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To cite this version:
Shuyan Liu, Johan Segers. Multivariate MA(∞) processes with heavy tails and random coefficients. 2010. <hal-00624123>

HAL Id: hal-00624123
https://hal.archives-ouvertes.fr/hal-00624123
Submitted on 15 Sep 2011

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Multivariate MA(∞) processes with heavy tails and random coefficients

SHUYAN LIU and JOHAN SEGERS

Abstract

Many interesting processes share the property of multivariate regular variation. This property is equivalent to existence of the tail process introduced by B. Basrak and J. Segers [1] to describe the asymptotic behavior for the extreme values of a regularly varying time series. We apply this theory to multivariate MA(∞) processes with random coefficients.

Key words : extremes, heavy tails, regular variation, tail process.

1 Introduction

We interested in multivariate infinite-order moving average process with random coefficient matrices, defined for \( t \in \mathbb{Z} \) by

\[
X_t = \sum_{i=0}^{\infty} C_i(t)\xi_{t-i},
\]

where \( (\xi_t)_{t \in \mathbb{Z}} \) is a sequence of i.i.d. regularly varying random vector in \( \mathbb{R}^q \) and \( \{C_i(t), i \geq 0, t \in \mathbb{Z}\} \) is an array of random \( d \times q \) matrices. The tail behavior of such process is usually controlled by a moment condition on the matrix \( C_i(t) \), this has been shown by Hult and Samorodnitsky [4]. We adapt the conditions as follows.

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3 Research supported by IAP research network grant nr. P6/03 of the Belgian government (Belgian Science Policy).
Condition 1.1 Suppose that $P(\bigcap_{i \geq 0}\{\|C_i(0)\| = 0\}) = 0$, and there is some $\varepsilon \in (0, \alpha)$ such that

$$\sum_i E\|C_i(0)\|^{\alpha - \varepsilon} < \infty \text{ and } \sum_i E\|C_i(0)\|^{\alpha + \varepsilon} < \infty, \text{ if } \alpha \in (0,1) \cup (1,2);$$

$$\mathbb{E}\left(\sum_i \|C_i(0)\|^{\alpha - \varepsilon}\right)^{\frac{\alpha + \varepsilon}{\alpha - \varepsilon}} < \infty, \text{ if } \alpha \in \{1,2\};$$

$$\mathbb{E}\left(\sum_i \|C_i(0)\|^{2}\right)^{\frac{\alpha + \varepsilon}{2}} < \infty, \text{ if } \alpha \in (2,\infty);$$

To apply the results in Hult and Samorodnitsky [4], a 'predictability' assumption is required.

Condition 1.2 Suppose there is a filtration $(\mathcal{F}_j, j \in \mathbb{Z})$ such that for all $t \in \mathbb{Z}$ and $i \in \mathbb{N} \cup \{0\}$

$$C_i(t) \in \mathcal{F}_{i-t}, \ \xi_{t-i} \in \mathcal{F}_{i-t+1}, \text{ (or } \xi_j \in \mathcal{F}_{i-j}),$$

$$\mathcal{F}_j \text{ is independent of } \sigma(\xi_{j-1}, \xi_{j-2}, \ldots).$$

If the sequence $(C_i(t))$ is independent of the sequence $(\xi_i)$, we can set $
\mathcal{F}_j = \sigma((C_k(t))_{k \geq 0, t \in \mathbb{Z}}, (\xi_k)_{k \geq 1-j}).$

Throughout the paper we will use the following notations, $\mathbb{E}^q = [-\infty, \infty] \setminus \{0\}$, $\mathbb{E}^q_u = \{x \in \mathbb{E}^q|\|x\| > u\}$ and $S^q = \{x \in \mathbb{R}^d|\|x\| = 1\}$. A function $f$ is called regularly varying at infinity with index $\tau \in \mathbb{R}$, if $\lim_{x \to \infty} f(ux)/f(x) = u^\tau$ for all $u > 0$.

Definition 1.1. A random vector $X$ in $\mathbb{R}^d$ is regularly varying with index $\alpha > 0$ if there exists a non-null Radon measure $\mu$ on $\mathbb{E}^d$ and a regularly varying function $V$ of index $-\alpha$ such that, as $x \to \infty$,

$$(1.7) \quad \frac{P\{x^{-1}X \in \cdot\}}{V(x)} \xrightarrow{v} \mu(\cdot),$$

where $\xrightarrow{v}$ denote the vague convergence of measures.

Relation (1.7) is equivalent to

$$\frac{P\{\|X\| > ux, x^{-1}X \in \cdot\}}{P\{\|X\| > x\}} \xrightarrow{v} u^{-\alpha}P\{\Theta \in \cdot\},$$

as $x \to \infty$. The law of $\Theta$ is called spectral measure.

