Visco-penalization of the sum of two monotone operators

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Abstract

A new type of approximating curve for finding a particular zero of the sum of two maximal monotone operators in a Hilbert space is investigated. This curve consists of the zeros of perturbed problems in which one operator is replaced with its Yosida approximation and a viscosity term is added. As the perturbation vanishes, the curve is shown to converge to the zero of the sum that solves a particular strictly monotone variational inequality. As an offspring of this result, we obtain an approximating curve for finding a particular zero of the sum of several maximal monotone operators. Applications to convex optimization are discussed.

1 Problem statement

A central problem which arises in various areas of nonlinear analysis and its applications is the inclusion problem

$$\text{find} \quad x \in \operatorname{zer}(A+B) = \big\{z \in \mathcal{H} \mid 0 \in Az + Bz\big\}, \tag{1.1}$$

where A and B are maximal monotone operators from a real Hilbert space $\mathcal{H}$ to its power set $2^{\mathcal{H}}$, e.g., [6, 13, 15, 17, 24, 26]. In many instances, (1.1) admits multiple solutions and one can select a particular point $x_0 \in \operatorname{zer}(A+B)$ by solving the variational inequality

$$x_0 \in \operatorname{zer}(A+B) \quad \text{and} \quad (\exists\, v_0 \in Vx_0)(\forall z \in \operatorname{zer}(A+B)) \quad \langle x_0 - z \mid v_0\rangle \leq 0, \tag{1.2}$$

where $V\colon \mathcal{H} \to 2^{\mathcal{H}}$ is a strictly monotone operator referred to as a viscosity operator. Bringing into play the normal cone operator (see (1.8)), we can conveniently rewrite (1.2) as

$$0 \in N_{\mathrm{zer}(A+B)} x_0 + V x_0. \tag{1.3}$$

We shall investigate the problem of solving (1.3) under the following standing assumptions (see Section 1.1 for notation).

Assumption 1.1

(i) A and B are maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ such that $A+B$ is maximal monotone and $\mathrm{zer}(A+B) \neq \varnothing$.

(ii) $V\colon \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator which satisfies the following properties.

  (a) V is uniformly monotone in the sense that there exists an increasing function $c\colon [0,+\infty[\to [0,+\infty[$ that vanishes only at 0 such that $\lim_{t\to+\infty} c(t)/t = +\infty$ and

$$(\forall (x,u) \in \operatorname{gr} V)(\forall (y,v) \in \operatorname{gr} V) \quad \langle x-y \mid u-v\rangle \geq c(\|x-y\|). \tag{1.4}$$

  (b) V maps every bounded subset of $\mathcal{H}$ into a bounded set.

It follows from [14, Theorem 3.10] that, under Assumption 1.1, the solution x_0 to (1.3) is uniquely defined and so is the approximating curve $(x_\varepsilon)_{\varepsilon\in]0,1[}$ defined by

$$(\forall \varepsilon \in]0,1[) \quad 0 \in Ax_\varepsilon + Bx_\varepsilon + \varepsilon V x_\varepsilon. \tag{1.5}$$

Moreover, $x_\varepsilon \to x_0$ when $\varepsilon \downarrow 0$ (historically, the earliest result in this direction was obtained in [11] with $A = 0$ and $V = \mathrm{Id}$, in which case x_0 is the zero of B of minimum norm). The asymptotic behavior of approximating curves plays a central role in proving the convergence of parent discrete or continuous dynamical systems for solving (1.3), e.g., [1, 5, 11, 12, 21]. However, inclusions involving, as in (1.5), several set-valued operators are not easily dealt with and neither are the associated dynamical systems. A common relaxation of Problem (1.1) is obtained by replacing A with its Yosida approximation (see (1.7)), which is a better-behaved, single-valued, Lipschitz continuous operator. In the context of discrete dynamical systems, such relaxations lead to splitting algorithms that have been studied in several places, e.g., [8, 13, 16, 17]. The objective of the present paper is to investigate the asymptotic behavior of an approximating curve obtained by replacing A with Yosida approximations in (1.5). More precisely, our main result (Theorem 3.1) establishes the strong convergence to the solution x_0 to (1.3) of the inexact approximating curve $(x_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$ defined by

$$(\forall \varepsilon \in]0,1[) \quad 0 \in {}^{\phi(\varepsilon)}A x_{\varepsilon,\phi(\varepsilon)} + Bx_{\varepsilon,\phi(\varepsilon)} + \varepsilon V x_{\varepsilon,\phi(\varepsilon)} + e_\varepsilon, \tag{1.6}$$

under suitable conditions on the function $\phi\colon]0,1[\to]0,1[$ and the error process $(e_\varepsilon)_{\varepsilon\in]0,1[}$.

The outline of the remainder of the paper is as follows. In Section 2, we provide the preliminary results that will be required to obtain our main result on the asymptotic behavior of (1.6) in Section 3. Finally, in Section 4, we address the case of more than two operators. Applications to convex optimization are discussed.

1.1 Notation

Throughout, $\mathcal{H}$ is a real Hilbert space with scalar product $\langle \cdot \mid \cdot \rangle$, norm $\|\cdot\|$, and identity operator Id. The symbols $\to$ and $\rightharpoonup$ denote, respectively, strong and weak convergence.

Let $M\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. Then $\operatorname{dom} M = \big\{x \in \mathcal{H} \mid Mx \neq \varnothing\big\}$ is the domain of M, $\operatorname{ran} M = \big\{u \in \mathcal{H} \mid (\exists\, x \in \mathcal{H})\ u \in Mx\big\}$ its range, $\operatorname{zer} M = \big\{x \in \mathcal{H} \mid 0 \in Mx\big\}$ its set of zeros, and $\operatorname{gr} M = \big\{(x,u) \in \mathcal{H} \times \mathcal{H} \mid u \in Mx\big\}$ its graph. The inverse of M is the operator $M^{-1}\colon \mathcal{H} \to 2^{\mathcal{H}}$ with graph $\big\{(u,x) \in \mathcal{H} \times \mathcal{H} \mid u \in Mx\big\}$, the resolvent of M is $J_M = (\mathrm{Id} + M)^{-1}$, and the Yosida approximation of M of index $\phi \in \left]0,+\infty\right[$ is

$$^{\phi}M = \frac{1}{\phi}(\mathrm{Id} - J_{\phi M}). \tag{1.7}$$

Moreover, M is γ-strongly monotone for some $\gamma \in \left]0,+\infty\right[$ if $M - \gamma\,\mathrm{Id}$ is monotone. For background on monotone operators, see [7] and [26].

The projection operator onto a nonempty closed convex subset C of $\mathcal{H}$ is denoted by P_C, its distance function by d_C, and its normal cone operator by N_C, i.e.,

$$N_C\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \begin{cases} \big\{u \in \mathcal{H} \mid (\forall y \in C)\ \ \langle y - x \mid u\rangle \leq 0\big\}, & \text{if } x \in C;\\ \varnothing, & \text{otherwise.}\end{cases} \tag{1.8}$$

A function $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ is proper if $\operatorname{dom} f = \big\{x \in \mathcal{H} \mid f(x) < +\infty\big\} \neq \varnothing$; in this case, its subdifferential is

$$\partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \ \langle y - x \mid u\rangle + f(x) \leq f(y)\big\}. \tag{1.9}$$

Moreover, f is γ-strongly convex for some $\gamma \in \left]0,+\infty\right[$ if $f - \gamma\|\cdot\|^2/2$ is convex. The class of proper lower semicontinuous convex functions from $\mathcal{H}$ to $\left]-\infty,+\infty\right]$ is denoted by $\Gamma_0(\mathcal{H})$. Now let $f \in \Gamma_0(\mathcal{H})$. The conjugate of f is the function $f^* \in \Gamma_0(\mathcal{H})$ defined by $f^*\colon u \mapsto \sup_{x\in\mathcal{H}} \langle x \mid u\rangle - f(x)$ and the Moreau envelope of index $\phi \in \left]0,+\infty\right[$ of f is the finite and continuous convex function $^{\phi}f\colon x \mapsto \inf_{y\in\mathcal{H}} f(y) + \|x-y\|^2/(2\phi)$. For every $x \in \mathcal{H}$, the function $y \mapsto f(y) + \|x-y\|^2/2$ admits a unique minimizer, which is denoted by $\operatorname{prox}_f x$. We have $\operatorname{prox}_f = J_{\partial f}$ and [18]

$$(\forall \phi \in \left]0,+\infty\right[)\quad \nabla\big({}^{\phi}f\big) = {}^{\phi}(\partial f) = \frac{1}{\phi}(\mathrm{Id} - \operatorname{prox}_{\phi f}) = \operatorname{prox}_{f^*/\phi}(\cdot/\phi). \tag{1.10}$$

For background on convex analysis, see [25].

