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Ahmad Fino, Hassan Ibrahim

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Ahmad Z. FINO[a,b], Hassan IBRAHIM[c,d]

[a]*LaMA-Liban, Lebanese University, P.O.Box 37 Tripoli, Lebanon*
[b]*School of Arts and Sciences, Lebanese International University (LIU), Tripoli Campus, Dahr el Ain Road, Tripoly, Lebanon*
[c]*Lebanese University, Faculty of Sciences-I, Hadath, Lebanon*
[d]*School of Arts and Sciences, Lebanese International University (LIU), Beirut Campus, Al-Mouseitbeh, P.B. Box 14-6404, Beirut, Lebanon*

Abstract

In this paper, we consider a nonhomogeneous space-time fractional telegraph equation defined in a bounded space domain, which is obtained from the standard telegraph equation by replacing the first- or second-order time derivative by the Caputo fractional derivative D_t^α, $\alpha > 0$; and the Laplacian operator by the fractional Laplacian $(-\Delta)^{\beta/2}$, $\beta \in (0, 2]$. We discuss and derive the analytical solutions under nonhomogeneous Dirichlet and Neumann boundary conditions by using the method of separation of variables. The obtained solutions are expressed through multivariate Mittag-Leffler type functions. Special cases of solutions are also discussed.

Keywords: Fractional telegraph equation, Fractional Laplacian, Caputo fractional derivative, Multivariate Mittag-Leffler type functions, Method of separating variables
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1. Introduction and main results

Fractional differential equations (FDEs) have attracted in the recent years a considerable interest both in mathematics and in applications. For example, fractional derivatives have been used successfully to model frequency dependent damping behavior of many viscoelastic materials, modeling of anomalous diffusive and subdiffusive systems and description of fractional random walk. They are also used in modeling of many chemical processed, mathematical biology and many other problems in engineering (see e.g. [2, 4, 7, 10, 11, 19, 20]). We offer a simple heuristic discussion to motivate the use of fractional derivatives. Let $X_1, \ldots, X_n, \ldots$ be independent, identically distributed symmetric random variables with the property that for each $n = 1, 2, \ldots$ and for suitable positive constants a_n, $a_n \sum_{j=1}^{n} X_j$ has the same distribution as X_1, and this holds for all n. Thus the distribution of $\sum_{i=1}^{n} X_i$, suitably scaled, is that of X_1. This is a kind of self-similarity, independent of the sample size. Necessarily (see [16]) the distribution of X_1 has Fourier transform $e^{-c|\xi|^\beta}$, for $\xi \in \mathbb{R}$, for certain constants $c > 0$ and $\beta \in (0, 2]$. These are precisely the symmetric stable laws, including the normal distribution ($\beta = 2$) and the Cauchy distribution ($\beta = 1$). The infinitesimal generator of the corresponding Feller-Markov semigroup is given by a positive multiple of the fractional Laplacian $-(-d^2/dx^2)^{\beta/2}$. The history and a comprehensive treatment of FDEs are provided by Podlubny [25] and a review of some applications of FDEs are given by Mainardi [20].

Roughly speaking, FDEs can be classified into three principal kinds: space-fractional equation, time-fractional equation and space-time fractional equation. One of the interest FDEs is the study of the *space-time fractional telegraph equation* which is a linear integro partial differential equation obtained from the classical telegraph equation by replacing the time and space derivative terms by fractional derivatives, for example by replacing the first- or second-order time derivative by the Caputo fractional derivative D_t^α, $\alpha > 0$ and the Laplacian operator by the fractional Laplacian $(-\Delta)^{\beta/2}$, $\beta \in (0, 2]$. This kind of equations has recently been considered by many authors. Cascaval et al. [8] discussed the time-fractional telegraph equations, dealing with well-posedness and presenting a study involving

