Remark 1.3. *The quantity* $\left(r_t + \lambda_t^i\right)_{t\in[0,T]}$ *can be seen as a default-adjusted interest rate process. The part* $\left(\lambda_t^i\right)_{t\in[0,T]}$ *is the risk-neutral mean loss rate of the instrument due to the default of the firm* $i \in \{A, B\}$*. The quantity* $\left(r_t + \lambda_t^i\right)_{t\in[0,T]}$ *therefore represents the probability and the timing of default, as well as for the effect of losses on default. This model allows us to capture an economic health of each firm since for each firm* $i \in \{A, B\}$*, the stochastic process* (λ_t^i) *has parameters whose values depend on the credit notation of the firm. And by the construction of the migration process* X*, we have correlation between each firm notation. This allows the model to capture financial health correlation between each firm, like the impact of the default of one firm against the others.*

Our aim is so to obtain explicit formulas of (1.9). This is done by the following Theorem using two different methods to evaluate the conditional Laplace transform of λ^i. The first one uses a Ricatti approach and the second one an analytical approximation.

Theorem 1.1. *Under Assumptions 1.1 and assuming that X is independent of W and that the risk-free interest rate r is a deterministic function, then we have for $i \in \{A, B\}$ that the defaultable bond price can be obtained by two formulas:*

1. *Riccati Approach:*

$$D^i_{0,T} = \mathbb{E}\left[\exp\left(-\int_0^T r_s ds\right)\exp\left\{-\sum_{j=1}^{M} B_{M-j}(\Delta_{t_{j-1}})\right\}\exp\left(-A_0(\Delta_{t_0}, i_0)\lambda\right)\right] \tag{1.10}$$

where

$$A_0(\Delta_{t_0}) = \frac{2}{\gamma^1+\kappa^1} - \frac{4\gamma^1}{\gamma^1+\kappa^1}\frac{1}{(\gamma^1+\kappa^1)\exp(\gamma^1\Delta_{t_0})+\gamma^1-\kappa^1}$$

$$\begin{aligned} B_{M-j}(\Delta_{t_{j-1}}) &= -\frac{\kappa^{M-j+1}\theta^{M-j+1}(\gamma^{M-j+1}+\kappa^{M-j+1})}{(\sigma^{M-j+1})^2}\Delta_{t_{j-1}} \\ &+2\frac{\kappa^{M-j+1}\theta^{M-j+1}}{(\sigma^{M-j+1})^2}\ln\left((\gamma^{M-j+1}+\kappa^{M-j+1})\exp(\gamma^{M-j+1}\Delta_{t_{j-1}})+\gamma^{M-j+1}-\kappa^{M-j+1}\right) \\ &-2\frac{\kappa^{M-j+1}\theta^{M-j+1}}{(\sigma^{M-j+1})^2}\ln\left(2\gamma^{M-j+1}\right) \end{aligned}$$

$$\gamma^{M-j+1} = \sqrt{(\kappa^{M-j+1})^2+2(\sigma^{M-j+1})^2}$$

where we denote for simplicity $\kappa^j = \kappa(X_{t_j})$, $\theta^j = \theta(X_{t_j})$ and $\sigma^j = \sigma(X_{t_j})$

2. *Analytic Approximation:*

$$D^i_{0,T} = \mathbb{E}(\exp\left(-\int_0^T r_s ds\right)\exp\left\{-\frac{u}{2}\sum_{k=1}^{n} h^2_{n-k+1}a_{n-k+1}\kappa_{n-k}\theta_{n-k} - \frac{u}{2}h_1\lambda_0\left[1+a_1\left(1-\kappa_0 h_1\right)\right]\right\}$$
$$\times\exp\left\{\sum_{k=1}^{n}\ln\left(\mathbb{E}^{t_0}_{\lambda_0,X}\left[\exp\left(\frac{h^3_{n-k+1}}{8}u^2\sigma^2_{n-k}a^2_{n-k+1}\left[\lambda_0+\sum_{i=0}^{n-k}\kappa_i(\theta_i-\lambda_i)h_{i+1}+\sum_{i=0}^{n-k}\sigma_i\sqrt{\lambda_i}\Delta W_i\right]\right)\right]\right)\right\}) \tag{1.11}$$

where the sequence a is given by

$$a_{n-1} = 1+\frac{h_n}{h_{n-1}}+\frac{h_n}{h_{n-1}}a_n\left(1-h_n\kappa_{n-1}\right) \quad \text{and} \quad a_n = 1.$$

Remark 1.4. *The hypothesis that X is independent of W has an economic sense since for example $X = (X^A, X^B)$ could represent the credit notation of two countries given by an exogenous entity like a credit rating agencies.*

2 Conditional Laplace transform formulas

We are now going to prove the Theorem 1.1. More precisely, we will find two explicit formulas to evaluate the conditional Laplace transform of λ with respect to X denoted by Φ. It is given, for all $u \in \mathbb{C}$, by

$$\Phi_{0,T,\lambda,X}(u) \quad = \quad \mathbb{E}\left[\exp\left(-u\int_0^T \lambda_s ds\right)|\lambda_0 = \lambda, \mathcal{F}_T^X\right] = \mathbb{E}_{\lambda,X}\left[\exp\left(-u\int_0^T \lambda_s ds\right)\right]. \qquad (2.12)$$

Hence, our defaultable bond price formulas will be obtained as a particular case of this equation by taking $u = 1$.

2.1 A Ricatti approach

By Remark 1.1, there exists an increasing sequence of $\mathbb{F}^X$-stopping times in interval $[0,T]$, where the value of the Markov chain changes. We denote by Γ this subdivision

$$0 = \tau_0 < \tau_1 < \cdots < \tau_M = T$$

So in each time interval $[\tau_k, \tau_{k+1}[$, $k \in \{1, \dots n\}$ the process X is constant. And so the CIR regime switching process λ has constant parameters on this each time interval.

Proposition 2.2. *The conditional Laplace transform of the regime switching CIR process (for $u = 1$) between time $[\tau_k, \tau_{k+1}[$ with $\lambda_{\tau_k} = \lambda$ and $X_{\tau_{k+1}} = j \in \mathcal{S}^d$ is given by*

$$\Phi_{\tau_k,\tau_{k+1},j} := \mathbb{E}\left[\exp\left(-\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds\right)|\lambda_{\tau_k} = \lambda, X_{\tau_{k+1}} = j\right] = \exp\left\{-A(\Delta_{t_k}, j)\lambda - B(\Delta_{t_k}, j)\right\} \qquad (2.13)$$

where $\Delta_{t_k} = \tau_{k+1} - \tau_k$ and

$$A(\Delta_{t_k}, j) = \frac{2}{\gamma_j + \kappa_j} - \frac{4\gamma_j}{\gamma_j + \kappa_j}\frac{1}{(\gamma_j + \kappa_j)\exp(\gamma_j \Delta_{t_k}) + \gamma_j - \kappa_j}, \qquad (2.14)$$

$$B(\Delta_{t_k}, j) = -\frac{\kappa_j\theta_j(\gamma_j + \kappa_j)}{\sigma_j^2}\Delta_{t_k} + 2\frac{\kappa_j\theta_j}{\sigma_j^2}\ln\left((\gamma_j + \kappa_j)\exp(\gamma_j\Delta_{t_k}) + \gamma_j - \kappa_j\right) - 2\frac{\kappa_j\theta_j}{\sigma_j^2}\ln\left(2\gamma_j\right), \quad (2.15)$$

$$\gamma_j = \sqrt{\kappa_j^2 + 2\sigma_j^2}. \qquad (2.16)$$

Proof. We recall that the constant parameter CIR process defined in (1.6) is an affine process (see Duffie and al. [8]). So as in each step of time $[\tau_k, \tau_{k+1}[$, the stochastic process X is constant. So the process λ is a classical CIR with constant parameters on each step. So on each time interval $[\tau_k, \tau_{k+1}[$, the process λ is affine, hence we can assume that the expression of $\Phi_{\tau_k,\tau_{k+1},j}$ is given by the form

$$\exp\left\{-A(\Delta_{t_k}, j)\lambda_{\tau_k} - B(\Delta_{t_k}, j)\right\} \qquad (2.17)$$

for some functions $A(\Delta_{t_k}, j)$ and $B(\Delta_{t_k}, j)$ solution of a system of Riccati equation. Then the expected result is well known and can be found for instance in Cox and al. [7].

