PainlevéGullstrand synchronizations in spherical symmetry

Alicia Herrero, Juan Antonio, Morales -Lladosa

▶ To cite this version:

Alicia Herrero, Juan Antonio, Morales -Lladosa. PainlevéGullstrand synchronizations in spherical symmetry. Classical and Quantum Gravity, 2010, 27 (17), pp.175007. 10.1088/0264-9381/27/17/175007 . hal-00619938

HAL Id: hal-00619938
https://hal.science/hal-00619938

Submitted on 7 Sep 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Painlevé-Gullstrand synchronizations in spherical symmetry

Alicia Herrero[1] **and Juan Antonio Morales-Lladosa**[2]

[1] Institut de Matemàtica Multidisciplinar and Departament de Matemàtica Aplicada, Universitat Politècnica de València,
E-46220 València, Spain.

[2] Departament d'Astronomia i Astrofísica, Universitat de València,
E-46100 Burjassot, València, Spain.

E-mail: `aherrero@mat.upv.es; antonio.morales@uv.es`

Abstract. A Painlevé-Gullstrand synchronization is a slicing of the space-time by a family of flat spacelike 3-surfaces. For spherically symmetric space-times, we show that a Painlevé-Gullstrand synchronization only exists in the region where $(dr)^2 \leq 1$, r being the curvature radius of the isometry group orbits (2-spheres). This condition says that the Misner-Sharp gravitational energy of these 2-spheres is not negative and has an intrinsic meaning in terms of the norm of the mean extrinsic curvature vector. It also provides an algebraic inequality involving the Weyl curvature scalar and the Ricci eigenvalues. We prove that the energy and momentum densities associated with the Weinberg complex of a Painlevé-Gullstrand slice vanish in these curvature coordinates, and we give a new interpretation of these slices by using semi-metric Newtonian connections. It is also outlined that, by solving the vacuum Einstein's equations in a coordinate system adapted to a Painlevé-Gullstrand synchronization, the Schwarzschild solution is directly obtained in a whole coordinate domain that includes the horizon and both its interior and exterior regions.

PACS numbers: 04.20.Cv, 04.20.-q

1. Introduction

It is known that Painlevé [1], Gullstrand [2] and (some years later) Lemaître [3] used a non-orthogonal curvature coordinate system which allows to extend the Schwarzschild solution inside its horizon, see Eq. (58) below. In this coordinate system, from now on called a Painlevé-Gullstrand (PG) coordinate system, the metric is not diagonal, but asymptotically flat, and regular across the horizon, and then, everywhere non-singular up to $r = 0$. Furthermore, one has a very simple spatial 3-geometry: the space-time appears foliated by a synchronization of flat instants (hereafter called PG synchronizations).‡

‡ We use here the abbreviations 'PG coordinates' and 'PG synchronization' by sake of simplicity, without any intention of misplacing the relevant contribution by Lemaître [3], who clarified the

Nowadays, there is an increasing interest in the study of this type of synchronizations. For instance: (i) in connection with astrophysical applications, by taking into account that the dynamics of the gravitational collapse should be pursued beyond the Schwarzschild radius in a PG coordinate system [6, 7, 8, 9], (ii) in spherically symmetric space-times (SSSTs), as a convenient initial condition preserved under time evolution [10, 11], (iii) in relativistic hydrodynamics, when an effective Lorentzian metric is introduced [12], (iv) in non-relativistic situations admiting a Lorentzian description, namely, in 'analog gravity models' (see, for example, [13, 14, 15, 16, 17, 18]), and also (v) in modeling the black hole geometry and its associated physics, or to describe some quantum effects by starting from a Hamiltonian formulation [19, 20, 21, 22]. More physical issues about the use of PG coordinates and their interpretation can be found in [4, 5, 23, 24]. For a description of the causal character of the geometric elements (coordinate lines, coordinate 2-surfaces and coordinate 3-surfaces) associated with PG coordinates see [25, 26].

The existence of PG synchronizations in SSSTs has been studied in the static case considering that the induced metric has vanishing Ricci tensor [27], and some specific constructions are presented in [28]. In more general cases, this existence is usually taken for granted but, recently, a limitation to this ansatz has been pointed out [29]. However, as far as the authors are aware, a definitive interpretation of this limitation as well as the analysis of the domains where a PG synchronization exists have not been done up to now. Then, some related questions arise: does every SSST admits a region of physical interest where a synchronization by flat instants exists? and, what are the advantages of adapting coordinates to a flat spatial 3-geometry? The main contribution of this paper is to state the above limitation clearly, in a form that it is coordinate independent, and to provide its physical interpretation.

Generalized (but, in general, non-flat) PG synchronizations have been constructed in Schwarzschild [30], Reissner-Nordström [29], and Kerr [31, 32] geometries, and also in non-vacuum SSSTs, where new insights in the study of gravitational collapse scenarios are achieved (by evolving an initial 3-geometry [33, 34]). However, here we restrict ourselves to flat synchronizations in order to discuss their existence in SSSTs.

The paper is structured as follows. Section 2 is devoted to introduce some general formulae for the induced geometry on space-like hypersurfaces and surfaces in SSSTs. In Sec. 3 the condition for the existence of a PG synchronization in SSSTs is analyzed and physically interpreted. In Sec. 4 we write the components of the Weinberg pseudotensor [35] with respect to a PG synchronization, and we prove that the energy and momentum densities of each PG slice vanish. In Sec. 5 we consider the semi-metric connection (see [36]) associated with a spherically symmetric metric expressed in PG coordinates and provide new insights on the Newtonian interpretation of the properties exhibited by the Schwarzschild field in these coordinates. Sec. 6 deals with the $3+1$ decomposition

coordinate character of the 'Schwarzschild singularity' obtaining an extended metric form for the Schwarzschild solution. In fact, the main motivation in papers [1, 2, 3] is rather different. For historical remarks about this subject, and some physical interpretations see [4, 5].

of the Einstein equations with respect to a PG synchronization. By integration of the vacuum equations one recovers the extended Schwarzschild metric in PG coordinates, including the region inside the horizon. Finally, in Sec. 7 we discuss the role that our results can play for a better understanding of the geometry and physics in SSSTs. Some preliminary results of this work were presented at the Spanish Relativity meeting ERE-2009 [37].