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asymptotic by using the Riemann-Liouville approach. Orsingher and Beghin [23] studied the fundamental solutions to time-fractional telegraph equations of order 2α. They obtained the Fourier transforms of the solutions for any α and gave a representation of their inverses in terms of stable densities. For the special case $\alpha = 1/2$, they also showed that the fundamental solution is the probability density of a telegraph process with Brownian time. Chen et al. [9] also discussed and derived the analytical solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions, by the method of separating variables. The analytical solution of the time-fractional telegraph equation considered by Chen et al. [9] is derived by Huang [13] in the whole space domain. Orsingher and Zhao [24] considered the space-fractional telegraph equations, obtaining the Fourier transform of its fundamental solution and presenting a symmetric process with discontinuous trajectories, whose transition function satisfies the space-fractional telegraph equation. Momani [22] discussed analytic and approximate solutions of the space-time fractional telegraph differential equations, with some special initial and boundary conditions, by means of the so-called Adomian decomposition method. Camargo et al. [6] discussed the so-called general space-time fractional telegraph equations by the methods of differential and integral calculus, discussing the solution by means of the Laplace and Fourier transforms in variables t and x, respectively.

We consider the following generalized space-time fractional telegraph equation

$$\begin{cases} D_t^\alpha u(x,t) + 2\lambda D_t^\delta u(x,t) = -c^2(-\Delta)^{\beta/2} u(x,t) + f(x,t), & x \in \Omega, \ t > 0, \\ \partial_t^{(k)} u(x,0) = \phi_k(x), & k = 0, \ldots, m_\alpha - 1, & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, is a smooth and bounded domain; $\partial_t^{(k)} := \frac{\partial^m}{\partial t^m}$, and

$$m_\alpha - 1 < \alpha \leq m_\alpha, \quad m_\delta - 1 < \delta \leq m_\delta, \quad \alpha > \delta \geq 0, \quad m_\alpha \in \mathbb{N}^*, \quad m_\delta \in \mathbb{N}, \quad 0 < \beta \leq 2, \quad \lambda \geq 0, \quad c \in \mathbb{R}.$$

We also suppose that f and ϕ_k, $k = 0, \ldots, m_\alpha - 1$, are continuous functions. For $\gamma \geq 0$, D^γ is the Caputo fractional derivative defined by

$$D_t^\gamma u(t) = \begin{cases} \dfrac{1}{\Gamma(m-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma+m-1} u^{(m)}(s)\, ds, & m-1 < \gamma < m, \\ u^{(m)}(t), & \gamma = m, \end{cases}$$

where $u^{(m)} := \frac{d^m}{dt^m} u$, $m \in \mathbb{N}$, and Γ is the Euler gamma function. The fractional Laplacian operator $(-\Delta)^{\beta/2}$ in a bounded domain is defined by (see Theorem 6 below)

$$(-\Delta)^{\beta/2} v(x) := \sum_{k=1}^\infty \lambda_k^{\beta/2} \int_\Omega v(y)\varphi_k(y)\, dy\, \varphi_k(x), \quad v \in D((-\Delta)^{\beta/2}),$$

where $\{\lambda_n\}_{n\geq 1}$ and $\{\varphi_n\}_{n\geq 1}$ are the eigenvalues and eigenvectors associated to $-\Delta$ in $L^2(\Omega)$, and

$$D((-\Delta)^{\beta/2}) = \left\{ v \in L^2(\Omega); \ \sum_{k=1}^\infty \left| \lambda_k^{\beta/2} \int_\Omega v(y)\varphi_k(y)\, dy \right|^2 < \infty \right\}.$$

The goal of this paper is to provide analytical solutions of equation (1.1) under nonhomogeneuous Dirichlet and Neumann boundary conditions. The existence is ensured via the method of separation of variables. Precisely, we prove the following theorems:

Theorem 1. *(Telegraph equation with Dirichlet conditions)*
Let $\Omega \subseteq \mathbb{R}^N$ be a smooth bounded domain. There exists an analytical solution u of problem (1.1) under the following Dirichlet boundary conditions

$$u(x,t) = g_1(x)g_2(t), \quad x \in \partial\Omega, \ t \geq 0, \tag{1.2}$$

where g_1 and g_2 are nonzero smooth functions defined, respectively, on $\Omega^c = \mathbb{R}^N \setminus \Omega$ and $[0, \infty)$, and satisfying the compatibility condition $\varphi_0(x) = g_1(x)g_2(0)$ for all $x \in \partial\Omega$.

