Homogenization of high-contrast two-phase conductivities perturbed by a magnetic field. Comparison between dimension two and dimension three.

Marc Briane, Laurent Pater

To cite this version:
Homogenization of high-contrast two-phase conductivities perturbed by a magnetic field. Comparison between dimension two and dimension three.

Marc BRIANE 
Institut de Recherche Mathématique de Rennes 
INSARennes 
mbriane@insa-rennes.fr

Laurent PATER 
Institut de Recherche Mathématique de Rennes 
Université de Rennes 1 
laurant.pater@ens-cachan.org

December 19, 2011

Abstract

Homogenized laws for sequences of high-contrast two-phase non-symmetric conductivities perturbed by a parameter $h$ are derived in two and three dimensions. The parameter $h$ characterizes the antisymmetric part of the conductivity for an idealized model of a conductor in the presence of a magnetic field. In dimension two an extension of the Dykhne transformation to non-periodic high conductivities permits to prove that the homogenized conductivity depends on $h$ through some homogenized matrix-valued function obtained in the absence of a magnetic field. This result is improved in the periodic framework thanks to an alternative approach, and illustrated by a cross-like thin structure. Using other tools, a fiber-reinforced medium in dimension three provides a quite different homogenized conductivity.

Keywords: homogenization, high-contrast conductivity, magneto-transport, strong field, two-phase composites.

AMS classification: 35B27, 74Q20

1 Introduction

The mathematical theory of homogenization for second-order elliptic partial differential equations has been widely studied since the pioneer works of Spagnolo on $G$-convergence [40], of Murat, Tartar on $H$-convergence [37, 38], and of Bensoussan, Lions, Papanicolaou on periodic structures [2], in the framework of uniformly bounded (both from below and above) sequences of conductivity matrix-valued functions. It is also known since the end of the seventies [24, 31] (see also the extensions [1, 22, 11, 32]) that the homogenization of the sequence of conductivity problems, in a bounded open set $\Omega$ of $\mathbb{R}^3$,

$$\begin{cases}
\text{div} (\sigma_n \nabla u_n) = f & \text{in } \Omega \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.1)

with a uniform boundedness from below but not from above for $\sigma_n$, may induce nonlocal effects. However, the situation is radically different in dimension two since the nature of problem (1.1) is shown [10, 13] to be preserved in the homogenization process provided that the sequence $\sigma_n$ is uniformly bounded from below.

$H$-convergence theory includes the case of non-symmetric conductivities in connection with the Hall effect [28] in electrodynamics (see, e.g., [33, 39]). Indeed, in the presence of a constant magnetic field the conductivity matrix is modified and becomes non-symmetric. Here, we consider an idealized model of an isotropic conductivity $\sigma(h)$ depending on a parameter $h$ which characterizes the antisymmetric part of the conductivity in the following way:
• in dimension two,
\[ \sigma(h) = \alpha I_2 + \beta h J, \quad J := \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \]  
where \( \alpha, \beta \) are scalar an \( h \in \mathbb{R} \),

• in dimension three,
\[ \sigma(h) = \alpha I_3 + \beta \mathcal{E}(h), \quad \mathcal{E}(h) := \left( \begin{array}{ccc} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{array} \right), \]  
where \( \alpha, \beta \) are scalar and \( h \in \mathbb{R}^3 \).

Since the seminal work of Bergman [3] the influence of a low magnetic field in composites has been studied for two-dimensional composites [34, 4, 17], and for columnar composites [7, 5, 8, 26, 27]. The case of a strong field, namely when the symmetric part and the antisymmetric part of the conductivity are of the same order, has been also investigated [6, 9]. Moreover, dimension three may induce anomalous homogenized Hall effects [20, 18, 19] which do not appear in dimension two [17].

In the context of high-contrast problems the situation is more delicate when the conductivities are not symmetric. An extension in dimension two of H-convergence for non-symmetric and non-uniformly bounded conductivities was obtained in [14] thanks to an appropriate div-curl lemma. More recently, the Keller, Dykhne [30, 23] two-dimensional duality principle which claims that the mapping
\[ A \mapsto \frac{A^T}{\det A} \]  
is stable under homogenization, was extended to high-contrast conductivities in [16]. However, the homogenization of both high-contrast and non-symmetric conductivities has not been precisely studied in the context of the strong field magneto-transport especially in dimension three. In this paper we establish an effective perturbation law for a mixture of two high-contrast isotropic phases in the presence of a magnetic field. The two-dimensional case is performed in a general way for non-periodic and periodic microstructures. It is then compared to the case of a three-dimensional fiber-reinforced microstructure.

In dimension two, following the modelization (1.2), consider a sequence \( \sigma_n(h) \) of isotropic two-phase matrix-valued conductivities perturbed by a fixed constant \( h \in \mathbb{R} \), and defined by
\[ \sigma_n(h) := (1 - \chi_n)(\alpha_1 I_2 + \beta_1 h J) + \chi_n(\alpha_{2,n} I_2 + \beta_{2,n} h J), \]  
where \( \chi_n \) is the characteristic function of phase 2, with volume fraction \( \theta_n \to 0, \alpha_1 > 0, \beta_1 \) are the constants of the low conducting phase 1, and \( \alpha_{2,n} \to \infty, \beta_{2,n} \) are real sequences of the highly conducting phase 2 where \( \beta_{2,n} \) is possibly unbounded. The coefficients \( \alpha_1, \beta_1 \), respectively \( \alpha_{2,n} \) and \( \beta_{2,n} \) also have the same order of magnitude according to the strong field assumption. Assuming that the sequence \( \theta_n^{-1} \chi_n \) converges weakly-* in the sense of the Radon measures to a bounded function, and that \( \theta_n \alpha_{2_n}, \theta_n \beta_{2,n} \) converge respectively to constants \( \alpha_2 > 0, \beta_2 \), we prove (see Theorem 2.2) that the perturbed conductivity \( \sigma_n(h) \) converges in an appropriate sense of H-convergence (see Definition 1.1) to the homogenized matrix-valued function
\[ \sigma_*(h) = \sigma_*^0(\alpha_1, \alpha_2 + \alpha_2^{-1} \beta_2^2 h^2) + \beta_1 h J, \]  
for some matrix-valued function \( \sigma_*^0 \) which depends uniquely on the microstructure \( \chi_n \) in the absence of a magnetic field, and is defined for a subsequence of \( n \). The proof of the result is based on a Dykhne transformation of the type
\[ A_n \mapsto \left( (p_n A_n + q_n J)^{-1} + r_n J \right)^{-1}, \]  
which permits to change the non-symmetric conductivity \( \sigma_n(h) \) into a symmetric one. Then, extending the duality principle (1.4) established in [16], we prove that transformation (1.7) is also stable under high-contrast conductivity homogenization.
In the periodic case, i.e. when $\sigma_n(h)(\cdot) = \Sigma_n(\cdot/\varepsilon_n)$ with $\Sigma_n$ $Y$-periodic and $\varepsilon_n \to 0$, we use an alternative approach based on an extension of Theorem 4.1 of [13] to $\varepsilon_n Y$-periodic but non-symmetric conductivities (see Theorem 3.1). So, it turns out that the homogenized conductivity $\sigma_\ast(h)$ is the limit as $n \to \infty$ of the constant $H$-limit $(\sigma_n)_\ast$ associated with the periodic homogenization (see, e.g., [2]) of the oscillating sequence $\Sigma_n(\cdot/\varepsilon)$ as $\varepsilon \to 0$ and for a fixed $n$. Finally, the Dykhne transformation performed by Milton [34] (see also [35], Chapter 4) applied to the local periodic conductivity $\Sigma_n$ and its effective conductivity $(\sigma_n)_\ast$, allows us to recover the perturbed homogenized formula (1.6). An example of a periodic cross-like thin structure provides an explicit computation of $\sigma_\ast(h)$ (see Proposition 3.2).

To make a comparison with dimension three we restrict ourselves to the $\varepsilon_n Y$-periodic fiber-reinforced structure introduced by Fenchelko, Khruslov [24] to derive a nonlocal effect in homogenization. However, in the present context the fiber radius $r_n$ is chosen to be super-critical, i.e. $r_n \to 0$ and $\varepsilon_n^2 |\ln r_n| \to 0$, in order to avoid such an effect. Similarly to (1.5) and following the modelization (1.3), the perturbed conductivity is defined for $h \in \mathbb{R}^3$, by

$$
\sigma_n(h) := (1 - \chi_n)(\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) + \chi_n(\alpha_2 I_3 + \beta_2 \mathcal{E}(h)),
$$

where $\chi_n$ is the characteristic function of the fibers which are parallel to the direction $e_3$. The form of (1.8) ensures the rotational invariance of $\sigma_n(h)$ for those orthogonal transformations which leave $h$ invariant. Under the same assumptions on the conductivity coefficients as in the two-dimensional case, with $\theta_n = \pi r_n^2$, but using a quite different approach, the homogenized conductivity is given by (see Theorem 4.1)

$$
\sigma_\ast(h) = \alpha_1 I_3 + \left(\frac{\alpha_2^2 + \alpha_2 \beta_2^2 |h|^2}{\alpha_2 + \beta_2^2 |h|^2}\right) e_3 \otimes e_3 + \beta_1 \mathcal{E}(h).
$$

The difference between formulas (1.6) and (1.9) provides a new example of gap between dimension two and dimension three in the high-contrast homogenization framework. As former examples of dimensional gap, we refer to the works [17, 20] about the 2d positivity property, versus the 3d non-positivity, of the effective Hall coefficient, and to the works [13, 24] concerning the 2d lack, versus the 3d appearance, of nonlocal effects in the homogenization process.

The paper is organized as follows. Section 2 and 3 deal with dimension two. In Section 2 we study the two-dimensional general (non-periodic) case thanks to an appropriate div-curl lemma. In Section 3 an alternative approach is performed in the periodic framework. Finally, Section 4 is devoted to the three-dimensional case with the fiber-reinforced structure.

Notations

- $\Omega$ denotes a bounded open subset of $\mathbb{R}^d$;
- $I_d$ denotes the unit matrix in $\mathbb{R}^{d \times d}$, and $J := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$;
- for any matrix $A$ in $\mathbb{R}^{d \times d}$, $A^T$ denotes the transposed of the matrix $A$, $A^s$ denotes its symmetric part;
- for $h \in \mathbb{R}^3$, $\mathcal{E}(h)$ denotes the antisymmetric matrix in $\mathbb{R}^{3 \times 3}$ defined by $\mathcal{E}(h) x := h \times x$, for $x \in \mathbb{R}^3$;
- for any $A, B \in \mathbb{R}^{d \times d}$, $A \leq B$ means that for any $\xi \in \mathbb{R}^d$, $A \xi \cdot \xi \leq B \xi \cdot \xi$; we will use the fact that for any invertible matrix $A \in \mathbb{R}^{d \times d}$, $A \geq \alpha I_d \Rightarrow A^{-1} \leq \alpha^{-1} I_d$;
- $| \cdot |$ denotes both the euclidean norm in $\mathbb{R}^d$ and the subordinate norm in $\mathbb{R}^{d \times d}$;
- for any locally compact subset $X$ of $\mathbb{R}^d$, $\mathcal{M}(X)$ denotes the space of the Radon measures defined on $X$;
• for any $\alpha, \beta > 0$, $\mathcal{M}(\alpha, \beta; \Omega)$ is the set of the invertible matrix-valued functions $A : \Omega \to \mathbb{R}^{d \times d}$ such that

$$\forall \xi \in \mathbb{R}^d, \quad A(x) \xi \cdot \xi \geq \alpha |\xi|^2 \quad \text{and} \quad A^{-1}(x) \xi \cdot \xi \geq \beta^{-1} |\xi|^2 \quad \text{a.e. in} \ \Omega; \quad (1.10)$$

• $C$ denotes a constant which may vary from a line to another one.

In the sequel, we will use the following extension of $H$-convergence and introduced in [16]:

Definition 1.1 Let $\alpha_n$ and $\beta_n$ be two sequences of positive numbers such that $\alpha_n \leq \beta_n$, and let $A_n$ be a sequence of matrix-valued functions in $\mathcal{M}(\alpha_n, \beta_n; \Omega)$ (see (1.10)). The sequence $A_n$ is said to $H(\mathcal{M}(\Omega)^2)$-converge to the matrix-valued function $A_*$ if for any distribution $f$ in $H^{-1}(\Omega)$, the solution $u_n$ of the problem

$$\begin{cases}
\text{div } (A_n \nabla u_n) = f & \text{in } \Omega \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}$$

satisfies the convergences

$$\begin{cases}
A_n \nabla u_n \rightharpoonup A_\ast \nabla u & \text{in } H^{1}_0(\Omega) \\
u_n \rightharpoonup u & \text{weakly-* in } \mathcal{M}(\Omega)^2,
\end{cases}$$

where $u$ is the solution of the problem

$$\begin{cases}
\text{div } (A_\ast \nabla u) = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

We now give a notation for $H(\mathcal{M}(\Omega)^2)$-limits of high-contrast two-phase composites. We consider the characteristic function $\chi_n$ of the highly conducting phase, and denote $\omega_n := \{\chi_n = 1\}$.

Notation 1.1 A sequence of isotropic two-phase conductivities in the absence of a magnetic field is denoted by

$$\sigma_n^0(\alpha_{1,n}, \alpha_{2,n}) := (1 - \chi_n)\alpha_{1,n}I_2 + \chi_n\alpha_{2,n}I_2; \quad (1.11)$$

with

$$\lim_{n \to \infty} \alpha_{1,n} = \alpha_1 > 0 \quad \text{and} \quad \lim_{n \to \infty} |\omega_n| \alpha_{2,n} = \alpha_2 > 0, \quad (1.12)$$

and its $H(\mathcal{M}(\Omega)^2)$-limit is denoted by $\sigma^0_\ast(\alpha_1, \alpha_2)$.

2 A two-dimensional non-periodic medium

2.1 A div-curl approach

We extend the classical div-curl lemma.

Lemma 2.1 Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$. Let $\alpha > 0$, let $\bar{a} \in L^\infty(\Omega)$ and let $A_n$ be a sequence of matrix-valued functions in $L^\infty(\Omega)^{2 \times 2}$ (not necessarily symmetric) satisfying

$$A_n \geq \alpha I_2 \quad \text{a.e. in } \Omega \quad \text{and} \quad \frac{\det A_n}{|A_n|} \rightharpoonup \bar{a} \in L^\infty(\Omega) \quad \text{weakly-* in } \mathcal{M}(\Omega). \quad (2.1)$$

Let $\xi_n$ be a sequence in $L^2(\Omega)^2$ and $v_n$ a sequence in $H^1(\Omega)$ satisfying the following assumptions:

(i) $\xi_n$ and $v_n$ satisfy the estimate

$$\int_{\Omega} A_n^{-1} \xi_n : \xi_n \, dx + ||v_n||_{H^1(\Omega)} \leq C; \quad (2.2)$$
(ii) $\xi_n$ satisfies the classical condition
\[
\text{div } \xi_n \text{ is compact in } H^{-1}(\Omega).
\] (2.3)

Then, there exist $\xi$ in $L^2(\Omega)^2$ and $v$ in $H^1(\Omega)$ such that the following convergences hold true up to a subsequence
\[
\xi_n \rightharpoonup^* \xi \text{ weakly-* in } \mathcal{M}(\Omega)^2 \quad \text{and} \quad \nabla v_n \rightharpoonup \nabla v \text{ weakly in } L^2(\Omega)^2.
\] (2.4)

Moreover, we have the following convergence in the distribution sense
\[
\xi_n \cdot \nabla v_n \rightharpoonup \xi \cdot \nabla v \text{ weakly in } \mathcal{D}'(\Omega).
\]

Proof of Lemma 2.1. The proof consists in considering the "good-divergence" sequence $\xi_n$ as a sum of a compact sequence of gradients $\nabla u_n$ and a sequence of divergence-free functions $J\nabla z_n$. We then use Lemma 3.1 of [16] to obtain the strong convergence of $z_n$ in $L^2_{\text{loc}}(\Omega)$. Finally, replacing $\xi_n$ by $\nabla u_n + J\nabla z_n$, we conclude owing to integration by parts.

First step: Proof of convergences (2.4).