□

We would like now to give an explicit form of the conditional Laplace transform of the CIR process between time 0 and T. This is done by the following Theorem.

Theorem 2.2. *Assume that the intensity process (λ_t) follows a regime switching CIR, then we have for all $\lambda_0 = \lambda > 0$ and $X_{\tau_1} = i_0 \in \mathcal{S}^d$ that*

$$\begin{aligned}\Phi_{0,T,\lambda,X}(1) = \mathbb{E}\left[\exp\left(-\int_0^T \lambda_s ds\right) |\lambda_0 = \lambda, \mathcal{F}_T^X\right] &:= \mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] \\ &= \exp\left\{-\sum_{j=1}^M B_{M-j}(\Delta_{t_{j-1}})\right\}\exp\left(-A_0(\Delta_{t_0}, i_0)\lambda\right)\end{aligned}$$

where

$$A_0(\Delta_{t_0}) = \frac{2}{\gamma^1+\kappa^1} - \frac{4\gamma^1}{\gamma^1+\kappa^1}\frac{1}{(\gamma^1+\kappa^1)\exp(\gamma^1\Delta_{t_0})+\gamma^1-\kappa^1}, \tag{2.18}$$

$$\begin{aligned}B_{M-j}(\Delta_{t_{j-1}}) = & -\frac{\kappa^{M-j+1}\theta^{M-j+1}(\gamma^{M-j+1}+\kappa^{M-j+1})}{(\sigma^{M-j+1})^2}\Delta_{t_{j-1}} \\ & +2\frac{\kappa^{M-j+1}\theta^{M-j+1}}{(\sigma^{M-j+1})^2}\ln\left((\gamma^{M-j+1}+\kappa^{M-j+1})\exp(\gamma^{M-j+1}\Delta_{t_{j-1}})+\gamma^{M-j+1}-\kappa^{M-j+1}\right) \\ & -2\frac{\kappa^{M-j+1}\theta^{M-j+1}}{(\sigma^{M-j+1})^2}\ln\left(2\gamma^{M-j+1}\right),\end{aligned} \tag{2.19}$$

$$\gamma^{M-j+1} = \sqrt{(\kappa^{M-j+1})^2 + 2(\sigma^{M-j+1})^2}, \tag{2.20}$$

where we denote for simplicity $\kappa^j = \kappa(X_{t_j})$, $\theta^j = \theta(X_{t_j})$ and $\sigma^j = \sigma(X_{t_j})$.

Proof. We have a sequence of increasing times $0 = \tau_0 < \tau_1 < \cdots < \tau_M = T$ where the Markov chain X changes its value. Hence

$$\begin{aligned}\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] &= \mathbb{E}_{\lambda,X}\left[\exp\left(-\sum_{k=0}^{M-1}\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds\right)\right] = E_{\lambda,X}\left[\exp\left(-\sum_{k=0}^{M-1}\int_{\tau_k}^{\tau_k+\Delta_{t_k}} \lambda_s ds\right)\right] \\ &= \mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-1}\exp\left(-\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds\right)\right]\end{aligned}$$

By hypothesis, X is independent of W, then conditioning with respect to $\overline{\mathcal{F}}_{\tau_{M-1}} := \overline{\mathcal{F}}_{M-1}$, we obtain

$$\begin{aligned}\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] &= \mathbb{E}_{\lambda,X}\left[\mathbb{E}\left[\prod_{k=0}^{M-1}\exp\left(-\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds\right)|\overline{\mathcal{F}}_{M-1}\right]\right] \\ &= \mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-2}\exp\left(-\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds\right)\mathbb{E}\left[\exp\left(-\int_{\tau_{M-1}}^{\tau_M} \lambda_s ds\right)|\overline{\mathcal{F}}_{M-1}\right]\right].\end{aligned} \tag{2.21}$$

Moreover, we know that $\mathbb{E}\left[\exp\left(-\int_{\tau_{M-1}}^{\tau_M}\lambda_s ds\right)|\overline{\mathcal{F}}_{M-1}\right]$ is equal to $\Phi(\tau_{M-1},\tau_M,X_M)$, where X_M means X_{τ_M}. So applying Proposition 2.2, we get

$$\mathbb{E}\left[\exp\left(-\int_{\tau_{M-1}}^{\tau_M}\lambda_s ds\right)|\overline{\mathcal{F}}_{M-1}\right] = \exp\left\{-A_{M-1}(\Delta_{t_{M-1}},X_M)\lambda_{\tau_{M-1}} - B_{M-1}(\Delta_{t_{M-1}},X_M)\right\}.$$

We recall that the quantities $A_{M-1}(\Delta_{t_{M-1}},X_M)$ and $B_{M-1}(\Delta_{t_{M-1}},X_M)$ are constants. Hence replacing this result in the expectation (2.21) gives

$$\begin{aligned}\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T\lambda_s ds\right)\right] &= \mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-2}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds\right)\exp\left\{-A_{M-1}(\Delta_{t_{M-1}},X_M)\lambda_{\tau_{M-1}} - B_{M-1}(\Delta_{t_{M-1}},X_M)\right\}\right] \\ &= \exp\left\{-B_{M-1}(\Delta_{t_{M-1}},X_M)\right\}\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-2}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds - A_{M-1}(\Delta_{t_{M-1}},X_M)\lambda_{\tau_{M-1}}\right)\right].\end{aligned}$$

To simplify the notation of the calculus we will denote by A_{k-1} (resp. B_{k-1}) the quantity $A_{k-1}(\Delta_{t_{k-1}},X_k)$ (resp. $B_{k-1}(\Delta_{t_{k-1}},X_k)$). Hence

$$\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T\lambda_s ds\right)\right] = \exp\left\{-B_{M-1}\right\}\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-2}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds - A_{M-1}\lambda_{\tau_{M-1}}\right)\right].$$

We condition again with respect to $\overline{\mathcal{F}}_{M-2}$ to obtain

$$\begin{aligned}\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T\lambda_s ds\right)\right] &= \exp\left\{-B_{M-1}\right\}\mathbb{E}_{\lambda,X}\left[\mathbb{E}\left[\prod_{k=0}^{M-2}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds - A_{M-1}\lambda_{\tau_{M-1}}\right)|\overline{\mathcal{F}}_{M-2}\right]\right] \\ &= \exp\left\{-B_{M-1}\right\}\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-3}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds\right)\mathbb{E}\left[\exp\left(-\int_{\tau_{M-2}}^{\tau_{M-1}}\lambda_s ds - A_{M-1}\lambda_{\tau_{M-1}}\right)|\overline{\mathcal{F}}_{M-2}\right]\right].\end{aligned}$$

To continue, we need to evaluate the conditional expectation:

$$\varphi_{\tau_{M-2},\Delta_{t_{M-2}}} := \mathbb{E}\left[\exp\left(-\int_{\tau_{M-2}}^{\tau_{M-1}}\lambda_s ds - A_{M-1}\lambda_{\tau_{M-1}}\right)|\overline{\mathcal{F}}_{M-2}\right].$$

Lemma 2.1. *Assume for all $k\in\{1,\dots,M\}$ that the conditional expectation $\varphi_{\tau_{M-k},\Delta_{t_{M-k}}}$ has an exponential affine structure form given by*

$$\varphi_{\tau_{M-k},\Delta_{t_{M-k}}} = \exp\left(-A_{M-k}(\Delta_{t_{M-k}},X_{M-k+1})\lambda_{\tau_{M-k}} - B_{M-k}(\Delta_{t_{M-k}},X_{M-k+1})\right). \tag{2.22}$$

Then we can find explicit forms for functions $A_{M-k}(\Delta_{t_{M-k}},X_{M-k+1})$ and $B_{M-k}(\Delta_{t_{M-k}},X_{M-k+1})$ which are given explicitly by equations (2.14) and (2.15) under the conditions that $A_{M-k}(0) = A_{M-k+1}$ and $B_{M-k}(0)=0$.