Theorem 2. *(Telegraph equation with Neumann conditions)*
Let $\Omega \subseteq \mathbb{R}^N$ be a smooth, convex and bounded domain. There exists an analytical solution u of problem (1.1) under the following Neumann boundary conditions

$$\frac{\partial u}{\partial n}(x, t) = g_1(x)g_2(t), \quad x \in \partial\Omega, \ t \geq 0, \tag{1.3}$$

where $n = \vec{n}$ is the outward-pointing unit normal vector on the boundary $\partial\Omega$, g_1 and g_2 are nonzero smooth functions defined, respectively, on $\partial\Omega$ and $[0, \infty)$.

The organization of this paper is as follows. In Section 2, we present some definitions and basic theorems that are needed in the proofs of Theorems 1 and 2. The proofs of Theorem 1 and Theorem 2 are presented, respectively, in Section 3 and Section 4. Finally, some conclusions are given in Section 5.

2. Basic concepts

In this section we present the main tools concerning the fractional Laplacian and fractional derivatives that will be used throughout our analysis. We start with some definitions.

Definition 1 (see [18]). *A real or complex-valued function $f(t)$, $t > 0$, is said to be in the space C_α, $\alpha \in \mathbb{R}$, if there exists a real number $p > \alpha$ such that*

$$f(t) = t^p f_1(t)$$

for some function f_1 in $C[0, \infty)$.

Clearly, C_α is a vector space and the set of spaces C_α is ordered by inclusion according to

$$C_\alpha \subseteq C_\beta \Leftrightarrow \alpha \geq \beta.$$

Definition 2 (see [18]). *A function $f(t)$, $t > 0$, is said to be in the space C_α^m, $m \in \mathbb{N}$, if and only if $f^{(m)} \in C_\alpha$.*

For example, the function f defined by $f(t) = 1$, $t > 0$, is in C_0^1.

Definition 3 (see [15, 18]). *A multivariate Mittag-Leffler function $E_{(a_1,\ldots,a_n),b}(z_1,\ldots,z_n)$ of n complex variables $z_1,\ldots,z_n \in \mathbb{C}$ with complex parameters $a_1,\ldots,a_n,b \in \mathbb{C}$ (of positive real parts), is defined as*

$$E_{(a_1,\ldots,a_n),b}(z_1,\ldots,z_n) = \sum_{k=0}^{\infty} \sum_{\substack{l_1 + \ldots + l_n = k \\ l_1,\ldots,l_n \geq 0}} (k; l_1,\ldots,l_n) \frac{\prod_{i=1}^{n} z_i^{l_i}}{\Gamma(b + \sum_{i=1}^{n} a_i l_i)},$$

in terms of the multinomial coefficients

$$(k; l_1,\ldots,l_n) := \frac{k!}{l_1! \ldots l_n!} \quad (k, l_1,\ldots,l_n \in \mathbb{N}).$$

In particular, if $n = 1$, the multivariate Mitteg-Leffler function is reduced to the Mitteg-Leffler function

$$E_{a_1,b}(z_1) = \sum_{k=0}^{\infty} \frac{z_1^k}{\Gamma(b + ka_1)}, \qquad a_1, b, z_1 \in \mathbb{C}; \ \mathfrak{R}(a_1), \mathfrak{R}(b) > 0,$$

where $\mathfrak{R}(z)$, $z \in \mathbb{C}$, is the complex real part of z.

The separation of variables method used in this paper leads to two FDEs in time and space. The next theorem ensures the existence of the fractional differential equation involving the time variable.

