Multivariate MA(∞) processes with heavy tails and random coefficients
Shuyan Liu, Johan Segers

To cite this version:
Shuyan Liu, Johan Segers. Multivariate MA(∞) processes with heavy tails and random coefficients. 2010. hal-00624123
Multivariate MA(∞) processes with heavy tails and random coefficients

SHUYAN LIU\textsuperscript{1} and JOHAN SEGERS\textsuperscript{2}

Abstract

Many interesting processes share the property of multivariate regular variation. This property is equivalent to existence of the tail process introduced by B. Basrak and J. Segers \cite{BasrakSegers1} to describe the asymptotic behavior for the extreme values of a regularly varying time series. We apply this theory to multivariate MA(∞) processes with random coefficients.

Key words: extremes, heavy tails, regular variation, tail process.

1 Introduction

We interested in multivariate infinite-order moving average process with random coefficient matrices, defined for $t \in \mathbb{Z}$ by

\begin{equation}
X_t = \sum_{i=0}^{\infty} C_i(t) \xi_{t-i},
\end{equation}

where $(\xi_t)_{t \in \mathbb{Z}}$ is a sequence of i.i.d. regularly varying random vector in $\mathbb{R}^q$ and $\{C_i(t), i \geq 0, t \in \mathbb{Z}\}$ is an array of random $d \times q$ matrices. The tail behavior of such process is usually controlled by a moment condition on the matrix $C_i(t)$, this has been shown by Hult and Samorodnitsky \cite{HultSamorodnitsky}. We adapt the conditions as follows.

\textsuperscript{1} Equipe SAMM, Université Paris I, 90 rue de Tolbiac, 75013 Paris, France.
\textsuperscript{2} Institut de statistique, Université Catholique de Louvain, Voie du Roman Pays, 20, B-1348 Louvain-la-Neuve, Belgium.
\textsuperscript{3} Research supported by IAP research network grant nr. P6/03 of the Belgian government (Belgian Science Policy).
1 INTRODUCTION

Condition 1.1 Suppose that $\mathbb{P}(\bigcap_{i \geq 0} \{\|C_i(0)\| = 0\}) = 0$, and there is some $\varepsilon \in (0, \alpha)$ such that
\begin{equation}
\sum_i E\|C_i(0)\|^\alpha - \varepsilon < \infty \text{ and } \sum_i E\|C_i(0)\|^\alpha + \varepsilon < \infty, \text{ if } \alpha \in (0, 1) \cup (1, 2);
\end{equation}
\begin{equation}
E \left( \sum_i \|C_i(0)\|^{\alpha - \varepsilon} \right) < \infty, \text{ if } \alpha \in \{1, 2\};
\end{equation}
\begin{equation}
E \left( \sum_i \|C_i(0)\|^{2} \right) < \infty, \text{ if } \alpha \in (2, \infty);
\end{equation}

To apply the results in Hult and Samorodnitsky [4], a 'predictability' assumption is required.

Condition 1.2 Suppose there is a filtration $\{\mathcal{F}_j, j \in \mathbb{Z}\}$ such that for all $t \in \mathbb{Z}$ and $i \in \mathbb{N} \cup \{0\}$
\begin{equation}
C_i(t) \in \mathcal{F}_{i-t}, \ \xi_{t-i} \in \mathcal{F}_{i-t+1}, \ (\text{or } \xi_j \in \mathcal{F}_{1-j}),
\end{equation}
\begin{equation}
\mathcal{F}_j \text{ is independent of } \sigma(\xi_{-j}, \xi_{-j-1}, \ldots).
\end{equation}

If the sequence $(C_i(t))$ is independent of the sequence $(\xi_i)$, we can set $\mathcal{F}_j = \sigma((C_k(t))_{k \geq 0, t \in \mathbb{Z}}, (\xi_k)_{k \geq 1-j})$.

Throughout the paper we will use the following notations, $\mathbb{E}^q = [-\infty, \infty]^q \setminus \{0\}$, $\mathbb{E}^q_u = \{x \in \mathbb{E}^q \mid \|x\| > u\}$ and $S^q_1 = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$. A function $f$ is called regularly varying at infinity with index $\tau \in \mathbb{R}$, if $\lim_{x \to \infty} f(ux)/f(x) = u^\tau$ for all $u > 0$.

Definition 1.1. A random vector $X$ in $\mathbb{R}^d$ is regularly varying with index $\alpha > 0$ if there exists a non-null Radon measure $\mu$ on $\mathbb{E}^d$ and a regularly varying function $V$ of index $-\alpha$ such that, as $x \to \infty$,
\begin{equation}
\frac{\mathbb{P}\{x^{-1}X \in \cdot\}}{V(x)} \stackrel{v}{\to} \mu(\cdot),
\end{equation}
where $\stackrel{v}{\to}$ denote the vague convergence of measures.

Relation (1.7) is equivalent to
\begin{equation}
\frac{\mathbb{P}\{|X| > ux, x^{-1}X \in \cdot\}}{\mathbb{P}\{|X| > x\}} \stackrel{v}{\to} u^{-\alpha} \mathbb{P}\{\Theta \in \cdot\},
\end{equation}
as $x \to \infty$. The law of $\Theta$ is called spectral measure.