Proof. Let $\varphi_{\tau_{M-k},\Delta_{t_{M-k}}} := \mathbb{E}\left[\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k+1}}\lambda_s ds - A_{M-k+1}\lambda_{\tau_{M-k+1}}\right)|\overline{\mathcal{F}}_{M-k}\right]$ then

$$\varphi_{\tau_{M-k},\Delta_{t_{M-k}}} = \mathbb{E}_{M-k}\left[\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k+1}}\lambda_s ds - A_{M-k+1}\lambda_{\tau_{M-k}+\Delta_{t_{M-k}}}\right)\right].$$

Taking a small time interval $dt \ll \Delta_{t_{M-2}}$ to obtain

$$\begin{aligned} &\mathbb{E}_{M-k}\left[\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k+1}} \lambda_s ds - A_{M-k+1}\lambda_{\tau_{M-k}+\Delta_{t_{M-k}}}\right)\right] \\ &= \mathbb{E}_{M-k}\left[\mathbb{E}_{M-k+dt}\left[\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k+1}} \lambda_s ds - A_{M-k+1}\lambda_{\tau_{M-k}+\Delta_{t_{M-k}}}\right)\right]\right]. \end{aligned}$$

Thus $\mathbb{E}_{M-k+dt}\left[\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k+1}} \lambda_s ds - A_{M-k+1}\lambda_{\tau_{M-k+1}}\right)\right]$

$$= \mathbb{E}_{M-k}\left[\varphi(\tau_{M-k}+dt, \Delta_{t_{M-k}}-dt)\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k}+dt} \lambda_s ds\right)\right].$$

We now use the hypothesis on the form of φ to get

$$\mathbb{E}_{M-k}\left[\exp\left(-A_{M-k}(\Delta_{t_{M-k}}-dt, X_{M-k+1})\lambda_{\tau_{M-k}+d_t} - B_{M-k}(\Delta_{t_{M-k}}-dt, X_{M-k+1})\right)\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k}+dt} \lambda_s ds\right)\right].$$

Then using our simplified notations we obtain

$$= \mathbb{E}_{M-k}\left[\exp\left(-A_{M-k}(\Delta_{t_{M-k}}-dt)\lambda_{\tau_{M-k}+d_t} - B_{M-k}(\Delta_{t_{M-k}}-dt)\right)\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k}+dt} \lambda_s ds\right)\right].$$

For small dt and using the stochastic differential equation of λ, we get

$$\begin{aligned} = \;& \mathbb{E}_{M-k}[\exp\{-A_{M-k}(\Delta_{t_{M-k}}-dt)\left[\lambda_{\tau_{M-k}} + \kappa^{M-k+1}\left(\theta^{M-k+1}-\lambda_{\tau_{M-k}}\right)dt + \sigma^{M-k+1}\sqrt{\lambda_{\tau_{M-k}}}dW_t\right] \\ & -B_{M-k}(\Delta_{t_{M-k}}-dt) - \lambda_{\tau_{M-k}}dt\}] \end{aligned}$$

where $\kappa^{M-k+1} = \kappa(X_{\tau_{M-k+1}})$, $\theta^{M-k+1} = \theta(X_{\tau_{M-k+1}})$ and $\sigma^{M-k+1} = \delta(X_{\tau_{M-k+1}})$.

$$\begin{aligned} = \;& \exp\left\{-A_{M-k}(\Delta_{t_{M-k}}-dt)\lambda_{\tau_{M-k}} - A_{M-k}(\Delta_{t_{M-k}}-dt)\kappa^{M-k+1}\left(\theta^{M-k+1}-\lambda_{\tau_{M-k}}\right)dt\right\} \\ & \times\exp\left\{-B_{M-k}(\Delta_{t_{M-k}}-dt) - \lambda_{\tau_{M-k}}d_t\right\}\mathbb{E}_{M-k}\left[\exp\left(-A_{M-k}(\Delta_{t_{M-k}}-dt)\sigma^{M-k+1}\sqrt{\lambda_{\tau_{M-k}}}dW_t\right)\right] \\ = \;& \exp\left\{-A_{M-k}(\Delta_{t_{M-k}}-dt)\lambda_{\tau_{M-k}} - A_{M-k}(\Delta_{t_{M-k}})\kappa^{M-k+1}\left(\theta^{M-k+1}-\lambda_{\tau_{M-k}}\right)dt\right\} \\ & \times\exp\left\{-B_{M-k}(\Delta_{t_{M-k}}-dt) - \lambda_{\tau_{M-k}}dt\right\}\exp\left(\frac{1}{2}A^2_{M-k}(\Delta_{t_{M-k}})(\sigma^{M-k+1})^2\lambda_{\tau_{M-k}}dt\right). \end{aligned}$$

By identifying with the assumed expression of φ in (2.22), we get

$$\begin{cases} A_{M-k}(\Delta_{t_{M-k}}) = A_{M-k}(\Delta_{t_{M-k}}-dt) - A_{M-k}(\Delta_{t_{M-k}})\kappa^{M-k+1}dt - \frac{1}{2}A^2_{M-k}(\Delta_{t_{M-k}})(\sigma^{M-k+1})^2dt + dt \\ B_{M-k}(\Delta_{t_{M-k}}) = B_{M-k}(\Delta_{t_{M-k}}-dt) + A_{M-k}(\Delta_{t_{M-k}})\kappa^{M-k+1}\theta^{M-k+1}dt \end{cases}$$

Taking dt close to zero,

$$\begin{cases} \frac{\partial A_{M-k}(\Delta_{t_{M-k}})}{\partial \Delta_{t_{M-k}}} = -A_{M-k}(\Delta_{t_{M-k}})\kappa^{M-k+1} - \frac{1}{2}A^2_{M-k}(\Delta_{t_{M-k}})(\sigma^{M-k+1})^2 + 1 \\ \frac{\partial B_{M-k}(\Delta_{t_{M-k}})}{\partial \Delta_{t_{M-k}}} = A_{M-k}(\Delta_{t_{M-k}})\kappa^{M-k+1}\theta^{M-k+1} \end{cases}$$

with conditions for $\Delta_{t_{M-k}} \equiv 0$, $A_{M-k}(0) = A_{M-k+1}$ and $B_{M-k}(0) = 0$.

Hence by Proposition 2.2, we know the explicit forms of $A_{M-k}(\Delta_{t_{M-k}})$ and $B_{M-k}(\Delta_{t_{M-k}})$ which are given by equations (2.14), (2.15) with the recursive condition that $A_{M-k}(0) = A_{M-k+1}$ and initial condition $B_{M-k}(0) = 0$ □

We continue the proof of the Theorem 2.2, by applying the Lemma 2.1 with $k = 2$, we obtain

$$\mathbb{E}\left[\exp\left(-\int_{\tau_{M-2}}^{\tau_{M-1}} \lambda_s ds - A_{M-1}\lambda_{\tau_{M-1}}\right) |\overline{\mathcal{F}}_{M-2}\right] = \exp\left(-A_{M-2}(\Delta_{t_{M-2}})\lambda_{\tau_{M-2}} - B_{M-2}(\Delta_{t_{M-2}})\right)$$

with deterministic function $A_{M-2}(\Delta_{t_{M-2}})$ and $B_{M-2}(\Delta_{t_{M-2}})$. Hence

$$\begin{aligned}
\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T r_s ds\right)\right] &= \exp\{-B_{M-1}\}\,\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-3}\exp\left(-\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds\right)\exp\left(-A_{M-2}(\Delta_{t_{M-2}})\lambda_{t_{M-2}} - B_{M-2}(\Delta_{t_{M-2}})\right)\right] \\
&= \exp\{-B_{M-1} - B_{M-2}\}\,\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-3}\exp\left(-\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds - A_{M-2}(\Delta_{t_{M-2}})\lambda_{\tau_{M-2}}\right)\right]
\end{aligned}$$

Conditioning an other time with respect to $\overline{\mathcal{F}}_{M-3}$, we obtain

$$\begin{aligned}
\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] &= \exp\{-B_{M-1} - B_{M-2}\}\,\mathbb{E}_{\lambda,X}\left[\mathbb{E}\left[\prod_{k=0}^{M-3}\exp\left(-\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds - A_{M-2}\lambda_{\tau_{M-2}}\right)|\overline{\mathcal{F}}_{M-3}\right]\right] \\
&= \exp\left\{-\sum_{j=1}^{2} B_{M-j}\right\}\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-4}\exp\left(-\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds\right)\mathbb{E}\left[\exp\left(-\int_{\tau_{M-3}}^{\tau_{M-2}} \lambda_s ds - A_{M-2}\lambda_{\tau_{M-2}}\right)|\overline{\mathcal{F}}_{M-3}\right]\right]
\end{aligned}$$