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Theorem 3 (see [18]). *Let $\mu > \mu_1 > \ldots > \mu_n \geq 0$, $m_i - 1 < \mu_i \leq m_i$, $m_i \in \mathbb{N}$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, n$. Consider the initial value problem*

$$\begin{cases} D_t^{\mu} y(t) - \sum_{i=1}^{n} \lambda_i D_t^{\mu_i} y(t) = g(t), \\[2mm] y^{(k)}(0) = c_k \in \mathbb{R}, \quad k = 0, \ldots, m - 1, \quad m - 1 < \mu \leq m, \end{cases} \tag{2.1}$$

where g is assumed to lie in C_{-1} if $\mu \in \mathbb{N}^*$ or in C_{-1}^1 if $\mu \notin \mathbb{N}^*$. Then (2.1) has a unique solution in the space C_{-1}^m of the form

$$y(t) = \int_0^t s^{\mu-1} E_{(\mu-\mu_1,\ldots,\mu-\mu_n),\mu}(\lambda_1 s^{\mu-\mu_1}, \ldots, \lambda_n s^{\mu-\mu_n}) g(t-s) \, ds + \sum_{k=0}^{m-1} c_k u_k(t), \quad t \geq 0,$$

with

$$u_k(t) = \frac{t^k}{k!} + \sum_{i=l_k+1}^{n} \lambda_i t^{k+\mu-\mu_i} E_{(\mu-\mu_1,\ldots,\mu-\mu_n),k+1+\mu-\mu_i}(\lambda_1 t^{\mu-\mu_1}, \ldots, \lambda_n t^{\mu-\mu_n}), \quad k = 0, \ldots, m-1,$$

fulfills the initial conditions $u_k^{(l)}(0) = \delta_{kl}$, $k, l = 0, \ldots, m - 1$.

Remark 1. In Theorem 3, the natural numbers l_k, $k = 0, \ldots, m-1$, are determined from the condition $m_{l_k} \geq k + 1$ and $m_{l_k+1} \leq k$. In the case $m_i \leq k$, $i = 0, \ldots, m-1$, we set $l_k := 0$, and if $m_i \geq k + 1$, $i = 0, \ldots, m-1$, then $l_k := n$.

The existence of the fractional differential equation involving the space variable is guaranteed through the following theorems.

Theorem 4. *([3, Theorem 3.6.1]) Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded domain of class C^1. Then there exists a complete system $\{\varphi_j\}_{j=1}^{\infty}$ of eigenvectors, orthonormal in $L^2(\Omega)$ and orthogonal in $H_0^1(\Omega)$, and of eigenvalues λ_j for the following problem*

$$\begin{cases} -\Delta u = \lambda u, & in \ \Omega, \\[2mm] u = 0, & on \ \partial\Omega, \end{cases} \tag{2.2}$$

where $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. Moreover, $\lambda_j > 0$, $j = 1, \ldots, \infty$, and $\lim_{j\to\infty} \lambda_j = +\infty$.

Theorem 5. *([3, Theorem 3.6.2]) Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain of class C^2. Then there exists an orthonormal complete system $\{\varphi_j\}_{j=0}^{\infty}$ of eigenvectors in $L^2(\Omega)$ and a corresponding sequence λ_j of eigenvalues for the following problem*

$$\begin{cases} -\Delta u = \lambda u, & in \ \Omega, \\[2mm] \dfrac{\partial u}{\partial n} = 0, & in \ \Omega^c, \end{cases} \tag{2.3}$$

where $D(-\Delta) = \{u \in H^2(\Omega); \ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$. Moreover, $\lambda_j \geq 0$, $j = 0, \ldots, \infty$, and $\lim_{j\to\infty} \lambda_j = +\infty$.

The next theorem is already known in the literature (eg. [21]). However, for the sake of completeness, we hereby present the proof.

Theorem 6. *Let H be a Hilbert space. If $\{\lambda_n\}_{n\geq 1}$ and $\{\varphi_n\}_{n\geq 1}$ are the eigenvalues and eigenvectors associated to an operator A in H, then $\{\lambda_n^{\alpha}\}_{n\geq 1}$ and $\{\varphi_n\}_{n\geq 1}$ are the eigenvalues and eigenvectors to the fractional operator A^{α}, $-1 < \alpha \leq 1$.*