Let us precise the used notation. The curvature tensor ${R^k}_{lij}$ of a symmetric connection ∇ is defined according with the identity $\nabla_i\nabla_j\xi^k - \nabla_j\nabla_i\xi^k = {R^k}_{lij}\xi^l$ for the vector field ξ, and $R_{ij} \equiv {R^k}_{ikj}$ is the Ricci tensor. We take natural units in which $c = G = 1$ and the Einstein constant is $\kappa = 8\pi$. We say that $\{t, r, \theta, \varphi\}$ is a curvature coordinate system if for constant t and r the line element is $dl^2 = r^2 d\Omega^2$ (with $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\varphi^2$ the metric of the unit sphere). In these coordinates, the metric form is, in general, non-diagonal. When, in addition, the 3-surfaces defined by $t = constant$ are flat, the curvature coordinate system is called a PG coordinate system. From now on, it will be understood that the title of the sections and the results of this work always concern space-times with spherical symmetry. The agreement for space-time signature is $(- + ++)$.

2. Some geometrical relations

In this section we present the geometric background needed in the following sections: expressions for the Ricci tensor and the extrinsic curvature of a spherically symmetric synchronization, as well as, the mean curvature vector and the Gauss identity for the 2-spheres of a SSST. Of course, this material is not new and it may be bypassed or used as a glossary of formulae, which are conveniently referred throughout the text of the remaining sections. For an account on 2+2 warped space-times properties allowing to intrinsically characterize SSST see [38, 39].

Let (V_4, g) be a SSST, and let us consider a canonical coordinate system $\{T, R, \theta, \varphi\}$ of V_4, which is adapted to the symmetries of the metric g. Then we may express the metric in the following general form [40, 41, 42]

$$g = A\, dT \otimes dT + B\, dR \otimes dR + C(dT \otimes dR + dR \otimes dT) + D\,\sigma, \tag{1}$$

with $A(T, R), B(T, R), C(T, R)$ verifying the condition $\delta \equiv AB - C^2 < 0$, $D(T, R) \neq 0$, and $\sigma \equiv d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi$ being the metric on the unit 2-sphere.

2.1. Ricci tensor

The induced metric γ on the 3-surfaces $T = constant$, is written as

$$\gamma = B dR \otimes dR + D\sigma, \tag{2}$$

with $B \neq 0$. The Ricci tensor of γ, $\mathcal{R}ic(\gamma)$, is given by

$$\mathcal{R}ic(\gamma) = \left(\frac{B}{2}\mathcal{R} - \frac{B}{D}F\right) dR \otimes dR + \left(\frac{D}{4}\mathcal{R} + \frac{F}{2}\right)\sigma \tag{3}$$

where

$$F = 1 - \frac{(\partial_R D)^2}{4BD} \tag{4}$$

and $\mathcal{R} \equiv \mathcal{R}(\gamma)$, the scalar curvature of γ, results

$$\begin{aligned} \mathcal{R} &= \frac{2}{D}\left(1 + \frac{\partial_R B \partial_R D}{2B^2} + \frac{(\partial_R D)^2}{4BD} - \frac{\partial_R^2 D}{B}\right) \\ &= \begin{cases} \dfrac{2F}{D} + \dfrac{4\partial_R F}{\partial_R D} & \text{if} \quad \partial_R D \neq 0, \\ \dfrac{2}{D} & \text{if} \quad \partial_R D = 0. \end{cases} \end{aligned} \tag{5}$$

Notice that the 3-surfaces $T = \textit{constant}$ are conformally flat§ but, in general, they are not flat. From Eqs. (3), (4) and (5), we see that γ is a flat metric if, and only if, $F = 0$, that is

$$\mathcal{R}ic(\gamma) = 0 \quad \Longleftrightarrow \quad 4BD = (\partial_R D)^2. \tag{6}$$

2.2. Extrinsic curvature

The extrinsic curvature of the slicing $T = \textit{constant}$ is defined as $K = -\frac{1}{2}\mathcal{L}_n\gamma$ where $\mathcal{L}_n$ is the Lie derivative along the unit normal vector n,

$$n = \frac{1}{\alpha}\left(\frac{\partial}{\partial T} - \frac{C}{B}\frac{\partial}{\partial R}\right) \tag{7}$$

with $\alpha^2 = -\dfrac{\delta}{B}$. From (2), we obtain

$$K = \Psi B \, dR \otimes dR + \Phi D \, \sigma \tag{8}$$

where

$$\Psi = K_R^R = \frac{1}{2B\alpha}\left(2\partial_R C - \frac{C}{B}\partial_R B - \partial_T B\right) \tag{9}$$

$$\Phi = K_\theta^\theta = K_\varphi^\varphi = \frac{1}{2D\alpha}\left(\frac{C}{B}\partial_R D - \partial_T D\right) \tag{10}$$

are the eigenvalues of K. Developing the Lie derivative of K with respect to the shift vector $\beta = \frac{C}{B}\frac{\partial}{\partial R}$ we arrive to the expression

$$\mathcal{L}_\beta K = \frac{B}{C}\,\partial_R\left[\left(\frac{C}{B}\right)^2 \Psi B\right] dR \otimes dR + \frac{C}{B}\,\partial_\theta(\Phi D)\,\sigma, \tag{11}$$

which will be needed in Sec. 6 to split the Einstein evolution equations with respect to a PG synchronization.

§ Notice that the Cotton tensor of γ, $\mathcal{C}_{ijk}(\gamma) \equiv D_i Q_{jk} - D_j Q_{ik}$, ($D_i$ is the covariant derivative with the Levi-Civita connection of γ, and $Q_{ij} \equiv \mathcal{R}_{ij} - \frac{\mathcal{R}}{4}\gamma_{ij}$) identically vanishes, $\mathcal{C}_{ijk}(\gamma) = 0$. This is the result that one could expect ought to the algebraic properties of the Cotton tensor and the assumed spherical symmetry. This means that γ is a conformally flat metric. Then, γ may always be written in isotropic form by making a coordinate change on each 3-surface $T = \textit{constant}$. An interesting summary on the Cotton tensor properties is given in [43].