The random sequence $(X_i)$ given by (1.1) is built from two components which satisfy the following basic assumptions.

1 INTRODUCTION
The $q$-dimensional random vector $\xi_t$ is regularly varying of index $\alpha \in (0, \infty)$ and with spectral measure $\mathcal{L}(\Theta)$ on $S^{q-1}$, i.e. there exists a non-null Radon measure $\mu$ on $\mathbb{E}^q$ such that, as $x \to \infty$,

$$\frac{\mathbb{P}\left\{x^{-1}\xi_0 \in \cdot\right\}}{\mathbb{P}\left\{\|\xi_0\| > x\right\}} \overset{v}{\to} \mu(\cdot),$$

where the measure $\mu$ has the following representation, for all $u > 0$, 

$$\mu\left\{x \mid \|x\| > u, \frac{x}{\|x\|} \in \cdot\right\} = u^{-\alpha} \mathbb{P}\{\Theta \in \cdot\}.$$

The array of random matrices $\{C_i(t), i \geq 0, t \geq \mathbb{Z}\}$ of dimension $d \times q$ is independent of the sequence $(\xi_t)$ and stationary when index over $t \in \mathbb{Z}$.

2 Joint regular variation

The following theorem is adapted version of Theorem 3.1 in [4].

**Theorem 2.1** [4] Suppose that Condition 1.1 and 1.2 hold, then the series $X_t$ given by (1.1) converges a.s. and

$$0 < \sum_{i=0}^{\infty} \mathbb{E}\left[\|C_i(0)\Theta\|^\alpha\right] < \infty$$

as $x \to \infty$ on $\mathbb{E}^q$.

Observe that by assumption A1, the function $u \mapsto \mathbb{P}\{\|\xi_0\| > u\}$ is regularly varying with index $-\alpha < 0$. Therefore if the limit on the right-hand side in (2.8) is a non-null Radon measure, then vector $X_0$ is regularly varying with index $\alpha$. As we shall see slightly stronger condition is needed to ensure regular variation of vector $X_0$. Consider for $u > 0$

$$\mathbb{E}\left[\mu \circ C_i(0)^{-1}(\mathbb{E}^d_u)\right] = \mathbb{E}\left[\mu \{x \mid \|C_i(0)x\| > u\}\right]$$

$$= \mathbb{E}\int \int \mathbb{1}_{\{\|C_i(0)r\Theta\| > u\}} d(-r^{-\alpha}) \mathbb{P}_{\Theta}(d\Theta)$$

$$= u^{-\alpha} \mathbb{E}\|C_i(0)\Theta\|^\alpha.$$

Thus if

$$0 < \sum_{i=0}^{\infty} \mathbb{E}\|C_i(0)\Theta\|^\alpha < \infty$$

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the random vector $X_0$ is regularly varying.

Let us define a random vector of length $(t - s + 1)d$ for $s, t \in \mathbb{Z}$ with $s \leq t$

$$
\tilde{X}(s, t) = (X_s, \ldots, X_t)'.
$$

By the definition of the series $X_t$ (1.1), we have

(2.10) \hspace{1cm} \tilde{X}(s, t) = \sum_{i=0}^{\infty} \tilde{C}_i(s, t)\xi_{t-i},

where $\{\tilde{C}_i(s, t)\}$ is an array of random matrices of dimension $(t - s + 1)d \times q$ defined by

(2.11) \hspace{1cm} \tilde{C}_i(s, t) = (C_{i-t+s}(s), C_{i-t+s+1}(s + 1), \ldots, C_{i-1}(t-1), C_i(t))', \ i \geq 0,

where $C_i(\cdot) = 0$, if $i < 0$. By condition (1.5) and the definition of the sequence $\{\tilde{C}_i(s, t)\}$ (2.11), we have

$$
\tilde{C}_i(s, t) \in \mathcal{F}_{i-t}, \text{ for all } s, t \in \mathbb{Z}, i \in \mathbb{N} \cup \{0\}.
$$

Hence Condition 1.2 holds for the sequence $\{\tilde{C}_i(s, t)\}$ and $(\xi_i)$. In order to obtain the convergence of type (2.8) for the series $\tilde{X}(s, t)$ defined by (2.10), it is sufficient to show that the array $\{\tilde{C}_i(s, t)\}$ satisfy Condition 1.1. For this, the following lemma is needed.