We can now again apply Lemma 2.1 with $k = 3$ and we obtain again that

$$\mathbb{E}\left[\exp\left(-\int_{\tau_{M-3}}^{\tau_{M-2}} \lambda_s ds - A_{M-2}\lambda_{\tau_{M-2}}\right) |\overline{\mathcal{F}}_{M-3}\right] = \exp\left(-A_{M-3}(\Delta_{t_{M-3}})\lambda_{\tau_{M-3}} - B_{M-3}(\Delta_{t_{M-3}})\right)$$

And so

$$\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] = \exp\left\{-\sum_{j=1}^{3} B_{M-j}\right\}\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-4}\exp\left(-\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds - A_{M-3}(\Delta_{t_{M-3}})\lambda_{\tau_{M-3}}\right)\right]$$

By iterating the conditioning with respect to $\overline{\mathcal{F}}_{M-k}$, k going to 4 to M and applying the Lemma 2.1 we finally obtain

$$\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] \quad = \quad \exp\left\{-\sum_{j=1}^{M} B_{M-j}\right\}\exp\left(-A_0(\Delta_{t_0})\lambda_{\tau_0}\right)$$

with by hypothesis $\lambda_{\tau_0} = \lambda$ and $A_0(\Delta_{t_0}) = A_0(\Delta_{t_0}, X_{\tau_1})$ with $X_{\tau_1} = i_0 \in \mathcal{S}^d$. □

We can obtain the general expression of the conditional Laplace transform of the regime switching CIR process using Theorem 2.2.

Corollary 2.2. *For all $u \in \mathbb{C}$, we have that the conditional Laplace transform of the regime switching CIR process with $\lambda_0 = \lambda$ and $X_{\tau_1} = i_0 \in \mathcal{S}^d$ is given by*

$$\begin{aligned}\Phi_{0,T,\lambda,X}(u) &:= \mathbb{E}_{\lambda,X}\left[\exp\left(-u\int_0^T \lambda_s ds\right)\right] \\ &= \exp\left\{-\sum_{j=1}^{M} \tilde{B}_{M-j}\left(\Delta_{t_{M-j}}\right)\right\}\exp\left(-\tilde{A}_0\left(\Delta_{t_0}, i_0\right)\lambda\right) \qquad (2.23)\end{aligned}$$

where the functions $\tilde{B}_{M-j}$ for $j = \{1,\ldots,M\}$ and $\tilde{A}_0$ are given by equations (2.18) and (2.19) taking parameters $\tilde{\kappa}^j := \kappa(\tilde{X}_{\tau_j}) = \kappa^j$, $\tilde{\theta}^j := \theta(\tilde{X}_{\tau_j}) = u\theta^j$ and $\tilde{\sigma}^j := \sigma(\tilde{X}_{\tau_j}) = \sqrt{u}\sigma^j$.

Proof. Since $\mathbb{E}\left[\exp\left(-u\int_0^T \lambda_s ds\right)|\lambda_0 = \lambda, \mathcal{F}_T^X\right] = \mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T (u\lambda_s)ds\right)\right]$. This is the conditional Laplace transform of a process $(u\lambda)_t$ which is still a CIR process with new parameters $\tilde{\kappa}_t = \kappa_t$, $\tilde{\theta}_t = u\theta_t$ and $\tilde{\sigma}_t = \sqrt{u}\sigma_t$, for all $t \in [0,T]$. Hence applying Theorem 2.2 with this set of parameters gives the expected result. □

2.2 Analytic approximation

We give now a second way to evaluate the defaultable bond. In fact, we give now an analytical approximation to evaluate the conditional Laplace transform of a regime switching CIR.

2.2.1 Construction of the new times grid

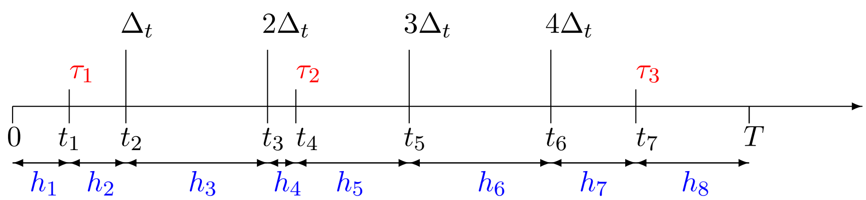
Let Δ_t be a fixed time step, then starting in time 0 we partition the time interval $[0,T]$ in time steps of

- size Δ_t if there is no jump of the Markov process between time 0 to Δ_t.
- size τ_1 if there is the first jump of the Markov process at stopping time τ_1 less than Δ_t.

Hence we denote by h_1 the first time step of size Δ_t or τ_1. Then we will proceed as the following: at time t_k, corresponding of the time after the step h_k, we construct the step h_{k+1} of size

- Δ_t if there is no jump of the Markov process between time t_k to $t_k + \Delta_t$.
- τ_i if there is the i jumps of the Markov process at stopping time τ_i less than $t_k + \Delta_t$.

As an example of this construction



This construction implies that $h_k = t_k - t_{k-1} \leq \Delta_t$ and that the parameters of the regime switching CIR are constants (and bounded) in these each time intervals $[t_k, t_{k+1}[$, $k \in \{0, 1, \ldots, n-$

1}. It follows as an application of the tree property of conditional expectation that the conditional Laplace transform of λ is given by

$$\begin{aligned}\Phi_{0,T,\lambda,X}(u) &:= \mathbb{E}\left[\exp\left(-u\int_0^T \lambda_s ds\right)|\lambda_0,\mathcal{F}_T^X\right] = \mathbb{E}_{\lambda_0,X}\left[\exp\left(-u\int_0^T \lambda_s ds\right)\right] \\ &= \mathbb{E}^{t_0}_{\lambda_0,X}\mathbb{E}^{t_1}_{\lambda,X}\dots\mathbb{E}^{t_{n-1}}_{\lambda,X}\left[\exp\left(-u\int_0^T \lambda_s ds\right)\right]. \end{aligned}\tag{2.24}$$

Proposition 2.3. *Let for all $k\in\{1,\dots,n-1\}$,*

$$F_k = \exp\left(\frac{h^3_{n-k+1}}{8}u^2\sigma^2_{n-k}a^2_{n-k+1}\lambda_{n-k}\right).\tag{2.25}$$

Then, we have

$$\mathbb{E}^{t_0}_{\lambda_0,X}\left[\frac{\exp\left(-u\int_0^T\lambda_s ds\right)}{\prod_{i=1}^{n-1}F_i}\right] = \exp\left(-\frac{u}{2}\sum_{k=1}^{n}h_k^2a_k\kappa_{k-1}\theta_{k-1} - \frac{u}{2}h_1\lambda_0\left[1+a_1\left(1-\kappa_0h_1\right)\right]\right)F_n\tag{2.26}$$

where

$$a_{n-1} = 1+\frac{h_n}{h_{n-1}}+\frac{h_n}{h_{n-1}}a_n\left(1-h_n\kappa_{n-1}\right)\quad \textit{and}\quad a_n=1.\tag{2.27}$$

Proof. Using trapezoidal rule, we obtain that the expectation at time t_{n-1} is given by

$$\begin{aligned}\mathbb{E}^{t_{n-1}}_{\lambda_0,X}\left[\exp\left(-u\int_0^T\lambda_s ds\right)\right] &= \mathbb{E}^{t_{n-1}}_{\lambda_0,X}\left[\exp\left(-u\sum_{i=1}^{n}\left(\frac{\lambda_i+\lambda_{i-1}}{2}h_i\right)\right)\right]\\ = \exp\left(-u\sum_{i=1}^{n-2}\left(\frac{\lambda_i+\lambda_{i-1}}{2}h_i\right)-u\frac{\lambda_{n-2}}{2}h_{n-1}\right)&\mathbb{E}^{t_{n-1}}_{\lambda,X}\left[\exp\left(-\frac{u}{2}\left[h_n\lambda_n+h_n\lambda_{n-1}+h_{n-1}\lambda_{n-1}\right]\right)\right].\end{aligned}$$