2 Preliminary results

Lemma 2.1 *Let u and v be points in $\mathcal{H}$, and let ϕ and ρ be real numbers in $\left[0,+\infty\right[$. Then*

$$\langle \phi u - \rho v \mid v - u\rangle \leq \frac{1}{4}\big(\phi\|v\|^2 + \rho\|u\|^2\big). \tag{2.1}$$

Proof. We have $0 \leq \|2u - v\|^2 = 4\|u\|^2 - 4\langle u \mid v\rangle + \|v\|^2$. Hence, $\langle \phi u \mid v\rangle \leq \phi(\|u\|^2 + \|v\|^2/4)$. Likewise, $\langle \rho v \mid u\rangle \leq \rho(\|v\|^2 + \|u\|^2/4)$. Adding these two inequalities yields (2.1). ☐

Lemma 2.2 *Let $M\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator and suppose that Assumption* 1.1(ii) *is satisfied. Then the inclusion $0 \in Mx + Vx$ possesses exactly one solution.*

Proof. Assumption 1.1(ii)(b) implies that V is locally bounded. Consequently, it results from [26, Theorem 32.G] that V^{-1} is surjective. Thus,

$$\operatorname{dom} M \subset \operatorname{dom} V = \operatorname{ran} V^{-1} = \mathcal{H} \tag{2.2}$$

and, since [26, Theorem 32.I] implies that $M + V$ is maximal monotone, the conclusion follows from [14, Lemma 3.8(ii)]. ☐

Definition 2.3 [14, Definition 3.1] *Let $(M_\varepsilon)_{\varepsilon\in]0,1[}$ be a family of maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ and let $(x_\varepsilon)_{\varepsilon\in]0,1[}$ be a family in $\mathcal{H}$. Then $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is* $\mathcal{A}$-focused *with respect to $(M_\varepsilon)_{\varepsilon\in]0,1[}$ if, for every $x \in \mathcal{H}$ and every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $]0,1[$ such that $\varepsilon_n \downarrow 0$,*

$$\big[\, x_{\varepsilon_n} \rightharpoonup x \quad \textit{and} \quad {}^1M_{\varepsilon_n}x_{\varepsilon_n} \to 0 \,\big] \quad \Rightarrow \quad (\forall \varepsilon \in\,]0,1[)\;\; x \in \operatorname{zer} M_\varepsilon. \tag{2.3}$$

The following result is an extension of [14, Theorem 3.10] which allows for inexact inclusions.

Theorem 2.4 *Let $(M_\varepsilon)_{\varepsilon\in]0,1[}$ be a family of maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ such that $C = \bigcap_{\varepsilon\in]0,1[} \operatorname{zer} M_\varepsilon \neq \varnothing$, and suppose that Assumption* 1.1(ii) *is satisfied. Then there exists a unique point $x_0 \in C$ such that $0 \in N_C x_0 + V x_0$. Moreover, the inclusions*

$$(\forall \varepsilon \in\,]0,1[) \quad 0 \in M_\varepsilon x_\varepsilon + \varepsilon V x_\varepsilon + e_\varepsilon, \quad \textit{where} \quad e_\varepsilon \in \mathcal{H}, \tag{2.4}$$

define a unique family $(x_\varepsilon)_{\varepsilon\in]0,1[}$. Now suppose that $\|e_\varepsilon\|/\varepsilon \to 0$ as $\varepsilon \downarrow 0$ and that $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is $\mathcal{A}$-focused with respect to $(M_\varepsilon)_{\varepsilon\in]0,1[}$. Then $x_\varepsilon \to x_0$ as $\varepsilon \downarrow 0$.

Proof. By maximal monotonicity of the operators $(M_\varepsilon^{-1})_{\varepsilon\in]0,1[}$, the sets $(\operatorname{zer} M_\varepsilon)_{\varepsilon\in]0,1[}$ are closed and convex [7, Proposition 3.5.6.1]. Thus C is nonempty, closed, and convex, and N_C is therefore maximal monotone [26, Example 32.15]. Consequently, it follows from Lemma 2.2 that x_0 is uniquely defined. Since the operators $\big(\varepsilon^{-1}(M_\varepsilon + e_\varepsilon)\big)_{\varepsilon\in]0,1[}$ are maximal monotone, it also follows from Lemma 2.2 that the family $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is uniquely defined. The same argument also shows that the auxiliary family of inclusions

$$(\forall \varepsilon \in\,]0,1[) \quad 0 \in M_\varepsilon y_\varepsilon + \varepsilon V y_\varepsilon \tag{2.5}$$

defines a unique approximating curve $(y_\varepsilon)_{\varepsilon\in]0,1[}$. Moreover, [14, Theorem 3.10] asserts that

$$(y_\varepsilon)_{\varepsilon\in]0,1[} \text{ is } \mathcal{A}\text{-focused with respect to } (M_\varepsilon)_{\varepsilon\in]0,1[} \;\Rightarrow\; y_\varepsilon \to x_0 \text{ as } \varepsilon \downarrow 0. \tag{2.6}$$

It follows from (2.4) and (2.5) that, for every $\varepsilon \in\,]0,1[$, there exist points $v_\varepsilon \in V x_\varepsilon$ and $w_\varepsilon \in V y_\varepsilon$ such that

$$-\varepsilon v_\varepsilon - e_\varepsilon \in M_\varepsilon x_\varepsilon \quad \text{and} \quad -\varepsilon w_\varepsilon \in M_\varepsilon y_\varepsilon. \tag{2.7}$$

Using the monotonicity of the operators $(M_\varepsilon)_{\varepsilon\in]0,1[}$ and the uniform monotonicity of V, we obtain

$$(\forall \varepsilon \in\,]0,1[) \quad 0 \leq \langle x_\varepsilon - y_\varepsilon \mid w_\varepsilon - \varepsilon^{-1} e_\varepsilon - v_\varepsilon \rangle \tag{2.8}$$

and

$$(\forall \varepsilon \in\,]0,1[) \quad c(\|x_\varepsilon - y_\varepsilon\|) \leq \langle x_\varepsilon - y_\varepsilon \mid v_\varepsilon - w_\varepsilon \rangle, \tag{2.9}$$

respectively. Adding (2.8) to (2.9), and then using Cauchy-Schwarz, we obtain

$$(\forall \varepsilon \in\,]0,1[) \quad c(\|x_\varepsilon - y_\varepsilon\|) \leq -\frac{1}{\varepsilon}\langle x_\varepsilon - y_\varepsilon \mid e_\varepsilon \rangle \leq \|x_\varepsilon - y_\varepsilon\| \frac{\|e_\varepsilon\|}{\varepsilon}. \tag{2.10}$$

Now suppose that $\|e_\varepsilon\|/\varepsilon \to 0$ as $\varepsilon \downarrow 0$. Then it follows from (2.10) that there exists $\beta \in\,]0,+\infty[$ such that

$$(\forall \varepsilon \in\,]0,1[) \quad x_\varepsilon \neq y_\varepsilon \quad \Rightarrow \quad \frac{c(\|x_\varepsilon - y_\varepsilon\|)}{\|x_\varepsilon - y_\varepsilon\|} \leq \beta. \tag{2.11}$$

Hence, since $\lim_{t\to+\infty} c(t)/t = +\infty$, we infer from (2.11) that $(\|x_\varepsilon - y_\varepsilon\|)_{\varepsilon\in]0,1[}$ is bounded and, in turn, from (2.10) that

$$\|x_\varepsilon - y_\varepsilon\| \to 0 \quad \text{as} \quad \varepsilon \downarrow 0. \tag{2.12}$$

In addition, since the operators $({}^1M_\varepsilon)_{\varepsilon\in]0,1[}$ are Lipschitz continuous [7, Theorem 3.5.9(ii)], we obtain $\|{}^1M_\varepsilon x_\varepsilon - {}^1M_\varepsilon y_\varepsilon\| \to 0$ as $\varepsilon \downarrow 0$. Altogether, if $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is $\mathcal{A}$-focused with respect to $(M_\varepsilon)_{\varepsilon\in]0,1[}$, $(y_\varepsilon)_{\varepsilon\in]0,1[}$ is likewise. In view of (2.6) and (2.12), we conclude that $x_\varepsilon \to x_0$ as $\varepsilon \downarrow 0$. □

The following theorem, which is of interest in its own right, will also be required. It is a natural extension of the well-known Brézis-Crandall-Pazy condition [9].

Theorem 2.5 *Let M_1 and M_2 be maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$. Suppose that Assumption* 1.1(ii) *is satisfied and consider the inclusions*

$$(\forall \rho \in\,]0,1[) \quad 0 \in {}^\rho M_1 z_\rho + M_2 z_\rho + V z_\rho. \tag{2.13}$$

Then the following hold.

(i) *The family $(z_\rho)_{\rho\in]0,1[}$ is uniquely defined.*

(ii) *The following conditions are equivalent:*

  (a) *There exists a unique point $z_0 \in \mathcal{H}$ such that $0 \in M_1 z_0 + M_2 z_0 + V z_0$.*

  (b) *The family $({}^\rho M_1 z_\rho)_{\rho\in]0,1[}$ is bounded.*

(iii) *If one of the conditions in* (ii) *is satisfied, then $z_\rho \to z_0$ as $\rho \downarrow 0$.*

Proof. (i): It follows from [7, Theorem 3.5.9] that the operators $({}^\rho M_1)_{\rho\in]0,1[}$ are maximal monotone with domain $\mathcal{H}$. In turn, [26, Theorem 32.I] asserts that the operators $({}^\rho M_1 + M_2)_{\rho\in]0,1[}$ are maximal monotone. Thus, we obtain the desired conclusion through Lemma 2.2.