**Lemma 2.2** Let $A = (A_1, \ldots, A_n)'$ be a block matrix of dimension $nd \times q$ with $n$ blocks, $A_i$ be a $d \times q$ matrix with entries in $\mathbb{R}$, $i = 1, \ldots, n$. Then

(2.12) \hspace{1cm} \|A\| \leq \sum_{i=1}^{n} \|A_i\|,

where $\|\cdot\|$ denote the matrix norm.

**Proof.** Without loss of generality, we consider the vector norm $\|(x_1, \ldots, x_d)'\|_d = \max_{1 \leq i \leq d} \{\|x_i\|\}$ on $\mathbb{R}^d$. We may decompose the matrix $A$ as

(2.13) \hspace{1cm} A = \sum_{i=1}^{n} B_i

with $B_i$ is a $nd \times q$ matrix defined by

$$
B_i = (0, \ldots, A_i, \ldots, 0)'
$$

where $A_i$ is in the $i$th position. We have, for all $x \in \mathbb{R}^d$

$$
\|B_i x\|_{nd} = \|(0, \ldots, A_i x, \ldots, 0)'\|_{nd} = \|A_i x\|_d, \ i = 1, \ldots, n.
$$
Therefore

\[ (2.14) \quad \|B_i\| = \sup_{\|x\|_q \leq 1} \{\|B_i x\|_{nd}\} = \sup_{\|x\|_a \leq 1} \{\|A_i x\|_d\} = \|A_i\|. \]

Inequality (2.12) follows from (2.13) and (2.14). □

By Lemma 2.2 and the definition of the sequence \( \{\tilde{C}_i(s, t)\} \) (2.11), we obtain

\[ (2.15) \quad \|\tilde{C}_i(s, t)\| \leq \sum_{j=(i-t+s)\lor 0}^i \|\tilde{C}_j(t - i + j)\|. \]

Using (2.15) and the following inequality, for \( a_i > 0, i = 1, \ldots, n \),

\[ (2.16) \quad \left( \sum_{i=1}^n a_i \right)^p \leq c \sum_{i=1}^n a_i^p, \]

where \( c = 1 \) if \( p \leq 1 \), \( c = n^{p-1} \) if \( p > 1 \), we obtain, if \( \alpha \in (0, 1) \cup (1, 2) \), for some \( 0 < \varepsilon < \alpha \),

\[ (2.17) \quad \sum_{i=0}^\infty \mathbb{E}\|\tilde{C}_i(s, t)\|^\alpha - \varepsilon \leq c_1 \sum_{i=0}^\infty \mathbb{E} \left( \sum_{j=(i-t+s)\lor 0}^i \|\tilde{C}_j(t - i + j)\| \right)^\alpha - \varepsilon \]

\[ = c_1 \sum_{j=0}^{t-s} \sum_{i=0}^\infty \mathbb{E}\|\tilde{C}_i(t - j)\|^\alpha - \varepsilon, \]

where \( c_1 \) is a constant depending on \( \alpha - \varepsilon \) and \( t - s \). The latter inequality in combination with the condition (1.2) and the stationarity of the sequence \( \{C_{i}(t)\} \) implies

\[ \sum_{i=0}^\infty \mathbb{E}\|\tilde{C}_i(s, t)\|^\alpha - \varepsilon < \infty. \]

By the similar method for (2.17), we obtain, for some \( 0 < \varepsilon < \alpha \),

\[ \sum_{i=0}^\infty \mathbb{E}\|\tilde{C}_i(s, t)\|^\alpha + \varepsilon \leq c_2 \sum_{j=0}^{t-s} \sum_{i=0}^\infty \mathbb{E}\|C_{i}(t - j)\|^\alpha + \varepsilon, \quad \text{if } \alpha \in (0, 1) \cup (1, 2), \]

\[ \mathbb{E} \left( \sum_{i=0}^\infty \|\tilde{C}_i(s, t)\|^{\alpha - \varepsilon} \right)^{\frac{\alpha + \varepsilon}{\alpha - \varepsilon}} \leq c_3 \sum_{j=0}^{t-s} \mathbb{E} \left( \sum_{i=0}^\infty \|C_{i}(t - j)\|^{\alpha - \varepsilon} \right)^{\frac{\alpha + \varepsilon}{\alpha - \varepsilon}}, \quad \text{if } \alpha \in \{1, 2\} \]
where $c_2$, $c_3$ and $c_4$ are the constants depending on $\alpha$, $\varepsilon$ and $t - s$. As a consequence, under the conditions of Theorem 2.1, it is not only the marginal distribution of $(X_t)$ that is regularly varying, the same holds for all finite dimensional distributions.