An easy computation gives
\[
\left( (A_n^{-1})^s \right)^{-1} = \frac{\det A_n}{\det A_n^s}. \tag{2.5}
\]

The sequence $\xi_n$ is bounded in $L^1(\Omega)^2$ since the Cauchy-Schwarz inequality combined with the weak-* convergence of (2.1), (2.2) and (2.5) yields
\[
\left( \int_\Omega |\xi_n| \, dx \right)^2 \leq \int_\Omega \left( (A_n^{-1})^s \right)^{-1} \, dx \int_\Omega (A_n^{-1})^s \xi_n \cdot \xi_n \, dx = \int_\Omega \frac{\det A_n}{\det A_n^s} |A_n^s| \, dx \int_\Omega A_n^{-1} \xi_n \cdot \xi_n \, dx \leq C.
\]

Therefore, $\xi_n$ converges up to a subsequence to some $\xi \in \mathcal{M}(\Omega)^2$ in the weak-* sense of the measures. Let us prove that the vector-valued measure $\xi$ is actually in $L^2(\Omega)^2$. Again by the Cauchy-Schwarz inequality combined with (2.1), (2.2) and (2.5) we have, for any $\Phi \in C_0^\infty(\Omega)^2$,
\[
\left| \int_\Omega \xi(dx) \cdot \Phi \right| = \lim_{n \to \infty} \left| \int_\Omega \xi_n \cdot \Phi \, dx \right| \\
\leq \limsup_{n \to \infty} \left( \int_\Omega \frac{\det A_n}{\det A_n^s} |A_n^s| |\Phi|^2 \, dx \right)^{\frac{1}{2}} \cdot \left( \int_\Omega A_n^{-1} \xi_n \cdot \xi_n \, dx \right)^{\frac{1}{2}} \leq C \left( \int_\Omega a |\Phi|^2 \, dx \right)^{\frac{1}{2}},
\]

which implies that $\xi$ is absolutely continuous with respect to the Lebesgue measure. Since $a \in L^\infty(\Omega)$, we also get that
\[
\left| \int_\Omega \xi \cdot \Phi \, dx \right| \leq ||\Phi||_{L^2(\Omega)^2}
\]

hence $\xi \in L^2(\Omega)^2$. Therefore, the first convergence of (2.4) holds true with its limit in $L^2(\Omega)^2$. The second one is immediate.


By (2.3), the sequence $u_n$ in $H^1_0(\Omega)$ defined by $u_n := \Delta^{-1} (\text{div } \xi_n)$ strongly converges in $H^1_0(\Omega)$:
\[
u_n \to u \quad \text{in } H^1_0(\Omega). \tag{2.6}
\]

Let $\omega$ be a regular simply connected open set such that $\omega \subset \subset \Omega$. Since by definition $\xi_n - \nabla u_n$ is a divergence-free function in $L^2(\Omega)^2$, there exists (see, e.g., [25]) a unique stream function $z_n \in H^1(\omega)$ with zero $\omega$-average such that
\[
\xi_n = \nabla u_n + J\nabla z_n \quad \text{a.e. in } \omega. \tag{2.7}
\]
Third step: Convergence of the stream function $z_n$.

Since $\nabla u_n$ is bounded in $L^2(\Omega)^2$ by the second step, $\xi_n$ is bounded in $L^1(\Omega)^2$ by the first step and $z_n$ has a zero $\omega$-average, the Sobolev embedding of $W^{1,1}(\omega)$ into $L^2(\omega)$ combined with the Poincaré-Wirtinger inequality in $\omega$ implies that $z_n$ is bounded in $L^2(\omega)$ and thus converges, up to a subsequence still denoted by $n$, to a function $z$ in $L^2(\omega)$. Moreover, let us define

$$S_n := (J^{-1}(A_n^{-1})^s J)^{-1}.$$ 

The Cauchy-Schwarz inequality gives

$$\int_{\omega} S_n^{-1} \nabla z_n \cdot \nabla z_n \, dx = \int_{\omega} J^{-1}(A_n^{-1})^s J \nabla z_n \cdot \nabla z_n \, dx$$

$$= \int_{\omega} (A_n^{-1})^s J \nabla z_n \cdot J \nabla z_n \, dx$$

$$= \int_{\omega} (A_n^{-1})^s [\xi_n - \nabla u_n] \cdot [\xi_n - \nabla u_n] \, dx$$

$$\leq 2 \int_{\omega} (A_n^{-1})^s \xi_n \cdot \xi_n \, dx + 2 \int_{\omega} (A_n^{-1})^s \nabla u_n \cdot \nabla u_n \, dx$$

$$= 2 \int_{\omega} A_n^{-1} \xi_n \cdot \xi_n \, dx + 2 \int_{\omega} A_n^{-1} \nabla u_n \cdot \nabla u_n \, dx.$$ 

The first term is bounded by (2.2) and the last term by the inequality $A_n^{-1} \leq \alpha^{-1} I_2$ and the convergence (2.6). Therefore, the sequences $v_n := z_n$ and, by (2.14), $S_n$ satisfy all the assumptions of Lemma 3.1 of [16] since, by (2.5),

$$S_n = \frac{\det A_n}{\det A_s} J^{-1} A_s^s J.$$ 

Then, we obtain the convergence

$$z_n \rightarrow z \quad \text{strongly in } L^2_{\text{loc}}(\omega). \quad (2.8)$$ 

Moreover, the convergence (2.6) gives

$$\xi = \nabla u + J \nabla z \quad \text{in } \mathcal{D}'(\omega). \quad (2.9)$$ 

Fourth step: Integration by parts and conclusion.

We have, as $J \nabla v_n$ is a divergence-free function,

$$\xi_n \cdot \nabla v_n = (\nabla u_n + J \nabla z_n) \cdot \nabla v_n = \nabla u_n \cdot \nabla v_n - \text{div} (z_n j \nabla v_n). \quad (2.10)$$

The strong convergence of $\nabla u_n$ in (2.6), the second weak convergence of (2.4) justified in the first step and (2.8) give

$$\nabla u_n \cdot \nabla v_n - \text{div} (z_n J \nabla v_n) \rightarrow \nabla u \cdot \nabla v - \text{div} (z J \nabla v) \quad \text{in } \mathcal{D}'(\omega). \quad (2.11)$$

We conclude, by combining this convergence with (2.10), (2.9) and integrating by parts, to the convergence

$$\xi_n \cdot \nabla v_n \rightarrow \nabla u \cdot \nabla v - \text{div} (z J \nabla v) = (\nabla u + J \nabla z) \cdot \nabla v = \xi \cdot \nabla v \quad \text{weakly in } \mathcal{D}'(\omega).$$ 

for an arbitrary open subset $\omega$ of $\Omega$. 

For the reader’s convenience, we first recall in Theorem 2.1 below the main result of [16] concerning the Keller duality for high contrast conductivities. Then, Proposition 2.1 is an extension of this result to a more general transformation.
Theorem 2.1 ([16]) Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$ such that $|\partial \Omega| = 0$. Let $\alpha > 0$, let $\beta_n, n \in \mathbb{N}$ be a sequence of real numbers such that $\beta_n \geq \alpha$, and let $A_n$ be a sequence of matrix-valued functions (not necessarily symmetric) in $\mathcal{M}(\alpha, \beta; \Omega)$. Assume that there exists a function $\tilde{a} \in L^\infty(\Omega)$ such that
\[
 \frac{\det A_n}{\det A_n^T} |A_n^*| \to \tilde{a} \text{ weakly-* in } \mathcal{M}(\Omega).
\] (2.12)

Then, there exist a subsequence of $n$, still denoted by $n$, and a matrix-valued function $A_*$ in $\mathcal{M}(\alpha, \beta; \Omega)$, with $\beta = 2||\tilde{a}||_{L^\infty(\Omega)}$, such that
\[
 A_n \overset{H(\mathcal{M}(\Omega)^2)}{\longrightarrow} A_* \quad \text{and} \quad \frac{A_n^T}{\det A_n} \overset{H(\mathcal{M}(\Omega)^2)}{\longrightarrow} \frac{A_*^T}{\det A_*}.
\] (2.13)

Proposition 2.1 Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$ such that $|\partial \Omega| = 0$. Let $p_n, q_n$ and $r_n$, $n \in \mathbb{N}$ be sequences of real numbers converging respectively to $p > 0$, $q$ and $0$. Let $\alpha > 0$, let $\beta_n$, $n \in \mathbb{N}$ be a sequence of real numbers such that $\beta_n \geq \alpha$, and let $A_n$ be a sequence of matrix-valued functions in $\mathcal{M}(\alpha, \beta_n; \Omega)$ (not necessarily symmetric) satisfying
\[
r_n A_n \text{ is bounded in } L^\infty(\Omega)^{2 \times 2} \quad \text{and} \quad \frac{\det A_n}{\det A_n^T} |A_n^*| \to \tilde{a} \in L^\infty(\Omega) \text{ weakly-* in } \mathcal{M}(\Omega),
\] (2.14)

and that
\[
 B_n = ((p_n A_n + q_n J)^{-1} + r_n J)^{-1} \text{ is a sequence of symmetric matrices.}
\] (2.15)

Then, there exist a subsequence of $n$, still denoted by $n$, and a matrix-valued function $A_*$ in $\mathcal{M}(\alpha, \beta; \Omega)$, with $\beta = 2||\tilde{a}||_{L^\infty(\Omega)}$, such that
\[
 A_n \overset{H(\mathcal{M}(\Omega)^2)}{\longrightarrow} A_* \quad \text{and} \quad ((p_n A_n + q_n J)^{-1} + r_n J)^{-1} \overset{H(\mathcal{M}(\Omega)^2)}{\longrightarrow} pA_* + qJ.
\] (2.16)

Remark 2.1 Proposition 2.1 completes Theorem 2.1 performed with the transformation
\[
 A \mapsto \frac{A^T}{\det A} = J^{-1} A^{-1} J,
\] (2.17)

to other Dykhne transformations of type (see [35], Section 4.1):
\[
 A \mapsto ((pA + qJ)^{-1} + rJ)^{-1} = (pA + qJ)((1 - rq)I_2 + rpJA)^{-1}
\] (2.18)

Remark 2.2 The convergence of $r_n$ to $r = 0$ is not necessary but sufficient for our purpose. If $r \neq 0$, the different convergences are conserved but lead us to the expression
\[
 pA_* + qJ = B_* ((1 - qr)I_2 + p rJ A_*).
\] (2.19)

Proof of Proposition 2.1. The proof is divided into two steps. In the first step, we use Lemma 2.1 to show the $H(\mathcal{M}(\Omega)^2)$-convergence of $\tilde{A}_n := p_n A_n + q_n J$ to $pA_* + qJ$. In the second step, we build a matrix $Q_n$ which will be used as a corrector for $B_n$ and then use again Lemma 2.1.

First step: $\tilde{A}_* = pA_* + qJ$.

First of all, thanks to Theorem 2.2 [16], we already know that, up to a subsequence still denoted by $n$, $A_n$ $H(\mathcal{M}(\Omega)^2)$-converges to $A_*$. We consider a corrector $P_n$ associated with $A_n$ in the sense of Murat-Tartar (see, e.g., [38]), such that, for $\lambda \in \mathbb{R}^2$, $P_n \lambda = \nabla w_n^\lambda$ is defined by
\[
 \begin{cases}
 \text{div}(A_n \nabla w_n^\lambda) = \text{div}(A_* \nabla (\lambda \cdot x)) & \text{in } \Omega \\
 w_n^\lambda = \lambda \cdot x & \text{on } \partial \Omega
\end{cases}
\] (2.20)
Again with Theorem 2.2 of [16] and Definition 1.1, we know that $P_n\lambda$ converges weakly in $L^2(\Omega)^2$ to $\lambda$ and $A_n P_n \lambda$ converges weakly-* in $\mathcal{M}(\Omega)$ to $A_s \lambda$.

Since, for any $\lambda, \mu \in \mathbb{R}^2$,
\[
\alpha \| \nabla w_n^\alpha \|_{L^2(\Omega)}^2 \leq \int_{\Omega} A_n \nabla w_n^\alpha \cdot \nabla w_n^\alpha \, dx = \int_{\Omega} A_n \mu \cdot \nabla w_n^\alpha \, dx \leq 2 \| \bar{u} \|_{L^\infty(\Omega)} |\mu| |\Omega|^{1/2} \| \nabla w_n^\alpha \|_{L^2(\Omega)}^2
\]
and
\[
\int_{\Omega} A_n^{-1} A_n \nabla w_n^\lambda \cdot A_n \nabla w_n^\lambda \, dx = \int_{\Omega} A_n \nabla w_n^\lambda \cdot \nabla w_n^\lambda \, dx,
\]

the sequences $\xi_n := A_n \nabla w_n^\lambda$ and $v_n := w_n^\alpha$ satisfy (2.2) and (2.3). This combined with (2.14) implies that we can apply Lemma 2.1 to obtain
\[
\forall \lambda, \mu \in \mathbb{R}, \quad A_n P_n \lambda \cdot P_n \mu \longrightarrow A_s \lambda \cdot \mu \quad \text{in} \quad \mathcal{D}'(\Omega).
\]

We denote $\tilde{A}_n := p_n A_n + q_n J$ and consider $\delta_n$ such that $\delta_n J := A_n - A_n^s$. Then, the matrix $\tilde{A}_n$ satisfies
\[
\tilde{A}_n \xi \cdot \xi = p_n A_n \xi \cdot \xi \geq p_n \alpha |\xi|^2.
\]

Moreover,
\[
det \tilde{A}_n = p_n^2 \det A_n^s + (p_n \delta_n + q_n)^2 \leq p_n^2 (\det A_n^s + 2\delta_n^2) + 2q_n^2 \leq 2p_n^2 \det A_n + 2q_n^2 \leq C \det A_n,
\]

the last inequality being a consequence of $A_n \geq \alpha I_2$. This inequality gives, by (2.14),
\[
\frac{\det \tilde{A}_n}{\det A_n^s} \tilde{A}_n^s = \frac{\det \tilde{A}_n}{p_n^2 \det A_n^s} p_n A_n^s \leq C \frac{\det A_n}{\det A_n^s} |A_n^s| \leq C.
\]

Then by (2.22), (2.23) and [16] again, up to a subsequence still denoted by $n$, $\tilde{A}_n H(\mathcal{M}(\Omega)^2)$-converges to $A_s$ and we have, by the classical div-curl lemma of [38] for $J P_n \lambda \cdot P_n \mu$ and (2.21),
\[
\forall \lambda, \mu \in \mathbb{R}, \quad (p_n A_n + q_n J) P_n \lambda \cdot P_n \mu = p_n A_n P_n \lambda \cdot P_n \mu + q_n J P_n \lambda \cdot P_n \mu \quad \mathcal{D}'(\Omega) \longrightarrow pA^s \lambda \cdot \mu + qJ \lambda \cdot \mu,
\]

that can be rewritten
\[
\tilde{A}_s = pA_s + qJ.
\]

**Second step: $B_s = \tilde{A}_s$.**

Let $\theta \in C^1_c(\Omega)$ and $\tilde{P}_n$ a corrector associated with $\tilde{A}_n$, such that, for $\lambda \in \mathbb{R}^2$, $\tilde{P}_n \lambda = \nabla \tilde{w}_n^\lambda$ is defined by
\[
\begin{cases}
\text{div}(\tilde{A}_n \nabla \tilde{w}_n^\lambda) = \text{div}(\tilde{A}_s \nabla (\theta \lambda \cdot x)) & \text{in} \quad \Omega \\
\tilde{w}_n^\lambda = 0 & \text{on} \quad \partial \Omega.
\end{cases}
\]

By Definition 1.1, we have
\[
\begin{cases}
\tilde{w}_n^\lambda \rightharpoonup \theta \lambda \cdot x & \text{weakly in} \quad H^1_0(\Omega), \\
\tilde{A}_n \nabla \tilde{w}_n^\lambda \rightharpoonup \tilde{A}_s \nabla (\theta \lambda \cdot x) & \text{weakly-* in} \quad \mathcal{M}(\Omega)^2.
\end{cases}
\]

Let us now consider $B_n = (\tilde{A}_n^{-1} + r_n J)^{-1}$. $B_n$ is symmetric and so is its inverse : 
\[
B_n^{-1} = \tilde{A}_n^{-1} + r_n J = (\tilde{A}_n^{-1} + r_n J)^s = (\tilde{A}_n^{-1})^s.
\]

We then have, by a little computation (like in (2.5)) and (2.23),
\[
\frac{\det B_n}{\det B_n^s} |B_n| = |B_n| = \left| \left( (\tilde{A}_n^{-1})^s \right)^{-1} \right| = \frac{\det \tilde{A}_n}{\det A_n^s} |A_n^s| \leq C.
\]

8
For any $\xi \in \mathbb{R}^2$, the sequence $\nu_n := (I + r_n J \tilde{A}_n)^{-1} \xi$ satisfies, by (2.14),

$$|\xi| \leq \left( 1 + ||r_n \tilde{A}_n||_{L^\infty(\Omega)^2}^2 \right) |\nu_n| \leq \left( 1 + p_n ||r_n A_n||_{L^\infty(\Omega)^2}^2 + q_n r_n \right) |\nu_n| \leq (1 + C) |\nu_n|,$$

hence

$$B_n \xi \cdot \nu_n = \tilde{A}_n \nu_n \cdot (I + r_n J \tilde{A}_n) \nu_n = \tilde{A}_n \nu_n \cdot \nu_n = p_n A_n \nu_n \cdot \nu_n \geq p_n \alpha |\nu_n|^2 \geq \frac{p_n}{1 + C^2} |\xi|^2 \geq C |\xi|^2 \ (2.27)$$

with $C > 0$. Therefore, with (2.27) and (2.26), again by Theorem 2.2 of [16], up to a subsequence still denoted by $n$, $B_n H(M(\Omega)^2)$-converges to $B_\ast$.