2.3. Mean curvature vector

The mean curvature vector H of a 2-sphere S defined by constant T and R is given by

$$H = -\frac{1}{D\delta}\Big[(B\partial_T D - C\partial_R D)\frac{\partial}{\partial T} + (A\partial_R D - C\partial_T D)\frac{\partial}{\partial R}\Big]. \tag{12}$$

This expression directly follows by taking the trace (with respect to the induced metric $D\sigma$) of the extrinsic curvature tensor $\mathcal{K}$ of each S, which is defined by

$$\mathcal{K}(e_a, e_b) \equiv -\left(\nabla_{e_a} e_b\right)^{\perp} = -\left(\Gamma^T_{ab}\,\frac{\partial}{\partial T} + \Gamma^R_{ab}\,\frac{\partial}{\partial R}\right) \tag{13}$$

where the minus sign is taken as a matter of convention. ∇ is the Levi-Civita connection of g, $\{e_i \equiv \frac{\partial}{\partial x^i}\}$ is a coordinate basis of S, $i = a, b = \theta, \varphi$, and the symbol $^{\perp}$ stands for the projection on $S^{\perp}$: the time-like 2-surface orthogonal to S. For a detailed study of $\mathcal{K}$ with applications in relativity see, for example, [44, 45, 46]. Then, it results

$$\mathcal{K} = \frac{1}{2} D\sigma \otimes H. \tag{14}$$

The one form Γ metrically equivalent to H, $\Gamma_\alpha = g_{\alpha\beta}H^\beta$, is written as

$$\Gamma = -\frac{1}{D}(\partial_T D dT + \partial_R D dR) = -d\ln D, \tag{15}$$

which can be also obtained from the general expressions presented in [47].

2.4. Gauss identity

Given a space-like 2-surface Σ of a space-time (V_4, g), the Gauss identity provides a scalar relation involving the background geometry and the intrinsic and extrinsic properties of Σ, and it may be expressed as‖

$$R(h) = \frac{2}{3}R(g) + g(H,H) + 2tr(K_l \times K_k) + 2Ric(l,k) - 2W(l,k,l,k) \tag{16}$$

where $R(h)$ is the scalar curvature of the induced metric h on Σ, $R(g)$, Ric and W are, respectively, the scalar curvature, the Ricci and the Weyl tensor of g, and l and k are two independent future pointing null vectors normal to Σ satisfying $g(l,k) = -1$. Now, H is the mean curvature of Σ, that is $H = tr\mathcal{K}$, with $\mathcal{K}$ the extrinsic curvature tensor of Σ (as defined by the first equality of (13), taking e_a and e_b tangent to Σ); K_l and K_k are, respectively, the second fundamental forms of Σ relative to l and k, that is, $K_l(e_a, e_b) = g(l, \mathcal{K}(e_a, e_b))$, and similarly for K_k. The trace is taken with respect to h, that is, $tr(K_l \times K_k) = h^{ad}h^{bc}(K_l)_{ab}(K_k)_{cd}$.

‖ The Gauss identity is usually given in terms of the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ (see, for instance [45, 46]) from which the expression (16) follows by taking into account the algebraic decomposition:

$$R_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha} - \frac{R}{3}\, g_{\alpha[\gamma}g_{\delta]\beta}$$

where the bracket stands for antisymmetrization of index pairs, $T_{[\alpha\beta]} \equiv \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha})$.

For $\Sigma = S$, a 2-sphere (orbit) in a SSST, the Gauss identity may be written as

$$\rho = 2\lambda + \frac{1}{3}(4\mu - \mu_1 - \mu_2) + \frac{1}{2}H^2. \tag{17}$$

with $\rho = 2/D$ the scalar curvature of $D\sigma$, λ the simple eigenvalue¶ of W (provided that $W \neq 0$), and $\{\mu, \mu_1, \mu_2\}$ stand for the Ricci eigenvalues. More precisely, the tangent space to S is the eigenplane associated with the eigenvalue μ. The eigenvalues μ_1 and μ_2 may be real (simple or multiple) or complex, and have associated the invariant 2-plane tangent to $S^{\perp}$. Here, we have considered the algebraic classification of Churchill-Plebański for a symmetric 2-tensor in Lorentzian geometry, and its peculiarities in spherical symmetry (see [48], and also [49] for an intrinsic approach). We have taken into account that $R(g) = 2\mu + \mu_1 + \mu_2$, and $2Ric(l,k) = -(\mu_1 + \mu_2)$. In addition $H^2 \equiv g(H,H) = -4tr(K_l \times K_k)$ because, according to (14), the second fundamental forms K_l and K_k are both proportional to $h = D\sigma$.

3. Painlevé-Gullstrand slicings

In this section we find the condition ensuring the existence of a PG-synchronization in spherical symmetry, and we discuss its invariant meaning in terms of the eigenvalues of the Weyl and Ricci tensors. Also, using a radial curvature coordinate r, we provide a physical interpretation of this condition in terms of the Misner-Sharp gravitational energy of a 2-sphere of radius r. As commented below, this result shows an interesting interconnection between the study and classification of trapped surfaces (see [50]) and the existence of flat slicings.

3.1. Existence condition

Let us start from the general metric form (1). Exploring the gauge freedom to make coordinate transformations of the form $T = T(t,r)$, $R = R(t,r)$, we look for a function $t(T,R)$ whose level hypersurfaces $t = constant$ are Euclidean, i. e., the induced metric is positive and flat. Under such a transformation, the metric (1) is expressed as

$$ds^2 = \xi^2\, dt^2 + \chi^2 dr^2 + 2\xi\cdot\chi\, dt\, dr + \mathcal{D}(t,r) d\Omega^2, \tag{18}$$

with $\mathcal{D}(t,r) \equiv D(T(t,r), R(t,r))$, and the vector fields ξ and χ defined by

$$\xi \equiv \dot{T}\frac{\partial}{\partial T} + \dot{R}\frac{\partial}{\partial R}, \qquad \chi \equiv T'\frac{\partial}{\partial T} + R'\frac{\partial}{\partial R}. \tag{19}$$

Over-dot and prime stand for partial derivative with respect t and r, respectively, and $J \equiv \dot{T}R' - T'\dot{R} \neq 0$ assures coordinate regularity.