Proof. Since H is a Hilbert space, we can extract an orthonormal complete system $\{\varphi_i\}_{i\geq 1}$ in H. Let $\{\lambda_n\}_{n\geq 1} \subseteq \mathbb{R}$ such that $\inf_{k\geq 1} \lambda_k > 0$. For $x \in H$, $t \geq 0$, we define a new operator $S(t)$ as follows

$$S(t)x = \sum_{k=1}^{\infty} e^{-\lambda_k t} < x, \varphi_k > \varphi_k.$$

Then $S(t)$ is a strongly continuous semigroup on H whose infinitesimal generator A is defined by

$$Ax = \sum_{k=1}^{\infty} \lambda_k < x, \varphi_k > \varphi_k$$

and

$$D(A) = \left\{ x \in H; \ \|Ax\|_H^2 < \infty \right\}.$$

By [3, Theorem 1.4.1], we have

$$\|Ax\|_H^2 = \sum_{k=1}^{\infty} |\lambda_k < x, \varphi_k > |^2,$$

which implies that

$$D(A) = \left\{ x \in H; \ \sum_{k=1}^{\infty} |\lambda_k < x, \varphi_k > |^2 < \infty \right\}.$$

Let $x = \varphi_l$, $l \geq 1$, then, as $\{\varphi_l\}_{l \geq 1}$ is an orthonormal system,

$$A\varphi_l = \sum_{k=1}^{\infty} \lambda_k < \varphi_l, \varphi_k > \varphi_k = \lambda_l \varphi_l,$$

hence $\{\lambda_l\}_{l \geq 1}$ and $\{\varphi_l\}_{l \geq 1}$ are the eigenvalues and eigenvectors associated to A in H. Let $\alpha \in (-1, 1]$, we have two cases to distinguish:

• If $-1 < \alpha < 0$, we have (see [1])

$$A^\alpha x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} S(t)x \, dt, \qquad \forall \ x \in D(A^\alpha),$$

where $D(A^\alpha) := \{x \in H; \ A^{\alpha-1}x \in D(A)\}$. Therefore

$$
\begin{aligned}
< A^\alpha x, \varphi_k > &= \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} < S(t)x, \varphi_k > dt \\
&= \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{\lambda_k t} < x, \varphi_k > dt \\
&= \frac{< x, \varphi_k >}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{\lambda_k t} \, dt \\
&= \lambda_k^\alpha < x, \varphi_k > .
\end{aligned}
$$

This implies that

$$A^\alpha x = \sum_{k=1}^{\infty} \lambda_k^\alpha < x, \varphi_k > \varphi_k.$$

Take $x = \varphi_l$, it follows

$$A^\alpha \varphi_l = \sum_{k=1}^{\infty} \lambda_k^\alpha < \varphi_l, \varphi_k > \varphi_k = \lambda_l^\alpha \varphi_l.$$

• If $0 < \alpha < 1$,

$$A^\alpha \varphi_l = A^{-(1-\alpha)} A\varphi_l = A^{-(1-\alpha)} \lambda_l \varphi_l = \lambda_l A^{\alpha-1} \varphi_l.$$

Using the first case, we get

$$A^{\alpha-1} \varphi_l = \lambda_l^{\alpha-1} \varphi_l$$

which implies

$$A^\alpha \varphi_l = \lambda_l^\alpha \varphi_l.$$

The last formula still true for the cases $\alpha = 0$ and $\alpha = 1$. As a conclusion, $\{\lambda_l^\alpha\}_{l\geq 1}$ and $\{\varphi_l\}_{l\geq 1}$ are the eigenvalues and eigenvectors associated to A^α, $\forall \alpha \in (-1, 1]$. Moreover, using [3, Theorem 1.4.1], we obtain

$$\|A^\alpha x\|_H^2 = \sum_{k=1}^\infty |\lambda_k^\alpha < x, \varphi_k >|^2,$$

which yields

$$D(A^\alpha) = \left\{ x \in H; \ \sum_{k=1}^\infty |\lambda_k^\alpha < x, \varphi_k >|^2 < \infty \right\}.$$

$\square$

3. Proof of Theorem 1

This section is devoted to derive the analytical solution of problem (1.1) with nonhomogeneous Dirichlet boundary conditions (1.2). Indeed, we present the

Proof of Theorem 1. We proceed in three steps.