The random sequence $(X_t)$ given by (1.1) is built from two components which satisfy the following basic assumptions.

Throughout the paper we will use the following notations,
The \( q \)-dimensional random vector \( \xi_t \) is regularly varying of index \( \alpha \in (0, \infty) \) and with spectral measure \( \mathcal{L}(\Theta) \) on \( S^{q-1} \), i.e. there exists a non-null Radon measure \( \mu \) on \( \mathbb{E}^q \) such that, as \( x \to \infty \),

\[
\frac{P\{x^{-1}\xi_0 \in \cdot \}}{P\{\|\xi_0\| > x\}} \xrightarrow{v} \mu(\cdot),
\]

where the measure \( \mu \) has the following representation, for all \( u > 0 \),

\[
\mu\left\{ x \mid \|x\| > u, \frac{x}{\|x\|} \in \cdot \right\} = u^{-\alpha} P\{\Theta \in \cdot\}.
\]

The array of random matrices \( \{C_i(t), i \geq 0, t \geq \mathbb{Z}\} \) of dimension \( d \times q \) is independent of the sequence \( (\xi_t) \) and stationary when index over \( t \in \mathbb{Z} \).

### 2 Joint regular variation

The following theorem is adapted version of Theorem 3.1 in [4].

**Theorem 2.1** [4] Suppose that Condition A1 and A2 hold, then the series \( X_t \) given by (1.1) converges a.s. and

\[
\frac{P\{x^{-1}X_0 \in \cdot \}}{P\{\|X_0\| > x\}} \xrightarrow{v} \sum_{i=0}^{\infty} E\left[ \mu \circ C_i(0)^{-1}(\cdot) \right],
\]

as \( x \to \infty \) on \( \mathbb{E}^q \).

Observe that by assumption A1, the function \( u \mapsto P\{\|\xi_0\| > u\} \) is regularly varying with index \( -\alpha < 0 \). Therefore if the limit on the right-hand side in (2.8) is a non-null Radon measure, then vector \( X_0 \) is regularly varying with index \( \alpha \). As we shall see slightly stronger condition is needed to ensure regular variation of vector \( X_0 \). Consider for \( u > 0 \)

\[
E\left[ \mu \circ C_i(0)^{-1}(\mathbb{E}_u^d) \right] = E\{x \mid \|C_i(0)x\| > u\}
\]

\[
= E \int \int 1_{\{\|C_i(0)r\Theta\| > u\}} d(-r^{-\alpha}) P_\Theta(d\theta)
\]

\[
= u^{-\alpha} E\|C_i(0)\Theta\|^\alpha.
\]

Thus if

\[
0 < \sum_{i=0}^{\infty} E\|C_i(0)\Theta\|^\alpha < \infty
\]
the random vector \( X_0 \) is regularly varying.

Let us define a random vector of length \((t - s + 1)d\) for \(s, t \in \mathbb{Z}\) with \(s \leq t\)

\[ \tilde{X}(s, t) = (X_s, \ldots, X_t)' \]

By the definition of the series \( X_t \) (1.1), we have

\[ (2.10) \quad \tilde{X}(s, t) = \sum_{i=0}^{\infty} \tilde{C}_i(s, t)\xi_{t-i}, \]

where \( \{\tilde{C}_i(s, t)\} \) is an array of random matrices of dimension \((t - s + 1)d \times q\) defined by

\[ (2.11) \quad \tilde{C}_i(s, t) = (C_i - t + s(s), C_i - t + s + 1(s + 1), \ldots, C_i - 1(t - 1), C_i(t))', \quad i \geq 0, \]

where \( C_i(\cdot) = 0 \), if \( i < 0 \). By condition (1.5) and the definition of the sequence \( \{\tilde{C}_i(s, t)\} \)

(2.11), we have

\[ \tilde{C}_i(s, t) \in F_{i-t}, \quad \text{for all} \quad s, t \in \mathbb{Z}, i \in \mathbb{N} \cup \{0\}. \]

Hence Condition 1.2 holds for the sequence \( \{\tilde{C}_i(s, t)\} \) and \((\xi_i)\). In order to obtain the convergence of type (2.8) for the series \( \tilde{X}(s, t) \) defined by (2.10), it is sufficient to show that the array \( \{\tilde{C}_i(s, t)\} \) satisfy Condition 1.1. For this, the following lemma is needed.

**Lemma 2.2** Let \( A = (A_1, \ldots, A_n)' \) be a block matrix of dimension \( nd \times q \) with \( n \) blocks, \( A_i \) be a \( d \times q \) matrix with entries in \( \mathbb{R} \), \( i = 1, \ldots, n \). Then

\[ (2.12) \quad \|A\| \leq \sum_{i=1}^{n} \|A_i\|, \]

where \( \|\cdot\| \) denote the matrix norm.