Let $\psi \in C_c^1(\Omega)$ and $R_n$ be a corrector associated to $B_n$, such that, for $\mu \in \mathbb{R}^2$, $R_n \mu = \nabla v_{n}^\mu$ is defined by

$$\begin{align*}
\text{div} (B_n \nabla v_{n}^\mu) &= \text{div} (B_\ast \nabla (\psi \mu \cdot x)) \quad \text{in } \Omega \\
\n_{v_{n}^\mu} &= 0 \quad \text{on } \partial \Omega.
\end{align*}$$

(2.28)

By Definition 1.1, we have the convergences

$$\begin{align*}
\text{weakly in } H^1_0(\Omega), \\
_{B_n \nabla v_{n}^\mu} \rightarrow B_\ast \nabla (\psi \mu \cdot x) \quad \text{weakly-* in } M(\Omega)^2.
\end{align*}$$

(2.29)

Let us define the matrix $Q_n := (I + r_n J \tilde{A}_n) \tilde{P}_n$. We have

$$B_n Q_n = (\tilde{A}_n^{-1} + r_n J)^{-1} (I + r_n J \tilde{A}_n) \tilde{P}_n = (\tilde{A}_n^{-1} + r_n J)^{-1} (\tilde{A}_n^{-1} + r_n J) \tilde{A}_n \tilde{P}_n = \tilde{A}_n \tilde{P}_n. \quad (2.30)$$

We are going to pass to the limit in $\mathcal{D}'(\Omega)$ the equality given by (2.30) and the symmetry of $B_n$:

$$\tilde{A}_n \tilde{P}_n \lambda \cdot R_n \mu = B_n Q_n \lambda \cdot R_n \mu = Q_n \lambda \cdot B_n R_n \mu. \quad (2.31)$$

On the one hand, $\tilde{A}_n$ satisfies (2.1) by (2.22) and (2.23). The sequences $\xi_n := \tilde{A}_n \tilde{P}_n \lambda$ and $v_n := v_{n}^\mu$ satisfy the hypothesis (2.3) by (2.24) and (2.2) because

$$\int_{\Omega} (\tilde{A}_n)^{-1} \xi_n \cdot \xi_n \ dx + ||v_n||_{H^1_0(\Omega)} = \int_{\Omega} \tilde{A}_n \tilde{P}_n \lambda \cdot \tilde{P}_n \lambda \ dx + ||v_{n}^\mu||_{H^1_0(\Omega)} \ dx \leq C$$

by (2.24) and the convergences (2.29) and (2.25). The application of Lemma 2.1, (2.25) and (2.29) give the convergence

$$\tilde{A}_n \tilde{P}_n \lambda \cdot R_n \mu \rightarrow A^* \nabla (\theta \lambda \cdot x) \cdot \nabla (\psi \mu \cdot x) \quad \text{in } \mathcal{D}'(\Omega).$$

(2.32)

On the other hand, we have the equality

$$Q_n \lambda \cdot B_n R_n \mu = B_n R_n \mu \cdot \tilde{P}_n \lambda + B_n R_n \mu \cdot r_n J \tilde{A}_n \tilde{P}_n. \quad (2.33)$$

The matrix $B_n$ satisfies (2.1) by (2.27) and (2.26). The sequences $\xi_n := B_n R_n \mu$ and $v_n := \tilde{w}_{n}^\lambda$ satisfy the hypothesis (2.3) by (2.28) and (2.2) of Lemma 2.1 because

$$\int_{\Omega} (B_n)^{-1} \xi_n \cdot \xi_n \ dx + ||v_n||_{H^1_0(\Omega)} = \int_{\Omega} B_n R_n \mu \cdot R_n \mu \ dx + ||\tilde{w}_{n}^\lambda||_{H^1_0(\Omega)} \ dx \leq C$$

by (2.28) and the convergences (2.25) and (2.29). The application of Lemma 2.1, (2.25) and (2.29) give the convergence

$$B_n R_n \mu \cdot \tilde{P}_n \lambda \rightarrow B_\ast \nabla (\psi \mu \cdot x) \cdot \nabla (\theta \lambda \cdot x) \quad \text{in } \mathcal{D}'(\Omega).$$

(2.34)
The convergence of the right part of (2.33) is more delicate. The demonstration is the same as for Lemma 2.1. Let \( \omega \) be a simply connected open subset of \( \Omega \) such as \( \omega \subset\subset \Omega \). The function 
\[
\tilde{A}_n \mathbf{P}_n \lambda - \tilde{A}_n \nabla (\theta \lambda \cdot x)
\]
leads us, by \((\theta \lambda \cdot x)\), \(B_\lambda\), and \(J\), to 
\[
B_\lambda \mathbf{P}_n \lambda - B_\lambda \mathbf{P}_n \lambda = r_n B_\lambda \mathbf{P}_n \lambda - r_n B_\lambda \mathbf{P}_n \lambda
\]
and taking into account that \(B_\lambda\) is symmetric and \(\omega\) is arbitrary, that:
\[
B_\lambda = \tilde{A}_\lambda = p A_\lambda + q J.
\]

### 2.2 An application to isotropic two-phase media

In this section, we study the homogenization of a two-phase isotropic medium with high contrast and non-necessarily symmetric conductivities. The study of the symmetric case in Proposition 2.2 permits to obtain Theorem 2.2 by applying the transformation of Proposition 2.1. We use Notation 1.1.

**Proposition 2.2** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^2 \) such that \( |\partial \Omega| = 0 \). Let \( \omega_n, \ n \in \mathbb{N} \), be a sequence of open subsets of \( \Omega \) with characteristic function \( \chi_n \), satisfying \( \theta_n := |\omega_n| < 1 \), \( \theta_n \) converges to 0, and
\[
\frac{\chi_n}{\theta_n} \longrightarrow a \in L^\infty(\Omega) \quad \text{weakly-\ast \ in \ } \mathcal{M}(\Omega).
\]
We assume that there exists \( \alpha_1, \alpha_2 > 0 \) and two positive sequences \( \alpha_{1,n}, \alpha_{2,n} \geq a_0 > 0 \) verifying
\[
\lim_{n \to \infty} \alpha_{1,n} = \alpha_1 \quad \text{and} \quad \lim_{n \to \infty} \theta_n \alpha_{2,n} = \alpha_2,
\]
and that the conductivity takes the form
\[
\sigma_n^0(\alpha_{1,n}, \alpha_{2,n}) = (1 - \chi_n)\alpha_{1,n}I_2 + \chi_n \alpha_{2,n}I_2.
\]
Then, there exists a subsequence of \( n \), still denoted by \( n \), and a locally Lipschitz function
\[
\sigma_\alpha^0 : [0, \infty)^2 \longrightarrow \mathcal{M}(a_0, 2||a||_\infty; \Omega)
\]
such that
\[
\forall (\alpha_1, \alpha_2) \in (0, \infty)^2, \quad \sigma_n^0(\alpha_{1,n}, \alpha_{2,n}) \overset{H(\mathcal{M}(\Omega)^2)}{\longrightarrow} \sigma_\alpha^0(\alpha_1, \alpha_2).
\]
Proof of Proposition 2.2. The proof is divided into two parts. We first prove the theorem for 
\(\alpha_{1,n} = \alpha_1, \alpha_{2,n} = \theta_n^{-1} \alpha_2\), and then treat the general case.

**First step:** The case \(\alpha_{1,n} = \alpha_1, \alpha_{2,n} = \theta_n^{-1} \alpha_2\).

In this step we denote \(\sigma_n^0(\alpha) := \sigma_n^0(\alpha_1, \theta_n^{-1} \alpha_2)\), for \(\alpha = (\alpha_1, \alpha_2) \in (0, \infty)^2\). Theorem 2.2 of [16] implies that for any \(\alpha \in (0, \infty)^2\), there exists a subsequence of \(n\) such that \(\sigma_n^0(\alpha) H(\mathcal{M}(\Omega)^2)\)-converges in the sense of Definition 1.1 to some matrix-valued function in \(\mathcal{M}(a_0, 2||a||_\infty; \Omega)\).

By a diagonal extraction, there exists a subsequence of \(n\), still denoted by \(n\), such that

\[
\forall \alpha \in \mathbb{Q}^2 \cap (0, \infty)^2, \quad \sigma_n^0(\alpha) \xrightarrow{H(\mathcal{M}(\Omega)^2)} \sigma_\alpha^0(\alpha). \tag{2.41}
\]

We are going to show that this convergence is true any pair \(\alpha \in (0, \infty)^2\).

We have, by (2.38), for any \(\alpha \in \mathbb{Q}^2 \cap (0, \infty)^2\),

\[
|\sigma_n^0(\alpha)| = (1 - \chi_n)\alpha_1 + \chi_n \alpha_2 \theta_n \xrightarrow{\theta_n} \alpha_1 + \alpha_2 a \in L^\infty(\Omega) \quad \text{weakly-}^* \text{ in } \mathcal{M}(\Omega) \tag{2.42}
\]

and, since \(\theta_n \in (0, 1)\),

\[
\forall \xi \in \mathbb{R}^2, \quad \sigma_n^0(\alpha)\xi = \alpha_1(1 - \chi_n)|\xi|^2 + \chi_n \frac{\alpha_2}{\theta_n} |\xi|^2 \geq \min(\alpha_1, \alpha_2)|\xi|^2 \quad \text{a.e. in } \Omega. \tag{2.43}
\]

By applying Theorem 2.2 of [16] with (2.42), we have the inequality

\[
|\sigma_n^0(\alpha)\lambda| \leq 2|\lambda| (\alpha_1 + \alpha_2||a||_\infty). \tag{2.44}
\]

For any \(\alpha \in \mathbb{Q}^2 \cap (0, \infty)^2\) and \(\lambda \in \mathbb{R}^2\), consider the corrector \(w_n^{\alpha, \lambda}\) associated with \(\sigma_n^0(\alpha)\) defined by

\[
\begin{align*}
\text{div} \left( \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \right) &= \text{div} \left( \sigma_n^0(\alpha) \lambda \right) \quad \text{in } \Omega, \\
w_n^{\alpha, \lambda} &= \lambda \cdot x \quad \text{on } \partial \Omega,
\end{align*}
\tag{2.45}
\]

which depends linearly on \(\lambda\).

Let \(\alpha \in \mathbb{Q}^2 \cap (0, \infty)^2\). Let us show that the energies

\[
\int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx \tag{2.46}
\]

are bounded. We have, by (2.45), (2.44) and the Cauchy-Schwarz inequality

\[
\int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx \\
= \int_{\Omega} \sigma_n^0(\alpha) \lambda \cdot (\nabla w_n^{\alpha, \lambda} - \lambda) \, dx + \int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \lambda \, dx \\
= \int_{\Omega} \sigma_n^0(\alpha) \lambda \cdot \nabla w_n^{\alpha, \lambda} \, dx - \int_{\Omega} \sigma_n^0(\alpha) \lambda \cdot \lambda \, dx + \int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \lambda \, dx \\
\geq 0
\]

which leads us to

\[
\int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx \leq \int_{\Omega} |\sigma_n^0(\alpha) \lambda \cdot \nabla w_n^{\alpha, \lambda}| \, dx + \int_{\Omega} |\sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \lambda| \, dx. \tag{2.47}
\]

On the one hand, the Cauchy-Schwarz inequality gives

\[
\left( \int_{\Omega} |\sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \lambda| \, dx \right)^2 \leq |\lambda|^2 \int_{\Omega} |\sigma_n^0(\alpha)| \, dx \int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx
\]
that is
\[
\left( \int_{\Omega} |\sigma_n^0(\alpha) \nabla w_n^{\alpha,\lambda} \cdot \lambda| \, dx \right)^2 \leq |\lambda|^2 |\alpha| \int_{\Omega} |\sigma_n^0(\alpha) \nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha,\lambda}| \, dx. \tag{2.48}
\]

On the other hand, by (2.43) and the Cauchy-Schwarz inequality, we have
\[
\int_{\Omega} |\sigma_n^0(\alpha) \lambda \cdot \nabla w_n^{\alpha,\lambda}| \, dx \leq 2|\lambda|(\alpha_1 + \alpha_2||\alpha||_{\infty}) \sqrt{\int_{\Omega} |\nabla w_n^{\alpha,\lambda}|^2 \, dx}
\leq 2|\lambda|(\alpha_1 + \alpha_2||\alpha||_{\infty}) \sqrt{\frac{1}{\alpha_1} \int_{\Omega} |\sigma_n^0(\alpha) \nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha,\lambda}| \, dx}
\]
that is
\[
\int_{\Omega} |\sigma_n^0(\alpha) \lambda \cdot \nabla w_n^{\alpha,\lambda}| \, dx \leq C |\lambda|^2 |\alpha| \sqrt{\frac{1}{\alpha_1} \int_{\Omega} |\sigma_n^0(\alpha) \nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha,\lambda}| \, dx} \tag{2.49}
\]
where \( C \) does not depend on \( n \) nor \( \alpha \).

By combining (2.47), (2.48), and (2.49), we have
\[
\int_{\Omega} |\sigma_n^0(\alpha) \nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha,\lambda}| \, dx \leq C |\lambda|^2 (|\alpha| + |\alpha|^2 (\alpha_1^{-1} + \alpha_2^{-1})) \tag{2.50}
\]
where \( C \) does not depend on \( n \) nor \( \alpha \).