**Corollary 2.3** Under the conditions of Theorem 2.1, the stationary process $(X_t)$ given by (1.1) satisfies, for $s,t \in \mathbb{Z}$ with $s \leq t$

\[
P\left\{ x^{-1}(X_s,\ldots,X_t)' \in \cdot \right\} \Rightarrow \sum_{i=0}^{\infty} E\left[ \mu \circ \hat{C}_i(s,t)^{-1}(\cdot) \right] =: \nu_{s,t}(\cdot),
\]

as $x \to \infty$ on $\mathbb{R}^{(t-s+1)d}$, where the sequence of random matrices $\{\hat{C}_i(s,t)\}$ is defined by (2.11). In particular, if $X_0$ is regularly varying, then the process $(X_t)$ is jointly regularly varying, i.e. for all $s,t \in \mathbb{Z}$ with $s \leq t$, random vector $(X_s,\ldots,X_t)'$ are regularly varying.

### 3 Tail process

The joint regular variation of the sequence $(X_t)$ in particular means that there exists a process $(Y_t)_{t \in \mathbb{Z}}$ in $\mathbb{R}^d$ with $P\left\{ \|Y_0\| > y \right\} = y^{-\alpha}$ for $y \geq 1$ such that for all $s,t \in \mathbb{Z}$ with $s \leq t$ and as $x \to \infty$

\[
\mathcal{L}(x^{-1}(X_s,\ldots,X_t) \mid \|X_0\| > x) \sim \mathcal{L}(Y_s,\ldots,Y_t),
\]

see Theorem 2.1 in [1]. The process $(Y_t)$ is called the *tail process* of $(X_t)$. Moreover the so-called *spectral process* $\left( \Theta_t = \frac{Y_t}{\|Y_0\|} \right)_{t \in \mathbb{Z}}$ is independent of $\|Y_0\|$, see Theorem 3.1 in [1].

To understand the tail process in this case, it is helpful for instance to consider a set $B$ in $\mathbb{R}^{(t-s+1)d}$ of the form $B = \mathbb{E}_{b_s}^d \times \cdots \times \mathbb{E}_{b_t}^d$ for arbitrary real constant $b_i > 0$, $i = s,\ldots,t$. The limit of the probabilities, as $x \to \infty$,

\[
\frac{P\left\{ x^{-1}(X_s,\ldots,X_t)' \in B \right\}}{P\left\{ \|Y_0\| > x \right\}}
\]
is the sum
\[\sum_{i=0}^{\infty} \mathbb{E} \left[ \mu \circ \tilde{C}_i(s, t)^{-1}(B) \right] = \sum_{i=0}^{\infty} \mathbb{E} \left[ \mu \left\{ x \mid \tilde{C}_i(s, t)x \in B \right\} \right] = \sum_{i=t-s}^{\infty} \mathbb{E} \left[ \mu \left\{ x \mid \|C_{i-t+s}(s)x\| > b_s, \ldots, \|C_i(t)x\| > b_t \right\} \right] = \sum_{i=t-s}^{\infty} \mathbb{E} \left[ \int \mathbb{P}_\Theta(d\theta) \prod_{i \in \{t-s, \ldots, t\}} \mathbb{I}_{\{\|C_{i-t+s-(r-1)\theta}\| > b_i, \|C_i(r\theta)\| > b_t\}} d\left(-r^{-\alpha}\right) \right] = \sum_{i=t-s}^{\infty} \mathbb{E} \left[ \min \left\{ \frac{\|C_{i-t+s}(s)\Theta\|^\alpha}{b_s^\alpha}, \ldots, \frac{\|C_i(t)\Theta\|^\alpha}{b_t^\alpha} \right\} \right].\]

In particular, if \(s = t\) and \(b_s = 1\), one can apply this to obtain the following limit, as \(x \to \infty\),

\[(3.19) \quad \frac{\mathbb{P}\{\|X_0\| > x\}}{\mathbb{P}\{\|\xi_0\| > x\}} \to \sum_{i=0}^{\infty} \mathbb{E}\|C_i(0)\Theta\|^\alpha.\]

The following result is the multivariate version of Breiman’s lemma [3] which appears as Proposition A. 1 in [2].