**Proof.** Without loss of generality, we consider the vector norm \( \|(x_1, \ldots, x_d)'\|_d = \max_{1 \leq i \leq d} \|x_i\| \) on \( \mathbb{R}^d \). We may decompose the matrix \( A \) as

\[ (2.13) \quad A = \sum_{i=1}^{n} B_i \]

with \( B_i \) is a \( nd \times q \) matrix defined by

\[ B_i = (0, \ldots, A_i, \ldots, 0)' \]

where \( A_i \) is in the \( i \)th position. We have, for all \( x \in \mathbb{R}^d \)

\[ \|B_ix\|_{nd} = \|(0, \ldots, A_ix, \ldots, 0)'\|_{nd} = \|A_ix\|_d, \quad i = 1, \ldots, n. \]
Therefore

\[(2.14) \quad \|B_i\| = \sup_{\|x\|_q \leq 1} \{\|B_i x\|_q\} = \sup_{\|x\|_q \leq 1} \{\|A_i x\|_d\} = \|A_i\|.
\]

Inequality (2.12) follows from (2.13) and (2.14). □

By Lemma 2.2 and the definition of the sequence \(\{\tilde{C}_i(s, t)\} (2.11)\), we obtain

\[(2.15) \quad \|\tilde{C}_i(s, t)\| \leq \sum_{j=(i-t+s)\vee 0}^{i} \|C_j(t - i + j)\|.
\]

Using (2.15) and the following inequality, for \(a_i > 0, i = 1, \ldots, n\),

\[(2.16) \quad \left(\sum_{i=1}^{n} a_i\right)^p \leq c \sum_{i=1}^{n} a_i^p,
\]

where \(c = 1\) if \(p \leq 1\), \(c = n^{p-1}\) if \(p > 1\), we obtain, if \(\alpha \in (0, 1) \cup (1, 2)\), for some \(0 < \varepsilon < \alpha\),

\[
\sum_{i=0}^{\infty} E\|\tilde{C}_i(s, t)\|^\alpha - \varepsilon \leq \sum_{i=0}^{\infty} E \left( \sum_{j=(i-t+s)\vee 0}^{i} \|C_j(t - i + j)\| \right)^{\alpha - \varepsilon}
\]

\[
\leq c_1 \sum_{i=0}^{\infty} \sum_{j=(i-t+s)\vee 0}^{i} E\|C_j(t - i + j)\|^\alpha - \varepsilon
\]

\[(2.17) \quad = c_1 \sum_{j=0}^{\infty} \sum_{i=0}^{t-s} E\|C_i(t - j)\|^\alpha - \varepsilon,
\]

where \(c_1\) is a constant depending on \(\alpha - \varepsilon\) and \(t - s\). The latter inequality in combination with the condition (1.2) and the stationarity of the sequence \(\{C_i(t)\}\) implies

\[
\sum_{i=0}^{\infty} E\|\tilde{C}_i(s, t)\|^\alpha - \varepsilon < \infty.
\]

By the similar method for (2.17), we obtain, for some \(0 < \varepsilon < \alpha\),

\[
\sum_{i=0}^{\infty} E\|\tilde{C}_i(s, t)\|^\alpha + \varepsilon \leq c_2 \sum_{j=0}^{t-s} \sum_{i=0}^{\infty} E\|C_i(t - j)\|^\alpha + \varepsilon, \text{ if } \alpha \in (0, 1) \cup (1, 2),
\]

\[
E \left( \sum_{i=0}^{\infty} \|\tilde{C}_i(s, t)\|^{\alpha - \varepsilon} \right)^{\frac{\alpha + \varepsilon}{\alpha - \varepsilon}} \leq c_3 \sum_{j=0}^{t-s} E \left( \sum_{i=0}^{\infty} \|C_i(t - j)\|^{\alpha - \varepsilon} \right)^{\frac{\alpha + \varepsilon}{\alpha - \varepsilon}}, \text{ if } \alpha \in \{1, 2\}
\]
and
\[
E \left( \sum_{i=0}^{\infty} \left\| \tilde{C}_i(s,t) \right\|^2 \right)^{\frac{\alpha+\varepsilon}{2}} \leq c_4 \sum_{j=0}^{t-s} E \left( \sum_{i=0}^{\infty} \left\| C_i(t-j) \right\|^2 \right)^{\frac{\alpha+\varepsilon}{2}}, \quad \text{if } \alpha \in (2, \infty),
\]

where \( c_2, c_3 \) and \( c_4 \) are the constants depending on \( \alpha, \varepsilon \) and \( t-s \). As a consequence, under the conditions of Theorem 2.1, it is not only the marginal distribution of \((X_t)\) that is regularly varying, the same holds for all finite dimensional distributions.