Using the approximation $\lambda_n \simeq \lambda_{n-1}+\kappa_{n-1}\left(\theta_{n-1}-\lambda_{n-1}\right)h_n+\sigma_{n-1}\sqrt{\lambda_{n-1}}\Delta W_{n-1}$ where $\Delta W_{n-1} = W_n - W_{n-1}$ and denote by G_{n-2} the quantity $\exp\left(-u\sum_{i=1}^{n-2}\left(\frac{\lambda_i+\lambda_{i-1}}{2}h_i\right)-u\frac{\lambda_{n-2}}{2}h_{n-1}\right)$. We obtain that $\mathbb{E}^{t_{n-1}}_{\lambda_0,X}\left[\exp\left(-u\int_0^T\lambda_s ds\right)\right]$ is equal to

$$\begin{aligned}&G_{n-2}\mathbb{E}^{t_{n-1}}_{\lambda_0,X}\left[\exp\left(-\frac{u}{2}\left[h_n\left(\lambda_{n-1}+\kappa_{n-1}\left(\theta_{n-1}-\lambda_{n-1}\right)h_n+\sigma_{n-1}\sqrt{\lambda_{n-1}}\Delta W_{n-1}\right)+h_n\lambda_{n-1}+h_{n-1}\lambda_{n-1}\right]\right)\right]\\ =\ &G_{n-2}\exp\left(-\frac{u}{2}\left[h_n\lambda_{n-1}+h_n^2\kappa_{n-1}\theta_{n-1}-h_n^2\kappa_{n-1}\lambda_{n-1}+h_n\lambda_{n-1}+h_{n-1}\lambda_{n-1}\right]\right)\\ &\times\mathbb{E}^{t_{n-1}}_{\lambda_0,X}\left[\exp\left(-\frac{u}{2}h_n\sigma_{n-1}\sqrt{\lambda_{n-1}}\Delta W_{n-1}\right)\right].\end{aligned}$$

Moreover we have that $\epsilon\sim\mathcal{N}(0,1)$ then for a constant K we know that

$$\mathbb{E}\left[\exp\left(K\sqrt{T}\epsilon\right)\right] = \exp\left(\frac{K^2T}{2}\right).\tag{2.28}$$

Applying (2.28) and factorize by $-\frac{u\lambda_{n-1}h_{n-1}}{2}$, we obtain that $\mathbb{E}^{t_{n-1}}_{\lambda_0,X}\left[\exp\left(-u\int_0^T\lambda_s ds\right)\right]$ is equal to

$$\begin{aligned}
&G_{n-2}\exp\left(\frac{u^2}{8}h_n^3\sigma_{n-1}^2\lambda_{n-1}a_n^2\right)\exp\left(-\frac{u\lambda_{n-1}h_{n-1}}{2}\left[1+\frac{h_n}{h_{n-1}}+\frac{h_n}{h_{n-1}}a_n\left(1-h_n\kappa_{n-1}\right)\right]-\frac{u}{2}h_n^2a_n\kappa_{n-1}\theta_{n-1}\right)\\
=&G_{n-2}\exp\left(-\frac{u}{2}h_n^2a_n\kappa_{n-1}\theta_{n-1}\right)\exp\left(-\frac{u\lambda_{n-1}h_{n-1}}{2}a_{n-1}\right)F_1.
\end{aligned}$$

Hence

$$\mathbb{E}^{t_{n-1}}_{\lambda_0,X}\left[\frac{\exp\left(-u\int_0^T\lambda_s ds\right)}{F_1}\right] = G_{n-2}\exp\left(\frac{u^2}{8}h_n^3\sigma_{n-1}^2\lambda_{n-1}\right)\exp\left(-\frac{u\lambda_{n-1}h_{n-1}}{2}a_{n-1}\right)$$

Then denoting $G_{n-3}=\exp\left(-u\sum_{i=1}^{n-3}\left(\frac{\lambda_i+\lambda_{i-1}}{2}h_i\right)-u\frac{\lambda_{n-3}}{2}h_{n-2}\right)$, we get the conditional expectation based on the information until t_{n-2}

$$\begin{aligned}
\mathbb{E}^{t_{n-2}}_{\lambda_0,X}\left[\mathbb{E}^{t_{n-1}}_{\lambda,X}\left[\frac{\exp\left(-u\int_0^T\lambda_s ds\right)}{F_1}\right]\right] &= G_{n-3}\exp\left(-\frac{u}{2}h_n^2a_n^2\kappa_{n-1}\theta_{n-1}\right)\\
&\times\mathbb{E}^{t_{n-2}}_{\lambda_0,X}\left[\exp\left(-\frac{u}{2}\left[\lambda_{n-1}h_{n-1}a_{n-1}+h_{n-2}\lambda_{n-2}+h_{n-1}\lambda_{n-2}\right]\right)\right]\\
&= G_{n-3}\exp\left(-\frac{u}{2}h_n^2a_n\kappa_{n-1}\theta_{n-1}\right)\exp\left(\frac{u^2}{8}h_{n-1}^3\sigma_{n-2}^2\lambda_{n-2}a_{n-1}^2\right)\\
&\quad\exp\left(-\frac{u}{2}\lambda_{n-2}h_{n-1}a_{n-1}-\frac{u}{2}\kappa_{n-2}\left(\theta_{n-2}-\lambda_{n-2}\right)h_{n-1}^2a_{n-1}-h_{n-2}\lambda_{n-2}-h_{n-1}\lambda_{n-2}\right)\\
&= G_{n-3}\exp\left(-\frac{u}{2}h_n^2a_n\kappa_{n-1}\theta_{n-1}\right)\exp\left(\frac{u^2}{8}h_{n-1}^3\sigma_{n-2}^2\lambda_{n-2}a_{n-1}^2\right)\exp\left(-\frac{u}{2}h_{n-1}^2a_{n-1}\kappa_{n-2}\theta_{n-2}\right)\\
&\quad\times\exp\left(-\frac{u}{2}\lambda_{n-2}h_{n-1}a_{n-1}+\frac{u}{2}\kappa_{n-2}\lambda_{n-2}h_{n-1}^2a_{n-1}-h_{n-2}\lambda_{n-2}-h_{n-1}\lambda_{n-2}\right)\\
&= G_{n-3}\exp\left(-\frac{u}{2}h_n^2a_n\kappa_{n-1}\theta_{n-1}-\frac{u}{2}h_{n-1}^2a_{n-1}\kappa_{n-2}\theta_{n-2}\right)F_2\\
&\quad\times\exp\left(-\frac{u}{2}\lambda_{n-2}h_{n-2}\left[1+\frac{h_{n-1}}{h_{n-2}}+\frac{h_{n-1}}{h_{n-2}}a_{n-1}\left(1-\kappa_{n-2}h_{n-1}\right)\right]\right)\\
&= G_{n-3}\exp\left(-\frac{u}{2}h_n^2a_n\kappa_{n-1}\theta_{n-1}-\frac{u}{2}h_{n-1}^2a_{n-1}\kappa_{n-2}\theta_{n-2}\right)\exp\left(-\frac{u}{2}\lambda_{n-2}h_{n-2}a_{n-2}\right)F_2.
\end{aligned}$$

Hence by iterations, we obtain

$$\begin{aligned}
\mathbb{E}^{t_{n-k}}_{\lambda_0,X}\left[\frac{\exp\left(-u\int_0^T\lambda_s ds\right)}{\prod_{i=1}^{k-1}F_i}\right] =&G_{n-k-1}\exp\left(-\frac{u}{2}\sum_{i=1}^{k}h_{n-k+i}^2a_{n-k+i}\kappa_{n-k+i-1}\theta_{n-k+i-1}\right)\\
&\times\exp\left(-\frac{u}{2}\lambda_{n-k}h_{n-k}a_{n-k}\right)F_k.
\end{aligned}$$

Then until time t_0, we finally obtain the expected result. $\square$

Theorem 2.3. *For all $u \in \mathbb{C}$, the conditional Laplace transform Φ of the regime switching CIR process is given by*

$$\begin{aligned}\ln\left(\Phi_{0,T,\lambda,X}(u)\right) &= \ln\left(\mathbb{E}^{t_0}_{\lambda_0,X}\left[\exp\left(-u\int_0^T \lambda_s ds\right)\right]\right)\\ &= -\frac{u}{2}\sum_{k=1}^{n} h_k^2 a_k \kappa_{k-1}\theta_{k-1} - \frac{u}{2}h_1\lambda_0\left[1+a_1\left(1-\kappa_0 h_1\right)\right]\\ +\sum_{k=1}^{n}\ln&\left(\mathbb{E}^{t_0}_{\lambda_0,X}\left[\exp\left\{\frac{h^3_{n-k+1}}{8}u^2\sigma^2_{n-k}a^2_{n-k+1}\left[\lambda_0+\sum_{i=0}^{n-k}\kappa_i(\theta_i-\lambda_i)h_{i+1}+\sum_{i=0}^{n-k}\sigma_i\sqrt{\lambda_i}\Delta W_i\right]\right\}\right]\right) \quad (2.29)\end{aligned}$$

where the sequence a is defined in Proposition 2.3.