**Lemma 3.1** Let \(Z\) be a \(q\)-dimensional random vector and let \(A\) be a \(d \times q\) random matrix, independent of \(Z\). Assume that \(Z\) is multivariate regularly varying of index \(\alpha \in (0, \infty)\), i.e. there exists a non-null Radon measure \(\mu\) on \(\mathbb{R}^q\) such that, as \(x \to \infty\),

\[\mathbb{P}\{x^{-1}z \in \cdot\} \overset{v}{\to} \mu(\cdot).\]

If \(\mathbb{E}\|A\|^\beta < \infty\) for some \(\beta > \alpha\), then in \(\mathbb{E}^d\), as \(x \to \infty\),

\[\frac{\mathbb{P}\{x^{-1}AZ \in \cdot\}}{\mathbb{P}\{\|Z\| > x\}} \overset{v}{\to} \mathbb{E}\left[\mu \circ A^{-1}(\cdot)\right].\]

**Theorem 3.2** Let \((X_t)_{t \in \mathbb{Z}}\) be a stationary process given by (1.1). Suppose that Condition 1.I and 1.Z hold. If \(\mathbb{P}\{\cap_{i \geq 0} \{\|C_i(0)\Theta\| = 0\} = 0\}\), then for \(s, t \in \mathbb{Z}\) with \(s \leq 0 \leq t\)
and bounded and continuous function \( f : (\mathbb{R}^d)^{t-s+1} \to \mathbb{R} \),

\[
\mathbb{E} \left[ f \left( \frac{X_s}{\|X_0\|}, \ldots, \frac{X_t}{\|X_0\|} \right) \mid \|X_0\| > x \right] \to \frac{1}{\sum_{i=0}^{\infty} \mathbb{E}\|C_i(0)\|^\alpha} \sum_{i=0}^{\infty} \mathbb{E} \left[ f \left( \frac{C_{i+s}(s)\Theta}{\|C_i(0)\|^\alpha}, \ldots, \frac{C_{i+t}(t)\Theta}{\|C_i(0)\|^\alpha} \right) \|C_i(0)\|^\alpha \right],
\]

as \( x \to \infty \), where \( C_i(\cdot) = 0 \) if \( i < 0 \).

**Proof.** Let \( h : (\mathbb{R}^d)^{t-s+1} \to \mathbb{R} \) be bounded and continuous. In view of convergence (3.19) and Corollary 2.3, as \( x \to \infty \),

\[
\mathbb{E} \left[ f \left( X_s x, \ldots, X_t x \right) \mid \|X_0\| > x \right] \to \frac{1}{\sum_{i=0}^{\infty} \mathbb{E}\|C_i(0)\|^\alpha} \int h(y_s, \ldots, y_t) \mathbb{I}_{\{\|y_0\| > 1\}} \nu_{s,t}(dy).
\]

Note

\[
\tilde{X}(s,t) = (X_s, \ldots, X_t)' = \sum_{i=0}^{\infty} \tilde{C}_i(s,t)\xi_{t-i},
\]

where the sequence \( \{\tilde{C}_i(s,t)\} \) is defined by (2.11). By (2.15), (2.16), Condition 1.1 and the stationarity of sequence \( \{C_i(t)\} \), we have for some \( \beta > \alpha \),

\[
\mathbb{E}\|\tilde{C}_i(s,t)\|^\beta \leq c \sum_{j=(i-t+s)\vee 0}^{i} \mathbb{E}\|C_j(t-i+j)\|^\beta = c \sum_{j=(i-t+s)\vee 0}^{i} \mathbb{E}\|C_j(0)\|^\beta < \infty,
\]

where \( c \) is a constant depending on \( \beta \) and \( t-s \). Using Lemma 3.1, we obtain, for each \( i \geq 0 \)

\[
\mathbb{P} \left\{ x^{-1}\tilde{C}_i(s,t)\xi_{t-i} \in \cdot \right\} \to \mathbb{E} \left[ \mu \circ \tilde{C}_i(s,t)^{-1}(\cdot) \right] =: \nu_{s,t}^{(i)}(\cdot),
\]

as \( x \to \infty \), in \( \mathbb{P}(t-s+1)^d \). By the definition of the measure \( \nu_{s,t} \) in (2.18),

\[
\nu_{s,t}(\cdot) = \sum_{i=0}^{\infty} \nu_{s,t}^{(i)}(\cdot).
\]