**Corollary 2.3** Under the conditions of Theorem 2.1, the stationary process \((X_t)\) given by (1.1) satisfies, for \( s, t \in \mathbb{Z} \) with \( s \leq t \)
\[
\frac{P \left\{ x^{-1}(X_s, \ldots, X_t)' \in \cdot \right\}}{P \{\|\xi_0\| > x\}} \xrightarrow{v} \sum_{i=0}^{\infty} E \left[ \mu \circ \tilde{C}_i(s,t)^{-1}(\cdot) \right] =: \nu_{s,t}(\cdot),
\]
as \( x \to \infty \) on \( \mathbb{E}^{(t-s+1)d} \), where the sequence of random matrices \( \{\tilde{C}_i(s,t)\} \) is defined by (2.11). In particular, if \( X_0 \) is regularly varying, then the process \((X_t)\) is jointly regularly varying, i.e. for all \( s, t \in \mathbb{Z} \) with \( s \leq t \), random vector \((X_s, \ldots, X_t)'\) are regularly varying.

### 3 Tail process

The joint regular variation of the sequence \((X_t)\) in particular means that there exists a process \((Y_t)_{t \in \mathbb{Z}}\) in \( \mathbb{R}^d \) with \( P \{\|\xi_0\| > y\} = y^{-\alpha} \) for \( y \geq 1 \) such that for all \( s, t \in \mathbb{Z} \) with \( s \leq t \) and as \( x \to \infty \)
\[
\mathcal{L}(x^{-1}(X_s, \ldots, X_t) \mid \|X_0\| > x) \sim \mathcal{L}(Y_s, \ldots, Y_t),
\]
see Theorem 2.1 in [1]. The process \((Y_t)\) is called the tail process of \((X_t)\). Moreover the so-called spectral process \( \Theta_t = \frac{Y_t}{\|Y_0\|} \) is independent of \( \|X_0\| \), see Theorem 3.1 in [1].

To understand the tail process in this case, it is helpful for instance to consider a set \( B \) in \( \mathbb{E}^{(t-s+1)d} \) of the form \( B = \mathbb{E}_{b_s}^d \times \cdots \times \mathbb{E}_{b_t}^d \) for arbitrary real constant \( b_i > 0, \ i = s, \ldots, t \). The limit of the probabilities, as \( x \to \infty \),
\[
\frac{P \left\{ x^{-1}(X_s, \ldots, X_t)' \in B \right\}}{P \{\|\xi_0\| > x\}}
\]
is the sum
\[ \sum_{i=0}^{\infty} \mathbb{E} \left[ \mu \circ \tilde{C}_i(s, t)^{-1}(B) \right] \]
\[ = \sum_{i=t-s}^{\infty} \mathbb{E} \left[ \mu \left\{ x \mid \|C_i(t+s)x\| > b_s, \ldots, \|C_i(t)x\| > b_t \right\} \right] \]
\[ = \sum_{i=t-s}^{\infty} \mathbb{E} \left[ \int \mathbb{P}_\Theta (d\theta) \prod_{j=1}^q \mathbb{I}(\|C_{i-t+s}(s)\theta\| > b_s) \cdots \mathbb{I}(\|C_i(t)\theta\| > b_t) \, d(-r^\alpha) \right] \]
\[ = \sum_{i=t-s}^{\infty} \mathbb{E} \left[ \min \left\{ \frac{\|C_{i-t+s}(s)\Theta\|^\alpha}{b_s^\alpha}, \ldots, \frac{\|C_i(t)\Theta\|^\alpha}{b_t^\alpha} \right\} \right]. \]

In particular, if \( s = t \) and \( b_s = 1 \), one can apply this to obtain the following limit, as \( x \to \infty \),

\[ \frac{\mathbb{P}\{\|X_0\| > x\}}{\mathbb{P}\{\|\xi_0\| > x\}} \to \sum_{i=0}^{\infty} \mathbb{E}\|C_i(0)\Theta\|^\alpha. \]

The following result is the multivariate version of Breiman’s lemma \[3\] which appears as Proposition A. 1 in \[2\].

**Lemma 3.1** Let \( Z \) be a \( q \)-dimensional random vector and let \( A \) be a \( d \times q \) random matrix, independent of \( Z \). Assume that \( Z \) is multivariate regularly varying of index \( \alpha \in (0, \infty) \), i.e. there exists a non-null Radon measure \( \mu \) on \( E^q \) such that, as \( x \to \infty \),

\[ \mathbb{P}\{x^{-1}Z \in \cdot \} \mathbb{P}\{|Z| > x\} \to \mathbb{E}\|C_0(0)\Theta\|^\alpha. \]

If \( \mathbb{E}\|A\|^\beta < \infty \) for some \( \beta > \alpha \), then in \( E^d \), as \( x \to \infty \),

\[ \frac{\mathbb{P}\{x^{-1}AZ \in \cdot \}}{\mathbb{P}\{|Z| > x\}} \to \mathbb{E}\left[ \mu \circ A^{-1} \right]. \]

**Theorem 3.2** Let \((X_t)_{t \in \mathbb{Z}}\) be a stationary process given by (1.1). Suppose that Condition [I.1] and [I.2] hold. If \( \mathbb{P}\{\cap_{t \geq 0} \{\|C_i(0)\Theta\| = 0\} \} = 0 \), then for \( s, t \in \mathbb{Z} \) with \( s \leq 0 \leq t \)
and bounded and continuous function \( f : (\mathbb{R}^d)^{t-s+1} \to \mathbb{R} \),

\[
\begin{align*}
E \left[ f \left( \frac{X_s}{\|X_0\|}, \ldots, \frac{X_t}{\|X_0\|} \right) \bigg| \|X_0\| > x \right] \\
\to \frac{1}{\sum_{i=0}^{\infty} E\|C_i(0)\Theta\|^\alpha} \sum_{i=0}^{\infty} E \left[ f \left( \frac{C_{i+s}(s)\Theta}{\|C_i(0)\Theta\|}, \ldots, \frac{C_{i+t}(t)\Theta}{\|C_i(0)\Theta\|} \right) \|C_i(0)\Theta\|^\alpha \right],
\end{align*}
\]

as \( x \to \infty \), where \( C_i(\cdot) = 0 \) if \( i < 0 \).