Proof. As in [6], we see that it would be difficult to compute the expression $\mathbb{E}^{t_{n-k-1}}_{\lambda_0,X}\left[F_k\right]$ explicitly. That is why we simply approximate the expression F_k at time t_{n-k} by $\mathbb{E}^{0}_{\lambda_0,X}\left[F_k\right]$. Firstly, we can use the following approximation

$$\lambda_{n-k} \simeq \lambda_0 + \sum_{i=0}^{n-k}\kappa_i(\theta_i-\lambda_i)h_{i+1} + \sum_{i=0}^{n-k}\sigma_i\sqrt{\lambda_i}\Delta W_i.$$

Then

$$\begin{aligned}F_k &= \exp\left(\frac{h^3_{n-k+1}}{8}u^2\sigma^2_{n-k}a^2_{n-k+1}\lambda_{n-k}\right)\\ &= \exp\left(\frac{h^3_{n-k+1}}{8}u^2\sigma^2_{n-k}a^2_{n-k+1}\left[\lambda_0+\sum_{i=0}^{n-k}\kappa_i(\theta_i-\lambda_i)h_{i+1}+\sum_{i=0}^{n-k}\sigma_i\sqrt{\lambda_i}\Delta W_i\right]\right). \quad (2.30)\end{aligned}$$

Approximate the expression of F_k at time t_{n-k} by the expectation at time 0, we obtain

$$\begin{aligned}\ln\left(\mathbb{E}^{t_0}_{\lambda_0,X}\left[\frac{\exp\left(-u\int_0^T\lambda_s ds\right)}{\prod_{i=1}^{n-1}F_i}\right]\right) &\simeq \ln\left(\mathbb{E}^{t_0}_{\lambda_0,X}\left[\frac{\exp\left(-u\int_0^T\lambda_s ds\right)}{\prod_{k=1}^{n}\mathbb{E}^{t_0}_{\lambda_0,X}\left[F_k\right]}\right]\right)\\ &= \ln\left(\frac{\mathbb{E}^{t_0}_{\lambda_0,X}\left[\exp\left(-u\int_0^T\lambda_s ds\right)\right]}{\prod_{k=1}^{n}\mathbb{E}^{t_0}_{\lambda_0,X}\left[F_k\right]}\right)\\ &= \ln\left(\mathbb{E}^{t_0}_{\lambda_0,X}\left[\exp\left(-u\int_0^T\lambda_s ds\right)\right]\right) - \ln\left(\prod_{k=1}^{n}\mathbb{E}^{t_0}_{\lambda_0,X}\left[F_k\right]\right)\\ &= \ln\left(\Phi_{0,T,\lambda,X}(u)\right) - \sum_{k=1}^{n}\ln\left(\mathbb{E}^{t_0}_{\lambda_0,X}\left[F_k\right]\right).\end{aligned}$$

We conclude using (2.30).

□

Remark 2.5. *The analytic approximation formula given in previous Theorem could be also used as a way to simulate conditional Laplace transform.*

3 Simulations

3.1 Pricing zero coupon Bond in the two firms case with two regimes

We fixe the time maturity of zero coupon bond T equal to 10 (i.e. a ten years ahead maturity). We take a deterministic risk free interest rate equals to zero.

3.1.1 The model parameters and heuristic

The heuristic of the calculus of the defaultable Bond price is then done by a Monte Carlo approach with $MC \in \mathbb{N}$ steps:

1. We know the value of the infinitesimal generator Π^X of the credit migration process X. This one is given or estimated on some historical datas.

2. We generate a sequence of increasing stopping times and the times corresponding trajectory of X.

3. (a) We apply the formula (1.10) to calculate the price of this defaultable Bond price for the firm A or B.

   (b) We apply the construction of the time grid studied in subsection 2.2.1. Then we applied the formula (1.11).

4. We come back to step 2. until we will have done MC times this methods.

5. We evaluate the means of the MC values obtained in points 3 (a) and (b).

Hence assume that we have 2 regimes which represent a "normal" economic regime (regime 0) and a "crisis" regime (regime 1), then the credit migration process X is done in a set of four states: $\{(0,0);(1,0);(0,1);(1,1)\}$. For our simulation, in this part, we fix the infinitesimal generator Π^X of the credit migration process X equals to

$$\Pi^X = \begin{pmatrix} -0.1083 & 0.0455 & 0.0455 & 0.0174 \\ 0.0542 & -0.1644 & 0.0082 & 0.1004 \\ 0.0542 & 0.01 & -0.1644 & 0.1003 \\ 0.0542 & 0.01 & 0.01 & -0.0741 \end{pmatrix}$$

which corresponds to a transition matrix

$$P^X = \begin{pmatrix} 0.90 & 0.04 & 0.04 & 0.02 \\ 0.05 & 0.85 & 0.01 & 0.09 \\ 0.05 & 0.01 & 0.85 & 0.09 \\ 0.05 & 0.01 & 0.01 & 0.93 \end{pmatrix}$$

In other words, if we are in a state where only the firm A is on "crisis" (i.e. state (1,0)) the probability that the firm B goes into "crisis" in the next time step is 0.01. Figure 1 gives an example of the trajectory of the credit migration process X and of the sequence of stopping times τ where the credit migration process jumps.

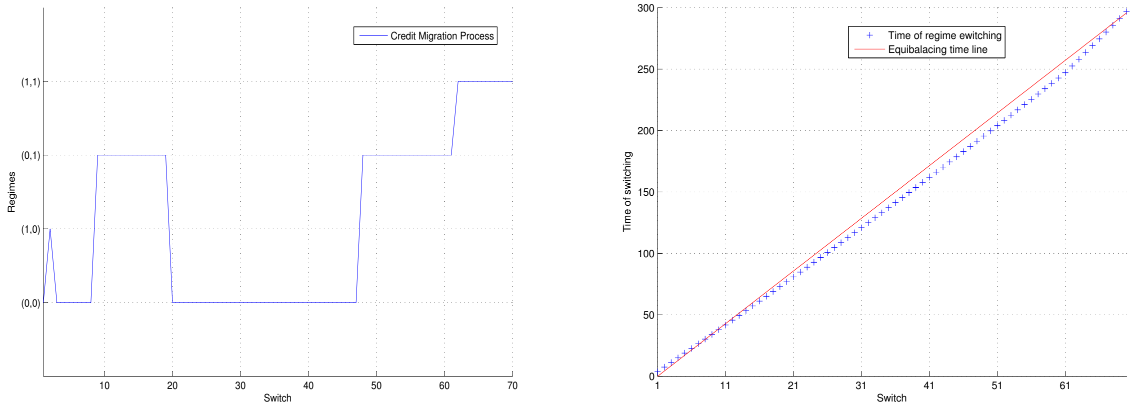


Figure 1: On left: Example of trajectory of the credit migration process X. On right: Example of instant of regime switching of the credit migration process X.

So we need to have four sets of CIR default intensity parameters. Let for $i \in \{A, B\}$, ν^i, ξ^i and ρ^i be real valued such that the set of parameters are given by Table 1

Parameters	κ_X	θ_X	σ_X
(0,0)	0.1	0.15	0.15
(1,0)	$0.1 + \nu^A$	$0.15 + \xi^A$	$0.15 + \rho^A$
(0,1)	$0.1 + \nu^B$	$0.15 + \xi^B$	$0.15 + \rho^B$
(1,1)	$0.1 + \nu^A + \nu^B$	$0.15 + \xi^A + \xi^B$	$0.15 + \rho^A + \rho^B$

Table 1: Parameters values of the CIR default intensity in the 2 regimes case.