We denote

\[
H(x) = \mathbb{E} \left[ h \left( x^{-1}\tilde{C}_i(s,t)\xi_{t-i} \right) \mathbb{I}_{\{\|C_{i-s}(0)\|\xi_{t-i} > x\}} \right] / \mathbb{P} \left\{ \|\xi_0\| > x \right\}.
\]
On the one hand, it follows from (3.22), as \( x \to \infty \),
\[
H(x) = \int h(y_{s}, \ldots, y_{t}) I_{\{\|y_{0}\| > 1\}} P x^{-1} \hat{C}_{i}(s, t) \xi_{t-i}^{-1} (dy) / P \{\|\xi_{0}\| > x\} 
\]
(3.24)
\[
\to \int h(y_{s}, \ldots, y_{t}) I_{\{\|y_{0}\| > 1\}} \nu_{s,t}^{(i)} (dy). 
\]
On the other hand, by independence between \( \hat{C}_{i}(s, t) \) and \( \xi_{t-i} \) and assumption A1, as \( x \to \infty \),
\[
H(x) = \int \mathbb{E} \left[ h \left( \hat{C}_{i}(s, t) y \right) I_{\{\|C_{i}(0)\| > 1\}} \right] P x^{-1} \xi_{t-i}^{-1} (dy) / P \{\|\xi_{0}\| > x\} 
\]
(3.25)
\[
\to \int_{0}^{\infty} \mathbb{E} \left[ h \left( \hat{C}_{i}(s, t) \Theta r \right) I_{\{\|C_{i}(0)\| > 1\}} \right] d(-r^{-\alpha}). 
\]
Since for \( i < t \), \( \nu_{s,t}^{(i)} \{ (y_{s}, \ldots, y_{t}) \mid y_{s} = \cdots = y_{0} = 0 \} = 1 \),
\[
\int h(y_{s}, \ldots, y_{t}) I_{\{\|y_{0}\| > 1\}} \nu_{s,t}^{(i)} (dy) = 0, \ i = 0, \ldots, t-1. 
\]
In combination with (3.23), (3.24), (3.25) and (3.26), it follows that
\[
\int h(y_{s}, \ldots, y_{t}) I_{\{\|y_{0}\| > 1\}} \nu_{s,t}^{(i)} (dy) = \sum_{i=t}^{\infty} \int_{0}^{\infty} \mathbb{E} \left[ h \left( \hat{C}_{i}(s, t) \Theta r \right) I_{\{\|C_{i}(0)\| > 1\}} \right] d(-r^{-\alpha}). 
\]
(3.27)
Considering (3.21) and (3.27), we have, as \( x \to \infty \),
\[
\mathbb{E} \left[ f \left( \frac{X_{s}}{x}, \ldots, \frac{X_{t}}{x} \right) \mid \|X_{0}\| > x \right] \to I
\]
where
\[
I = \frac{1}{\sum_{i=0}^{\infty} \mathbb{E}\|C_{i}(0)\|^{\alpha}} \sum_{i=0}^{\infty} \int_{0}^{\infty} \mathbb{E} \left[ h \left( C_{i-t+s}(s) \Theta r, \ldots, C_{i}(t) \Theta r \right) I_{\{\|C_{i-t+s}(0)\| > 1\}} \right] d(-r^{-\alpha}). 
\]
Applying this relation to the function \( h(x_{s}, \ldots, x_{t}) = f \left( \frac{x_{s}}{\|x_{0}\|}, \ldots, \frac{x_{t}}{\|x_{0}\|} \right) \) to see that, as \( x \to \infty \), the left-hand side of (3.20) converges to
\[
\frac{1}{\sum_{i=0}^{\infty} \mathbb{E}\|C_{i}(0)\|^{\alpha}} \sum_{i=0}^{\infty} \int_{0}^{\infty} \mathbb{E} \left[ f \left( \frac{C_{i-t+s}(s) \Theta}{\|C_{i-t+s}(0)\|}, \ldots, \frac{C_{i}(t) \Theta}{\|C_{i-t+s}(0)\|} \right) I_{\{\|C_{i-t+s}(0)\| > 1\}} \right] d(-r^{-\alpha}). 
\]
By Fubini’s theorem, this is equal to the right-hand side of (3.20). \( \square \)
4 Applied models

4.1 Heavy tailed multivariate ARMA(1,1) process

Assume that \((A_t, B_t, \xi_t)\) is an i.i.d. sequence of random vectors in \(\mathbb{R}^{d \times d} \times \mathbb{R}^{d} \times \mathbb{R}^{d}\), for some \(d \geq 1\). Throughout we assume that \(\xi_t\)'s are regularly varying random vectors.

Suppose that a stationary sequence \((X_t)_{t \in \mathbb{Z}}\) with value in \(\mathbb{R}^d\) satisfies a multivariate random coefficient ARMA(1,1) equation of the following form

\[
X_t = A_t X_{t-1} + B_t \xi_{t-1} + \xi_t.
\]

Iterating this equation backwards we arrive at the following MA(\(\infty\)) representation of this process

\[
X_t = \xi_t + \sum_{i \geq 1} (A_{t-i+1} + B_{t-i+1}) \prod_{j=0}^{i-2} A_{t-j} \xi_{t-i},
\]

namely, the stationary solution can be represented as (1.1) with

\[
C_i(t) = \begin{cases} 
  \text{Id}, & i = 0, \\
  (A_{t-i+1} + B_{t-i+1}) \prod_{j=0}^{i-2} A_{t-j}, & i \geq 1.
\end{cases}
\]

From Theorem 3.1 in Hult and Samorodnitsky [4] we obtain the following result.