(ii): We first suppose that there exists a point $z_0 \in \mathcal{H}$ such that

$$0 \in M_1 z_0 + M_2 z_0 + V z_0. \tag{2.14}$$

Note that, since $M_1 + M_2 + V$ is strictly monotone, this point is necessarily unique. Now fix $\rho \in \,]0,1[$. We deduce from (2.14), (2.13), and (i) that there exist $u_0 \in M_1 z_0$, $v_0 \in V z_0$, and $v_\rho \in V z_\rho$ such that

$$-u_0 - v_0 \in M_2 z_0 \quad \text{and} \quad -\,{}^{\rho}M_1 z_\rho - v_\rho \in M_2 z_\rho. \tag{2.15}$$

Hence, the monotonicity of M_2 yields

$$\langle z_\rho - z_0 \mid u_0 - {}^{\rho}M_1 z_\rho \rangle \geq \langle z_\rho - z_0 \mid v_\rho - v_0 \rangle, \tag{2.16}$$

and, in view of the monotonicity of V, we obtain

$$\langle z_\rho - z_0 \mid u_0 - {}^{\rho}M_1 z_\rho \rangle \geq 0. \tag{2.17}$$

On the other hand, the inclusions $u_0 \in M_1 z_0$ and ${}^{\rho}M_1 z_\rho \in M_1(J_{\rho M_1} z_\rho)$, together with the monotonicity of M_1, lead to the inequality

$$\langle z_0 - J_{\rho M_1} z_\rho \mid u_0 - {}^{\rho}M_1 z_\rho \rangle \geq 0. \tag{2.18}$$

Adding (2.17) to (2.18) results in

$$0 \leq \langle z_\rho - J_{\rho M_1} z_\rho \mid u_0 - {}^{\rho}M_1 z_\rho \rangle = \rho\big(\langle {}^{\rho}M_1 z_\rho \mid u_0 \rangle - \| {}^{\rho}M_1 z_\rho \|^2\big). \tag{2.19}$$

Consequently, Cauchy-Schwarz yields

$$\| {}^{\rho}M_1 z_\rho \| \leq \| u_0 \|. \tag{2.20}$$

Conversely, suppose that (ii)(b) is satisfied, i.e., there exists $\beta \in \,]0,+\infty[$ such that

$$\sup_{\rho \in]0,1[} \| {}^{\rho}M_1 z_\rho \| \leq \beta. \tag{2.21}$$

It suffices to show the existence of a point $z_0 \in \mathcal{H}$ such that $0 \in M_1 z_0 + M_2 z_0 + V z_0$ as its uniqueness will follow from the strict monotonicity of $M_1 + M_2 + V$. Let us first prove that $\|z_\varepsilon - z_\rho\| \to 0$ as $\varepsilon \downarrow 0$ and $\rho \downarrow 0$. To this end, take ε and ρ in $]0,1[$. By (2.13) and (i), there exist $v_\varepsilon \in V z_\varepsilon$ and $v_\rho \in V z_\rho$ such that

$$-\,{}^{\varepsilon}M_1 z_\varepsilon - v_\varepsilon \in M_2 z_\varepsilon \quad \text{and} \quad -\,{}^{\rho}M_1 z_\rho - v_\rho \in M_2 z_\rho. \tag{2.22}$$

On the one hand, the monotonicity of M_2 and the uniform monotonicity of V yield

$$c(\|z_\varepsilon - z_\rho\|) \leq \langle z_\varepsilon - z_\rho \mid {}^{\rho}M_1 z_\rho - {}^{\varepsilon}M_1 z_\varepsilon \rangle. \tag{2.23}$$

On the other hand, it follows from the monotonicity of M_1 and the inclusions ${}^{\varepsilon}M_1 z_\varepsilon \in M_1(J_{\varepsilon M_1} z_\varepsilon)$ and ${}^{\rho}M_1 z_\rho \in M_1(J_{\rho M_1} z_\rho)$ that

$$0 \leq \langle J_{\rho M_1} z_\rho - J_{\varepsilon M_1} z_\varepsilon \mid {}^{\rho}M_1 z_\rho - {}^{\varepsilon}M_1 z_\varepsilon \rangle. \tag{2.24}$$

Adding (2.23) to (2.24), and then using Lemma 2.1 and (2.21), we obtain

$$\begin{aligned} c(\|z_\varepsilon - z_\rho\|) &\leq \langle \varepsilon({}^\varepsilon M_1 z_\varepsilon) - \rho({}^\rho M_1 z_\rho) \mid {}^\rho M_1 z_\rho - {}^\varepsilon M_1 z_\varepsilon\rangle \\ &\leq \frac{1}{4}(\varepsilon+\rho)\beta^2. \end{aligned} \tag{2.25}$$

Thus,

$$\|z_\varepsilon - z_\rho\| \to 0 \quad \text{as} \quad \varepsilon \downarrow 0 \quad \text{and} \quad \rho \downarrow 0. \tag{2.26}$$

Now let $(\rho_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $]0,1[$ such that $\rho_n \downarrow 0$ as $n \to +\infty$. We deduce from (2.26) that $(z_{\rho_n})_{n\in\mathbb{N}}$ is a Cauchy sequence. Hence, there exists a point $z_0 \in \mathcal{H}$ such that $z_{\rho_n} \to z_0$ as $n \to +\infty$. Let us show that $0 \in M_1 z_0 + M_2 z_0 + V z_0$. First, since the sequence $({}^{\rho_n}M_1 z_{\rho_n})_{n\in\mathbb{N}}$ is bounded, there exist a point $u \in \mathcal{H}$ and a subsequence $(\rho_{k_n})_{n\in\mathbb{N}}$ of $(\rho_n)_{n\in\mathbb{N}}$ such that ${}^{\rho_{k_n}}M_1 z_{\rho_{k_n}} \rightharpoonup u$. Since $z_{\rho_{k_n}} \to z_0$, using the fact that $({}^{\rho_{k_n}}M_1)_{n\in\mathbb{N}}$ graph-converges to M_1 (see [2, p. 360]) and applying [2, Proposition 3.59], we obtain

$$u \in M_1 z_0. \tag{2.27}$$

Furthermore, since $(z_{\rho_n})_{n\in\mathbb{N}}$ is bounded, so is $(v_{\rho_n})_{n\in\mathbb{N}}$ in the light of Assumption 1.1(ii)(b). Therefore, passing to a further subsequence if necessary, we assume that $v_{\rho_{k_n}} \rightharpoonup v$ for some $v \in \mathcal{H}$. Since V is maximal monotone, $\operatorname{gr} V$ is sequentially closed in $\mathcal{H}^{\text{strong}} \times \mathcal{H}^{\text{weak}}$ [7, Proposition 3.5.6.2] and therefore, recalling that $z_{\rho_{k_n}} \to z_0$, we get

$$v \in V z_0. \tag{2.28}$$

Likewise, since $-v_{\rho_{k_n}} - {}^{\rho_{k_n}}M_1 z_{\rho_{k_n}} \rightharpoonup -v-u$, it follows from (2.22) and the sequential closedness of $\operatorname{gr} M_2$ in $\mathcal{H}^{\text{strong}} \times \mathcal{H}^{\text{weak}}$ that

$$-v-u \in M_2 z_0. \tag{2.29}$$

Altogether, (2.27), (2.28), and (2.29) imply that $0 \in M_1 z_0 + M_2 z_0 + V z_0$.

(iii): Suppose that there exists $z_0 \in \mathcal{H}$ such that $0 \in M_1 z_0 + M_2 z_0 + V z_0$ and let $\rho \in]0,1[$. Then there exist $u_0 \in M_1 z_0$, $v_0 \in V z_0$, and $v_\rho \in V z_\rho$ such that (2.15) holds. In turn, (2.16) is satisfied and the uniform monotonicity of V leads to

$$c(\|z_\rho - z_0\|) \leq \langle z_\rho - z_0 \mid u_0 - {}^\rho M_1 z_\rho\rangle. \tag{2.30}$$

Adding this inequality to (2.18), and then using Cauchy-Schwarz and (2.20), we obtain

$$c(\|z_\rho - z_0\|) \leq \langle z_\rho - J_{\rho M_1} z_\rho \mid u_0 - {}^\rho M_1 z_\rho\rangle \leq \rho\big(\|{}^\rho M_1 z_\rho\|\,\|u_0\| - \|{}^\rho M_1 z_\rho\|^2\big) \leq \rho\|u_0\|^2. \tag{2.31}$$

We conclude that $z_\rho \to z_0$ as $\rho \downarrow 0$. $\square$

Remark 2.6 In particular, if $V = \operatorname{Id} - h$, where $h \in \mathcal{H}$, then Assumption 1.1(ii) is satisfied and Theorem 2.5 reduces to the Hilbert space version of results found in [9, Section 2].