The scalar products $\xi^2 \equiv g(\xi,\xi)$, $\chi^2 \equiv g(\chi,\chi)$ and $\xi\cdot\chi \equiv g(\xi,\chi)$ can be written as

$$\delta(dD)^2\, \dot{T}^2 + 2\dot{\mathcal{D}}\, Z\, \dot{T} + B\dot{\mathcal{D}}^2 = (\partial_R D)^2\, \xi^2, \tag{20}$$

¶ One has

$$\frac{1}{2}W_{\mu\nu\alpha\beta}(l^\alpha k^\beta - l^\beta k^\alpha) = \lambda(l_\mu k_\nu - l_\nu k_\mu),$$

and then, $W(l,k,l,k) \equiv W_{\mu\nu\alpha\beta} l^\mu k^\nu l^\alpha k^\beta = -\lambda$.

$$\delta(dD)^2\, T'^2 + 2\mathcal{D}'\, Z\, T' + B\mathcal{D}'^2 = (\partial_R D)^2\, \chi^2, \tag{21}$$

$$(\partial_R D)^2\, \xi \cdot \chi = \delta(dD)^2\, \dot{T}\, T' + Z\,(\mathcal{D}'\dot{T} + \dot{\mathcal{D}}T') + B\dot{\mathcal{D}}\mathcal{D}', \tag{22}$$

where we have used the relations $\dot{\mathcal{D}} = \dot{T}\partial_T D + \dot{R}\partial_R D$ and $\mathcal{D}' = T'\partial_T D + R'\partial_R D$ to substitute $\dot{R}$ and R' in terms of $\dot{T}$ and T'. We have denoted $Z \equiv C\partial_R D - B\partial_T D$. Then, due to the Lorentzian character of the metric ($\delta < 0$), Eqs. (20) and (21) lead to real values for $\dot{T}$ and T' if, and only if, the inequalities

$$\xi^2 (d\mathcal{D})^2 \leq \dot{\mathcal{D}}^2, \quad \chi^2 (d\mathcal{D})^2 \leq \mathcal{D}'^2 \tag{23}$$

are satisfied. Now, looking for a flat synchronization, we have that the induced metric on the 3-surfaces $t = \textit{constant}$ is flat if, and only if,

$$4\,\mathcal{D}\,\chi^2 = \mathcal{D}'^2 \tag{24}$$

according to Eq. (6). Consequently, in the case of a flat synchronization, the second inequality in (23) is equivalent to

$$(d\sqrt{\mathcal{D}})^2 \leq 1. \tag{25}$$

So, under the assumed spherical symmetry, Eq. (25) provides the necessary and sufficient condition to be fulfilled for the existence of a flat slicing. The first inequality in (23) guarantees that the slices are space-like, that is, that the slicing is a PG synchronization.

3.2. Geometric interpretation

In terms of the scalar curvature $\rho = 2/\mathcal{D}$ of the metric $\mathcal{D}d\Omega^2$, the above condition (25) may be expressed as follows

$$(d\rho)^2 \leq 2\rho^3, \tag{26}$$

which involves the sole invariant ρ. On the other hand, according to Eq. (15), $H^2 = \Gamma^2 = (d\ln\mathcal{D})^2$, and then Eq. (25) gives an upper bound for the norm of the mean extrinsic curvature H of the group orbits,

$$H^2 \leq \frac{4}{\mathcal{D}} = 2\rho. \tag{27}$$

Moreover, from the Gauss relation (17), we arrive to the following result.

Proposition 1 *In a spherically symmetric space-time the following conditions are equivalent.*

(i) *There exists a Painlevé-Gullstrand synchronization.*

(ii) $(d\rho)^2 \leq 2\rho^3$, *where* ρ *is the scalar curvature of the 2-spheres.*

(iii) $H^2 \leq 2\rho$, *where* H *is the mean curvature vector of the 2-spheres.*

(iv) $\mu_1 + \mu_2 - 4\mu \leq 6\lambda$, *where* μ_1, μ_2, *and* μ *(double) are the Ricci eigenvalues, and* λ *is the simple eigenvalue of the Weyl tensor, or* $\lambda = 0$ *when the space-time is conformally flat.*

Notice that this is a geometric result, which will be physically interpreted in the next subsection. Taking into account the Einstein equations, in the above item (iv) the Ricci eigenvalues, $\{\mu_1, \mu_2, \mu\}$ may by substituted by the corresponding energy tensor eigenvalues, $\{e_1, e_2, e\}$, giving

$$e_1 + e_2 - e \leq 3\lambda. \tag{28}$$

3.3. Physical interpretation

By definition, see [41, 40], r is a coordinate of curvature for the spherically symmetric metric form (18) if $\mathcal{D}(t, r) = r^2$, so that $\mathcal{D}' = 2r$ and $\dot{\mathcal{D}} = 0$. Then, (25) says that $(dr)^2 \leq 1$, and taking into account that the Misner-Sharp gravitational energy E of a 2-sphere of radius r is expressed as (see [51, 50])

$$E = \frac{r}{2}\Big(1 - (dr)^2\Big), \tag{29}$$

we arrive to the following result.

Proposition 2 *Any spherically symmetric space-time admits a Painlevé-Gullstrand synchronization in the region where the Misner-Sharp gravitational energy is non-negative, $E \geq 0$.*

The Misner-Sharp energy has been painstakingly analyzed in [50], providing useful criteria to study trapped surfaces. The main novelty here has been to relate this concept and the existence of PG synchronizations.+

Moreover, the flatness condition (24) implies that $\chi^2 = 1$, and the metric (18) is written as

$$ds^2 = \xi^2\, dt^2 + 2\xi \cdot \chi\, dt\, dr + dr^2 + r^2 d\Omega^2. \tag{30}$$

Then, accordingly to (20) and (22), the following relations must occur

$$\begin{aligned} &\mathcal{A}(t, r) \equiv \xi^2 = J^2\, \delta (dr)^2 \\ &\mathcal{B}(t, r) \equiv \xi \cdot \chi = \varepsilon J \sqrt{\delta[(dr)^2 - 1]} \end{aligned} \tag{31}$$

where $\varepsilon = \pm 1$. So, the real function $\mathcal{B}$ exists in the region where $(dr)^2 \leq 1$, and we have the following result.

Proposition 3 *Let r be the radius of curvature of the orbits (2-spheres) of the isometry group of a spherically symmetric space-time with metric g. In the region defined by the condition*

$$(dr)^2 \equiv g^{\mu\nu} \partial_\mu r \partial_\nu r \leq 1, \tag{32}$$

the Misner-Sharp energy is not negative and a curvature coordinate system $\{t, r, \theta, \varphi\}$ exists in which the metric line element may be written as

$$ds^2 = \mathcal{A}(t, r)\, dt^2 + 2\mathcal{B}(t, r)\, dt\, dr + dr^2 + r^2 d\Omega^2, \tag{33}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$.

+ Marc Mars inspired us in the obtaining of this relation.