Let \( \alpha' \in \mathbb{Q}^2 \cap (0, \infty)^2 \). The sequences \( \xi_n := \sigma_n^0(\alpha) \nabla w_n^{\alpha,\lambda} \) and \( v_n := w_n^{\alpha',\lambda} \) satisfy the assumptions (2.2) and (2.3) of Lemma 2.1. By symmetry, we have the convergences
\[
\begin{align*}
\sigma_n^0(\alpha) \nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha,\lambda} &\longrightarrow \sigma_n^0(\alpha) \lambda \cdot \lambda \quad \text{weakly in } \mathcal{D}'(\Omega), \\
\sigma_n^0(\alpha') \nabla w_n^{\alpha',\lambda} \cdot \nabla w_n^{\alpha,\lambda} &\longrightarrow \sigma_n^0(\alpha') \lambda \cdot \lambda \quad \text{weakly in } \mathcal{D}'(\Omega). \tag{2.51}
\end{align*}
\]
As the matrices are symmetric, we have
\[
(\sigma_n^0(\alpha) \cdot \sigma_n^0(\alpha')) \nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha,\lambda} = \sigma_n^0(\alpha) \nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha',\lambda} - \sigma_n^0(\alpha') \nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha,\lambda},
\]
hence
\[
(\sigma_n^0(\alpha) \cdot \sigma_n^0(\alpha')) \nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha,\lambda} \longrightarrow (\sigma_n^0(\alpha) \cdot \sigma_n^0(\alpha')) \lambda \cdot \lambda \quad \text{weakly in } \mathcal{D}'(\Omega). \tag{2.52}
\]
Let \( \lambda \in \mathbb{R}^2 \). We have, by the Cauchy-Schwarz inequality, with the Einstein convention
\[
\int_{\Omega} \left| (\sigma_n^0(\alpha) - \sigma_n^0(\alpha')) \nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha',\lambda} \right| \, dx
= \int_{\Omega \setminus \omega_n} |\alpha_1 - \alpha_1'| |\nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha',\lambda}| \, dx + \int_{\omega_n} |\alpha_2 - \alpha_2'| |\nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha',\lambda}| \, dx
\leq |\alpha_1 - \alpha_1'| \int_{\Omega \setminus \omega_n} |\nabla w_n^{\alpha,\lambda}|^2 \, dx \sqrt{\int_{\Omega \setminus \omega_n} |\nabla w_n^{\alpha',\lambda}|^2 \, dx}
+ |\alpha_2 - \alpha_2'| \int_{\omega_n} |\nabla w_n^{\alpha,\lambda}|^2 \, dx \sqrt{\int_{\omega_n} |\nabla w_n^{\alpha',\lambda}|^2 \, dx}
\leq |\alpha_1 - \alpha_1'| \frac{1}{\alpha_1} \int_{\Omega} |\sigma_n^0(\alpha) \nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha,\lambda}| \, dx \sqrt{\frac{1}{\alpha_1} \int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha',\lambda} \cdot \nabla w_n^{\alpha',\lambda} \, dx}.
\]
This combined with (2.50) yields
\[
\int_{\Omega} \left| (\sigma_n^0(\alpha) - \sigma_n^0(\alpha')) \nabla w_n^{\alpha,\lambda} \cdot \nabla w_n^{\alpha',\lambda} \right| \leq C |\lambda|^2 \frac{|\alpha_1 - \alpha_1'|}{\sqrt{\alpha_1 ||\alpha||_{\infty}}} M(\alpha) M(\alpha')
\]

12
The sequence of (2.52) is thus bounded in $L^1(\Omega)^2$ which implies that (2.52) holds weakly-$*$ in $\mathcal{M}(\Omega)$.

Hence, we get, for any $\varphi \in \mathcal{C}_c(\Omega)$, that
\[
\int_{\Omega} \left| (\sigma_n^0(\alpha) - \sigma_n^0(\alpha')) \cdot \varphi \right| \, dx \leq C \left| \alpha \right|^2 \frac{\left| \alpha_i - \alpha'_i \right|}{\sqrt{|\alpha_i| |\alpha'_i|}} \, M(\alpha) \, M(\alpha') \, \| \varphi \|_{\infty}.
\] (2.53)

Then, the Riesz representation theorem implies that
\[
\left\| \sigma_n^0(\alpha) - \sigma_n^0(\alpha') \right\|_{L^1(\Omega)^2} \leq C \left| \alpha - \alpha' \right|.
\]

Therefore, by the definition of $M$ in (2.50), for any compact subset $K \subset (0, \infty)^2$,
\[
\exists C > 0, \quad \forall \alpha, \alpha' \in \mathbb{Q}^2 \cap K, \quad \left\| \sigma_n^0(\alpha) - \sigma_n^0(\alpha') \right\|_{L^1(\Omega)^2} \leq C \left| \alpha - \alpha' \right|.
\] (2.54)

This estimate permits to extend the definition (2.41) of $\sigma_*^0$ on $(0, \infty)^2$ by
\[
\forall \alpha \in (0, \infty)^2, \quad \sigma_*^0(\alpha) = \lim_{\alpha' \in \mathbb{Q}^2 \cap (0, \infty)^2} \sigma_n^0(\alpha') \quad \text{strongly in} \quad L^1(\Omega)^2.
\] (2.55)

Let $\alpha \in (0, \infty)^2$. Theorem 2.2 of [16] implies that there exists a subsequence of $n$, denoted by $n'$, and a matrix-valued function $\tilde{\sigma}_* \in \mathcal{M}(a_0, 2 ||a||_{\infty}; \Omega)$ such that
\[
\sigma_n(\alpha) \xrightarrow{H(\mathcal{M}(\Omega)^2)} \tilde{\sigma}_*.
\] (2.56)

Repeating the arguments leading to (2.54), for any positive sequence of rational pair $(\alpha^q)_{q \in \mathbb{N}}$ converging to $\alpha$, we have
\[
\exists C > 0, \quad \left\| \tilde{\sigma}_* - \sigma_*^0(\alpha^q) \right\|_{L^1(\Omega)^2} \leq C \left| \alpha - \alpha^q \right|,
\] (2.57)

hence, by (2.55), $\tilde{\sigma}_* = \sigma_*^0(\alpha)$. Therefore by the uniqueness of the limit in (2.56), we obtain for the whole sequence satisfying (2.41)
\[
\forall \alpha \in (0, \infty)^2, \quad \sigma_n(\alpha) \xrightarrow{H(\mathcal{M}(\Omega)^2)} \sigma_*^0(\alpha).
\] (2.58)

In particular, the function $\sigma_*^0$ satisfies (2.54) and (2.55), i.e. $\sigma_*^0$ is a locally Lipschitz function on $(0, \infty)^2$.

**Second step:** The general case.

We denote $\alpha^n = (\alpha_{1,n}, \alpha_{2,n})$ and $\sigma_n^0(\alpha^n) = \sigma_n^0(\alpha_{1,n}, \alpha_{2,n})$. Theorem 2.2 of [16] implies that there exists a subsequence of $n$, denoted by $n'$, such that $\sigma_n^0(\alpha^n') \xrightarrow{H(\mathcal{M}(\Omega)^2)}$-converges to some $\tilde{\sigma}_* \in \mathcal{M}(a_0, 2 ||a||_{\infty}; \Omega)$ in the sense of Definition 1.1.

As in the first step, for any $\alpha^n' \in (0, \infty)^2$ and $\lambda \in \mathbb{R}^2$, we can consider the corrector $w_{n'}^{\alpha^n', \lambda}$ associated with $\sigma_n^0(\alpha^n')$ defined by
\[
\begin{cases}
\text{div} \left( \sigma_n^0(\alpha^n') \nabla w_{n'}^{\alpha^n', \lambda} \right) = \text{div} (\tilde{\sigma}_* \lambda) & \text{in} \, \Omega, \\
\quad w_{n'}^{\alpha^n', \lambda} = \lambda \cdot x & \text{on} \, \partial \Omega,
\end{cases}
\] (2.59)

which depends linearly on $\lambda$. Proceeding as in the first step, we obtain like in (2.52), with $\alpha = (\alpha_1, \alpha_2)$ the limit of $\alpha^n$ according to (2.39),
\[
\left( \sigma_n^0(\alpha) - \sigma_n^0(\alpha') \right) \nabla w_{n'}^{\alpha^n', \lambda} \cdot \nabla w_{n'}^{\alpha^n', \lambda} \rightharpoonup \left( \sigma_*^0(\alpha) - \tilde{\sigma}_* \right) \lambda \cdot \lambda \quad \text{weakly in} \quad \mathcal{D}'(\Omega).
\] (2.60)
Moreover, by the energy bound (2.50), which also holds for $\alpha'\sigma$, we have, for any $\varphi \in \mathcal{D}(\Omega)$,
\[
\int_{\Omega} \left( \sigma_{n,\sigma'}(\alpha) - \sigma_{0}(\alpha') \right) \nabla w_{n,\lambda}^{\alpha'} \cdot \nabla w_{n,\lambda}^{\alpha'} \varphi \, dx \xrightarrow{n' \to \infty} 0.
\]
This combined with (2.60), yields
\[
\int_{\Omega} \left( \sigma_{\sigma}^{0}(\alpha) - \tilde{\sigma}_{\sigma} \right) \lambda \cdot \lambda \varphi \, dx = 0,
\]
which implies that $\sigma_{\sigma}^{0}(\alpha) = \tilde{\sigma}_{\sigma}$. We conclude by a uniqueness argument. 

We can now obtain a result for (perturbed) non-symmetric conductivities. Then, we will use a Dykhne transformation to recover the symmetric case following the Milton approach [35] (pp. 61–65). This will allow us to apply Proposition 2.2.

**Theorem 2.2** Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$ such that $|\partial \Omega| = 0$. Let $\omega_n$, $n \in \mathbb{N}$, be a sequence of open subsets of $\Omega$ and denote by $\chi_n$ their characteristic function. We assume that $\theta_n = |\omega_n| < 1$ converges to $0$ and
\[
\frac{\chi_n}{\theta_n} \rightarrow a \in L^{\infty}(\Omega) \text{ weakly-}\ast \text{ in } \mathcal{M}(\Omega).
\]  

Consider the conductivity defined by
\[
\sigma_{\sigma}(h) = (1 - \chi_n)\sigma_1(h) + \frac{\chi_n}{\theta_n} \sigma_2(h)
\]
where for $j = 1, 2$, $\sigma_j(h) = \alpha_j + h\beta_j \in \mathbb{R}^{2 \times 2}$ with $\alpha_1, \alpha_2 > 0$ and $(\beta_1, \beta_2) \neq (0, 0)$. Then, there exists a subsequence of $n$, still denoted by $n$, and a locally Lipschitz function
\[
\sigma_{\sigma}^{0} : (0, \infty)^2 \rightarrow \mathcal{M}\left(\min(\alpha_1, \alpha_2), 2(|\sigma_1| + |\sigma_2| ||a||_{\infty}) : \Omega\right)
\]
such that
\[
\sigma_{\sigma}(h) \xrightarrow{H(\mathcal{M}(\Omega)^2)} \sigma_{\sigma}^{0}(\alpha_1, \alpha_2 + \alpha_2^{-1}\beta_2^2 h^2) + h\beta_1 J.
\]

**Proof of Theorem 2.2.** We have
\[
\forall \xi \in \mathbb{R}^2, \quad \sigma_{\sigma}(h) \xi \cdot \xi = (1 - \chi_n)\alpha_1|\xi|^2 + \frac{\chi_n}{\theta_n} \alpha_2|\xi|^2 \geq \min(\alpha_1, \alpha_2)|\xi|^2 \text{ a.e. in } \Omega
\]
and, by (2.61),
\[
|\sigma_{\sigma}(h)| = (1 - \chi_n)|\sigma_1(h)| + \frac{\chi_n}{\theta_n} |\sigma_2(h)| \xrightarrow{} |\sigma_1(h)| + a|\sigma_2(h)| \in L^{\infty}(\Omega) \text{ weakly-}\ast \text{ in } \mathcal{M}(\Omega).
\]

In order to make a Dykhne transformation like in p.62 of [35], we consider two real coefficients $a_n$ and $b_n$ in such a way that
\[
B_n := (a_n\sigma_{\sigma}(h) + b_n J)(a_n I_2 + J\sigma_{\sigma}(h))^{-1} = \left( (p_n\sigma_{\sigma}(h) + q_n J)^{-1} + r_n J \right)^{-1}
\]
is symmetric. An easy computation shows that the previous equality holds when
\[
p_n := \frac{a_n^2}{a_n^2 + b_n}, \quad q_n := \frac{a_nb_n}{a_n^2 + b_n} \quad \text{and} \quad r_n := \frac{1}{a_n}.
\]
On the one hand, the estimates (3.39) and (3.40) with $\alpha_{2,n} = \theta_n^{-1}\alpha_2$, $\beta_{2,n} = \theta_n^{-1}\beta_2$, yield (note that they are independent of $\chi_n$)
\[
p_n \xrightarrow{n \to \infty} 1, \quad q_n \xrightarrow{n \to \infty} -h\beta_1, \quad r_n \xrightarrow{n \to \infty} 0 \quad \text{and} \quad ||r_n\sigma_{\sigma}(h)||_{\infty} \leq C(|\sigma_1(h)| + |\sigma_2(h)|). \quad (2.63)
\]
On the other hand, as in Section 3.2, with Notation 1.1 and (3.34), we have
\[ B_n = \sigma_n^0(\alpha_{1,n}(h), \alpha_{2,n}(h)), \]  
(2.64)
where
\[ \alpha_{1,n}(h) = \frac{a_n(\alpha_1 + ih\beta_1) + ib_n}{a_n + i(\alpha_1 + ih\beta_1)} \quad \text{and} \quad \alpha_{2,n}(h) = \frac{a_n(\alpha_2/\theta_n + ih\beta_2/\theta_n) + ib_n}{a_n + i(\alpha_2/\theta_n + ih\beta_2/\theta_n)}. \]  
(2.65)
Hence, like in (3.41), we have
\[ \lim_{n \to \infty} \alpha_{1,n}(h) = \alpha_1 \quad \text{and} \quad \lim_{n \to \infty} \theta_n \alpha_{2,n}(h) = \alpha_2 + \alpha_2^{-1}\beta^2 h^2. \]  
(2.66)
We can first apply Proposition 2.2 with the conditions (2.64) and (2.66) to have the \( H(M(\Omega)^2) \)-convergence of \( B_n \). Then, by virtue of Proposition 2.1, with (2.63) we get that
\[ \sigma_n(h) \xrightarrow{H(M(\Omega)^2)} \sigma_0^0(\alpha_1, \alpha_2 + \alpha_2^{-1}\beta^2 h^2) + h\beta_1 J. \]

\[ \square \]

3 A two-dimensional periodic medium

In this section we consider a sequence \( \Sigma_n \) of matrix valued functions (not necessarily symmetric) in \( L^\infty(\mathbb{R}^2)^{2 \times 2} \), which satisfies the following assumptions:

1. \( \Sigma_n \) is \( Y \)-periodic, where \( Y := (0,1)^2 \), i.e.,
\[ \forall n \in \mathbb{N}, \forall \kappa \in \mathbb{Z}^2, \quad \Sigma_n(., + \kappa) = \Sigma_n(.) \quad \text{a.e. in } \mathbb{R}^2, \]  
(3.1)

2. \( \Sigma_n \) is equi-coercive in \( \mathbb{R}^2 \), i.e.,
\[ \exists \alpha > 0 \quad \text{such that} \quad \forall n \in \mathbb{N}, \forall \xi \in \mathbb{R}^2, \quad \Sigma_n \xi \cdot \xi \geq \alpha |\xi|^2 \quad \text{a.e. in } \mathbb{R}^2. \]  
(3.2)

Let \( \varepsilon_n \) be a sequence of positive numbers which tends to 0. From the sequences \( \Sigma_n \) and \( \varepsilon_n \) we define the highly oscillating sequence of matrix-valued functions \( \sigma_n \) by
\[ \sigma_n(x) = \Sigma_n \left( \frac{x}{\varepsilon_n} \right), \quad \text{a.e. } x \in \mathbb{R}^2. \]  
(3.3)
By virtue of (3.1) and (3.2), \( \sigma_n \) is an equi-coercive sequence of \( \varepsilon_n \)-periodic matrix-valued functions in \( L^\infty(\mathbb{R}^2)^{2 \times 2} \). For a fixed \( n \in \mathbb{N} \), let \( (\sigma_n)_s \) be the constant matrix defined by
\[ \forall \lambda, \mu \in \mathbb{R}^2, \quad (\sigma_n)_s \lambda \cdot \mu = \int_Y \Sigma_n \nabla W^\lambda_n \cdot \nabla W^\mu_n \, dy, \]  
(3.4)
where, for any \( \lambda \in \mathbb{R}^2 \), \( W^\lambda_n \in H^1_2(Y) \), the set of \( Y \)-periodic functions belonging to \( H^1_{loc}(\mathbb{R}^2) \), is the solution of the auxiliary problem
\[ \int_Y (W^\lambda_n - \lambda \cdot y) \, dy = 0 \quad \text{and} \quad \text{div}(\Sigma_n \nabla W^\lambda_n) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2) \]  
(3.5)
or equivalently
\[ \left\{ \begin{array}{l}
\int_Y \Sigma_n \nabla W^\lambda_n \cdot \nabla \varphi \, dy = 0, \quad \forall \varphi \in H^1_2(Y) \\
\int_Y (W^\lambda_n(y) - \lambda \cdot y) \, dy = 0.
\end{array} \right. \]  
(3.6)
Set
\[ w^\lambda_n(x) := \varepsilon_n W^\lambda_n \left( \frac{x}{\varepsilon_n} \right), \quad \text{for } x \in \Omega, \]  
(3.7)
and
\[ w_n := (w_{n1}^1, w_{n2}^2) = (w_{n1}^1, w_{n2}^2). \]  
(3.8)
### 3.1 A uniform convergence result

**Theorem 3.1** Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$ with a Lipschitz boundary. Consider a highly oscillating sequence of matrix-valued functions $\sigma_n$ satisfying (3.1), (3.2), (3.3) and the constant matrix $(\sigma_n)_*$ defined by (3.4). We assume that

\[
(\sigma_n)_* \rightarrow \sigma_* \text{ in } \mathbb{R}^{2\times 2}.
\]

Consider, for $f \in H^{-1}(\Omega) \cap W^{-1,q}(\Omega)$ with $q > 2$, the solution $u_n$ of the problem

\[
P_n \left\{
-\text{div}(\sigma_n \nabla u_n) = f \quad \text{in } \Omega \\
u_n = 0 \quad \text{on } \partial \Omega.
\right.
\]

Then, $u_n$ converges uniformly to the solution $u \in H^1_0(\Omega)$ of

\[
P \left\{
-\text{div}(\sigma_* \nabla u) = f \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega.
\right.
\]

Moreover we have the corrector result, with the $\varepsilon_n Y$-periodic sequence $w_n$ defined in (3.8):

\[
\nabla u_n - \sum_{i=1}^{2} \partial_i u \nabla w_n^i \rightarrow 0 \text{ in } L^1(\Omega)^2.
\]

**Remark 3.1** The first point of Theorem 3.1 is an extension to the non-symmetric case of the results of [13] and [15]. The uniform convergence of $u_n$ is a straightforward consequence of Theorem 2.7 of [15] taking into account that in the present case $\sigma_n \in L^\infty(\Omega)^{2\times 2}$ for a fixed $n$. The fact that $f \in W^{-1,q}(\Omega)$ with $q > 2$ ensures the uniform convergence.