**Corollary 4.1** Suppose that \(\xi \in \text{RV}(\alpha, \mu)\) and there is some \(0 < \varepsilon < \alpha\) such that

\[
E\|A\|^{\alpha+\varepsilon} < 1, \quad E\|B\|^{\alpha+\varepsilon} < \infty.
\]

Then the series (4.29) converges a.s. and

\[
\frac{P\{u^{-1} X_0 \in \cdot\}}{P\{\|\xi_0\| > u\}} \xrightarrow{u \to \infty} E\left[\sum \mu \circ C_j(0)^{-1}(\cdot)\right]
\]

as \(u \to \infty\) on \(\mathbb{R}^d \setminus \{0\}\).

**Proof.** Since \(C_0(0) = \text{Id}\), we have \(P(\cap_{j \geq 0}\{\|C_j(0)\| = 0\}) = 0\).

For \(\alpha \in (0, 1) \cup (1, 2)\), we have

\[
\sum_i E\|C_i(0)\|^{\alpha-\varepsilon} = 1 + \sum_{i \geq 1} E\left\|A_{t-i+1} + B_{t-i+1} \prod_{j=0}^{i-2} A_{t-j}\right\|^{\alpha-\varepsilon} \leq 1 + \sum_{i \geq 1} E\|A + B\|^{\alpha-\varepsilon} \prod_{j=0}^{i-2} \|A_{t-j}\|^{\alpha-\varepsilon} = 1 + E\|A + B\|^{\alpha-\varepsilon} \sum_{i \geq 1} (E\|A\|^{\alpha-\varepsilon})^{i-1} \leq 1 + E\|A + B\|^{\alpha-\varepsilon} \sum_{i \geq 0} (E\|A\|^{\alpha-\varepsilon})^{i+\frac{\varepsilon}{\alpha}} = 1 + E\|A + B\|^{\alpha-\varepsilon} \sum_{i \geq 0} (E\|A\|^{\alpha-\varepsilon})^{i+\frac{\varepsilon}{\alpha}}.
\]
4.1 Heavy tailed multivariate ARMA(1,1) process

By Jensen’s inequality, the last term is bounded by

\[(4.34) \quad 1 + E\|A + B\|^{\alpha-\varepsilon} \sum_{i \geq 0} (E\|A\|^{\alpha+\varepsilon})^i \frac{1}{\alpha+\varepsilon}.\]

In the case of \(\alpha \in (0, 1),\)

\[(4.35) \quad E\|A + B\|^{\alpha-\varepsilon} < E\|A\|^{\alpha-\varepsilon} + E\|B\|^{\alpha-\varepsilon}.\]

In the case of \(\alpha \in (1, 2),\)

\[(4.36) \quad E\|A + B\|^{\alpha-\varepsilon} \leq (E\|A + B\|^{\alpha+\varepsilon})^\frac{1}{\alpha+\varepsilon} \leq \left(\left(\frac{1}{\alpha} + \frac{1}{\alpha+\varepsilon}\right)\right)^{\alpha-\varepsilon}.\]

Combining (4.33), (4.34), (4.35) and (4.36) proves

\[
\sum E\|C_i(0)\|^{\alpha-\varepsilon} < \infty.
\]

Similarly as in (4.33), we have

\[(4.37) \quad \sum E\|C_i(0)\|^{\alpha+\varepsilon} \leq 1 + E\|A + B\|^{\alpha+\varepsilon} \sum_{i \geq 0} (E\|A\|^{\alpha+\varepsilon})^i.\]

If \(\alpha + \varepsilon < 1,\) then

\[(4.38) \quad E\|A + B\|^{\alpha+\varepsilon} < E\|A\|^{\alpha+\varepsilon} + E\|B\|^{\alpha+\varepsilon},\]

else

\[(4.39) \quad E\|A + B\|^{\alpha+\varepsilon} \leq \left(\frac{1}{\alpha} + \frac{1}{\alpha+\varepsilon}\right)^{\alpha+\varepsilon}.\]

Combining (4.37), (4.38) and (4.39) proves

\[
\sum E\|C_i(0)\|^{\alpha+\varepsilon} < \infty.
\]