**Proof.** Let \( h : (\mathbb{R}^d)^{t-s+1} \to \mathbb{R} \) be bounded and continuous. In view of convergence (3.19) and Corollary 2.3, as \( x \to \infty \),

\[
\begin{align*}
E & \left[ f \left( \frac{X_s}{\|X_0\|}, \ldots, \frac{X_t}{\|X_0\|} \right) \bigg| \|X_0\| > x \right] \\
\to & \frac{1}{\sum_{i=0}^{\infty} E\|C_i(0)\Theta\|^\alpha} \int h(y_s, \ldots, y_t) I\{\|\xi_0\| > 1\} \nu_{s,t}(dy).
\end{align*}
\]

Note

\[
\tilde{X}(s, t) = (X_s, \ldots, X_t)' = \sum_{i=0}^{\infty} \tilde{C}_i(s, t)\xi_{t-i},
\]

where the sequence \( \{\tilde{C}_i(s, t)\} \) is defined by (2.11). By (2.15), (2.16), Condition 1.1 and the stationarity of sequence \( \{C_i(t)\} \), we have for some \( \beta > \alpha \),

\[
E\|\tilde{C}_i(s, t)\|^\beta \leq c \sum_{j=(i-t+s)\vee 0}^{i} E\|C_j(t-i+j)\|^\beta = c \sum_{j=(i-t+s)\vee 0}^{i} E\|C_j(0)\|^\beta < \infty,
\]

where \( c \) is a constant depending on \( \beta \) and \( t-s \). Using Lemma 3.1, we obtain, for each \( i \geq 0 \)

\[
\frac{\mathbb{P}\left\{ x^{-1}\tilde{C}_i(s, t)\xi_{t-i} \in \cdot \right\}}{\mathbb{P}\{\|\xi_0\| > x\}} \to E \left[ \mu \circ \tilde{C}_i(s, t)^{-1}(\cdot) \right] =: \nu_{s,t}^{(i)}(\cdot),
\]

as \( x \to \infty \), in \( \mathbb{P}^{(t-s+1)d} \). By the definition of the measure \( \nu_{s,t} \) in (2.18),

\[
\nu_{s,t}(\cdot) = \sum_{i=0}^{\infty} \nu_{s,t}^{(i)}(\cdot).
\]

We denote

\[
H(x) = E \left[ h \left( x^{-1}\tilde{C}_i(s, t)\xi_{t-i} \right) I\{\|C_{t-i}(0)\xi_{t-i}\| > x\} \right] / \mathbb{P}\{\|\xi_0\| > x\}.
\]
On the one hand, it follows from (3.22), as $x \to \infty$,
\[
H(x) = \int h(y_s, \ldots, y_t) I_{\{\|y_0\|>1\}} P_{x^{-1} \tilde{C}_i(s,t) \xi_{t-i}}(dy) / P_{\{\|\xi_0\|> x\}}
\]
(3.24) \quad \to \quad \int h(y_s, \ldots, y_t) I_{\{\|y_0\|>1\}} \nu_s^{(i)}(dy).

On the other hand, by independence between $\tilde{C}_1(s,t)$ and $\xi_{t-i}$ and assumption A1, as $x \to \infty$,
\[
H(x) = \int E \left[ h \left( \tilde{C}_1(s,t) y \right) I_{\{\|C_{i-t}(0)y\|>1\}} P_{x^{-1} \xi_{t-i}}(dy) / P_{\{\|\xi_0\|> x\}} \right]
\]
(3.25) \quad \to \quad \int_0^\infty E \left[ h \left( \tilde{C}_1(s,t) \Theta r \right) I_{\{\|C_{i-t}(0)\Theta r\|>1\}} \right] d(-r^{-\alpha}).

Since for $i < t$, $\nu_{s,t}^{(i)} \{ (y_s, \ldots, y_t) \mid y_s = \ldots = y_0 = 0 \} = 1$,
\[
(3.26) \quad \int h(y_s, \ldots, y_t) I_{\{\|y_0\|>1\}} \nu_s^{(i)}(dy) = 0, \quad i = 0, \ldots, t-1.
\]

In combination with (3.23), (3.24), (3.25) and (3.26), it follows that
\[
\int h(y_s, \ldots, y_t) I_{\{\|y_0\|>1\}} \nu_s^{(i)}(dy)
\]
\[
= \sum_{i=t}^\infty \int h(y_s, \ldots, y_t) I_{\{\|y_0\|>1\}} \nu_s^{(i)}(dy)
\]
(3.27) \quad = \sum_{i=t}^\infty \int_0^\infty E \left[ h \left( \tilde{C}_i(s,t) \Theta r \right) I_{\{\|C_{i-t}(0)\Theta r\|>1\}} \right] d(-r^{-\alpha}).