3 The visco-penalization approximating curve

Theorem 3.1 *Suppose that Assumption* 1.1 *is satisfied. Then there exists a unique point* $x_0 \in \operatorname{zer}(A+B)$ *such that*

$$0 \in N_{\operatorname{zer}(A+B)}x_0 + Vx_0. \tag{3.1}$$

Moreover, given $\phi\colon\,]0,1[\, \to\,]0,1[$, *the inclusions*

$$(\forall \varepsilon \in\,]0,1[) \quad 0 \in {}^{\phi(\varepsilon)}\!Ax_{\varepsilon,\phi(\varepsilon)} + Bx_{\varepsilon,\phi(\varepsilon)} + \varepsilon Vx_{\varepsilon,\phi(\varepsilon)} + e_\varepsilon, \quad \text{where} \quad e_\varepsilon \in \mathcal{H}, \tag{3.2}$$

define a unique family $(x_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$. *Now suppose that* c *is continuous, that* $(\phi(\varepsilon)+\|e_\varepsilon\|)/\varepsilon \to 0$ *as* $\varepsilon \downarrow 0$, *and that one of the following holds:*

(i) $x_0 \in \operatorname{int}\operatorname{dom} A$.

(ii) $x_0 \in \operatorname{int}\operatorname{dom} B$.

(iii) *A and B satisfy the "angle property"*

$$(\exists\, \sigma_1 \in \mathbb{R})(\exists\, \sigma_2 \in [0,+\infty[)(\exists\, \sigma_3 \in [0,+\infty[)(\forall \rho \in\,]0,1[)(\forall (x,u) \in \operatorname{gr} B)$$
$$\langle {}^{\rho}\!Ax \mid u\rangle \geq -\|\sigma_1({}^{\rho}\!Ax) + \sigma_2 u\| - \sigma_3, \tag{3.3}$$

and one of the following holds:

(a) $\operatorname{dom} B$ *is bounded.*

(b) V *is Lipschitz continuous and strongly monotone.*

Then $x_{\varepsilon,\phi(\varepsilon)} \to x_0$ *as* $\varepsilon \downarrow 0$.

Proof. The set $\operatorname{zer}(A+B)$ is nonempty and, since $(A+B)^{-1}$ is maximal monotone, it is also closed and convex [7, Proposition 3.5.6.1]. Hence, $N_{\operatorname{zer}(A+B)}$ is maximal monotone [26, Example 32.15]. The existence and uniqueness of x_0 in (3.1) therefore follow from Lemma 2.2. On the other hand, arguing as in the proof of Theorem 2.5(i), we obtain the maximal monotonicity of the operators $\big(\varepsilon^{-1}({}^{\phi(\varepsilon)}\!A+B+e_\varepsilon)\big)_{\varepsilon\in]0,1[}$ and, in turn, the existence and uniqueness of $(x_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$ via Lemma 2.2. Using once again Lemma 2.2, we observe that the inclusions

$$(\forall \varepsilon \in\,]0,1[) \quad 0 \in (A+B)y_\varepsilon + \varepsilon Vy_\varepsilon + e_\varepsilon \tag{3.4}$$

also define a unique approximating curve $(y_\varepsilon)_{\varepsilon\in]0,1[}$. Now suppose that $\|e_\varepsilon\|/\varepsilon \to 0$ as $\varepsilon \downarrow 0$. Then, upon setting $M_\varepsilon \equiv A+B$ in Theorem 2.4, we get

$$y_\varepsilon \to x_0 \quad \text{as} \quad \varepsilon \downarrow 0, \tag{3.5}$$

since the $\mathcal{A}$-focusing property of $(y_\varepsilon)_{\varepsilon\in]0,1[}$ follows at once from the sequential closedness of $\operatorname{gr}(A+B)$ in $\mathcal{H}^{\text{weak}} \times \mathcal{H}^{\text{strong}}$, which is guaranteed by the maximal monotonicity of $A+B$ [7, Proposition 3.5.6.2]. Next, we shall show that there exist $\eta \in\,]0,1]$ and $\tau \in\,]0,+\infty[$ such that

$$(\forall \varepsilon \in\,]0,\eta[) \quad c(\|x_{\varepsilon,\phi(\varepsilon)} - y_\varepsilon\|) \leq \tau \frac{\phi(\varepsilon)}{\varepsilon}. \tag{3.6}$$

For this purpose, take ε and ρ in $]0,1[$. As seen above, there exists a unique point $x_{\varepsilon,\rho} \in \operatorname{dom} B$ such that

$$0 \in {}^{\rho}Ax_{\varepsilon,\rho} + Bx_{\varepsilon,\rho} + \varepsilon V x_{\varepsilon,\rho} + e_\varepsilon. \tag{3.7}$$

It follows from (3.7) and (3.2) that there exist points $v_{\varepsilon,\rho} \in V x_{\varepsilon,\rho}$ and $w_\varepsilon \in V x_{\varepsilon,\phi(\varepsilon)}$ such that

$$-\big({}^{\rho}Ax_{\varepsilon,\rho} + \varepsilon v_{\varepsilon,\rho} + e_\varepsilon\big) \in Bx_{\varepsilon,\rho} \tag{3.8}$$

and

$$-\big({}^{\phi(\varepsilon)}Ax_{\varepsilon,\phi(\varepsilon)} + \varepsilon w_\varepsilon + e_\varepsilon\big) \in Bx_{\varepsilon,\phi(\varepsilon)}. \tag{3.9}$$

Consequently, the monotonicity of B yields

$$0 \leq \langle x_{\varepsilon,\phi(\varepsilon)} - x_{\varepsilon,\rho} \mid {}^{\rho}Ax_{\varepsilon,\rho} - {}^{\phi(\varepsilon)}Ax_{\varepsilon,\phi(\varepsilon)}\rangle - \varepsilon\langle x_{\varepsilon,\phi(\varepsilon)} - x_{\varepsilon,\rho} \mid w_\varepsilon - v_{\varepsilon,\rho}\rangle, \tag{3.10}$$

and we deduce from (1.4) that

$$\varepsilon c(\|x_{\varepsilon,\phi(\varepsilon)} - x_{\varepsilon,\rho}\|) \leq \langle x_{\varepsilon,\phi(\varepsilon)} - x_{\varepsilon,\rho} \mid {}^{\rho}Ax_{\varepsilon,\rho} - {}^{\phi(\varepsilon)}Ax_{\varepsilon,\phi(\varepsilon)}\rangle. \tag{3.11}$$

On the other hand, the inclusions

$${}^{\rho}Ax_{\varepsilon,\rho} \in A(J_{\rho A}x_{\varepsilon,\rho}) \quad \text{and} \quad {}^{\phi(\varepsilon)}Ax_{\varepsilon,\phi(\varepsilon)} \in A(J_{\phi(\varepsilon)A}x_{\varepsilon,\phi(\varepsilon)}), \tag{3.12}$$

and the monotonicity of A lead to the inequality

$$0 \leq \langle J_{\rho A}x_{\varepsilon,\rho} - J_{\phi(\varepsilon)A}x_{\varepsilon,\phi(\varepsilon)} \mid {}^{\rho}Ax_{\varepsilon,\rho} - {}^{\phi(\varepsilon)}Ax_{\varepsilon,\phi(\varepsilon)}\rangle. \tag{3.13}$$

Adding (3.11) to (3.13), and then using Lemma 2.1, we obtain

$$\begin{aligned}
\varepsilon c(\|x_{\varepsilon,\phi(\varepsilon)} - x_{\varepsilon,\rho}\|) &\leq \langle \phi(\varepsilon)({}^{\phi(\varepsilon)}Ax_{\varepsilon,\phi(\varepsilon)}) - \rho({}^{\rho}Ax_{\varepsilon,\rho}) \mid {}^{\rho}Ax_{\varepsilon,\rho} - {}^{\phi(\varepsilon)}Ax_{\varepsilon,\phi(\varepsilon)}\rangle \\
&\leq \frac{1}{4}\big(\phi(\varepsilon)\|{}^{\rho}Ax_{\varepsilon,\rho}\|^2 + \rho\|{}^{\phi(\varepsilon)}Ax_{\varepsilon,\phi(\varepsilon)}\|^2\big).
\end{aligned} \tag{3.14}$$

Note that, since V satisfies Assumption 1.1(ii), so does $V_\varepsilon = \varepsilon V + e_\varepsilon$. Hence, applying Theorem 2.5 with $M_1 = A$, $M_2 = B$, and V_ε instead of V, we deduce from the existence of y_ε in (3.4) that the family $\big({}^{\rho}Ax_{\varepsilon,\rho}\big)_{\rho\in]0,1[}$ is bounded and that

$$x_{\varepsilon,\rho} \to y_\varepsilon \quad \text{as} \quad \rho \downarrow 0. \tag{3.15}$$