**Proof of Theorem 3.1.**

**Derivation of the limit problem $P$.**

We only have to show that $u$ is the solution of $P$ in (3.11). We consider a corrector $D\tilde{w}_n : \mathbb{R}^2 \rightarrow \mathbb{R}^{2\times 2}$ associated with $\sigma_n^T$ defined by

\[
\tilde{w}_n(x) := \varepsilon_n \tilde{W}_n \left( \frac{x}{\varepsilon_n} \right) = \left( \varepsilon_n \tilde{W}_n^1 \left( \frac{x}{\varepsilon_n} \right), \varepsilon_n \tilde{W}_n^2 \left( \frac{x}{\varepsilon_n} \right) \right)
\]

where for $i = 1, 2$, $\tilde{W}_n^i \in H^1_\varepsilon(Y)$ is the solution of the auxiliary problem

\[
\int_Y (\tilde{W}_n^i - e_i \cdot x) \, dx = 0 \quad \text{and} \quad \text{div} \left( \Sigma_n^T \nabla \tilde{W}_n^i \right) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^2).
\]

Again, thanks to Theorem 2.7 of [15], $\tilde{w}_n$ converges uniformly to the identity in $\Omega$ by the integral condition (3.13). Let $\varphi \in \mathcal{D}(\Omega)$. We have, using the Einstein convention, by integrating by parts...
This leads us to the equality

\[
\int_{\Omega} \sigma_n \nabla u_n \cdot \nabla (\varphi(\tilde{w}_n)) \ dx = - \int_{\Omega} \nabla u_n \cdot \sigma_n^T \nabla \tilde{w}_n^j (\partial_i \varphi(\tilde{w}_n)) \ dx
\]

Moreover, we have, again with the Einstein convention

\[
= - \int_{\Omega} \sigma_n^T \nabla \tilde{w}_n^j \cdot \nabla (u_n \partial_i \varphi(\tilde{w}_n)) \ dx - \int_{\Omega} \sigma_n^T \nabla \tilde{w}_n^j \cdot \nabla \tilde{w}_n^j \partial_{i,j}^2 \varphi(\tilde{w}_n) \ u_n \ dx
\]

This leads us to the equality

\[
\langle f, \varphi(\tilde{w}_n) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_{\Omega} \sigma_n \nabla u_n \cdot \nabla (\varphi(\tilde{w}_n)) \ dx = - \int_{\Omega} \sigma_n^T \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j \partial_{i,j}^2 \varphi(\tilde{w}_n) \ u_n \ dx \quad \text{(3.14)}
\]

To study the convergence of the last term of (3.14), we first show that \(\sigma_n^T \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j\) is bounded in \(L^1(\Omega)\). We have, by periodicity and the Cauchy-Schwarz inequality

\[
\int_{\Omega} |\sigma_n^T \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j| \ dx = \int_{\Omega} |\Sigma_n \nabla \tilde{W}_n^i \cdot \nabla \tilde{W}_n^j| \left( \frac{x}{\varepsilon_n} \right) \ dx \leq C \int_Y \left| \Sigma_n \nabla \tilde{W}_n^i \cdot \nabla \tilde{W}_n^j \right| \ dx
\]

\[
\leq C \sqrt{\int_Y \left| \Sigma_n \nabla \tilde{W}_n^i \cdot \nabla \tilde{W}_n^j \right| \ dx} \sqrt{\int_Y \left| \Sigma_n \nabla \tilde{W}_n^j \cdot \nabla \tilde{W}_n^j \right| \ dx}
\]

which is bounded by the hypothesis (3.9). Therefore,

\[
\sigma_n^T \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j \text{ is bounded in } L^1(\Omega).
\quad \text{(3.15)}
\]

Due to the periodicity, we know that for \(i, j = 1, 2\),

\[
2\sigma_n^T \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j = \sigma_n^T \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j + \sigma_n^T \nabla \tilde{w}_n^j \cdot \nabla \tilde{w}_n^i \rightarrow (\sigma^*)^T e_i \cdot e_j + (\sigma^*)^T e_j \cdot e_i = 2 (\sigma^*)^T e_i \cdot e_j
\]

weakly-* in \(M(\Omega)\). Hence, we get that

\[
\sigma_n^T \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j \rightarrow (\sigma^*)^T e_i \cdot e_j \quad \text{weakly-* in } M(\Omega).
\quad \text{(3.16)}
\]

Moreover, \(\partial_{i,j}^2 \varphi(\tilde{w}_n) \ u_n\) converges uniformly to \(\partial_{i,j}^2 \varphi \ u\). Thus, by passing to the limit in (3.14), we have, again with the Einstein convention

\[
\langle f, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = - \int_{\Omega} (\sigma^*)^T e_i \cdot e_j \partial_{i,j}^2 \varphi \ u \ dx = - \int_{\Omega} \sigma^T \nabla^2 \varphi \ u \ dx.
\]
Therefore, by integrating by parts and using \( \varphi = 0 \) on \( \partial \Omega \),

\[
\int_{\Omega} \sigma_n \nabla u \cdot \nabla \varphi \, dx = (f, \varphi)_{H^{-1}(\Omega), H^1_0(\Omega)}.
\] (3.17)

**Proof of the corrector result**

First of all, we show that the corrector function \( w_n \) is bounded in \( H^1(\Omega)^2 \). By the definition (3.8) of \( w_n \), the \( Y \)-periodicity of \( W_n \) and the equi-coercivity of \( \Sigma_n \), we have, for \( i = 1, 2 \),

\[
\alpha \| \nabla w_n^i \|^2_{L^2(\Omega)^2} \leq C\alpha \| \nabla W_n^i \|^2_{L^2(\Omega)^2} \leq C \int_Y \Sigma_n \nabla W_n^i \cdot \nabla W_n^i \, dx = C (\sigma_n)_{e_i, e_i}
\] (3.18)

which is bounded. This inequality combined with the uniform convergence of \( w_n \) yields to the boundedness of \( w_n \) in \( H^1(\Omega)^2 \).

Let us consider an approximation \( u^\delta \in \mathcal{D}(\Omega) \) of \( u \) such that

\[
||u - u^\delta||_{H^1_0(\Omega)} \leq \delta.
\] (3.19)

On the one hand, we have

\[
\int_{\Omega} \sigma_n \nabla u_n \cdot \nabla (u_n - u^\delta(w_n)) \, dx = (f, (u_n - u^\delta(w_n)))_{H^{-1}(\Omega), H^1_0(\Omega)}.
\]

Since \( w_n \) converges uniformly to identity on \( \Omega \) and is bounded in \( H^1(\Omega) \) (see (3.18)), with \( u^\delta \in \mathcal{D}(\Omega) \), \( u^\delta(w_n) \) converges weakly to \( u^\delta \) in \( H^1_0(\Omega) \). Hence, by the weak convergence of \( u_n \) to \( u \) in \( H^1_0(\Omega) \) and (3.19), we can pass to the limit the previous inequality and obtain, for any \( \delta > 0 \),

\[
\limsup_{n \to \infty} \left| \int_{\Omega} \sigma_n \nabla u_n \cdot \nabla (u_n - u^\delta(w_n)) \, dx \right| = \left| (f, u - u^\delta)_{H^{-1}(\Omega), H^1_0(\Omega)} \right| \leq C\delta.
\] (3.20)

On the other hand, similarly to the proof of the first point (3.14), we are led to the equality

\[
\int_{\Omega} \sigma_n \nabla (u^\delta(w_n)) \cdot \nabla (u_n - u^\delta(w_n)) \, dx = -\int_{\Omega} \sigma_n \nabla u_n^i \cdot \nabla u_n^j \, \partial_{ij}^2 u^\delta(w_n) \, (u_n - u^\delta(w_n)) \, dx.
\] (3.21)

As in the first point, \( \sigma_n \nabla u_n^i \cdot \nabla u_n^j \) is bounded in \( L^1(\Omega) \) (see (3.15)), \( u_n \) converges uniformly to \( u \) and \( \partial_{ij} u^\delta(w_n) \) converges uniformly to \( \partial_{ij} u^\delta \) because \( u^\delta \) is a \( \mathcal{D}(\Omega) \) function. By passing to the limit in (3.21)

\[
\int_{\Omega} \sigma_n \nabla (u^\delta(w_n)) \cdot \nabla (u_n - u^\delta(w_n)) \, dx \xrightarrow{n \to \infty} \int_{\Omega} (\sigma_s)_{e_i, e_j} \, \partial_{ij}^2 u^\delta \, (u - u^\delta) \, dx.
\] (3.22)

Moreover, like in (3.17) we have

\[
\int_{\Omega} (\sigma_s)_{e_i, e_j} \, \partial_{ij}^2 u^\delta \, (u - u^\delta) \, dx = \int_{\Omega} \sigma_s \nabla u^\delta \cdot \nabla (u - u^\delta) \, dx.
\] (3.23)

By combining this equality with the convergence (3.22), we obtain the inequality

\[
\lim_{n \to \infty} \left| \int_{\Omega} \sigma_n \nabla (u^\delta(w_n)) \cdot \nabla (u_n - u^\delta(w_n)) \, dx \right| \leq \int_{\Omega} \sigma_n \nabla u^\delta \cdot \nabla (u - u^\delta) \, dx \leq C |\sigma_s| \| u^\delta \|_{L^2(\Omega)^2} \| \nabla (u - u^\delta) \|_{L^2(\Omega)^2} \leq C\delta.
\] (3.24)

(3.25)

Thus, by adding (3.20) and (3.25), we have

\[
\limsup_{n \to \infty} \int_{\Omega} \sigma_n \nabla (u_n - u^\delta(w_n)) \cdot \nabla (u_n - u^\delta(w_n)) \, dx \leq C\delta.
\]
which leads us, by equi-coercivity, to

$$\limsup_{n \to \infty} \alpha \left| \nabla(u_n - u^\delta(w_n)) \right|_{L^2(\Omega)}^2 \leq \limsup_{n \to \infty} \left| \int_\Omega \sigma_n \nabla(u_n - u^\delta(w_n)) \cdot \nabla(u_n - u^\delta(w_n)) \, dx \right| \leq C\delta. \tag{3.26}$$

Thus, the Cauchy-Schwarz inequality, the boundedness of $\nabla w^i_n$ in $L^2(\Omega)^2$ (3.18) and the Einstein convention give, for any $\delta > 0$,

$$\left| \nabla u_n - \nabla w^i_n \partial_i u \right|_{L^1(\Omega)^2} \leq \left| \nabla u_n - \nabla w^i_n \partial_i u^\delta \right|_{L^1(\Omega)^2} + \left| \nabla w^i_n \partial_i (u^\delta - u) \right|_{L^1(\Omega)^2}$$

$$\leq \left| \nabla u_n - \nabla w^i_n \partial_i u^\delta \right|_{L^1(\Omega)^2} + ||\nabla w^i_n||_{L^2(\Omega)^2} ||\partial_i (u^\delta - u)||_{L^2(\Omega)}$$

$$\leq \left| \nabla u_n - \nabla w^i_n \partial_i u^\delta \right|_{L^1(\Omega)^2} + ||\nabla w^i_n||_{L^2(\Omega)^2} ||\partial_i (u^\delta - u)||_{L^2(\Omega)} + C\delta$$

$$\leq \left| \nabla u_n - \nabla w^i_n \partial_i u^\delta \right|_{L^1(\Omega)^2} + ||\nabla w^i_n||_{L^2(\Omega)^2} \left( ||\partial_i (u^\delta - u)\partial_i u^\delta(w_n)||_{L^2(\Omega)} + C\delta \right)$$

$$\leq \left| \nabla u_n - \nabla w^i_n \partial_i u^\delta \right|_{L^1(\Omega)^2} + \left( ||\nabla w^i_n||_{L^2(\Omega)^2} + C||\partial_i u^\delta - \partial_i u^\delta(w_n)||_{L^2(\Omega)} + C\delta \right).$$

Since $u^\delta \in \mathcal{D}(\Omega)$ and $w_n$ converges uniformly to the identity on $\Omega$, the second term of the last inequality converges to 0. Hence, we get that

$$\limsup_{n \to \infty} \left| \nabla u_n - \nabla w^i_n \partial_i u \right|_{L^1(\Omega)^2} \leq \limsup_{n \to \infty} \left| \nabla u_n - \nabla w^i_n \partial_i u^\delta(w_n) \right|_{L^1(\Omega)^2} + C\delta. \tag{3.27}$$

Finally, this inequality combined with (3.26) gives, for any $\delta > 0$,

$$0 \leq \limsup_{n \to \infty} \left| \nabla u_n - \nabla w^i_n \partial_i u \right|_{L^1(\Omega)^2} \leq C\sqrt{\delta} + C\delta,$n

which implies the corrector result (3.12). \hfill \Box

**Remark 3.2** If the solution $u$ is a $C^2$ function, then the convergence (3.12) holds true in $L^2(\Omega)$ since we may take $u = u^\delta$.

### 3.2 A two-phase result

Here, we recall a two-phase result due to G.W. Milton (see [35] pp. 61–65) using the Dykhne transformation.

In order to apply the previous theorem, we reformulate Milton’s calculus in such a way that every coefficient depends on $n$. We then consider, for a fixed $n$, the periodic homogenization of a conductivity $\sigma_n(h)$ to obtain $(\sigma_n)_h(h)$ through the link between the homogenization of the transformed conductivity and $(\sigma_n)_h$ given by formula (4.16) in [35]. Finally, we study the limit of $(\sigma_n)_h(h)$ through the asymptotic behavior of the coefficients of the transformation, and apply Theorem 3.1 in the example Section 3.3.