Considering (3.21) and (3.27), we have, as $x \to \infty$,
\[
E \left[ f \left( \frac{x_s}{x}, \ldots, \frac{x_t}{x} \right) \mid \|X_0\|> x \right] \to I
\]
where
\[
I = \frac{1}{\sum_{i=0}^\infty E \|C_i(0)\Theta\|^\alpha} \sum_{i=0}^\infty \int_0^\infty E \left[ h \left( C_{t-i+s}(s) \Theta r, \ldots, C_t(t) \Theta r \right) I_{\{\|C_{i-t}(0)\Theta r\|>1\}} \right] d(-r^{-\alpha}).
\]

Applying this relation to the function $h(x_s, \ldots, x_t) = f \left( \frac{x_s}{\|x_0\|}, \ldots, \frac{x_t}{\|x_0\|} \right)$ to see that, as $x \to \infty$, the left-hand side of (3.20) converges to
\[
\frac{1}{\sum_{i=0}^\infty E \|C_i(0)\Theta\|^\alpha} \sum_{i=0}^\infty \int_0^\infty E \left[ f \left( \frac{C_{t-i+s}(s) \Theta r}{\|C_{i-t}(0)\Theta\|}, \ldots, \frac{C_t(t) \Theta r}{\|C_{i-t}(0)\Theta\|} \right) I_{\{\|C_{i-t}(0)\Theta r\|>1\}} \right] d(-r^{-\alpha}).
\]
By Fubini’s theorem, this is equal to the right-hand side of (3.20). \qed
4 Applied models

4.1 Heavy tailed multivariate ARMA(1,1) process

Assume that \((A_t, B_t, \xi_t)\) is an i.i.d. sequence of random vectors in \(\mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \times \mathbb{R}^d\), for some \(d \geq 1\). Throughout we assume that \(\xi_t\)'s are regularly varying random vectors.

Suppose that a stationary sequence \((X_t)_{t \in \mathbb{Z}}\) with value in \(\mathbb{R}^d\) satisfies a multivariate random coefficient ARMA(1,1) equation of the following form

\[
X_t = A_t X_{t-1} + B_t \xi_{t-1} + \xi_t.
\]

Iterating this equation backwards we arrive at the following MA(\(\infty\)) representation of this process

\[
X_t = \xi_t + \sum_{i \geq 1} (A_{t-i+1} + B_{t-i+1}) \prod_{j=0}^{i-2} A_{t-j} \xi_{t-i},
\]

namely, the stationary solution can be represented as (1.1) with

\[
C_i(t) = \begin{cases} 
Id, & i = 0, \\
(A_{t-i+1} + B_{t-i+1}) \prod_{j=0}^{i-2} A_{t-j}, & i \geq 1.
\end{cases}
\]

From Theorem 3.1 in Hult and Samorodnitsky [4] we obtain the following result.

**Corollary 4.1** Suppose that \(\xi \in RV(\alpha, \mu)\) and there is some \(0 < \varepsilon < \alpha\) such that

\[
E\|A\|^{\alpha + \varepsilon} < 1, \ E\|B\|^{\alpha + \varepsilon} < \infty.
\]

Then the series (4.29) converges a.s. and

\[
\frac{P\{u^{-1} X_0 \in \cdot \}}{P\{|\xi_0| > u\}} \to E \left[ \sum \mu \circ C_j(0)^{-1} (\cdot) \right]
\]

as \(u \to \infty\) on \(\mathbb{R}^d \setminus \{0\}\).

**Proof.** Since \(C_0(0) = Id\), we have \(P(\bigcap_{j \geq 0}\{|C_j(0)| = 0\}) = 0\).

For \(\alpha \in (0, 1) \cup (1, 2)\), we have

\[
\sum E\|C_i(0)\|^{\alpha - \varepsilon} = 1 + \sum_{i \geq 1} E \left\| \left( A_{i-1} + B_{i-1} \right) \prod_{j=0}^{i-2} A_{t-j} \right\|^{\alpha - \varepsilon} \leq 1 + \sum_{i \geq 1} E\|A + B\|^{\alpha - \varepsilon} \prod_{j=0}^{i-2} E\|A_{t-j}\|^{\alpha - \varepsilon} = 1 + E\|A + B\|^{\alpha - \varepsilon} \sum_{i \geq 1} (E\|A\|^{\alpha - \varepsilon})^{i-1} = 1 + E\|A + B\|^{\alpha - \varepsilon} \sum_{i \geq 0} (E\|A\|^{\alpha - \varepsilon})^{\frac{\alpha - \varepsilon}{\alpha - \varepsilon}} \frac{\alpha - \varepsilon}{\alpha - \varepsilon}.
\]