The Lorentzian character of the metric impose that the functions $\mathcal{A}$ and $\mathcal{B}$ must satisfy the sole restriction $\mathcal{A} < \mathcal{B}^2$, which is implied by (31).

The Misner-Sharp energy is a geometric invariant that may be physically interpreted as an effective gravitational energy whose origin is the interaction between the energetic content and its associated field (see [50]). Given that the intrinsic and extrinsic scalar curvatures of the 2-spheres are $\rho = 2/r^2$ and $H^2 = 4(d \ln r)^2$, and according with (29) one has the invariant expression

$$E = \frac{1}{\sqrt{2\rho}}\Big(1 - \frac{1}{2\rho}H^2\Big). \tag{34}$$

Finally, notice that Eq. (27) does not constraint the causal character of the mean curvature vector H, which might be time-like, light-like or space-like. This is a remarkable property, because a 2-sphere is said to be trapped, marginal or untrapped if H is, respectively, time-like, light-like or space-like (see e. g. [45, 46, 50, 52]).

4. Energy and momenta densities of a Painlevé-Gullstrand slice

In this section we establish the following result.

Proposition 4 *In any spherically symmetric space-time, the Weinberg energy and momenta densities vanish for every Painlevé-Gullstrand synchronization.*

Of course, to find a coordinate system in which the Weinberg densities vanish is not a surprising property, due to the non-tensorial character of them. However, the novelty here is to show that, for every SSST, such a vanishing property occurs in PG coordinates.

In order to proof the above result, let us consider the metric (33) written in a quasi-Minkowskian form, that is $g = \eta + h$ with η the Minkowski metric, $h_{00} = 1 + \mathcal{A}$, $h_{0i} = \mathcal{B}x_i/r$, and $h_{ij} = 0$.

We start from the expression of the Weinberg pseudo-tensor [35],

$$2Q^{i0\lambda} = \frac{\partial h^\mu_\mu}{\partial x_0}\eta^{i\lambda} - \frac{\partial h^\mu_\mu}{\partial x_i}\eta^{0\lambda} - \frac{\partial h^{\mu 0}}{\partial x^\mu}\eta^{i\lambda} + \frac{\partial h^{\mu i}}{\partial x^\mu}\eta^{0\lambda} + \frac{\partial h^{0\lambda}}{\partial x_i} - \frac{\partial h^{i\lambda}}{\partial x_0},$$

where Latin and Greek indexes go from 1 to 3 and from 0 to 3, respectively, and all indexes are raised and lowered with the flat metric η. In this case, it results $Q^{i00} = 0$ (according with [53]) and

$$2Q^{i0j} = \left(\frac{\mathcal{B}}{r} + \mathcal{B}'\right)\delta_{ij} + \left(\frac{\mathcal{B}}{r} - \mathcal{B}'\right)\frac{x_i}{r}\frac{x_j}{r} \tag{35}$$

The derivative of this expression leads to

$$2\,\frac{\partial Q^{i0j}}{\partial x^k} = \left(\frac{\mathcal{B}'}{r} - \frac{\mathcal{B}}{r^2} + \mathcal{B}''\right)\delta_{ij}\,\frac{x_k}{r} + \left(\frac{\mathcal{B}}{r^2} - \frac{\mathcal{B}'}{r}\right)\left(\delta_{ik}\,\frac{x_j}{r} + \delta_{jk}\,\frac{x_i}{r}\right)$$
$$+\left(3\frac{\mathcal{B}'}{r} - 3\frac{\mathcal{B}}{r^2} - \mathcal{B}''\right)\frac{x_i}{r}\,\frac{x_j}{r}\,\frac{x_k}{r}.$$

and by contraction of the indexes, directly follows that $\frac{\partial Q^{i0j}}{\partial x^i} = 0$. Then, the four-momentum density vanishes,

$$\tau^{0\lambda} \equiv -\frac{1}{8\pi G}\frac{\partial Q^{i0\lambda}}{\partial x^i} = 0 \tag{36}$$

and hence, the angular momentum densities $\mathrm{j}^{i\lambda} = x^i\tau^{0\lambda} - x^\lambda\tau^{0i}$ also vanish, according with the announced conclusion.

For the special case of the Schwarzschild geometry, the vanishing of the energy density may be intuitively understood invoking the Einstein equivalence principle. Taking $\epsilon = 1$ in the extended form (58) of the Schwarzschild metric, t represents the proper time of a radial geodesic observer which initially stays, in $r = \infty$, at rest with respect to a static observer. Locally, such an observer does not feel any gravitational effect.

5. Painlevé-Gullstrand slicings and semi-metric connections

In the eighties, Bel proposed an extended Newtonian theory of gravitation based on a semi-metric connection associated with an observer congruence and a flat spatial 3-metric [36]. In a space-time, with metric $g_{\mu\nu}$, which admits a spatially flat slicing given by the coordinate hypersurfaces $x^0 = \mathit{constant}$, the connection coefficients of the aforementioned semi-metric connection are written as [36],

$$\Lambda^k = -\Gamma^k_{00} = \frac{1}{2}\delta^{ki}(\partial_i g_{00} - 2\partial_0 g_{0i}) \tag{37}$$

$$\Omega^k_j = -2\Gamma^k_{0j} = \delta^{ki}(\partial_i g_{0j} - 2\partial_j g_{0i}). \tag{38}$$

Consequently, a SSST metric admits a Newtonian interpretation when it is written in PG coordinates and it is considered in the above context. In fact, taking into account the expression (33) of the metric, we have $g_{00} = \mathcal{A}$, $g_{0i} = \mathcal{B}x_i/r$, and then

$$\partial_i g_{0j} = \frac{\mathcal{B}}{r}\delta_{ij} + \left(\mathcal{B}' - \frac{\mathcal{B}}{r}\right)\frac{x_i}{r}\,\frac{x_j}{r} = \partial_j g_{0i}.$$

Then, the connection coefficients result

$$\Lambda^k = \frac{1}{2}\left(\mathcal{A}' - 2\dot{\mathcal{B}}\right)\frac{x^k}{r} \tag{39}$$

$$\Omega^k_j = 0, \tag{40}$$

which means that, in the region of a SSST where a PG synchronization exist, the gravitational field may be interpreted as an inertial field of radial accelerations and vanishing rotation.