In this section we consider a two-phase periodic isotropic medium. Let $\chi_n$ be a sequence of characteristic functions of subsets of $Y$. We define for any $\alpha_1 > 0$, $\beta_1 \in \mathbb{R}$, any sequences $\alpha_{2,n} > 0$, $\beta_{2,n} \in \mathbb{R}$ and any $h \in \mathbb{R}$, a parametrized conductivity $\Sigma_n(h)$:

$$\Sigma_n(h) = (1 - \chi_n)(\alpha_1 I_2 + h\beta_1 J) + \chi_n(\alpha_{2,n} I_2 + h\beta_{2,n} J) \quad \text{in } Y. \tag{3.28}$$

We still denote by $\Sigma_n(h)$ the periodic extension to $\mathbb{R}^2$ of $\Sigma_n(h)$ (which satisfies (3.1)). We assume that $\Sigma_n(h)$ satisfies (3.2), and define $\sigma_n(h)$ by (3.3) and $(\sigma_n)_h(h)$ by (3.4).

We have the following result based on an analysis of [35] (pp. 61–65).

19
Proposition 3.1 Let $\chi_n$ be a sequence of characteristic functions of subsets of $Y$, $\alpha_1, \alpha_2 > 0$, a positive sequence $\alpha_{2,n}, \beta_1, \beta_2 \in \mathbb{R}$, and a sequence $\beta_{2,n}$ such that

$$\lim_{n \to \infty} \alpha_{2,n} = \infty, \quad \liminf_{n \to \infty} |\beta_{2,n} - \beta_1| > 0, \quad \text{and} \quad \lim_{n \to \infty} \frac{\beta_{2,n}}{\alpha_{2,n}} = \frac{\beta_2}{\alpha_2}. \quad (3.29)$$

Assume that the effective conductivity in the absence of a magnetic field

$$(\sigma_n^0)_{\gamma_1,n, \gamma_2,n} \text{ is bounded when } \lim_{n \to \infty} \gamma_1,n = \alpha_1 \text{ and } \lim_{n \to \infty} \gamma_{2,n} = \gamma_2 > 0. \quad (3.30)$$

Then, there exist two parametrized positive sequences $\alpha_1', n(h), \alpha_{2,n}'(h)$ such that

$$\lim_{n \to \infty} \alpha_1,n(h) = \alpha_1 \text{ and } \alpha_2,n'(h) \sim \frac{a^2 + 2h\beta_1^2}{\alpha_2} \alpha_{2,n}, \quad (3.31)$$

and

$$(\sigma_n)_*(h) = (\sigma_n^0)_*(\alpha_1', n(h), \alpha_{2,n}'(h)) + h\beta_1 J + o(n \to \infty) (1) \quad (3.32)$$

where $(\sigma_n^0)_*(\alpha_1,n(h), \alpha_{2,n}'(h))$ is bounded.

Remark 3.3 In view of condition (3.29), the case where $\beta_{2,n}$ tends to $\beta_1$ corresponds to perturb the symmetric conductivity

$$\sigma_n^s = (1 - \chi_n)\alpha_1 I_2 + \chi_n \alpha_{2,n} I_2$$

by

$$\sigma_n^s + \beta_1 J + o(n \to \infty) (1).$$

Then it is clear that

$$(\sigma_n)_*(h) = (\sigma_n^s)_* + \beta_1 J + o(n \to \infty) (1).$$

Proof of Proposition 3.1. The proof is divided into two parts. After applying Milton’s computation (pp. 61–64 of [35]), we study the asymptotic behavior of the different coefficients.

First step: Applying Dykhne’s transformation through Milton’s computations.

In order to make the Dykhne’s transformation following Milton [35] (pp. 62–64), we consider two real coefficients $a_n, b_n$ such that

$$\sigma_n' := (a_n \sigma_n(h) + b_n J) (a_n I_2 + J \sigma_n(h))^{-1} = a_n (\sigma_n(h) + (a_n)^{-1} b_n J) (a_n I_2 + J \sigma_n(h))^{-1} \quad (3.33)$$

is symmetric and, more precisely, according to Notation 1.1, reads as

$$\sigma_n' = (1 - \chi_n)\alpha_1', n(h) I_2 + \chi_n \alpha_{2,n}'(h) I_2 = \sigma_n^0(\alpha_1', n(h), \alpha_{2,n}'(h)). \quad (3.34)$$

Then, using the complex representation

$$\alpha I_2 + \beta J \longleftrightarrow \alpha + \beta i \quad (3.35)$$

suggested by Tartar [41], the constants $a_n, b_n$ must satisfy

$$\alpha_1', n(h) = \frac{a_n (\alpha_1 + ih\beta_1) + ib_n}{a_n + i(\alpha_1 + ih\beta_1)} \in \mathbb{R} \quad \text{and} \quad \alpha_{2,n}'(h) = \frac{a_n (\alpha_{2,n} + ih \beta_{2,n}) + ib_n}{a_n + i(\alpha_{2,n} + ih \beta_{2,n})} \in \mathbb{R}, \quad (3.36)$$

which implies that

$$b_n = \frac{-a_n^2 h \beta_1 + a_n \Delta_1}{a_n - h \beta_1^2} = \frac{-a_n^2 h \beta_{2,n} + a_n \Delta_{2,n}}{a_n - h \beta_{2,n}}. \quad (3.37)$$
Denoting $\Delta_1 := \alpha_1^2 + h^2\beta_1^2$ and $\Delta_2,n := \alpha_2^2,n + h^2\beta_2^2,n$ (thanks to (3.29), $n$ is considered to be larger enough such that $\beta_{2,n} - \beta_1 \neq 0$ and $a_n$ is real), the equality (3.37) provides two non-zero solutions for $a_n$:

$$a_n = \frac{\Delta_{2,n} - \Delta_1 + \sqrt{(\Delta_{2,n} - \Delta_1)^2 + 4h^2(\beta_{2,n} - \beta_1)(\beta_{2,n}\Delta_1 - \beta_1\Delta_{2,n})}}{2h(\beta_{2,n} - \beta_1)},$$

(3.38)

and

$$a_n^- = \frac{\Delta_{2,n} - \Delta_1 - \sqrt{(\Delta_{2,n} - \Delta_1)^2 + 4h^2(\beta_{2,n} - \beta_1)(\beta_{2,n}\Delta_1 - \beta_1\Delta_{2,n})}}{2h(\beta_{2,n} - \beta_1)}.$$

The value (3.38) is associated with a positive matrix $\sigma'_n$, while $a_n^- \cdot h$ leads us to the negative matrix $\sigma_n^- = -J(\sigma'_n)^{-1}J^{-1}$ to exclude (see [34] for more details).

**Second step**: asymptotic behavior of the coefficients and the homogenized matrix.

One the one hand, by the equality (3.38) combined with (3.29), we have

$$\lim_{n \to \infty} a_n \frac{h(\beta_{2,n} - \beta_1)}{\alpha_{2,n}^2} = \frac{\alpha_2^2 + h^2\beta_2^2}{\alpha_2^2}$$

which clearly implies that

$$a_n \sim \frac{\alpha_2^2 + h^2\beta_2^2}{\alpha_2^2} \frac{\alpha_{2,n}^2}{h(\beta_{2,n} - \beta_1)} \quad \text{and} \quad a_n - h\beta_{2,n} \sim \frac{\alpha_{2,n}^2}{h(\beta_{2,n} - \beta_1)}.$$

(3.39)

On the other hand, (3.29), (3.39) and the first equality of (3.37) give

$$b_n = -a_nh\beta_1 + \Delta_1 + o_{n \to \infty}(1).$$

(3.40)

From (3.29), (3.38), (3.39) and (3.40) we deduce the following asymptotic behavior for the modified phases:

$$\lim_{n \to \infty} \alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \lim_{n \to \infty} \frac{\alpha'_{2,n}(h)}{\alpha_{2,n}} = \alpha_2^2 + h^2\beta_2^2.$$  

(3.41)

To consider $(\sigma'_n)_*$, we need to verify that $\sigma'_n$ is equi-coercive. We have, by denoting for any $\xi \in \mathbb{R}^2$, $\nu_n = (a_nI_2 + J\sigma_n(h))^{-1}\xi$,

$$\forall \xi \in \mathbb{R}^2, \quad \sigma'_n \cdot \xi = (a_n\sigma_n(h) + b_nJ)\nu_n \cdot (a_nI_2 + J\sigma_n(h))\nu_n = (a_n^2 + b_n)\sigma_n(h)\nu_n \cdot \nu_n$$

and, because $a_n^{-1}\sigma_n(h)$ is bounded in $L^\infty(\Omega)^{2\times 2}$ by (3.39),

$$\forall \xi \in \mathbb{R}^2, \quad |\xi| = |a_n\nu_n + J\sigma_n(h)\nu_n| \leq a_n(1 + C)|\nu_n|.$$  

The equi-coercivity of $\sigma_n(h)$ gives

$$\exists C > 0, \quad \forall \xi \in \mathbb{R}^2, \quad \sigma'_n \cdot \xi \geq \frac{C}{(1 + C)^2} \frac{a_n^2 + b_n}{a_n^2}|\xi|^2$$

(3.42)

that is, for $n$ larger enough, by (3.39) and (3.40), $\sigma'_n$ is equi-coercive.

We can now apply the Keller-Dykhne duality theorem (see, e.g., [30, 23]) to equality (3.33) to obtain

$$(\sigma'_n)_* = (a_n(\sigma_n)_* + b_nJ)(a_nI_2 + J(\sigma_n)_*)^{-1}.$$  

(3.43)

Moreover, by inverting this transformation, we have

$$(\sigma_n)_*(h) = (a_nI_2 - (\sigma'_n)_*J)^{-1}(a_n(\sigma'_n)_* - b_nJ).$$

21
Figure 3.1: The period of the cross-like thin structure

Considering \((3.29)\), \((3.39)\), \((3.40)\), and the boundedness of \((\sigma'_n)_*\) (as a consequence of the bound \((3.30)\)) we get that

\[
(\sigma)_*(h) = (\sigma'_n)_* - \frac{b_n}{a_n} J + o_{n \to \infty}(1) = (\sigma'_n)_* + h \beta_1 J + o_{n \to \infty}(1),
\]

which concludes the proof taking into account \((3.34)\). □

To derive the limit of \((\sigma'_n)_* (\alpha'_{1,n}(h), \alpha'_{2,n}(h))\), we need more information on the geometry of the high conductive phase. To this end, we study the following example.

### 3.3 A cross-like thin structure

We consider a bounded open subset \(\Omega \subset \mathbb{R}^2\) with a Lipschitz boundary, a real sequence \(\varepsilon_n\) converging to 0, and \(f \in H^{-1}(\Omega) \cap W^{-1,q}(\Omega)\) with \(q > 2\). We define, for any \(h \in \mathbb{R}\), \(\alpha_1, \beta_1 > 0\) and positive sequences \(t_n \in (0, 1/2), \alpha_2, n, \beta_2, n\), a parametrized matrix-valued function \(\Sigma_n(h)\) from the unit rectangular cell period \(Y := (-\frac{\ell}{2}, \frac{\ell}{2}) \times (-\frac{1}{2}, \frac{1}{2})\), with \(\ell \geq 1\), to \(\mathbb{R}^{2 \times 2}\), by (cf. figure 3.1)

\[
\Sigma_n(h) := \begin{cases} 
\alpha_2, n, I_2 + \beta_2, n, h J & \text{in } \omega_n := \{(x_1, x_2) \in Y \mid |x_1|, |x_2| \leq t_n\} \\
\alpha_1, I_2 + \beta_1, h J & \text{in } Y \setminus \omega_n
\end{cases}
\]

Denoting again by \(\Sigma_n(h)\) its periodic extension to \(\mathbb{R}^2\), we finally consider the conductivity

\[
\sigma_n(h)(x) = \Sigma_n(h) \left( \frac{x}{\varepsilon_n} \right), \quad x \in \Omega,
\]

and the associated homogenization problem:

\[
P_n \begin{cases} 
-\text{div}(\sigma_n(h) \nabla u_n) = f & \text{in } \Omega \\
u_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]

By virtue of Theorem 3.1 and Proposition 3.1, we focus on the study of the limit of \((\sigma'_n)_* (\alpha'_{1,n}(h), \alpha'_{2,n}(h))\).

**Proposition 3.2** Let \(\sigma_n(h)\) be the conductivity defined by \((3.45)\) and \((3.46)\) and its homogenization problem \((3.47)\). We assume that:

\[
2t_n(\ell + 1)\alpha_2, n \xrightarrow{n \to \infty} \alpha_2 > 0 \quad \text{and} \quad 2t_n(\ell + 1)\beta_2, n \xrightarrow{n \to \infty} \beta_2 > 0.
\]
Then, the homogenized conductivity is given by

\[
\sigma_*(h) = \begin{pmatrix}
\alpha_1 + \frac{\alpha_2 + \beta_2 h^2}{(\ell + 1)\alpha_2} & -h\beta_1 \\
-h\beta_1 & \alpha_1 + \frac{\alpha_2 + \beta_2 h^2}{\ell(\ell + 1)\alpha_2}
\end{pmatrix}.
\]

Remark 3.4 The previous proposition does not respect exactly the framework defined at the beginning of this section because the period cell is not the unit square \(Y = (0,1)^2\): we can nevertheless extend all this section to any type of period cells.

Remark 3.5 The condition (3.48) is a condition of boundedness in \(L^1(\Omega)^{2\times 2}\) of \(\sigma_n\) because

\[
|\omega_n| = 2t_n(\ell + 1) - 4t_n^2 \sim 2t_n(\ell + 1),
\]

which will ensure the convergence of \((\sigma^0_n)_*\).

Proof of Proposition 3.2. In order to apply Proposition 3.1, we consider two positive sequences \(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\) satisfying

\[
\lim_{n \to \infty} \alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \alpha'_{2,n}(h) \sim \frac{\alpha_2 + h^2\beta_2}{\alpha_2} \alpha_{2,n}.
\]  

(3.49)

We will study the homogenization of \(\sigma'_n := \sigma^0_n(\alpha'_{1,n}(h), \alpha'_{2,n}(h))\).

To this end, consider a corrector \(W^\lambda_n = \lambda \cdot x - X^\lambda_n\) in the Murat-Tartar sense (see, e.g., [38]) associated with

\[
\Sigma'_n := \begin{cases}
\alpha'_{2,n}(h) I_2 & \text{in } \omega_n = \{(x_1, x_2) \in Y \mid |x_1|, |x_2| \leq t_n\} \\
\alpha'_{1,n}(h) I_2 & \text{in } Y \setminus \omega_n
\end{cases}
\]  

(3.50)

and defined by

\[
\begin{cases}
\text{div}(\Sigma'_n \nabla X^\lambda_n) = \text{div}(\Sigma'_n \lambda) \quad \text{in } \mathcal{D}'(\mathbb{R}^2) \\
X^\lambda_n \text{ is } Y \text{- periodic} \\
\int_Y X^\lambda_n \, dy = 0.
\end{cases}
\]  

(3.51)

On one hand, the extra diagonal coefficients of \((\sigma'_n)_*\) are equal to 0 because, as \(\Sigma'_n\) is an even function on \(Y\), we have, for \(i = 1, 2\),

\[
\begin{cases}
y_i \mapsto W^{e_i}_n(y) & \text{is an odd function}, \\
y_i \mapsto W^{e_j}_n(y) & \text{is an even function for } i \neq j,
\end{cases}
\]

which implies that \(y_i \mapsto \Sigma'_n \nabla W^{e_i}_n \cdot \nabla W^{e_j}_n\) is an odd function. Then, by symmetry of \(Y\) with respect to 0,

\[
(\sigma'_n)_* e_i \cdot e_j = \int_Y \Sigma'_n \nabla W^{e_i}_n \cdot \nabla W^{e_j}_n \, dy = 0.
\]

On the other hand, as \(\Sigma'_n\) is isotropic, for the diagonal coefficients, we use the Voigt-Reuss inequalities (see, e.g., [29] p.44 or [36]): for any \(i = 1, 2\) and \(j \neq i\),

\[
\langle (\Sigma'_n e_i \cdot e_i)^{-1} \rangle_i^{-1} \leq (\sigma'_n)_* e_i \cdot e_i \leq \langle (\Sigma'_n e_i \cdot e_i)^{-1} \rangle_i^{-1}
\]

(3.52)

where \(\langle \cdot \rangle_i\) denotes the average with respect to \(y_i\) at a fixed \(y_j\) for \(j \neq i\).
An easy computation gives, for the direction $e_1$,
\[
(1 - 2t_n) \left( \frac{\ell - 2t_n}{\ell\alpha_{1,n}(h)} + \frac{2t_n}{\ell\alpha_{2,n}(h)} \right)^{-1} + 2t_n \left( \frac{\ell}{\ell\alpha_{2,n}(h)} \right)^{-1} \leq (\sigma'_n)_* e_1 \cdot e_1
\]
and
\[
(\sigma'_n)_* e_1 \cdot e_1 \leq \ell \left( \frac{\ell - 2t_n}{(1 - 2t_n)\alpha_{1,n}(h) + 2t_n\alpha_{2,n}(h)} + \frac{2t_n}{\alpha_{2,n}(h)} \right)^{-1}.
\]
By (3.48) and (3.49), we have the convergence
\[
\lim_{n \to \infty} (\sigma'_n)_* e_1 = \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{(\ell + 1)\alpha_2}.
\]
A similar computation on the direction $e_2$ gives the asymptotic behavior:
\[
\lim_{n \to \infty} (\sigma'_n)_* = \lim_{n \to \infty} (\sigma^0)_* (\alpha'_{1,n}(h), \alpha'_{2,n}(h)) = \begin{pmatrix} \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{(\ell + 1)\alpha_2} & 0 \\ 0 & \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{\ell(\ell + 1)\alpha_2} \end{pmatrix}.
\]
Moreover, the matrix $\sigma_n(h)$ clearly satisfies all the hypothesis of Theorem 3.1. By Theorem 3.1 and (3.53), we have
\[
\lim_{n \to \infty} (\sigma_n)_* = \lim_{n \to \infty} (\sigma^0)_* (\alpha'_{1,n}(h), \alpha'_{2,n}(h)) + \beta_1 h J = \begin{pmatrix} \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{(\ell + 1)\alpha_2} & -h\beta_1 \\ h\beta_1 & \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{\ell(\ell + 1)\alpha_2} \end{pmatrix}
\]
We finally apply Theorem 3.1 to get that $\sigma_*(h) = \lim_{n \to \infty} (\sigma_n)_*$. 