Remark 3.6.
- *For $i \in \{A, B\}$, the constant ν^i, ξ^i and ρ^i are choosen such that the CIR condition holds, i.e. $2\kappa_X \theta_X \geq \sigma_X^2$.*
- *The state $(0, 0)$ can be seen as a standard economic state where nor firm A nor firm B are in crisis.*

3.1.2 Comparison of the different formulas to evaluate defaultable bond price.

Convergence:

We know that the formula of the conditional survey probability with respect to $\mathbb{G}$ is given by equation (1.4). We would like now to compare the different formulas to pricing defaultable zero coupon bond (i.e. formulas (1.4), (1.10) and (1.11)). In tables 2, 3 and 4, we resume the convergence results in the case of a four states regime parameters defined as in Table 1.

Remark 3.7. *We take for the time step parameter Δ_t (appearing in subsection 2.2.1 for the calculus of (1.11)) the value* 0.01.

Parameters:	ν^A	ν^B	ξ^A	ξ^B	ρ^A	ρ^B
Values:	0.2	0	0	0.3	0	0.1

Table 2: Values of the constant parameters defined in Table 1.

Regimes:	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
Bond price values:	0.6086	0.3777	0.2740	0.0668

Table 3: Values of the Bond price standard formula in $t = 0$ in each regime with a maturity $T = 10$ years.

Bond Price	Ricatti: 1.10 (std)	C.T.(sec.)	Analytic: 1.11 (Std)	C.T.	MC: 1.4	C.T.
$MC = 100$	0.5619 (0.1110)	1.94	0.5585 (0.1699)	15.98	0.6500	1.95
$MC = 300$	0.5692 (0.1015)	5.34	0.5587 (0.1602)	52.52	0.6233	6.61
$MC = 400$	0.5736 (0.0949)	6.87	0.5649 (0.1505)	60.48	0.6400	9.58
$MC = 500$	0.5748 (0.0927)	7.97	0.5658 (0.1511)	78.32	0.6360	12.44
$MC = 1000$	0.5738 (0.0961)	16.31	0.5654 (0.1533)	146.51	0.6220	33.23
$MC = 2000$	0.5727 (0.0995)	27.33	0.5646 (0.1533)	221.22	0.5770	96.78

Table 4: Results for the formulas convergence in $t = 0$ with initial regime the regime $(0,0)$ and maturity $T = 10$ years.

Bond Price	Analytic: 1.11
$\Delta_t = 1$	0.5612
$\Delta_t = 0.1$	0.5648
$\Delta_t = 0.01$	0.5649

In Table 4, we can see that all formulas converge when the number of Monte Carlo simulations increases. Whereas the bond price value given by formula (1.10) based on Riccati approach or formula (1.11) based on analytic approach converges quicker than the value given by formula (1.4). Indeed, it is sufficient to take 400 Monte Carlo simulations to converge while it is necessary to take at least 2000 Monte Carlo simulations with formula (1.4). The difference of 10^{-1} on the value given by (1.10) and (1.11) could be due to the error approximation of the conditional expectation at time t_{n-k} of F_k (see. proof of Theorem 2.3). Hence our two formulas need less simulations than formula (1.4) to converge. Moreover we observe that the Riccati approach formula (1.10) need a smaller computation time. Only 6.87 sec while formula (1.11) needs 60.48 sec and formula (1.4) needs 96.78 sec. Hence formula based on Riccati approach needs ten times less times than Analytic approach to converge. Whereas, we know that we used CIR model for intensity modeling since there exists explicit formula for bond. Hence as we said before, the analytic approach is interesting to obtain an explicit easy scheme to simulate defaultable bond.

Bond price with respect to the maturity T:

We observe in Table 5 and Figure 2 that the three formulas give similar results. Whereas, firstly, we made this simulation taking 2000 Monte Carlo simulations for the Probabilistic approach (formula

Bond Price	Ricatti: 1.10 ($MC = 400$)	Analytic: 1.11 ($MC = 400$)	MC: 1.4 ($MC = 2000$)
$T = 1$	0,9926	0,9923	0,9940
$T = 2$	0,9709	0,9696	0,9770
$T = 5$	0,8458	0,8405	0,8480
$T = 7$	0,7376	0,7261	0,7365
$T = 10$	0,5736	0,5649	0,5770
$T = 15$	0,3579	0,3948	0,3505

Table 5: Value of the Defaultable zero coupon bond price at time 0 with respect to the maturity time T with $\Delta_t = 0.01$.

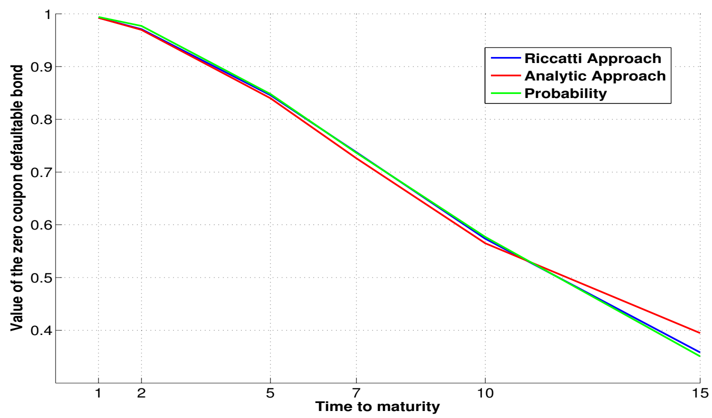


Figure 2: Graphs of the value of the Defaultable zero coupon bond price at time 0 with respect to the maturity time T with $\Delta_t = 0.01$ (MC=400 for the two first formulas and 2000 for the Probability approach).

(1.4)). Secondly, we remark, when the maturity T is greater than 10, that the result given by the analytic approximation is not better than the other. This relative mispricing was observed in the non regime switching case and uniform step time model discretization in [6] as soon as the maturity T is greater than 10.

3.1.3 Other simulations with Riccati approach formula.

Bond Price all over time $t \in [0, T]$

Taking parameters as in Table 2, we can draw the value of a defaultable zero coupon bond price over time $t \in [0, T]$ using formula (1.10). An example is given in Figure 3.

Bond Price in function of probability that B goes to crisis:

Taking parameters as in Table 2, we evaluate the price of a defaultable zero coupon bond in function of the probability $P\left(X_{t+\Delta_t} = (0,1) | X_t = (0,0)\right)$ and $P\left(X_{t+\Delta_t} = (1,1) | X_t = (0,0)\right)$. This

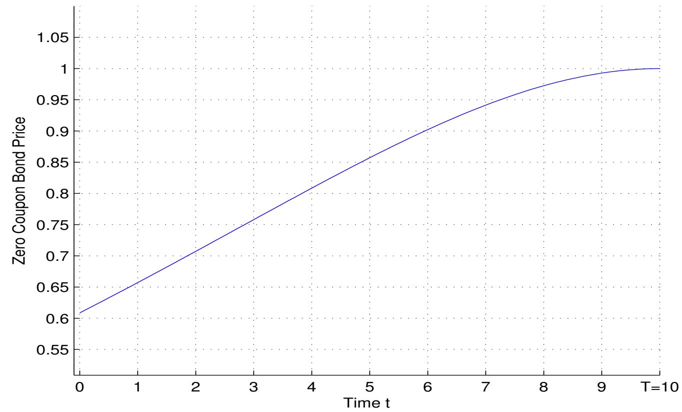


Figure 3: Price of a defaultable zero coupon bond price in each time t between time 0 to maturity T.

is $p^X_{1,3}$ and $p^X_{1,4}$. Hence we take a parametric transition matrix of the form:

$$P^X = \begin{pmatrix} 1-a-3b & a & 2b & b \\ 0.05 & 0.85 & 0.01 & 0.09 \\ 0.05 & 0.01 & 0.85 & 0.09 \\ 0.05 & 0.01 & 0.01 & 0.93 \end{pmatrix}$$

where $a, b \in [0,1]$. We obtain the following result: Hence we observe in Figure 4 that when b