In particular, for the case of the Schwarzschild metric, we have $\mathcal{A} = -\left(1 - \frac{2m}{r}\right)$, $\mathcal{B} = \epsilon\sqrt{\frac{2m}{r}}$, and then the vector component of the connection reduces to

$$\vec{\Lambda} = -\frac{m}{r^2}\vec{e_r}, \tag{41}$$

where $\vec{e_r}$ is the unit vector in the radial direction. The above expression (41) gives the acceleration of a unit mass particle radially falling in the Newtonian field of a mass m. Similar Newtonian interpretations have been considered from a different point of view (see, for example, [4, 5, 23]).

6. Painlevé-Gullstrand slicings and Einstein equations

In General Relativity, when dealing with the evolution (or $3+1$) formalism (see [54, 55], and [56] for a recent review) one introduces a vorticity free observer n, $n^2 = -1$, and Einstein equations are decomposed in the following set of constraint equations (κ is the Einstein constant),

$$\mathcal{R}(\gamma) + (\mathrm{tr}K)^2 - \mathrm{tr}K^2 = 2\,\kappa\,\tau \tag{42}$$

$$\nabla\cdot(K - \mathrm{trK}\,\gamma) = \kappa\,q \tag{43}$$

and this other system of evolution equations

$$\partial_t\gamma = -2\alpha K + \mathcal{L}_\beta\gamma \tag{44}$$

$$\partial_t K = -\nabla\nabla\alpha - \kappa\alpha[\Pi + \frac{1}{2}(\tau - p)\gamma] + \alpha[\mathcal{R}ic(\gamma) + \mathrm{tr}K\;K - 2K^2] + \mathcal{L}_\beta K\,. \tag{45}$$

Here, γ and K are, respectively, the metric and the extrinsic curvature of the space-like slices whose normal vector is n; ∇ is the Levi-Civita connection of γ, and the Ricci tensor and scalar curvature of γ are denoted by $\mathcal{R}ic(\gamma)$ and $\mathcal{R}(\gamma)$, respectively; the trace operator associated with γ is denoted by tr, so that, $(\nabla\cdot K)_a \equiv (\mathrm{tr}\nabla K)_a \equiv \gamma^{ij}\nabla_i K_{ja}$ is the divergence of K with respect to γ. In the usual evolution formalism notation, n is written as $n = \alpha^{-1}(\frac{\partial}{\partial t} - \beta)$, where α is the lapse function and β is the shift vector.

The energy content $\mathcal{T} \equiv \{\tau, q, p, \Pi\}$ has been decomposed relatively to n, that is

$$\mathcal{T} = \tau n\otimes n + n\otimes q + q\otimes n + \Pi + p\gamma, \tag{46}$$

with $\tau \equiv \mathcal{T}(n,n)$, $q \equiv -\perp\mathcal{T}(n,\cdot)$, p and Π being the energy density, the energy flux, the mean pressure and the traceless anisotropic pressure as measured by n, respectively; $\perp$ is the projector on the 3-space orthogonal to n associated with the 3-metric $\gamma \equiv g + n\otimes n$.

6.1. Spherical symmetry

In the case of a SSST, using the expression (8) of the extrinsic curvature, the constraint equations (42) and (43) are equivalent to

$$\Phi(\Phi + 2\Psi) = \kappa\tau - \frac{\mathcal{R}}{2} \tag{47}$$

$$2\partial_R\Phi + \frac{\partial_R D}{D}(\Phi - \Psi) = -\,\kappa q_R \tag{48}$$

where q_R is now the radial component of the energy flux. For the evolution equation (45), taking into account the expression (11), we have $\Pi_{\varphi\varphi} = \Pi_{\theta\theta}\sin^2\theta$ and

$$\begin{aligned}\partial_T(\Psi B) \;&=\; -\sqrt{B}\partial_R\left(\frac{\partial_R\alpha}{\sqrt{B}}\right) - \kappa\alpha\Big(\Pi_{RR} + \frac{1}{2}(\tau - p)B\Big)\\ &\quad +\alpha\Big(\frac{B}{2}\mathcal{R} - \frac{B}{D}F + B\Psi(2\Phi - \Psi)\Big) + \frac{B}{C}\partial_R\left[\left(\frac{C}{B}\right)^2 B\Psi\right]\end{aligned} \tag{49}$$

$$\begin{aligned}\partial_T(\Phi D) \;&=\; -\frac{\partial_R D}{2B}\partial_R\alpha - \kappa\alpha\Big(\Pi_{\theta\theta} + \tfrac{1}{2}(\tau - p)D\Big)\\ &\quad +\alpha\Big(\frac{D}{4}\mathcal{R} + \frac{F}{2} + \Phi\Psi D\Big) + \frac{C}{B}\partial_R(D\Phi).\end{aligned} \tag{50}$$

For the metric (1), Eqs. (47), (48), (49) and (50) are the $3+1$ splitting of the Einstein equations with respect to a vorticity free observer. The proper space of such an observer is Euclidean if, and only if, $F = 0$, and then $\mathcal{R} = 0$. The integration of these equations for simple energetic contents (for instance, a dust model) should provide the corresponding metric form in PG coordinates. In the next section, the vacuum case is considered: the extended form of Schwarzschild solution in PG coordinates is obtained from the sole consideration of the field equations.

6.2. Schwarzschild vacuum solution

The extended Painlevé-Gullstrand-Lemaître metric form of the Schwarzschild solution may be obtained assuming spherical symmetry and the existence of a flat synchronization, $\mathcal{R}ic(\gamma) = 0$, and then, solving the vacuum Einstein equations in a coordinate system adapted to such a synchronization. So, let us take $\tau = p = \Pi_{RR} = \Pi_{\theta\theta} = F = \mathcal{R} = 0$. Then, for the metric expression (33), the lapse function is given by $\alpha^2 = \mathcal{B}^2 - \mathcal{A}$ and the constraint equations (47) and (48) result

$$\Phi(\Phi + 2\Psi) \;= 0 \tag{51}$$

$$r\Phi' + \Phi - \Psi = 0 \tag{52}$$

with

$$\Phi = \frac{1}{\alpha}\frac{\mathcal{B}}{r}, \qquad \Psi = \frac{\mathcal{B}'}{\alpha}\,. \tag{53}$$