4 A three-dimensional fibered microstructure

In this section we study a particular two-phase composite in dimension three. One of the phases is composed by a periodic set of high conductivity fibers embedded in an isotropic medium (figure 4.1a). The conductivity $\sigma_n(h)$ is not symmetric due to the perturbation of a magnetic field.

First, describe the geometry of the microstructure. Let $Y := (-\frac{1}{2}, \frac{1}{2})^3$ be the unit cube centered at the origin of $\mathbb{R}^3$. For $r_n \in (0, \frac{1}{2})$, consider the closed cylinder $\omega_n$ parallel to the $x_3$-axis, of radius $r_n$ and centered in $Y$:
\[
\omega_n := \{ y \in Y \mid y_1^2 + y_2^2 \leq r_n^2 \}.
\]
Let $\Omega = \tilde{\Omega} \times (0, 1)$ be an open cylinder of $\mathbb{R}^3$, where $\tilde{\Omega}$ is a bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary. For $\varepsilon_n \in (0, 1)$, consider the closed subset $\Omega_n$ of $\Omega$ defined by the intersection of $\Omega$ with the $\varepsilon_n Y$-periodic network in $\mathbb{R}^3$ composed by the closed cylinders parallel to the $x_3$-axis, centered on the points $\varepsilon_n k$, $k \in \mathbb{Z}^3$, in the $x_1$-$x_2$ plane, and of radius $\varepsilon_n r_n$, namely:
\[
\Omega_n := \Omega \cap \bigcup_{\nu \in \mathbb{Z}^3} \varepsilon_n (\omega_n + \nu).
\]
The period cell of the microstructure is represented in figure 4.1b.
The fibers lattice

We then define the two-phase conductivity by

$$
\sigma_n(h) = \begin{cases} 
\alpha_1 I_3 + \beta_1 \mathcal{E}(h) & \text{in } \Omega \setminus \Omega_n \\
\alpha_{2,n} I_3 + \beta_{2,n} \mathcal{E}(h) & \text{in } \Omega_n,
\end{cases}
$$

where $\alpha_1 > 0$, $\beta_1 \in \mathbb{R}$, $\alpha_{2,n} > 0$ and $\beta_{2,n}$ are real sequences, and

$$
\mathcal{E}(h) := \begin{pmatrix} 0 & -h_3 & h_2 \\
h_3 & 0 & -h_1 \\
-h_2 & h_1 & 0 \end{pmatrix}, \quad \text{for } h = \begin{pmatrix} h_1 \\
h_2 \\
h_3 \end{pmatrix} \in \mathbb{R}^3.
$$

Our aim is to study the homogenization problem

$$
P_{\Omega,n} \begin{cases} 
-\operatorname{div}(\sigma_n(h) \nabla u_n) = f & \text{in } \Omega \\
u_n = 0 & \text{on } \partial \Omega.
\end{cases}
$$

Theorem 4.1 Let $\alpha_1 > 0$, $\beta_1 \in \mathbb{R}$, and let $\varepsilon_n, r_n, \alpha_{2,n}, \beta_{2,n}$, $n \in \mathbb{N}$, be real sequences such that $\varepsilon_n, r_n > 0$ converge to 0, $\alpha_{2,n} > 0$, and

$$
\lim_{n \to \infty} \varepsilon_n^2 |\ln r_n| = 0, \quad \lim_{n \to \infty} |\omega_n| \alpha_{2,n} = \alpha_2 > 0, \quad \lim_{n \to \infty} |\omega_n| \beta_{2,n} = \beta_2 \in \mathbb{R}.
$$

Consider, for $h \in \mathbb{R}^3$, the conductivity $\sigma_n(h)$ defined by (4.3).

Then, there exists a subsequence of $n$, still denoted by $n$, such that, for any $f \in H^{-1}(\Omega)$ and any $h \in \mathbb{R}^3$, the solution $u_n$ of $P_{\Omega,n}$ converges weakly in $H_0^1(\Omega)$ to the solution $u$ of

$$
P_{\Omega,*} \begin{cases} 
-\operatorname{div}(\sigma_*(h) \nabla u) = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$
where $\sigma(h)$ is given by

$$\sigma(h) = \alpha_1 I_3 + \left(\frac{\alpha_2^3 + \alpha_2 \beta_2 |h|^2}{\alpha_2^2 + \beta_2 h_3^2}\right) e_3 \otimes e_3 + \beta_1 E(h).$$

(4.7)

**Remark 4.1** Theorem 4.1 can be actually extended to fibers with a more general cross-section. More precisely, we can replace the disk $r_nD$ of radius $r_n$ by the homothetic $r_nQ$ of any connected open set $Q$ included in the unit disk $D$, such that the present fiber $\omega_n$ is replaced by the new fiber $r_nQ \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ in the period cell of the microstructure.

On the one hand, this change allows us to use the same test function $v_n$ (4.8) defined in the proof of Theorem 4.1, since $v_n$ remains equal to 1 in the new fibers due to the inclusion $Q \subset D$. On the other hand, Lemma 4.1 allows us to replace the disk $D$ by the open set $Q \subset D$.

**Remark 4.2** We can also extend the result of Theorem 4.1 to an isotropic fibered microstructure composed by three similar periodic fibers lattices arranged in the three orthogonal directions $e_1, e_2, e_3$, namely

$$\omega_n := \bigcup_{j=1}^3 \{y \in Y | \sum_{i \neq j} y_i^2 \leq r_n^2\} \quad \text{and} \quad \Omega_n := \Omega \cap \bigcup_{\nu \in \mathbb{Z}^3} \varepsilon_n(\omega_n + \nu),$$

as represented in figure 4.2. Then, we derive the following homogenization conductivity:

$$\sigma(h) = \alpha_1 I_3 + \sum_{i=1}^3 \left(\frac{\alpha_2^3 + \alpha_2 \beta_2 |h|^2}{\alpha_2^2 + \beta_2 h_3^2}\right) e_i \otimes e_i + \beta_1 E(h).$$

Figure 4.2: The period cell of the isotropic fibered structure in dimension 3

**Remark 4.3** We can check that when the volume fraction $\theta_n = \theta$ and the highly conducting phase of the conductivity $\alpha_{2,n} = \alpha_0$ and $\beta_{2,n} = \beta_0$ are independent of $n$, the explicit formula of [27] denoted by $\sigma_n(\theta, h)$, for the classical (since the period cell is now independent of $n$) periodically homogenized conductivity (see (3.4)) has a limit as $\theta \to 0$ when $\alpha_0$ and $\beta_0$ converge. Indeed, we may replace in the computations of [27] the optimal Vigdergauz shape by the circular cross-section in the previous asymptotic regime. Therefore, Theorem 4.1 validates the double process characterized by the homogenization at a fixed volume fraction $\theta$ combined with the limit as $\theta \to 0$, by one homogenization process in which both the period and the volume fraction $\theta_n = \pi r_n^2$ of the high conductivity phase tend to 0 as $n \to \infty$.

**Remark 4.4** The hypothesis on the convergence of $\varepsilon_n^2 |\ln r_n|$ (4.5) allows us to avoid nonlocal effects in dimension three (see [24, 1]). These effects do not appear in dimension two as shown in [12]. Therefore, we can make a comparison between dimension two and dimension three based on the strong field perturbation in the absence of nonlocal effects.
Remark 4.5 If \( h = h_3 e_3 \), the homogenized conductivity becomes
\[
\sigma_\ast(h) = \alpha_1 I_3 + \alpha_2 e_3 \otimes e_3 + \beta_1 h_3 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
which reduces to the simplified two-dimensional case when the symmetric part of the conductivity is independent of \( h_3 \) (i.e. \( \sigma_\ast^{\text{sym}} \) in (2.40) does not depend on its second argument).

**Proof of Theorem 4.1** The proof will be divided into four parts. We first prove the weak-* convergence in \( \mathcal{M}(\Omega) \) of \( \sigma_n(h) \nabla u_n \) in \( \Omega_n \). Then we establish a linear system satisfied by the limits defined by
\[
\mathbf{1}_{\Omega_n} \frac{\partial u_n}{|\omega_n| \partial x_i} \to \xi_i \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega).
\]
Moreover, we deduce from Lemma 4.1 that
\[
\mathbf{1}_{\Omega_n} \frac{\partial u_n}{|\omega_n| \partial x_3} \to \frac{\partial u}{\partial x_3} \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega).
\]
We finally calculate the homogenized matrix.

We first remark that, classically, the sequence of solutions \( u_n \) of \( P_{\Omega,n} \) (see (4.4)) is bounded in \( H^1_0(\Omega) \) because, since \( \alpha_{2,n} \) diverges to \( \infty \):
\[
||\nabla u_n||^2_{L^2(\Omega)^3} \leq C \int_\Omega (\alpha_1 \mathbf{1}_{\Omega_n} I_3 + \alpha_{2,n} \mathbf{1}_{\Omega_n} I_3) \nabla u_n \cdot \nabla u_n \, dx = \int_\Omega \sigma_n(h) \nabla u_n \cdot \nabla u_n \, dx.
\]
By the Poincaré inequality, the previous inequality and (4.4) lead us to
\[
||u_n||^2_{H^1_0(\Omega)} \leq C ||\nabla u_n||^2_{L^2(\Omega)^3} \leq C \langle f, u_n \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \leq C ||f||_{H^{-1}(\Omega)} ||u_n||_{H^1_0(\Omega)}
\]
and then
\[
||u_n||_{H^1_0(\Omega)} \leq C ||f||_{H^{-1}(\Omega)}.
\]
Thus, up to a subsequence still denoted by \( n \), \( u_n \) converges weakly to some function \( u \) in \( H^1_0(\Omega) \).

**First step:** Weak-* convergence in \( \mathcal{M}(\Omega) \) of the conductivity in the fibers \( \mathbf{1}_{\Omega_n} (\alpha_{2,n} I_3 + \beta_{2,n} \sigma(h)) \nabla u_n \).

We proceed as in [22] with a suitable oscillating test function. For \( R \in (0, 1/2) \), define the \( Y \)-periodic (independent of \( y_3 \)) function \( V_n \) by
\[
V_n(y_1, y_2, y_3) = \begin{cases} 
1 & \text{if } \sqrt{y_1^2 + y_2^2} \leq r_n \\
\frac{1}{\ln R - \ln \sqrt{y_1^2 + y_2^2}} & \text{if } r_n \leq \sqrt{y_1^2 + y_2^2} \leq R \\
0 & \text{if } \sqrt{y_1^2 + y_2^2} \geq R,
\end{cases}
\]
and the rescaled function
\[
v_n(x) = V_n \left( \frac{x}{\varepsilon_n} \right), \quad \text{for } x \in \mathbb{R}^3. \tag{4.8}
\]
In particular, by using the cylindrical coordinates and the fact that \( r_n \) converges to \( 0 \), this function satisfies the inequalities
\[
||v_n||^2_{L^2(\Omega)} \leq C ||V_n||^2_{L^2(\gamma)} = C \left[ \ln \frac{R}{r_n} \right]^{-2} \left( \pi r_n^2 + \int_0^{2\pi} \int_0^R r \ln^2 \frac{R}{r} \, dr \, d\theta \right) \]
\[
= C \left[ \ln \frac{R}{r_n} \right]^{-2} \left( \pi R^2 - r_n^2 - \pi r_n^2 \ln^2 \frac{R}{r_n} - \pi \ln \frac{R}{r_n} \right) \leq C \left[ \ln \frac{R}{r_n} \right]^{-2},
\]
\[
||\nabla v_n||^2_{L^2(\Omega)^3} \leq C \epsilon_n^2 ||\nabla V_n||^2_{L^2(\gamma)^3} = C \epsilon_n^2 \left[ \ln \frac{R}{r_n} \right]^{-2} \int_0^{2\pi} \int_{r_n}^R \frac{1}{r} \, dr \, d\theta \leq C \epsilon_n \left[ \ln \frac{R}{r_n} \right]^{-1}.
\]
and, consequently
\[
\|v_n\|_{L^2(\Omega)} + \varepsilon_n \|\nabla v_n\|_{L^2(\Omega)^3} \leq C \sqrt{\ln \frac{R}{r_n}} \rightarrow 0. \tag{4.9}
\]

Let \( \lambda \) be a vector in \( \mathbb{R}^3 \) perpendicular to the \( x_3 \)-axis. Define the \( Y \)-periodic function \( \tilde{X}_n \) by \( \nabla \tilde{X}_n = \lambda \) in \( \omega_n \), such that \( X_n \in \mathcal{D}(Y) \) and is \( Y \)-periodic, and the rescaled function \( X_n \) by

\[
X_n(x) = \varepsilon_n \tilde{X}_n \left( \frac{x}{\varepsilon_n} \right). \tag{4.10}
\]

In particular, \( X_n \) satisfies
\[
\|X_n\|_{\infty} = \varepsilon_n \|\tilde{X}_n\|_{\infty} \leq C \varepsilon_n, \quad \|\nabla X_n\|_{\infty} = \|\nabla \tilde{X}_n\|_{\infty} \leq C \quad \text{and} \quad \nabla X_n = \lambda \quad \text{in} \quad \Omega_n. \tag{4.11}
\]

We have, by (4.11) and (4.9),
\[
\|v_n X_n\|_{H^1(\Omega)} \leq \|X_n\|_{\infty}\|v_n\|_{L^2(\Omega)} + \|X_n\|_{\infty}\|\nabla v_n\|_{L^2(\Omega)^3} + \|\nabla X_n\|_{\infty}\|v_n\|_{L^2(\Omega)} \leq C (\|v_n\|_{L^2(\Omega)} + \varepsilon_n \|\nabla v_n\|_{L^2(\Omega)^3}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]
which gives
\[
\forall \varphi \in \mathcal{D}(\Omega), \quad \varphi v_n X_n \rightarrow 0 \quad \text{strongly in} \quad H^1_0(\Omega). \tag{4.12}
\]

Let \( \varphi \in \mathcal{D}(\Omega) \). By the strong convergence (4.12), we have
\[
\int_{\Omega} \sigma_n(h) \nabla u_n \cdot \nabla (\varphi v_n X_n) \; dx = \langle f, \varphi v_n X_n \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \tag{4.13}
\]