More precisely, it follows from (3.4) that there exists a point $u_\varepsilon \in Ay_\varepsilon$ such that

$$-u_\varepsilon \in By_\varepsilon + \varepsilon V y_\varepsilon + e_\varepsilon \tag{3.16}$$

and, proceeding as in (2.15)–(2.20), we obtain

$$(\forall \rho \in \,]0,1[) \quad \big\| {}^{\rho}\!A x_{\varepsilon,\rho} \big\| \leq \|u_\varepsilon\|. \tag{3.17}$$

We shall now show that if one of conditions (i)–(iii) holds, then

$$(\exists\, \eta \in \,]0,1])(\exists\, \tau \in \,]0,+\infty[) \quad \sup_{\varepsilon \in]0,\eta[} \; \sup_{\rho \in]0,1[} \big\| {}^{\rho}\!A x_{\varepsilon,\rho} \big\|^2 \leq 4\tau. \tag{3.18}$$

(i) Suppose that $x_0 \in \operatorname{int}\operatorname{dom} A$. Then, by [26, Proposition 32.33], A is locally bounded at x_0 and, therefore, there exists a bounded neighborhood X_1 of x_0 such that $A(X_1)$ is bounded. On the other hand, it follows from (3.5) that there exists $\eta \in \,]0,1]$ such that $(\forall \varepsilon \in \,]0,\eta[\,)\ y_\varepsilon \in X_1$. Hence, $(\forall \varepsilon \in \,]0,\eta[)\ u_\varepsilon \in A y_\varepsilon \subset A(X_1)$. We thus obtain the boundedness of $(u_\varepsilon)_{\varepsilon \in]0,\eta[}$ and therefore (3.18) via (3.17).

(ii) Suppose that $x_0 \in \operatorname{int}\operatorname{dom} B$. As in (i), there exists a bounded neighborhood X_2 of x_0 such that $B(X_2)$ is bounded. However, by (3.5), there exists $\eta \in \,]0,1]$ such that $(\forall \varepsilon \in \,]0,\eta[\,)\ y_\varepsilon \in X_2$. On the other hand, $(e_\varepsilon)_{\varepsilon \in]0,\eta[}$ lies in some bounded set U and, by Assumption 1.1(ii)(b), $V(X_2)$ is bounded. Altogether, we derive from (3.16) that $-(u_\varepsilon)_{\varepsilon \in]0,\eta[}$ lies in the bounded set $B(X_2) + \bigcup_{\varepsilon \in]0,\eta[} \varepsilon V(X_2) + U$. In view of (3.17), we obtain (3.18).

(iii) Suppose that (3.3) holds. We deduce from (3.3) and (3.8) that

$$\langle {}^{\rho}\!A x_{\varepsilon,\rho} \mid {}^{\rho}\!A x_{\varepsilon,\rho} + \varepsilon v_{\varepsilon,\rho} + e_\varepsilon \rangle \leq \|(\sigma_1 - \sigma_2)({}^{\rho}\!A x_{\varepsilon,\rho}) - \sigma_2(\varepsilon v_{\varepsilon,\rho} + e_\varepsilon)\| + \sigma_3. \tag{3.19}$$

Therefore, using Cauchy-Schwarz and setting $\omega = \sup_{\varepsilon \in]0,1[} \|e_\varepsilon\|$, we obtain

$$\begin{aligned} \|{}^{\rho}\!A x_{\varepsilon,\rho}\|^2 &\leq \|{}^{\rho}\!A x_{\varepsilon,\rho}\| \big(\|\varepsilon v_{\varepsilon,\rho} + e_\varepsilon\| + |\sigma_1 - \sigma_2|\big) + \sigma_2 \|\varepsilon v_{\varepsilon,\rho} + e_\varepsilon\| + \sigma_3 \\ &\leq \|{}^{\rho}\!A x_{\varepsilon,\rho}\| \big(\varepsilon \|v_{\varepsilon,\rho}\| + \kappa_1\big) + \sigma_2 \varepsilon \|v_{\varepsilon,\rho}\| + \kappa_2, \end{aligned} \tag{3.20}$$

where $\kappa_1 = \omega + |\sigma_1 - \sigma_2|$ and $\kappa_2 = \sigma_2 \omega + \sigma_3$. We now consider two cases.

(a) Suppose that $\operatorname{dom} B$ is bounded. Then (3.7) implies that $\sup_{\varepsilon \in]0,1[} \sup_{\rho \in]0,1[} \|x_{\varepsilon,\rho}\| < +\infty$ and Assumption 1.1(ii)(b) yields $\sup_{\varepsilon \in]0,1[} \sup_{\rho \in]0,1[} \|v_{\varepsilon,\rho}\| < +\infty$. Consequently, it follows from (3.20) that there exist constants κ_3 and κ_4 in $[0,+\infty[$, which are independent from ε and ρ, such that $\|{}^{\rho}\!A x_{\varepsilon,\rho}\|^2 \leq \kappa_3 \|{}^{\rho}\!A x_{\varepsilon,\rho}\| + \kappa_4$. Thus, (3.18) holds with $\eta = 1$.

(b) Suppose that V is β-Lipschitz continuous (hence single-valued) and γ-strongly monotone, for some β and γ in $]0,+\infty[$. Fix $z \in \operatorname{dom} A \cap \operatorname{dom} B$, $v \in Bz$, and set $w = V0$ and $r_\rho = {}^{\rho}\!A z$. Then

$$\|v_{\varepsilon,\rho}\| \leq \|v_{\varepsilon,\rho} - w\| + \|w\| \leq \beta \|x_{\varepsilon,\rho}\| + \|w\|. \tag{3.21}$$

Moreover, $r_\rho + v \in ({}^{\rho}\!A + B)z$ and we derive from (3.7) that $-(\varepsilon v_{\varepsilon,\rho} + e_\varepsilon) \in ({}^{\rho}\!A + B)x_{\varepsilon,\rho}$. Hence, by monotonicity of ${}^{\rho}\!A + B$,

$$\langle r_\rho + v + \varepsilon v_{\varepsilon,\rho} + e_\varepsilon \mid x_{\varepsilon,\rho} - z \rangle \leq 0. \tag{3.22}$$

Now let r_0 be the element of minimal norm in Az. Then $\|r_\rho\| \leq \|r_0\|$ [7, Theorem 3.5.9]. Hence, upon setting $\kappa_5 = \|r_0\| + \|v\| + \omega$ and $\kappa_6 = \kappa_5\|z\|$, we deduce from (3.22) and the Cauchy-Schwarz inequality that

$$\varepsilon\langle x_{\varepsilon,\rho} \mid v_{\varepsilon,\rho}\rangle \leq \kappa_5\|x_{\varepsilon,\rho}\| + \varepsilon\|z\|\,\|v_{\varepsilon,\rho}\| + \kappa_6. \tag{3.23}$$

On the other hand, the γ-strong monotonicity of V yields

$$\gamma\|x_{\varepsilon,\rho}\|^2 \leq \langle x_{\varepsilon,\rho} \mid v_{\varepsilon,\rho} - w\rangle \leq \langle x_{\varepsilon,\rho} \mid v_{\varepsilon,\rho}\rangle + \|w\|\,\|x_{\varepsilon,\rho}\|. \tag{3.24}$$

By first combining (3.23) and (3.24), and then using (3.21), we obtain

$$\begin{aligned}\gamma(\varepsilon\|x_{\varepsilon,\rho}\|)^2 &\leq \varepsilon^2\langle x_{\varepsilon,\rho} \mid v_{\varepsilon,\rho}\rangle + \|w\|(\varepsilon\|x_{\varepsilon,\rho}\|)\\ &\leq \kappa_5(\varepsilon\|x_{\varepsilon,\rho}\|) + \|z\|(\varepsilon\|v_{\varepsilon,\rho}\|) + \|w\|(\varepsilon\|x_{\varepsilon,\rho}\|) + \kappa_6\\ &\leq \kappa_5(\varepsilon\|x_{\varepsilon,\rho}\|) + \beta\|z\|(\varepsilon\|x_{\varepsilon,\rho}\|) + \|z\|\,\|w\| + \|w\|(\varepsilon\|x_{\varepsilon,\rho}\|) + \kappa_6.\end{aligned} \tag{3.25}$$

In other words, there exist constants κ_7 and κ_8 in $[0,+\infty[$, which are independent from ε and ρ, such that

$$(\varepsilon\|x_{\varepsilon,\rho}\|)^2 \leq \kappa_7(\varepsilon\|x_{\varepsilon,\rho}\|) + \kappa_8. \tag{3.26}$$

Accordingly, $\sup_{\varepsilon\in]0,1[}\sup_{\rho\in]0,1[}\varepsilon\|x_{\varepsilon,\rho}\| < +\infty$ and, in view of (3.21), we have

$$\sup_{\varepsilon\in]0,1[}\sup_{\rho\in]0,1[}\varepsilon\|v_{\varepsilon,\rho}\| < +\infty. \tag{3.27}$$

Therefore, it follows from (3.20) that there exist constants κ_9 and κ_{10} in $[0,+\infty[$, which are independent from ε and ρ, such that $\|{}^{\rho}Ax_{\varepsilon,\rho}\|^2 \leq \kappa_9\|{}^{\rho}Ax_{\varepsilon,\rho}\| + \kappa_{10}$. This shows that (3.18) holds with $\eta = 1$.