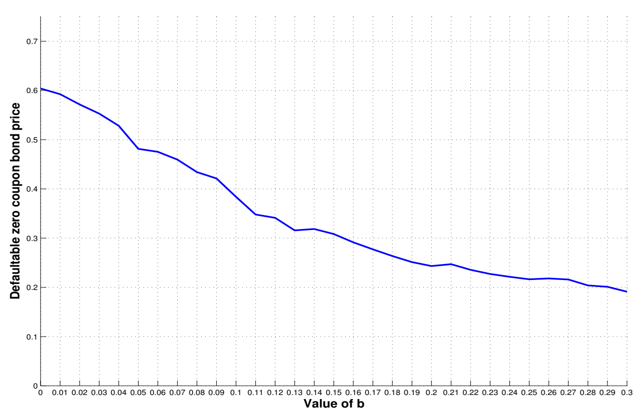


Figure 4: Price of a defaultable zero coupon bond price in $t = 0$ for maturity $T = 10$ and values of $a = 0.04$ in function of b.

grows up (i.e. the probability $P\left(X_{t+\Delta_t} = (1,1)|X_t = (0,0)\right)$), the price of the defaultable zero coupon bond price of the firm A decreases. This means that the economic status of the firm B (the probability to go in crisis) impacts the value of the defaultable zero coupon bond of the firm A.

3.2 Regime switching defaultable intensity estimation

We work now on real data. We would like to focus more on the modeling issue. We will show that our regime switching model could capture some market features or economics behavior. Hence, we can use this algorithm to estimate the intensities of the two firms and then construct the Markov copula as explained before.

3.2.1 Calibration on Greek sovereign spread between 19/10/2009 to 13/05/2010

First, Figure 5 shows the plot of the Greek sovereign spread between the 19/10/2009 to 13/05/2010. For the estimation, we use the estimation procedure developed and studied in Goutte

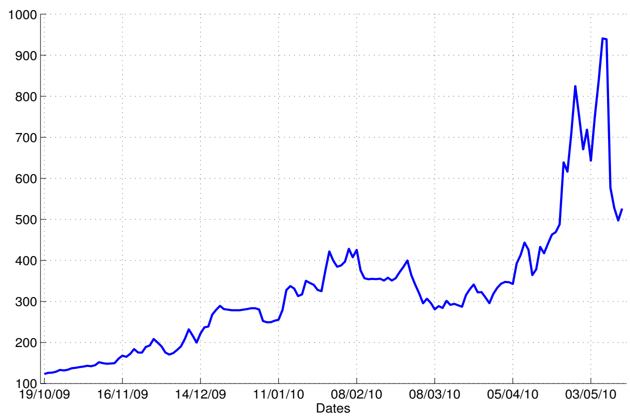


Figure 5: Greek Spread between the 19/10/2009 to 13/05/2010.

and Zou [15] for regime switching Cox Ingersoll Ross process applied to foreign exchange rate data. We assume that there are two regimes. This means that there is a "good" one and a "bad" one economies like a time crisis period and a "standard" economic period, or a spike time period and a non spike time period. Our results are stated in Table 6. Figure 6 gives a graphical representation

	Regime 1	Regime 2
$\hat{\kappa}$	0.022860	0.117918
$\hat{\theta}$	309.460660	620.721205
$\hat{\sigma}$	0.774675	3.092136
$\hat{\Pi}^X_{ii}$	0.974977	0.934452
π_i	0.723722	0.276278

Table 6: Maximum Likelihood estimation results.

of each time period in each regime.

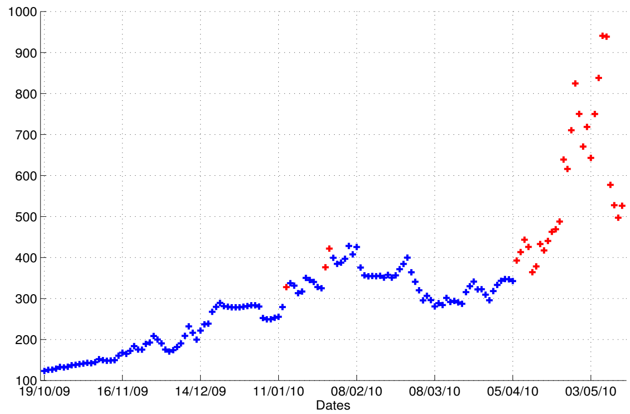


Figure 6: Greek Spread regime calibration between the 19/10/2009 to 13/05/2010. (The color blue is when we are in regime 1 and red for regime 2).

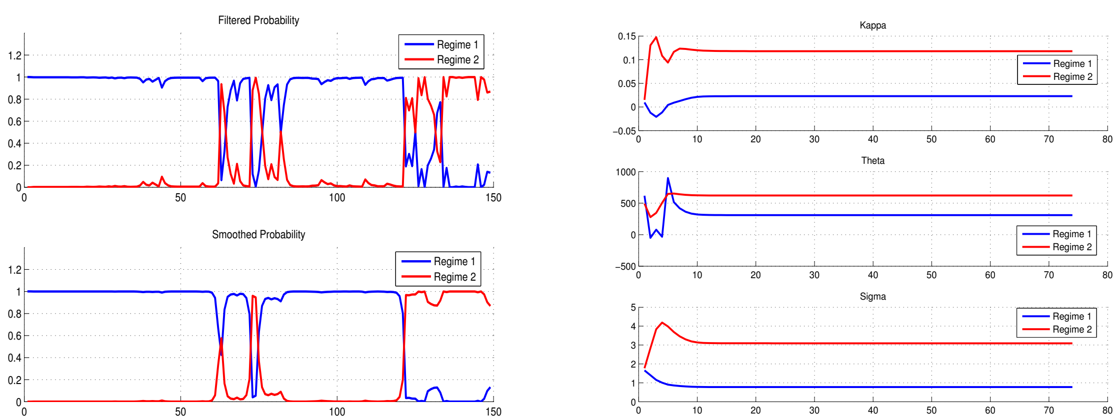


Figure 7: On left: Smoothed and Filtered probabilities. On right: Parameters convergence steps

3.2.2 Interpretations

We can see clearly in Figure 6 that there are two significant time periods. The first one between the 19/10/09 and april 2010, and the second one after april 2010. This period corresponds to the beginning of the economic world crisis . Hence we can see on the estimation results Table 6 that parameters values are very different in each regime. Before the crisis, we have a mean reverting parameter less than after crisis, $\hat{\kappa_1} = 0.02286$ against $\hat{\kappa_2} = 0.117918$. And in the time crisis period the volatility of the defaultable intensity is multiplied by 4 with respect to the volatility value before crisis.

We can see also in the right graph of Figure 7, the estimation process is fast, indeed only 15 iterations of our algorithm are sufficient for convergence to the true estimated parameters values.

Moreover, we can calculate the Regime classification measure (RCM) obtained by this regime switching model. In fact, let $K > 0$ be the number of regimes, the RCM statistics is then given by

$$RCM(K) = 100\left(1 - \frac{K}{K-1}\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{K}\left(P\left(X_t = i|\mathcal{F}_T^{\lambda};\hat{\Theta}\right) - \frac{1}{K}\right)^2\right) \tag{3.31}$$

where the quantity $P\left(X_t = i|\mathcal{F}_T^{\lambda};\hat{\Theta}\right)$ is the smoothed probability given in the left graph on Figure 7 and $\hat{\Theta}$ is the vector parameter estimation results $\left(i.e.\quad \hat{\Theta} := \left(\hat{\kappa}, \hat{\theta}, \hat{\sigma}, \hat{\Pi}^X\right)\right)$. The constant serves to normalize the statistic to be between 0 and 100. Good regime classification is associated with low RCM statistic value: a value of 0 means perfect regime classification and a value of 100 implies that no information about regimes is revealed. In our case we obtain a RCM equals to 8.41. Hence, it shows that this model with regime switching parameters captures very well two significant economics time period. And so this is a real add for the valuation of defaultable bond.

3.2.3 Methodology

Hence, we can apply this estimation method to find each estimated parameter for firms or countries A and B. Then using the copula construction theory defined in Corollary 1.1 and developed in section 1.2.1 we can apply pricing formulas given in Theorem 1.1 to obtain estimation of the price of defaultable bond regarding the correlation regime structure of each defaultable intensity regime switching estimations.

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