Let us decompose this integral which converges to 0, into the integral on the fibers set \( \Omega_n \) and the integral on its complementary:
\[
\int_{\Omega} \sigma_n(h) \nabla u_n \cdot \nabla (\varphi v_n X_n) \; dx = \int_{\Omega \setminus \Omega_n} \sigma_n(h) \nabla u_n \cdot \nabla (\varphi v_n X_n) \; dx \tag{4.14a}
\]
\[
+ \int_{\Omega_n} (\alpha_1 I_3 + \beta_1 \delta(h)) \nabla u_n \cdot \nabla (\varphi v_n X_n) \; dx. \tag{4.14b}
\]

The expression (4.14a) converges to 0 since, by the Cauchy-Schwarz inequality, the boundedness of \( u_n \) in \( H^1_0(\Omega) \) and (4.12), we have
\[
\left| \int_{\Omega \setminus \Omega_n} (\alpha_1 I_3 + \beta_1 \delta(h)) \nabla u_n \cdot \nabla (\varphi v_n X_n) \; dx \right| \leq |\alpha_1 I_3 + \beta_1 \delta(h)| \|v_n X_n\|_{L^2(\Omega)} \|\nabla u_n\|_{L^2(\Omega)^3} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \tag{4.15}
\]

Consequently, as \( v_n = 1 \) and \( \nabla X_n = \lambda \) on \( \Omega_n \), by (4.13), (4.14a), (4.14b) and (4.15), we have
\[
\int_{\Omega_n} \sigma_n(h) \nabla u_n \cdot \lambda \varphi \; dx + \int_{\Omega_n} \sigma_n(h) \nabla u_n \cdot \nabla \varphi X_n \; dx \rightarrow 0. \tag{4.16}
\]

To prove the convergence to 0 of the right term, we now show that \( \mathbf{1}_{\Omega_n} (\alpha_{2,n} I_3 + \beta_{2,n} \delta(h)) \nabla u_n \) is bounded in \( L^1(\Omega)^3 \). We have, by the Cauchy-Schwarz inequality, (4.5) and the classical equivalent \( |\Omega_n| \sim |\Omega| |\omega_n| \),
\[
\left( \int_{\Omega_n} |(\alpha_{2,n} I_3 + \beta_{2,n} \delta(h)) \nabla u_n| \; dx \right)^2 \leq |I_3 + \alpha_{2,n}^1 \beta_{2,n} \delta(h)|^2 |\Omega_n| \alpha_{2,n} \int_{\Omega_n} \alpha_{2,n} |\nabla u_n|^2 \; dx \leq C \int \sigma_n(h) \nabla u_n \cdot \nabla u_n \; dx \leq C \|f\|_{H^{-1}(\Omega)} \|u_n\|_{H^3(\Omega)}. \tag{4.17}
\]
This combined with the boundedness of \( u_n \) in \( H^1_0(\Omega) \) implies that \( \mathbb{I}_{\Omega_n}(\alpha_{2,n} I_3 + \beta_{2,n} e(h)) \nabla u_n \) is bounded in \( L^1(\Omega)^3 \). This bound and the uniform convergence to 0 of \( X_n \) (see (4.11)) imply the convergence to 0 of the right term of (4.16), hence
\[
\int_{\Omega_n} (\alpha_{2,n} I_3 + \beta_{2,n} e(h)) \nabla u_n \cdot \lambda \varphi \, dx \to 0 \quad \text{as} \quad n \to \infty.
\]
We rewrite this condition as
\[
\forall \lambda \perp e_3, \quad \mathbb{I}_{\Omega_n} (\alpha_{2,n} I_3 + \beta_{2,n} e(h)) \nabla u_n \cdot \lambda \to 0 \quad \text{weakly-* in} \quad \mathcal{M}(\Omega). \quad (4.17)
\]

**Second step:** Linear relations between weak-* limits of \( \frac{\mathbb{I}_{\Omega_n} \partial u_n}{|\omega_n| \partial x_i} \).

Thanks to the Cauchy-Schwarz inequality, we have
\[
\left\| \frac{\mathbb{I}_{\Omega_n} \partial u_n}{|\omega_n| \partial x_i} \right\|_{L^1(\Omega)} \leq \frac{1}{|\omega_n|} \int_{\Omega_n} |\nabla u_n| \, dx \leq \frac{1}{\sqrt{|\alpha_{2,n}| \omega_n}} \sqrt{\frac{|\Omega_n|}{|\omega_n|}} \sqrt{\int_{\Omega_n} \alpha_{2,n} |\nabla u_n|^2 \, dx}
\]
which leads us, by (4.5) and the asymptotic behavior \( |\Omega_n| \sim |\Omega| \, |\omega_n| \), to
\[
\left\| \frac{\mathbb{I}_{\Omega_n} \partial u_n}{|\omega_n| \partial x_i} \right\|_{L^1(\Omega)} \leq \frac{C}{\sqrt{|\alpha_{2,n}| \omega_n}} \int_{\Omega} \sigma_n(h) \nabla u_n \cdot \nabla u_n \, dx \leq C \langle f, u_n \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}
\]
which is bounded by the boundedness of \( u_n \) in \( H^1_0(\Omega) \). This allows us to define, up to a subsequence, the following limits
\[
\frac{\mathbb{I}_{\Omega_n} \partial u_n}{|\omega_n| \partial x_i} \to \xi_i \quad \text{weakly-* in} \quad \mathcal{M}(\Omega), \quad \text{for} \quad i = 1, 2, 3. \quad (4.18)
\]
Then, by (4.17) we have
\[
(\alpha_{2,n} I_3 + \beta_{2,n} e(h)) \mathbb{I}_{\Omega_n} \nabla u_n \cdot \lambda = (\alpha_{2,n} |\omega_n| I_3 + \beta_{2,n} |\omega_n| e(h)) \frac{\mathbb{I}_{\Omega_n}}{|\omega_n|} \nabla u_n \cdot \lambda \to 0 \quad \text{weakly-* in} \quad \mathcal{M}(\Omega).
\]
Therefore, putting \( \lambda = e_1, e_2 \) in this limit and using condition (4.5), we obtain the linear system
\[
\begin{cases}
\alpha_2 \xi_1 + \beta_2 h_2 \xi_3 - \beta_2 h_3 \xi_2 = 0 \\
\alpha_2 \xi_2 + \beta_2 h_3 \xi_1 - \beta_2 h_1 \xi_3 = 0
\end{cases}
\quad \text{in} \quad \mathcal{M}(\Omega),
\]
which is equivalent to
\[
\begin{cases}
\xi_1 = \frac{\beta_2 h_1 h_3 - \alpha_2 \beta_2 h_2}{\alpha_2^2 + \beta_2^2 h_3^2} \xi_3 \\
\xi_2 = \frac{\beta_2^2 h_2 h_3 + \alpha_2 \beta_2 h_1}{\alpha_2^2 + \beta_2^2 h_3^2} \xi_3
\end{cases}
\quad \text{in} \quad \mathcal{M}(\Omega). \quad (4.19)
\]

**Third step:** Proof of \( \xi_3 = \frac{\partial u}{\partial x_3} \).

We need the following result which is an extension of the estimate (3.13) of [21]. The statement of this lemma is more general than necessary for our purpose but is linked to Remark 4.1.

**Lemma 4.1** Let \( Q \) be a non-empty connected open subset of the unit disk \( D \). Then, there exists a constant \( C > 0 \) such that any function \( U \in H^1(Y) \) satisfies the estimate
\[
\left| \frac{1}{|r_n Q|} \int_{r_n Q \times (-\frac{1}{2}, \frac{1}{2})} U \, dy - \int_Y U \, dy \right| \leq C \sqrt{|\ln r_n| \, ||\nabla U||_{L^2(Y)}}. \quad (4.20)
\]
Proof of Lemma 4.1. Let $U \in H^1(Y)$. To prove Lemma 4.1, we compare the average value of $U$ on $r_nQ$ and $r_nD$. Denoting $\bar{y} = (y_1, y_2)$, we have, for any $y_3 \in (-\frac{1}{2}, \frac{1}{2})$,
\[
\left| \int_{r_nQ} U(\bar{y}, y_3) \, d\bar{y} - \int_{r_nD} U(\bar{y}, y_3) \, d\bar{y} \right| = \left| \int_{Q} U(r_n\bar{y}, y_3) \, d\bar{y} - \int_{D} U(r_n\bar{y}, y_3) \, d\bar{y} \right| 
\leq \int_{Q} U(r_n\bar{y}, y_3) - \int_{D} U(r_n\bar{y}, y_3) \, d\bar{y} 
\]
and, since $Q \subset D$,
\[
\left| \int_{r_nQ} U(\bar{y}, y_3) \, d\bar{y} - \int_{r_nD} U(\bar{y}, y_3) \, d\bar{y} \right| \leq \frac{|D|}{|Q|} \int_{D} U(r_n\bar{y}, y_3) - \int_{D} U(r_n\bar{y}, y_3) \, d\bar{y} 
\leq C \int_{D} r_n \left( \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} \right) (r_n\bar{y}, y_3) \, d\bar{y} 
\leq C |r_n| \int_{D} \left( \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} \right) (\bar{y}, y_3) \, d\bar{y},
\]
the last inequality being a consequence of the Poincaré-Wirtinger inequality. Hence, integrating the previous inequality with respect to $y_3 \in (-\frac{1}{2}, \frac{1}{2})$ and applying the Cauchy-Schwarz inequality, we obtain that
\[
\left| \int_{r_nQ} \times (-\frac{1}{2}, \frac{1}{2}) U(y) \, dy - \int_{r_nD \times (-\frac{1}{2}, \frac{1}{2})} U(y) \, dy \right| \leq \frac{C}{\pi r_n} \int_{r_nD \times (-\frac{1}{2}, \frac{1}{2})} |\nabla U| (y) \, dy 
\leq C \sqrt{\int_{r_nD \times (-\frac{1}{2}, \frac{1}{2})} |\nabla U|^2 (y) \, dy} 
\leq C ||\nabla U||_{L^2(Y)^3}.
\]
This combined with the estimate (3.13) of [21], i.e. (4.20) for $Q = D$, and the fact that $\sqrt{\ln r_n}$ diverges to $\infty$ give the thesis. \hfill \Box

Let $\varphi \in \mathcal{D}(\Omega)$. A rescaling of (4.20) with $Q = D$ implies the inequality
\[
\left| \frac{1}{|\omega_n|} \int_{\Omega_n} u_n \varphi \, dx - \int_{\Omega} u_n \varphi \, dx \right| \leq C \varepsilon_n \sqrt{\ln r_n} ||\nabla (u_n \varphi)||_{L^2(\Omega)^3}.
\]
Combining this estimate and the first condition of (4.5) with
\[
||\nabla (u_n \varphi)||_{L^2(\Omega)^3} \leq ||\nabla u_n||_{L^2(\Omega)^3} ||\varphi||_\infty + ||u_n||_{L^2(\Omega)} ||\nabla \varphi||_\infty \leq C,
\]
it follows that
\[
\frac{1}{|\omega_n|} u_n - u_n \to 0 \quad \text{in} \quad \mathcal{D}'(\Omega).
\]
This convergence does not hold true when $\varepsilon_n^2 \ln r_n$ converges to some positive constant. Under this critical regime, non-local effects appear (see Remark 4.4).

Finally, as $\mathbf{1}_{\Omega_n}$ does not depend on the $x_3$ variable, we have
\[
\frac{\mathbf{1}_{\Omega_n}}{|\omega_n|} \frac{\partial u_n}{\partial x_3} = \frac{\partial}{\partial x_3} \left( \frac{\mathbf{1}_{\Omega_n}}{|\omega_n|} u_n - u_n \right) + \frac{\partial u_n}{\partial x_3} - \frac{\partial u}{\partial x_3} = \xi_3 \quad \text{in} \quad \mathcal{D}'(\Omega).
\]

Fourth step: Derivation of the homogenized matrix.
We now study the limit of $\sigma_n(h) \nabla u_n$ in order to obtain $\sigma_\ast(h)$. We have
\[
\sigma_n(h) \nabla u_n \cdot e_1 = 1_{\Omega \setminus \Omega_n} \left( \alpha_1 \frac{\partial u_n}{\partial x_1} - \beta_1 h_3 \frac{\partial u_n}{\partial x_2} + \beta_1 h_2 \frac{\partial u_n}{\partial x_3} \right) + \alpha_2, n |\omega_n| \frac{\partial u_n}{\omega_n} |_{\Omega_n} \frac{\partial u_n}{\partial x_1} - \beta_2, n |\omega_n| \frac{\partial u_n}{\partial x_2} + \beta_2, n |\omega_n| \frac{\partial u_n}{\partial x_3}
\] (4.21)

Hence, passing to the weak-* limit in $\mathcal{M}(\Omega)$ this equality and using the linear system (4.19), $\sigma_n(h) \nabla u_n \cdot e_1$ weakly-* converges in $\mathcal{M}(\Omega)$ to
\[
\left( \alpha_1 \frac{\partial u}{\partial x_1} - \beta_1 h_3 \frac{\partial u}{\partial x_2} + \beta_1 h_2 \frac{\partial u}{\partial x_3} \right) + \alpha_2 \xi_1 - \beta_2 h_3 \xi_2 + \beta_2 h_2 \xi_3
\]
\[
= (\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) \nabla u \cdot e_1 + \alpha_2 \frac{\partial^2 u}{\partial x_2^2} h_3 - \alpha_2 \frac{\partial^2 h_2}{\partial x_2^2} \xi_3 - \beta_2 h_3 \frac{\partial^2 h_2}{\partial x_2^2} h_3 + \alpha_2 \frac{\partial^2 h_1}{\partial x_2^2} \xi_3
\]
\[
= (\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) \nabla u \cdot e_1 + \frac{\alpha_2 (\beta_2^2 h_1 - \alpha_2 \beta_2 h_2) - \beta_2 h_3 (\beta_2^2 h_2 h_3 + \alpha_2 \beta_2 h_1) + \beta_2 h_2 (\alpha_2^2 + \beta_2^2 h_3^2) \xi_3}{\alpha_2^2 + \beta_2^2 h_3^2} = 0
\]

that is
\[
\sigma_n(h) \nabla u_n \cdot e_1 \rightharpoonup (\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) \nabla u \cdot e_1 \text{ weakly-* in } \mathcal{M}(\Omega). \tag{4.22}
\]

The same calculus leads us to
\[
\sigma_n(h) \nabla u_n \cdot e_2 \rightharpoonup (\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) \nabla u \cdot e_2 \text{ weakly-* in } \mathcal{M}(\Omega). \tag{4.23}
\]

We have, for the last direction $e_3$,
\[
\sigma_n(h) \nabla u_n \cdot e_3 \rightharpoonup \left( \alpha_1 \frac{\partial u}{\partial x_3} - \beta_1 h_2 \frac{\partial u}{\partial x_2} + \beta_1 h_1 \frac{\partial u}{\partial x_2} \right) + \alpha_2 \xi_3 - \beta_2 h_2 \xi_1 - \beta_2 h_1 \xi_2 \text{ weakly-* in } \mathcal{M}(\Omega).
\]

Hence, again with the linear system (4.19),
\[
\left( \alpha_1 \frac{\partial u}{\partial x_3} - \beta_1 h_2 \frac{\partial u}{\partial x_1} + \beta_1 h_1 \frac{\partial u}{\partial x_2} \right) + \alpha_2 \xi_3 - \beta_2 h_2 \xi_1 - \beta_2 h_1 \xi_2
\]
\[
= (\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) \nabla u \cdot e_3 + \alpha_2 \xi_3 - \beta_2 h_2 \frac{\partial^2 h_1}{\partial x_2^2} h_3 + \alpha_2 \frac{\partial^2 h_2}{\partial x_2^2} h_3 + \beta_2 h_1 \frac{\partial^2 h_2}{\partial x_2^2} h_3 + \alpha_2 \beta_2 h_1 \xi_3.
\]

Finally, by the previous equality, (4.22) and (4.23), we get that
\[
\sigma_\ast(h) = \alpha_1 I_3 + \left( \frac{\alpha_2^2 + \alpha_2 \beta_2^2}{\alpha_2^2 + \beta_2^2 h_3^2} \right) e_3 \otimes e_3 + \beta_1 \mathcal{E}(h).
\]

Acknowledgments: The authors wish to thank the referees for suggestions which led to improvement.

References