For \(\alpha \in \{1, 2\},\) using Lemma 3.2.1 in Kwapien and Woyczynski [5] it follows that

\[
(4.40) \quad 1 + (E\|A + B\|^{\alpha+\varepsilon})^{\frac{\alpha+\varepsilon}{\alpha+2}} \sum_{i \geq 0} (E\|A\|^{\alpha+\varepsilon})^i \frac{1}{\alpha+2}.
\]
4.1 Heavy tailed multivariate ARMA(1,1) process

Considering (4.39) and (4.40) we get

$$E \left( \sum \|C_i(0)\|^{\alpha - \epsilon} \right)^{\frac{\alpha + \epsilon}{\alpha - \epsilon}} < \infty.$$  

For $\alpha \in (2, \infty)$, using Lemma 3.2.1 in Kwapien and Woyczynski [5] it follows that

$$E \left( \sum \|C_i(0)\|^2 \right)^{\frac{\alpha + \epsilon}{2}} \leq \left[ \sum \left( E \|C_i(0)\|^{\alpha + \epsilon} \right)^{\frac{2}{\alpha + \epsilon}} \right]^{\frac{\alpha + \epsilon}{2}} \leq 1 + \left( E \|A + B\|^{\alpha + \epsilon} \right)^{\frac{2}{\alpha + \epsilon}} \sum_{i \geq 0} \left( E \|A\|^{\alpha + \epsilon} \right)^{i} \frac{2}{\alpha + \epsilon}. $$

(4.41)

Considering (4.39) and (4.41) we get

$$E \left( \sum \|C_i(0)\|^2 \right)^{\frac{\alpha + \epsilon}{2}} < \infty.$$  

□

Here is another proof of Corollary 4.1

Proof. We denote

(4.42)  
$$\tilde{X}_t = (X_t, \xi_t, \xi_{t+1})',$$  

$$\tilde{A}_t = \begin{pmatrix} A_t & B_t & Id \\ 0 & 0 & Id \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$Z_t = (0, 0, \xi_{t+1})'.$$

Then the relation (4.28) can be wrote as

(4.43)  
$$\tilde{X}_t = \tilde{A}_t \tilde{X}_{t-1} + Z_t.$$

Iterating this equation we get

$$\tilde{X}_t = \sum_{i=0}^{\infty} C_i(t) Z_{t-i}$$

where $C_0(t) = Id$, $C_1(t) = \tilde{A}_t$ and

$$C_i(t) = \begin{pmatrix} \prod_{j=0}^{i-1} A_{t-j} & B_{t-i+1} \prod_{j=0}^{i-2} A_{t-j} & (A_{t-i+2} + B_{t-i+2}) \prod_{j=0}^{i-3} A_{t-j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i \geq 2.$$
4.1 Heavy tailed multivariate ARMA(1,1) process

It is equivalent to prove the convergence (4.32) of vector $\tilde{X}_t$ defined by (4.42).

By Lemma 2.2 and the properties of matrix norm, for $i \geq 1$

$$
\|C_i(t)\| \leq \prod_{j=0}^{i-1} \|A_{t-j}\| + \|B_{t-i+1}\| \prod_{j=0}^{i-2} \|A_{t-j}\| + \|A_{t-i+2}\| \prod_{j=0}^{i-3} \|A_{t-j}\|
$$

where $A_i = B_i = Id$ if $i > t$. It follows from (2.16), if $\alpha \in (0, 1) \cup (2, 1)$

$$
\sum_{i \geq 0} E\|C_i(0)\|^{\alpha - \varepsilon} < \infty.
$$

Similarly we can prove

$$
\sum_{i \geq 0} E\|C_i(0)\|^{\alpha + \varepsilon} < \infty.
$$

For $\alpha \in \{1, 2\}$, using Lemma 3.2.1 in Kwapien and Woyczynski [5] it follows that

$$
E \left( \sum_{i \geq 0} \|C_i(0)\|^{\alpha - \varepsilon} \right)^{\frac{a + \varepsilon}{a - \varepsilon}} \leq \left[ E \sum_{i \geq 0} \|C_i(0)\|^{\alpha + \varepsilon} \right]^{\frac{a + \varepsilon}{a - \varepsilon}}.
$$

By (2.16) and (4.44), $\sum (E\|C_i(0)\|^{\alpha + \varepsilon})^{\frac{a + \varepsilon}{a - \varepsilon}}$ is bounded by the right-hand side of (4.45) which is bounded. Hence

$$
E \left( \sum_{i \geq 0} \|C_i(0)\|^{\alpha - \varepsilon} \right)^{\frac{a + \varepsilon}{a - \varepsilon}} < \infty.
$$
By similar method of (4.48), we get
\[
\mathbb{E} \left( \sum \|C_i(0)\|^2 \right)^{\frac{\alpha+\epsilon}{2}} < \infty.
\]

References