Step 1 (Change of variables). In order to solve the problem with nonhomogeneous boundary conditions, we firstly transform it into a problem with homogeneous boundary conditions. Let

$$u(x, t) = v(x, t) + w(x, t),$$

where v is a new unknown function and

$$w(x, t) = y(x)g_2(t) \tag{3.1}$$

with y is a solution of the following β-harmonic equation

$$\begin{cases} (-\Delta)^{\beta/2} y(x) = 0, & \text{in } \Omega, \\ y(x) = g_1(x), & \text{in } \Omega^c. \end{cases} \tag{3.2}$$

The existence of y can be found in [12, 27]. However, in the case where $\Omega = B_r = B(0, r)$, the solution y can be expressed explicitly by [17]

$$y(x) = \int_{\mathbb{R}^N \setminus B_r} P(x, y)g_1(y)\, dy,$$

where

$$P(x, y) := C_{N,\beta} \left(\frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^{\beta/2} \frac{1}{|x - y|^N}, \quad \text{for } x \in B_r \text{ and } y \notin B_r,$$

is the Poisson Kernel with $C_{N,\beta} = \Gamma(N/2)\pi^{-N/2-1}\sin(\pi\beta/2)$.

The function w satisfies the boundary conditions

$$w(x, t) = y(x)g_2(t) = g_1(x)g_2(t) = u(x, t), \text{ for all } x \in \partial\Omega, \ t \geq 0,$$

which implies

$$v(x, t) = u(x, t) - w(x, t) = 0, \text{ for all } x \in \partial\Omega, \ t \geq 0.$$

Therefore $v(x, t)$ satisfies the following problem with homogeneous boundary conditions

$$\begin{cases} D_t^\alpha v(x, t) + 2\lambda D_t^\delta v(x, t) = -c^2(-\Delta)^{\beta/2} v(x, t) + \tilde{f}(t, x), & x \in \Omega, \ t > 0, \\ \partial_t^{(k)} v(x, 0) = \tilde{\phi}_k(x), \quad k = 0, \ldots, m_\alpha - 1, & x \in \overline{\Omega}, \\ v(x, t) = 0, & x \in \partial\Omega, \ t \geq 0, \end{cases} \tag{3.3}$$

where

$$\tilde{f}(t,x) = -D_t^\alpha w(x,t) - 2\lambda D_t^\delta w(x,t) + f(t,x),$$
$$\tilde{\varphi}_k(x) = \varphi_k(x) - \partial_t^{(k)} w(x,0), \quad k = 0,\ldots, m_\alpha - 1.$$

Step 2 (Separation of variables). We solve problem, (3.3) with zero function replacing $\tilde{f}(t,x)$, by the method of separation of variables. If we let $v(t,x) = X(x)T(t)$ and substitute it in (3.3), we obtain a fractional differential equation in $X(x)$:

$$\begin{cases} (-\Delta)^{\beta/2} X(x) = \dfrac{a}{c^2} X(x), & \text{in } \Omega, \\ X(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

and a fractional linear differential equation with the Caputo derivative in $T(t)$:

$$D_t^\alpha T(t) + 2\lambda D_t^\delta T(t) + aT(t) = 0, \tag{3.5}$$

where the parameter a is a positive constant. Applying Theorem 6 with $A = -\Delta$, $\alpha = \beta/2$, $H = L^2(\Omega)$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, there exists $\{\lambda_n^{\beta/2}\}_{n\geq 1}$, $\{\varphi_n\}_{n\geq 1}$ eigenvalues and eigenvectors of problem (3.4) where $\{\lambda_n\}_{n\geq 1}$, $\{\varphi_n\}_{n\geq 1}$ are the eigenvalues and eigenvectors of (3.4) with $\beta = 2$ (see Theorem 4). Note that for $a = a_n$, $n \geq 1$, we have $X = X_n = \varphi_n$ and $\frac{a_n}{c^2} = \lambda_n^{\beta/2}$.