To complete the proof, let us observe that (3.14) and (3.18) yield

$$(\forall\varepsilon \in\,]0,\eta[)(\forall\rho \in\,]0,1[) \quad c(\|x_{\varepsilon,\phi(\varepsilon)} - x_{\varepsilon,\rho}\|) \leq \frac{\tau}{\varepsilon}(\phi(\varepsilon) + \rho). \tag{3.28}$$

In view of (3.15), if c is continuous, passing to the limit when $\rho \downarrow 0$ in (3.28) yields (3.6). Thus, if $\phi(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \downarrow 0$, we obtain $x_{\varepsilon,\phi(\varepsilon)} - y_\varepsilon \to 0$ as $\varepsilon \downarrow 0$ and, in view of (3.5), we conclude that $x_{\varepsilon,\phi(\varepsilon)} \to x_0$ as $\varepsilon \downarrow 0$. □

Remark 3.2 (infeasible case) Suppose that Assumption 1.1 is satisfied, except that we now assume that $\mathrm{zer}(A + B) = \varnothing$. In addition, suppose that $\phi(\varepsilon) + \|e_\varepsilon\| \to 0$ as $\varepsilon \downarrow 0$ in (3.2). Then

$$\|x_{\varepsilon,\phi(\varepsilon)}\| \to +\infty \quad \text{as} \quad \varepsilon \downarrow 0. \tag{3.29}$$

Proof. Suppose that (3.29) is not true. Then there exists a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $]0,1[$ that converges to 0 and such that $(x_{\varepsilon_n,\phi(\varepsilon_n)})_{n\in\mathbb{N}}$ is bounded. In view of (3.2), there exists a sequence $(v_{\varepsilon_n})_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that

$$(\forall n \in \mathbb{N}) \quad v_{\varepsilon_n} \in Vx_{\varepsilon_n,\phi(\varepsilon_n)} \quad \text{and} \quad -(\varepsilon_n v_{\varepsilon_n} + e_{\varepsilon_n}) \in ({}^{\phi(\varepsilon_n)}A + B)x_{\varepsilon_n,\phi(\varepsilon_n)}. \tag{3.30}$$

Since Assumption 1.1(ii)(b) implies that $(v_{\varepsilon_n})_{n\in\mathbb{N}}$ is bounded, we have

$$\varepsilon_n v_{\varepsilon_n} + e_{\varepsilon_n} \to 0. \tag{3.31}$$

On the other hand, we can extract a subsequence $(x_{\varepsilon_{k_n},\phi(\varepsilon_{k_n})})_{n\in\mathbb{N}}$ such that

$$x_{\varepsilon_{k_n},\phi(\varepsilon_{k_n})} \rightharpoonup x, \tag{3.32}$$

for some $x \in \mathcal{H}$. Moreover, it follows from Assumption 1.1(i) and [3, Proposition 5.3] that the sequence $({}^{\phi(\varepsilon_{k_n})}A + B)_{n\in\mathbb{N}}$ graph-converges to $A+B$. Consequently, (3.30), (3.31), (3.32), and [2, Proposition 3.59] force $x \in \operatorname{zer}(A+B)$, which contradicts our assumption. □

Remark 3.3 Condition (i) in Theorem 3.1 is satisfied in particular when $\operatorname{dom} A$ is open. For instance, if $A = \partial f$, where $f \in \Gamma_0(\mathcal{H})$, then $\operatorname{int}\operatorname{dom} f \subset \operatorname{dom}\partial f \subset \operatorname{dom} f$ [25, Theorem 2.4.9] and therefore $\operatorname{dom} A$ is open if $\operatorname{dom} f$ is open. Regarding Condition (iii) in Theorem 3.1, the "angle property" (3.3) was first used in [10, Section 2.3] with $\sigma_1 = \sigma_2 = 0$.

By setting $V = \mathrm{Id}$ and $e_\varepsilon \equiv 0$ in Theorem 3.1, we obtain our first corollary.

Corollary 3.4 *Suppose that Assumption* 1.1(i) *is satisfied, let* $\phi\colon\,]0,1[\, \to\,]0,1[$ *be such that* $\phi(\varepsilon)/\varepsilon \to 0$ *as* $\varepsilon \downarrow 0$, *and set* $x_0 = P_{\operatorname{zer}(A+B)}(0)$. *Then the inclusions*

$$(\forall \varepsilon \in\,]0,1[) \quad 0 \in {}^{\phi(\varepsilon)}A x_{\varepsilon,\phi(\varepsilon)} + B x_{\varepsilon,\phi(\varepsilon)} + \varepsilon x_{\varepsilon,\phi(\varepsilon)} \tag{3.33}$$

define a unique family $(x_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$, *and* $x_{\varepsilon,\phi(\varepsilon)} \to x_0$ *as* $\varepsilon \downarrow 0$ *if one of the following holds:*

(i) $x_0 \in (\operatorname{int}\operatorname{dom} A) \cup (\operatorname{int}\operatorname{dom} B)$.

(ii) *A and B satisfy* (3.3).

Remark 3.5

(i) In [19, Theorem 3], the convergence of $(x_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$ in Corollary 3.4 is announced without any additional hypothesis such as (i) or (ii). However, it is not clear to us how (3.18) can be satisfied without such an hypothesis.

(ii) Suppose that $A = 0$ in Corollary 3.4. Then we obtain the strong convergence of the approximating curve $(x_\varepsilon)_{\varepsilon\in]0,1[}$ defined by

$$(\forall \varepsilon \in\,]0,1[) \quad 0 \in Bx_\varepsilon + \varepsilon x_\varepsilon \tag{3.34}$$

to the zero x_0 of B of minimum norm as $\varepsilon \downarrow 0$. This classical result is due to Bruck [11]. When $B = \partial f$ with $f \in \Gamma_0(\mathcal{H})$, we recover the standard Tikhonov regularization setting [23].

Our second corollary deals with a visco-penalization method for finding a specific minimizer of the sum of two convex functions. We require the following notion of an inexact minimizer.

Definition 3.6 *Let $f\colon \mathcal{H} \to \,]-\infty,+\infty]$ be a proper function and let $e \in \mathcal{H}$. Then* $\operatorname{Argmin}_{\approx e} f = \big\{x \in \mathcal{H} \mid -e \in \partial f(x)\big\}$.

Corollary 3.7 *Let f and g be functions in $\Gamma_0(\mathcal{H})$ such that the set Z of minimizers of $f+g$ is nonempty and such that the cone generated by $\operatorname{dom} f - \operatorname{dom} g$ is a closed vector subspace. Let $h \in \Gamma_0(\mathcal{H})$ be a finite function that maps every bounded subset of $\mathcal{H}$ into a bounded set, and which is uniformly convex in the sense that there exists an increasing function $c\colon [0,+\infty[\, \to [0,+\infty[$ that vanishes only at 0 such that $\lim_{t\to+\infty} c(t)/t = +\infty$ and*

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H})(\forall \alpha \in \,]0,1[) \quad h\big(\alpha x + (1-\alpha)y\big) + \alpha(1-\alpha)c(\|x-y\|) \\ \leq \alpha h(x) + (1-\alpha)h(y). \quad (3.35)$$

Then h admits a unique minimizer x_0 over Z. Moreover, given $\phi\colon\,]0,1[\, \to \,]0,1[$, the inexact minimization problems

$$(\forall \varepsilon \in \,]0,1[) \quad x_{\varepsilon,\phi(\varepsilon)} \in \operatorname{Argmin}_{\approx e_\varepsilon} \big({}^{\phi(\varepsilon)}f + g + \varepsilon h\big), \quad \text{where} \quad e_\varepsilon \in \mathcal{H}, \quad (3.36)$$

define a unique family $(x_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$. Now suppose that c is continuous, that $(\phi(\varepsilon) + \|e_\varepsilon\|)/\varepsilon \to 0$ as $\varepsilon \downarrow 0$, and that one of the following holds:

(i) $x_0 \in \operatorname{int}\operatorname{dom} f$.

(ii) $x_0 \in \operatorname{int}\operatorname{dom} g$.

(iii) *f and g satisfy*

$$(\exists\, \sigma_1 \in \mathbb{R})(\exists\, \sigma_2 \in [0,+\infty[)(\exists\, \sigma_3 \in [0,+\infty[)(\forall \rho \in \,]0,1[)(\forall (x,u) \in \operatorname{gr}\partial g) \\ g(\operatorname{prox}_{\rho f} x) \leq g(x) + \rho\|\sigma_1 \operatorname{prox}_{f^*/\rho}(x/\rho) + \sigma_2 u\| + \rho\sigma_3, \quad (3.37)$$

and one of the following holds:

(a) $\operatorname{dom} g$ *is bounded.*

(b) *h is strongly convex and differentiable with a Lipschitz continuous gradient.*

Then $x_{\varepsilon,\phi(\varepsilon)} \to x_0$ as $\varepsilon \downarrow 0$.