(4.33)
By Jensen’s inequality, the last term is bounded by

\[ 1 + E\|A + B\|^{\alpha - \varepsilon} \sum_{i \geq 0} (E\|A\|^{\alpha + \varepsilon})^i \frac{\alpha - \varepsilon}{\alpha + \varepsilon}. \tag{4.34} \]

In the case of \( \alpha \in (0, 1) \),

\[ E\|A + B\|^{\alpha - \varepsilon} < E\|A\|^{\alpha - \varepsilon} + E\|B\|^{\alpha - \varepsilon}. \tag{4.35} \]

In the case of \( \alpha \in (1, 2) \)

\[ E\|A + B\|^{\alpha - \varepsilon} \leq (E\|A + B\|^{\alpha + \varepsilon})^{1 - \frac{\alpha - \varepsilon}{\alpha + \varepsilon}} \leq \left[ (E\|A\|^{\alpha + \varepsilon}) \frac{1}{\alpha + \varepsilon} + (E\|B\|^{\alpha + \varepsilon}) \frac{1}{\alpha + \varepsilon} \right]^{\alpha - \varepsilon}. \tag{4.36} \]

Combining (4.33), (4.34), (4.35) and (4.36) proves

\[ \sum_i E\|C_i(0)\|^{\alpha - \varepsilon} < \infty. \]

Similarly as in (4.33), we have

\[ \sum_i E\|C_i(0)\|^{\alpha + \varepsilon} \leq 1 + E\|A + B\|^{\alpha + \varepsilon} \sum_{i \geq 0} (E\|A\|^{\alpha + \varepsilon})^i. \tag{4.37} \]

If \( \alpha + \varepsilon < 1 \), then

\[ E\|A + B\|^{\alpha + \varepsilon} < E\|A\|^{\alpha + \varepsilon} + E\|B\|^{\alpha + \varepsilon}, \tag{4.38} \]

else

\[ E\|A + B\|^{\alpha + \varepsilon} \leq \left[ (E\|A\|^{\alpha + \varepsilon}) \frac{1}{\alpha + \varepsilon} + (E\|B\|^{\alpha + \varepsilon}) \frac{1}{\alpha + \varepsilon} \right]^{\alpha + \varepsilon}. \tag{4.39} \]

Combining (4.37), (4.38) and (4.39) proves

\[ \sum_i E\|C_i(0)\|^{\alpha + \varepsilon} < \infty. \]

For \( \alpha \in \{1, 2\} \), using Lemma 3.2.1 in Kwapień and Woyczynski [5] it follows that

\[
\begin{align*}
E \left( \sum_i \|C_i(0)\|^{\alpha - \varepsilon} \right)^{\frac{\alpha + \varepsilon}{\alpha - \varepsilon}} \\
\leq \left[ \sum_i (E\|C_i(0)\|^{\alpha + \varepsilon})^{\frac{\alpha - \varepsilon}{\alpha + \varepsilon}} \right]^{\frac{\alpha + \varepsilon}{\alpha - \varepsilon}} \\
= \left[ 1 + \sum_{i \geq 1} \left\{ E \left( \|A_{-i+1} + B_{-i+1}\| \prod_{j=0}^{i-2} A_{-j} \|^{\alpha + \varepsilon} \right)^{\frac{\alpha - \varepsilon}{\alpha + \varepsilon}} \right\} \right]^{\frac{\alpha + \varepsilon}{\alpha - \varepsilon}} \\
\leq \left[ 1 + \sum_{i \geq 1} \left\{ E\|A + B\|^{\alpha + \varepsilon} \prod_{j=0}^{i-2} \|A_{-j}\|^{\alpha + \varepsilon} \right\} \right]^{\frac{\alpha + \varepsilon}{\alpha - \varepsilon}} \\
= \left[ 1 + (E\|A + B\|^{\alpha + \varepsilon})^{\frac{\alpha - \varepsilon}{\alpha + \varepsilon}} \sum_{i \geq 0} (E\|A\|^{\alpha + \varepsilon})^{\frac{\alpha - \varepsilon}{\alpha + \varepsilon}} \right]^{\frac{\alpha + \varepsilon}{\alpha - \varepsilon}}. \tag{4.40}
\end{align*}
\]
Considering \(4.39\) and \(4.40\) we get

\[
E \left( \sum \| C_i(0) \|^{\alpha - \varepsilon} \right)^{\frac{\alpha + \varepsilon}{\alpha - \varepsilon}} < \infty.
\]

For \(\alpha \in (2, \infty)\), using Lemma 3.2.1 in Kwapien and Woyczynski [5] it follows that

\[
E \left( \sum \| C_i(0) \|^2 \right)^{\frac{\alpha + \varepsilon}{2}} \leq \left[ \sum (E \| C_i(0) \|^{\alpha + \varepsilon}) \frac{2}{\alpha + \varepsilon} \right]^{\frac{\alpha + \varepsilon}{2}} \leq \left[ 1 + (E \| A + B \|^{\alpha + \varepsilon}) \frac{2}{\alpha + \varepsilon} \sum_{i \geq 0} (E \| A \|^{\alpha + \varepsilon})^i \frac{2}{\alpha + \varepsilon} \right]^{\frac{\alpha + \varepsilon}{2}}.
\]

\[4.41\]

Considering \(4.39\) and \(4.41\) we get

\[
E \left( \sum \| C_i(0) \|^2 \right)^{\frac{\alpha + \varepsilon}{2}} < \infty.
\]

\[\square\]

Here is another proof of Corollary 4.1.