When $\Phi = 0$, taking into account also the evolution equations, we recover the Minkowski space-time. In the generic case, $\Phi = -2\Psi \neq 0$, Eq. (53) leads to

$$\mathcal{B} = f(t)r^{-1/2} \tag{54}$$

with $f(t)$ an arbitrary function. Substituting Eq. (54) in the momentum constraint (52), it reduces to $\alpha' = 0$. Consequently, the lapse is a function of the sole variable t, $\alpha(t)$, and we can take $\alpha = 1$ by re-scaling the coordinate t. Then, we have

$$\Phi = f(t)r^{-3/2} = -2\Psi. \tag{55}$$

Next, the evolution equations (49) and (50) are written as

$$\dot{\Psi} = \Psi(2\Phi - \Psi) + \frac{1}{\mathcal{B}}(\mathcal{B}^2\Psi)'$$

$$\dot{\Phi} = \Psi\Phi + \frac{\mathcal{B}}{r^2}(r^2\Phi)'.$$

Given that $\Phi = -2\Psi$, these last equations are equivalent to

$$\dot{\Psi} = \Psi^2 + \frac{\mathcal{B}}{r^2}(r^2\Psi)' \tag{56}$$

$$3\Psi + \frac{\mathcal{B}}{r} - \mathcal{B}' = 0. \tag{57}$$

By using the expressions (54) and (55), the equation (56) leads to $f(t) = constant$ and the equation (57) is identically satisfied. Finally, by taking $f = \epsilon\sqrt{2m}$, we obtain

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + 2\epsilon\sqrt{\frac{2m}{r}}\, dt\, dr + dr^2 + r^2 d\Omega^2, \tag{58}$$

which is the extended form of the Schwarzschild solution obtained by Painlevé, Gullstrand and Lemaître [1, 2, 3]. The positive parameter m is the Schwarzschild energy. The sign ϵ provides two coordinate branches for the solution, the Kruskal-Szekeres black and white hole regions being described by the above metric with $\epsilon = 1$ and $\epsilon = -1$, respectively, see [21]. Note that the r coordinate can take any positive value, $0 < r < +\infty$. In fact, from (58), we have $(dr)^2 = g^{rr} = 1 - \frac{2m}{r} < 1$, and the domain of a Painlevé-Gullstrand chart extends for every value of $r \neq 0$. Notice that (29) implies that $E = m$, which provides the physical interpretation of the parameter m as an effective energy [50]. Moreover, writing $dt = dt_S + \epsilon\sqrt{\frac{2m}{r}}(1 - \frac{2m}{r})^{-1}dr$, one recovers the usual Schwarzschild metric form

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt_S^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{59}$$

where t_S is the coordinate time of the static observer ($-\infty < t_S < \infty$) and the rank of the r coordinate is restricted to be $r > 2m$. According to the Jebsen-Birkhoff theorem (see [57] an references therein), we recover the Schwarzschild metric as the sole spherically symmetric solution of the vacuum Einstein equations.

Other derivations of the Schwarzschild solution providing improvements of the original proof of the Jebsen-Birkhoff theorem have been achieved by solving the field equations in null coordinates (see [50, 58] and reference therein). From the conceptual point of view, any of these derivations make unnecessary to get a coordinate transformation allowing to extend the domain of Schwarzschild chart from the outer to the inner horizon regions.

7. Discussion

In this work we have analyzed the existence of flat synchronizations in SSSTs. The condition (27) provides an upper bound for the norm of the mean extrinsic curvature vector of the isometry group orbits which, using the Gauss identity, may be expressed in terms of curvature invariants. Moreover, the associated flat slices have vanishing Weinberg energy and momentum densities. We have seen that any spherically symmetric metric admits a Newtonian interpretation in the context of the Bel extended Newtonian theory of gravitation. Our study offers a new perspective about the meaning of the Painlevé-Gullstrand coordinates. This study applies for any SSST in the region where these coordinates exist. In this region, the gradient of the radial Painlevé-Gullstrand coordinate r may be space-like, light-like, or time-like, according to the condition $(dr)^2 \leq 1$, which means that the Misner-Sharp gravitational energy of a sphere of radius r is non-negative. This condition may be tested for any SSST, starting from the general metric form (1). For instance, it occurs elsewhere in the Schwarzschild geometry, as it has been pointed out at the end of Sec. 6. Moreover, one has that $(dr)^2 \leq 1$ everywhere for any Robertson-Walker metric with an energetic content which satisfies the usual energy conditions. In fact, if we put in (1) $A = -1$, $B = a^2(t)/(1 + \frac{k}{4}r^2)^2$ (with $k = 1, 0, -1$ the universe curvature index), $C = 0$ and $D = r^2 B$, we obtain that, in this case, (27) is equivalent to $k + \dot{a}^2 \geq 0$, which means that the proper energy density of the cosmological fluid is non negative. Consequently, any Robertson-Walker space-time that satisfies this energy condition admits a PG synchronization. This property is also obtained directly from the inequality (28). In this case, $\tau = -e_1$ and $p = e_2 = e$ are, respectively, the energy density and the pressure of the cosmological fluid, and $\lambda = 0$, because the Robertson-Walker metric is conformally flat. We leave for a future work the obtaining of the Painlevé-Gullstrand form of these Robertson-Walker cosmological models.

Finally, we have presented an improved proof of the Jebsen-Birkhoff theorem by expressing and solving the vacuum Einstein Equations in PG-coordinates. So, the extended Painlevé-Gullstrand-Lemaître metric form of the Schwarzschild solution is directly obtained.

Acknowledgments

We appreciate valuable aids provided by the tandem Joan J. Ferrando–Juan A. Sáez during the development of this paper, and the orientational comment by Marc Mars during the presentation of this subject at the Spanish Relativity Meeting ERE-2009. This work has been supported by the Spanish Ministerio de Ciencia e Innovación MICIN-FEDER project FIS2009-07705.