Proof. Set $A = \partial f$, $B = \partial g$, and $V = \partial h$. Our hypotheses on f and g, [25, Theorem 3.1.11], and the sum rule for subdifferentials [4] (see also [25, Theorem 2.8.7]) imply that $Z = \operatorname{zer}(A+B)$, and that A and B satisfy Assumption 1.1(i). Moreover, we infer from [25, Theorem 3.5.10] and [25, Theorem 2.4.13] that V satisfies Assumption 1.1(ii). Next, it follows from [25, Theorem 2.9.1] that a point $x_0 \in \mathcal{H}$ minimizes h over Z if and only if $0 \in N_Z x_0 + \partial h(x_0)$, i.e., if and only if (3.1) holds. Furthermore, it follows from Definition 3.6, [25, Theorem 2.8.7], and (1.10) that (3.36) reduces to (3.2). To apply Theorem 3.1, it remains to check that items (i)–(iii) imply their counterpart in Theorem 3.1. Since $\operatorname{int}\operatorname{dom} f = \operatorname{int}\operatorname{dom}\partial f$ and $\operatorname{int}\operatorname{dom} g = \operatorname{int}\operatorname{dom}\partial g$ [25, Theorem 2.4.9], this is clearly the case for (i) and (ii). Next, let us show that (3.37) $\Rightarrow$ (3.3). To this end, fix $(x,u) \in \operatorname{gr}\partial g$. Then it follows from (1.10), (1.9), and (3.37) that

$$(\forall \rho \in \left]0,1\right]) \quad \langle {}^{\rho}(\partial f)x \mid u\rangle = \langle x - \operatorname{prox}_{\rho f} x \mid u\rangle/\rho$$
$$\geq \big(g(x) - g(\operatorname{prox}_{\rho f} x)\big)/\rho$$
$$\geq -\|\sigma_1 \operatorname{prox}_{f^*/\rho}(x/\rho) + \sigma_2 u\| - \sigma_3$$
$$= -\|\sigma_1({}^{\rho}(\partial f)x) + \sigma_2 u\| - \sigma_3, \tag{3.38}$$

and we obtain (3.3). Finally, if $\operatorname{dom} g$ is bounded, so is $\operatorname{dom}\partial g \subset \operatorname{dom} g$, while the conditions in (iii)(b) imply that $V = \nabla h$ is Lipschitz continuous and strongly monotone. $\square$

Remark 3.8 Consider the special case of Corollary 3.7 in which the following additional assumptions are made: f is the indicator function of a nonempty closed convex subset C of $\mathcal{H}$, g is Lipschitz continuous on $\mathcal{H}$, $h = \|\cdot\|^2/2$, $e_\varepsilon \equiv 0$, and $\phi\colon \varepsilon \mapsto \varepsilon^\theta$, where $\theta \in \left]1,+\infty\right[$. Then (3.36) becomes

$$(\forall \varepsilon \in \left]0,1\right]) \quad x_\varepsilon = \underset{y\in\mathcal{H}}{\operatorname{argmin}} \left(\frac{1}{2\varepsilon^\theta} d_C^2(y) + g(y) + \frac{\varepsilon}{2}\|y\|^2\right), \tag{3.39}$$

and Corollary 3.7 asserts that the unique curve $(x_\varepsilon)_{\varepsilon\in\left]0,1\right[}$ thus defined converges strongly to the minimizer of g over C of minimal norm as $\varepsilon \downarrow 0$. This result was established in [5, Example p. 531].

4 The case of m operators

In this section we derive from the results of Section 3 a visco-penalization approximating curve for the problem

$$\text{find} \quad x \in \operatorname{zer}\left(\sum_{i=1}^m A_i\right) \tag{4.1}$$

under the following set of assumptions.

Assumption 4.1

(i) $(A_i)_{1\leq i\leq m}$ is a finite family of maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ such that $\operatorname{zer}\left(\sum_{i=1}^m A_i\right) \neq \varnothing$ and $\operatorname{int}\bigcap_{i=1}^m \operatorname{dom} A_i \neq \varnothing$.

(ii) $V\colon \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator which is γ-strongly monotone for some $\gamma \in \left]0,+\infty\right[$, and which maps every bounded subset of $\mathcal{H}$ into a bounded set.

Theorem 4.2 *Suppose that Assumption* 4.1 *is satisfied. Then there exists a unique point* $x_0 \in \operatorname{zer}(\sum_{i=1}^m A_i)$ *such that*

$$0 \in N_{\operatorname{zer}(\sum_{i=1}^m A_i)} x_0 + V x_0. \tag{4.2}$$

Moreover, given $\phi\colon\ ]0,1[\ \to\ ]0,1[$*, the inclusions*

$$(\forall \varepsilon \in \,]0,1[) \quad 0 \in \sum_{i=1}^{m} {}^{\phi(\varepsilon)}A_i x_{\varepsilon,\phi(\varepsilon)} + \varepsilon V x_{\varepsilon,\phi(\varepsilon)} + e_\varepsilon, \quad \textit{where} \quad e_\varepsilon \in \mathcal{H}, \tag{4.3}$$

define a unique family $(x_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$*. Now suppose that* $(\phi(\varepsilon) + \|e_\varepsilon\|)/\varepsilon \to 0$ *as* $\varepsilon \downarrow 0$ *and that* $x_0 \in \operatorname{int} \bigcap_{i=1}^m \operatorname{dom} A_i$*. Then* $x_{\varepsilon,\phi(\varepsilon)} \to x_0$ *as* $\varepsilon \downarrow 0$.

Proof. We reformulate our m-operator problem as a 2-operator problem in a product space (similar setups are considered in [20] and [22]). Let $\boldsymbol{\mathcal{H}}$ be the Hilbert space obtained by endowing the Cartesian product $\mathcal{H}^m$ with the scalar product $\langle\langle \cdot \mid \cdot \rangle\rangle\colon (\boldsymbol{x},\boldsymbol{y}) \mapsto \sum_{i=1}^m \langle x_i \mid y_i\rangle$, where $\boldsymbol{x} = (x_i)_{1\leq i\leq m}$ and $\boldsymbol{y} = (y_i)_{1\leq i\leq m}$ denote generic elements in $\boldsymbol{\mathcal{H}}$. We shall denote by $|||\cdot|||$ the associated norm on $\boldsymbol{\mathcal{H}}$. Now set

$$\boldsymbol{A}\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}\colon \boldsymbol{x} \mapsto \bigtimes_{i=1}^{m} A_i x_i, \tag{4.4}$$

$\boldsymbol{D} = \big\{(x,\ldots,x) \in \boldsymbol{\mathcal{H}} \mid x \in \mathcal{H}\big\}$, and

$$\boldsymbol{V}\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}\colon \boldsymbol{x} \mapsto \frac{1}{m}\bigg(\bigtimes_{i=1}^{m} V x_i\bigg). \tag{4.5}$$

It is easily checked that $\boldsymbol{A}$ is maximal monotone with Yosida approximations

$$(\forall \phi \in \,]0,+\infty[) \quad {}^{\phi}\boldsymbol{A}\colon \boldsymbol{x} \mapsto \big({}^{\phi}A_i x_i\big)_{1\leq i\leq m}. \tag{4.6}$$

Moreover, since $\boldsymbol{D}$ is a closed vector subspace of $\boldsymbol{\mathcal{H}}$, (1.8) yields

$$(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{H}}) \quad N_{\boldsymbol{D}}\boldsymbol{x} = \begin{cases} \boldsymbol{D}^\perp = \big\{\boldsymbol{u} \in \boldsymbol{\mathcal{H}} \mid \sum_{i=1}^m u_i = 0\big\}, & \text{if } \boldsymbol{x} \in \boldsymbol{D};\\ \varnothing, & \text{otherwise.}\end{cases} \tag{4.7}$$

Now let us set $\boldsymbol{Z} = \operatorname{zer}\big(\boldsymbol{A} + N_{\boldsymbol{D}}\big)$ and $Z = \operatorname{zer}(\sum_{i=1}^m A_i)$. Then it follows at once from (4.4) and (4.7) that

$$\boldsymbol{Z} = \boldsymbol{D} \cap Z^m. \tag{4.8}$$

Consequently, Assumption 4.1(i) implies that $\boldsymbol{Z} \neq \varnothing$ and that $\operatorname{dom} N_{\boldsymbol{D}} \cap \operatorname{int}\operatorname{dom}\boldsymbol{A} = \boldsymbol{D} \cap \operatorname{int}\operatorname{dom}\boldsymbol{A} \neq \varnothing$. In turn, it follows from [26, Theorem 32.I] that $\boldsymbol{A} + N_{\boldsymbol{D}}$ is maximal monotone. Thus, Assumption 1.1(i) is satisfied by $\boldsymbol{A}$ and $N_{\boldsymbol{D}}$. On the other hand, it follows from Assumption 4.1(ii) that $\boldsymbol{V}$ satisfies Assumption 1.1(ii) with $c\colon t \mapsto \gamma t^2/m$ in (1.4). Now suppose that $\|e_\varepsilon\|/\varepsilon \to 0$ as $\varepsilon \downarrow 0$ and set $(\forall \varepsilon \in \,]0,1[)$ $\boldsymbol{e}_\varepsilon = m^{-1}(e_\varepsilon,\ldots,e_\varepsilon)$. Then $|||\boldsymbol{e}_\varepsilon|||/\varepsilon = m^{-1/2}\|e_\varepsilon\|/\varepsilon \to 0$ as $\varepsilon \downarrow 0$ and we can apply Theorem 3.1 in $\boldsymbol{\mathcal{H}}$ to the operators $\boldsymbol{A}$, $N_{\boldsymbol{D}}$, and $\boldsymbol{V}$ to obtain:

(a) the existence and uniqueness of a point $\boldsymbol{x}_0 \in \boldsymbol{Z}$ such that $0 \in N_{\boldsymbol{Z}}\boldsymbol{x}_0 + \boldsymbol{V}\boldsymbol{x}_0$;

(b) the existence and uniqueness of the curve $(\boldsymbol{x}_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$ defined by

$$(\forall \varepsilon \in \,]0,1[) \quad 0 \in {}^{\phi(\varepsilon)}\boldsymbol{A}\boldsymbol{x}_{\varepsilon,\phi(\varepsilon)} + N_{\boldsymbol{D}}\boldsymbol{x}_{\varepsilon,\phi(\varepsilon)} + \varepsilon\boldsymbol{V}\boldsymbol{x}_{\varepsilon,\phi(\varepsilon)} + \boldsymbol{e}_\varepsilon; \tag{4.9}$$

(c) the strong convergence of $(\boldsymbol{x}_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$ to $\boldsymbol{x}_0$ as $\varepsilon \downarrow 0$ if $\phi(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \downarrow 0$ and $\boldsymbol{x}_0 \in \operatorname{int}\operatorname{dom}\boldsymbol{A}$.

Now let $\boldsymbol{x} = (x, \ldots, x)$ be an arbitrary point in $\boldsymbol{Z}$. Then (1.8) and (4.8) yield

$$
\begin{aligned}
N_{\boldsymbol{Z}}\boldsymbol{x} &= \big\{\boldsymbol{u} \in \boldsymbol{\mathcal{H}} \mid (\forall \boldsymbol{z} \in \boldsymbol{D} \cap Z^m) \;\; \langle\langle \boldsymbol{z} - \boldsymbol{x} \mid \boldsymbol{u}\rangle\rangle \leq 0\big\} \\
&= \Big\{\boldsymbol{u} \in \boldsymbol{\mathcal{H}} \;\Big|\; (\forall z \in Z) \;\; \sum_{i=1}^m \langle z - x \mid u_i\rangle \leq 0\Big\} \\
&= \Big\{\boldsymbol{u} \in \boldsymbol{\mathcal{H}} \;\Big|\; \sum_{i=1}^m u_i \in N_Z x\Big\}. \qquad (4.10)
\end{aligned}
$$

Note that, by maximal monotonicity of V, the set Vx is convex [7, Proposition 3.5.6.1]. It therefore results from (4.10) and (4.5) that

$$
\begin{aligned}
0 \in N_{\boldsymbol{Z}}\boldsymbol{x} + \boldsymbol{V}\boldsymbol{x} \;\;&\Leftrightarrow\;\; (\exists\, \boldsymbol{v} \in \boldsymbol{V}\boldsymbol{x}) \quad -\boldsymbol{v} \in N_{\boldsymbol{Z}}\boldsymbol{x} \\
&\Leftrightarrow\;\; (\exists\, \boldsymbol{v} \in \boldsymbol{V}\boldsymbol{x}) \quad -\sum_{i=1}^m v_i \in N_Z x \quad \text{and} \quad \sum_{i=1}^m v_i \in Vx \\
&\Leftrightarrow\;\; 0 \in N_Z x + Vx. \qquad (4.11)
\end{aligned}
$$

In view of (a), we therefore have $\boldsymbol{x}_0 = (x_0, \ldots, x_0)$, where x_0 is the unique solution to (4.2). Next, we observe that (4.9) and (4.7) imply that $(\boldsymbol{x}_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$ lies in $\boldsymbol{D}$. Hence,

$$
(\forall \varepsilon \in]0,1[)(\exists\, x_{\varepsilon,\phi(\varepsilon)} \in \mathcal{H}) \quad \boldsymbol{x}_{\varepsilon,\phi(\varepsilon)} = (x_{\varepsilon,\phi(\varepsilon)}, \ldots, x_{\varepsilon,\phi(\varepsilon)}). \qquad (4.12)
$$

Let us show that the inclusions (4.9) in $\boldsymbol{\mathcal{H}}$ are equivalent to the inclusions (4.3) in $\mathcal{H}$. We derive from (4.9), (4.6), and (4.7) that, for every $\varepsilon \in]0,1[$, there exists $(v_{\varepsilon,i})_{1\leq i\leq m} \in (Vx_{\varepsilon,\phi(\varepsilon)})^m$ such that $\big({}^{\phi(\varepsilon)}A_i x_{\varepsilon,\phi(\varepsilon)} + \varepsilon m^{-1} v_{\varepsilon,i} + m^{-1} e_\varepsilon\big)_{1\leq i\leq m} \in \boldsymbol{D}^\perp$, i.e., $\sum_{i=1}^m {}^{\phi(\varepsilon)}A_i x_{\varepsilon,\phi(\varepsilon)} + \varepsilon m^{-1} \sum_{i=1}^m v_{\varepsilon,i} + e_\varepsilon = 0$ or, equivalently, $-\sum_{i=1}^m {}^{\phi(\varepsilon)}A_i x_{\varepsilon,\phi(\varepsilon)} - e_\varepsilon = \varepsilon m^{-1}\sum_{i=1}^m v_{\varepsilon,i} \in \varepsilon V x_{\varepsilon,\phi(\varepsilon)}$ since $Vx_{\varepsilon,\phi(\varepsilon)}$ is convex. This shows that $(x_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$ satisfies (4.3). Conversely, arguing along the same lines, we deduce that, if $(x_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$ satisfies (4.3), then $(\boldsymbol{x}_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[} = \big((x_{\varepsilon,\phi(\varepsilon)}, \ldots, x_{\varepsilon,\phi(\varepsilon)})\big)_{\varepsilon\in]0,1[}$ satisfies (4.9). Therefore, we derive from (b) the existence and uniqueness of the curve $(x_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$ in (4.3). Finally, if $\phi(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \downarrow 0$ and $x_0 \in \operatorname{int}\bigcap_{i=1}^m \operatorname{dom} A_i$, then $\boldsymbol{x}_0 \in \operatorname{int}\operatorname{dom}\boldsymbol{A}$ and (c) yields $\|x_{\varepsilon,\phi(\varepsilon)} - x_0\|^2 = m^{-1}|||\boldsymbol{x}_{\varepsilon,\phi(\varepsilon)} - \boldsymbol{x}_0|||^2 \to 0$ as $\varepsilon \downarrow 0$. □

Corollary 4.3 *Let $(f_i)_{1\leq i\leq m}$ be functions in $\Gamma_0(\mathcal{H})$ such that the sets $(\operatorname{dom} f_i)_{1\leq i\leq m}$ are open and satisfy $\bigcap_{i=1}^m \operatorname{dom} f_i \neq \varnothing$, and such that the set Z of minimizers of $\sum_{i=1}^m f_i$ is nonempty. Let $h \in \Gamma_0(\mathcal{H})$ be a finite function that maps every bounded subset of $\mathcal{H}$ into a bounded set, and which is γ-strongly convex for some $\gamma \in]0,+\infty[$. Then h admits a unique minimizer x_0 over Z. Moreover, given $\phi\colon]0,1[\to]0,1[$, the inexact minimization problems*

$$
(\forall \varepsilon \in]0,1[) \quad x_{\varepsilon,\phi(\varepsilon)} \in \operatorname{Argmin}_{\approx e_\varepsilon}\left(\sum_{i=1}^m {}^{\phi(\varepsilon)}f_i + \varepsilon h\right), \quad \textit{where} \quad e_\varepsilon \in \mathcal{H}, \qquad (4.13)
$$

define a unique family $(x_{\varepsilon,\phi(\varepsilon)})_{\varepsilon\in]0,1[}$. Now suppose that $(\phi(\varepsilon) + \|e_\varepsilon\|)/\varepsilon \to 0$ as $\varepsilon \downarrow 0$. Then $x_{\varepsilon,\phi(\varepsilon)} \to x_0$ as $\varepsilon \downarrow 0$.

Proof. Arguing as in the proof of Corollary 3.7, we learn that this is a special case of Theorem 4.2 with $(\forall i \in \{1, \ldots, m\})$ $A_i = \partial f_i$ and $V = \partial h$. Note that the hypotheses on $(f_i)_{1 \leq i \leq m}$ imply that $x_0 \in \bigcap_{i=1}^m \operatorname{dom} f_i = \bigcap_{i=1}^m \operatorname{int} \operatorname{dom} f_i = \operatorname{int} \bigcap_{i=1}^m \operatorname{dom} A_i$. ☐

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