**Proof.** We denote

\[4.42\]

\[
\tilde{X}_t = (X_t, \xi_t, \xi_{t+1})',
\]

\[
\tilde{A}_t = \begin{pmatrix}
A_t & B_t & Id \\
0 & 0 & Id \\
0 & 0 & 0
\end{pmatrix}
\]

and

\[
Z_t = (0, 0, \xi_{t+1})'.
\]

Then the relation \(4.28\) can be wrote as

\[4.43\]

\[
\tilde{X}_t = \tilde{A}_t \tilde{X}_{t-1} + Z_t.
\]

Iterating this equation we get

\[
\tilde{X}_t = \sum_{i=0}^{\infty} C_i(t) Z_{t-i}
\]

where \(C_0(t) = Id, C_1(t) = \tilde{A}_t\) and

\[
C_i(t) = \begin{pmatrix}
\prod_{j=0}^{i-1} A_{t-j} & B_{t-i+1} \prod_{j=0}^{i-2} A_{t-j} (A_{t-i+2} + B_{t-i+2}) \prod_{j=0}^{i-3} A_{t-j} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad i \geq 2.
\]
It is equivalent to prove the convergence \([4.32]\) of vector \(\tilde{X}_t\) defined by \([4.42]\).

By Lemma 2.2 and the properties of matrix norm, for \(i \geq 1\)
\[
\|C_i(t)\| \leq \prod_{j=0}^{i-1} \|A_{t-j}\| + \|B_{t-i+1}\| \prod_{j=0}^{i-2} \|A_{t-j}\| + \|A_{t-i+2} + B_{t-i+2}\| \prod_{j=0}^{i-3} \|A_{t-j}\|
\]
where \(A_i = B_i = Id\) if \(i > t\). It follows from \([2.16]\), if \(\alpha \in (0, 1) \cup (2, 1)\)
\[
\sum_{i \geq 0} E\|C_i(0)\|^{\alpha-\epsilon}
\]
\[
\leq 1 + c \sum_{i \geq 1} \left[ E \prod_{j=0}^{i-1} \|A_{t-j}\|^{\alpha-\epsilon} + E \left( \|B_{t-i+1}\| \prod_{j=0}^{i-2} \|A_{t-j}\| \right)^{\alpha-\epsilon} + \right]
\]
\[
+ E \left( \|A_{t-i+2} + B_{t-i+2}\| \prod_{j=0}^{i-3} \|A_{t-j}\| \right)^{\alpha-\epsilon}
\]
\[
= 1 + c \left( \sum_{i \geq 1} (E\|A\|^{\alpha-\epsilon})^i + E\|B\|^{\alpha-\epsilon} \sum_{i \geq 0} (E\|A\|^{\alpha-\epsilon})^i 
\right.
\]
\[
+ E\|A + B\|^{\alpha-\epsilon} \sum_{i \geq 0} (E\|A\|^{\alpha-\epsilon})^i + 2^{\alpha-\epsilon})
\]
where \(c\) is a constant depending on \(\alpha - \epsilon\). By Jensen’s inequality,
\[
\sum_{i \geq 0} E\|C_i(0)\|^{\alpha-\epsilon} < \infty.
\]
Similarly we can prove
\[
\sum_{i \geq 0} E\|C_i(0)\|^{\alpha+\epsilon} < \infty.
\]
For \(\alpha \in \{1, 2\}\), using Lemma 3.2.1 in Kwapien and Woyczynski \([5]\) it follows that
\[
\sum_{i \geq 0} E\|C_i(0)\|^{\alpha-\epsilon} \leq \left( \sum\E\|C_i(0)\|^{\alpha+\epsilon} \right)^{\frac{\alpha-\epsilon}{\alpha+\epsilon}} \leq \left( \sum\E\|C_i(0)\|^{\alpha+\epsilon} \right)^{\frac{\alpha+\epsilon}{\alpha-\epsilon}} \cdot \frac{\alpha+\epsilon}{\alpha-\epsilon}.
\]
By \([2.16]\) and \([4.44]\), \(\sum (E\|C_i(0)\|^{\alpha+\epsilon})^{\frac{\alpha+\epsilon}{\alpha-\epsilon}}\) is bounded by the right-hand side of \([4.45]\) which is bounded. Hence
\[
\E \left( \sum_{i \geq 0} \|C_i(0)\|^{\alpha-\epsilon} \right)^{\frac{\alpha+\epsilon}{\alpha-\epsilon}} < \infty.
\]
By similar method of (4.48), we get

$$E \left( \sum ||C_i(0)||^2 \right)^{\alpha + \varepsilon} < \infty.$$ 

□

References