References

[1] Painlevé P 1921 *C R Acad. Sc. Paris* **173** 677

[2] Gullstrand A 1922 *Archiv. Mat. Astron. Fys.* **16** 1

[3] Lemaître G 1933 *Ann. Soc. Sci. Bruxeles* **A 53** 51. Reprinted 1997 *Gen. Rel. Grav.* **29** 641

[4] Martel K and Poisson E 2001 *Am. J. Phys.* **69** 476

[5] Hamilton A J S and Lisle J P 2008 *Am. J. Phys.* **76** 519

[6] Eardley D M 1985 *Dynamical Spacetimes and Numerical Relativity* edited by J. M. Centrella (Cambridge: Cambridge University Press)

[7] Papadopoulos P and Font J A 1998 *Phys. Rev. D* **58** 024005

[8] Adler R J, Bjorken J D, Chen P and Liu J S 2005 *Am. J. Phys.* **73** 1148

[9] Ziprick J and Kunstatter G 2009 *Phys. Rev. D* **79** 101503(R)

[10] Guven J and Murchadha N Ó 1999 *Phys. Rev. D* **60** 104015

[11] Husain V, Qadir A and Siddiqui A A 2001 *Phys. Rev. D* **65** 027501

[12] Lichnerowicz A 1967 *Relativistic Hydrodymamics and Magnetohydrodynamics.* (Benjamin Inc., New York). See, for instance, p 36 and p 54

[13] Unruh W G 1981 *Phys. Rev. Lett.* **46** 1351

[14] Visser M 1998 *Class. Quantum Grav.* **15** 1767

[15] Bilić N 1999 *Class. Quantum Grav.* **16** 3953

[16] Barceló C, Liberati S, Sonego S and Visser M 2004 *New J Phys.* **6** 186

[17] Barceló C, Liberati S and Visser M 2005 Analogue Gravity *Living Rev. Rel.* **8** n^{o}12

[18] Perez Bergliaffa S E 2004 *Int. J. Mod. Phys. D* **13** 1469

[19] Kraus P and Wilczek F 1994 *Mod. Phys. Lett. A* **9** 3713

[20] Parikh M K and Wilczek F 2000 *Phys. Rev. Lett.* **85** 5042

[21] Schützhold R 2001 *Phys. Rev. D* **64** 024029

[22] Husain V and Winkler O 2005 *Phys. Rev. D* **71** 104001

[23] Visser M 2005 *Int. J. Mod. Phys. D* **14** 2051

[24] Nielsen A B and Visser M 2006 *Class. Quantum Grav.* **23** 4637

[25] Morales Lladosa J A 2006 Coordinates and frames from the causal point of view *A Century of Relativity Physics Proceeding of the XXVIII Spanish Relativity Meeting ERE-2005* ed Mornas L and Diaz Alonso J (AIP Conf. Proc No **841** New York) pp 537-541 (arXiv: gr-qc/0601138)

[26] Morales Lladosa J A 2008 Newtonian and relativistic location systems *Relativistic Astrophysics and Cosmology Proceeding of the XXX Spanish Relativity Meeting ERE-2007* ed Oscoz A, Mediavilla E and Serra-Ricart M *EAS Publications Series* Vol. **30** 841 (EAS EDP Sciences France) pp 119-124

[27] Beig R and Siddiqui A A 2007 *Class. Quantum Grav.* **24** 5435

[28] Qadir A and Siddiqui A A 2006 *Int. J. Mod. Phys. D* **15** 1419

[29] Lin C-Y and Soo C 2009 *Phys. Lett. B* **671** 493

[30] Lake K A class of quasi-stationary regular line elements for Schwarzschild geometry (arXiv:gr-qc/9407005)

[31] Doran C 2000 *Phys. Rev. D* **61** 067503

[32] Natário J 2009 *Gen. Rel. Grav.* **41** 2579

[33] Lasky P D and Lun A W C 2006 *Phys. Rev. D* **74** 084013. 2007 *Phys. Rev. D* **75** 024031. 2007 *Phys. Rev. D* **75** 104010

[34] Lasky P D, Lun A W C and Burston R B 2007 *The Anziam J.* **49** 53

[35] Weinberg S 1972 *Gravitation and Cosmology* (Wiley New York)

[36] Bel L 1990 Rigid motion invariance of Newtonian and Einstein's theories of General Relativity *Recent developments in gravitation Proceedings Relativity Meeting-89* ed Verdaguer E, Garriga J and Céspedes J (World Scientific Singapore) pp 47-119

[37] Herrero A and Morales-Lladosa J A 2010 Flat synchronizations in spherically symmetric space-times *Gravitation in the Large Proceeding XXXII Spanish Relativity Meeting ERE-2009* ed Lazkoz R and Vera R. To be published in *J. Phys. Conf. Series*

[38] Ferrando J J and Sáez J A *An intrinsic characterization of 2+2 warped spacetimes* arXiv:1005.1491 [gr-qc]

[39] Ferrando J J and Sáez J A *An intrinsic characterization of spherically symmetric spacetimes* arXiv:1005.1780 [gr-qc]

[40] Synge J L 1966 *Relativity: The General Theory* (Amsterdam: North-Holland Pub.)

[41] Takeno H 1966 The Theory of Spherically Symmetric Space-Times (Revised ed.) *Scientific Reports of the Research Institute for Theoretical Physics Hiroshima University No. 5*

[42] Plebański J and Krasiński A 2006 *An Introduction to General Relativity and Cosmology* (Cambridge: Cambridge University Press)

[43] García A A, Hehl F W, Heinicke C and Macías A 2004 *Class. Quantum Grav.* **21** 1099

[44] Ferrando J J and Sáez J A 2004 *J. Math. Phys. (N Y)* **45** 652

[45] Senovilla J M M 2007 *Class. Quantum Grav.* **24** 3091

[46] Mars M 2009 *Class. Quantum Grav.* **29** 193001

[47] Senovilla J M M 2002 *Class. Quantum Grav.* **19** L113

[48] Plebański J and Stachel J 1968 *J. Math. Phys. (N Y)* **9** 269

[49] Bona C, Coll B and Morales J A 1992 *J. Math. Phys. (N Y)* **33** 670

[50] Hayward S A 1996 *Phys. Rev. D* **53** 1938

[51] Misner C W and Sharp D H 1964 *Phys. Rev.* **136** B571

[52] Hayward S A 2010 *Phys. Rev. D* **81** 024037

[53] Mirshekari S and Abbassi A M 2009 *Mod. Phys. Lett.* **A24** 747

[54] Coll B 1980 *Isometries et conditions initials en Relativité Général* Thèse d' état (Paris: Univ. Pierre et Marie Curie)

[55] Bona C and Coll B 1985 *J. Math. Phys. (N Y)* **26** 1583

[56] Gourgoulhon É *3+1 formalism and bases of Numerical Relativity* (arXiv:gr-qc/0703035)

[57] Johansen N V and Ravndal F 2006 *Gen. Rel. Grav.* **38** 537

[58] Lake K 2006 *Class. Quantum Grav.* **23** 5883. See also this related preprint arXiv:gr-qz/1002.3600