We now seek a solution of (3.3) of the form

$$v(t,x) = \sum_{n=1}^\infty B_n(t)\varphi_n(x), \tag{3.6}$$

where we assume that the series can be differentiated term by term. In order to determine $B_n(t)$, we expand $\tilde{f}(t,x)$ in the orthonormal complete system $\{\varphi_n\}_{n\geq 1}$ (see proof of Theorem 6):

$$\tilde{f}(t,x) = \sum_{n=1}^\infty \tilde{f}_n(t)\varphi_n(x), \tag{3.7}$$

where

$$\tilde{f}_n(t) = <\tilde{f}(t,x),\varphi_n(x)> = \int_\Omega \tilde{f}(t,x)\varphi_n(x)\,dx.$$

Substituting (3.6), (3.7) into (3.3) yields

$$\sum_{n=1}^\infty \varphi_n(x)\left[D_t^\alpha B_n(t) + 2\lambda D_t^\delta B_n(t)\right] = \sum_{n=1}^\infty \varphi_n(x)\left[-a_n B_n(t) + \tilde{f}_n(t)\right], \tag{3.8}$$

where we have used the fact that $\varphi_n(x)$ is a solution of (3.4). Since $\{\varphi_n\}_{n\geq 1}$ is an orthonormal system, multiplying both members of (3.8) by φ_n and integrating on $\mathbb{R}^N$ lead to

$$D_t^\alpha B_n(t) + 2\lambda D_t^\delta B_n(t) + a_n B_n(t) = \tilde{f}_n(t). \tag{3.9}$$

On the other hand, since $v(t,x)$ satisfies the initial conditions in (3.3), we must have

$$\sum_{n=1}^\infty \partial_t^{(k)} B_n(0)\varphi_n(x) = \tilde{\varphi}_k(x), \quad k = 0,\ldots, m_\alpha - 1 \qquad x \in \overline{\Omega},$$

which implies

$$\partial_t^{(k)} B_n(0) = <\tilde{\varphi}_k(x),\varphi_n(x)> = \int_\Omega \tilde{\varphi}_k(x)\varphi_n(x)\,dx =: c_{k,n}, \quad n \geq 1, \ k = 0,\ldots, m_\alpha - 1. \tag{3.10}$$

Finally, for each value of n, (3.9) and (3.10) constitute a fractional initial value problem.

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Step 3 (Analytical solution). According to Theorem 3, the fractional initial value problem (3.9)-(3.10) has the analytical solution

$$B_n(t) = \int_0^t s^{\alpha-1} E_{(\alpha-\delta,\alpha),\alpha}(-2\lambda s^{\alpha-\delta}, -a_n s^{\alpha}) \tilde{f}_n(t-s) \, ds + \sum_{k=0}^{m_\alpha-1} c_{k,n} v_k(t), \quad t \geq 0, \tag{3.11}$$

where

$$v_k(t) = \frac{t^k}{k!} + \sum_{i=l_k+1}^{2} \xi_i t^{k+\alpha-\mu_i} E_{(\alpha-\delta,\alpha),k+1+\alpha-\mu_i}(\xi_1 t^{\alpha-\delta}, \xi_2 t^{\alpha}), \quad k = 0, \dots, m_\alpha - 1, \tag{3.12}$$

with $\xi_1 := -2\lambda$, $\xi_2 = -a_n$, $\mu_1 = \delta$ and $\mu_2 = 0$. Note that l_k, $k = 0, \dots, m_\alpha - 1$, are as in Remark 1. Hence we get the analytical solution of the initial boundary value problem (3.3) under the form

$$v(t,x) = \sum_{n=1}^{\infty} B_n(t)\varphi_n(x) = \sum_{n=1}^{\infty} \left[\int_0^t s^{\alpha-1} E_{(\alpha-\delta,\alpha),\alpha}(-2\lambda s^{\alpha-\delta}, -a_n s^{\alpha}) \tilde{f}_n(t-s) \, ds + \sum_{k=0}^{m_\alpha-1} c_{k,n} v_k(t) \right] \varphi_n(x), \tag{3.13}$$

where $v_k(t)$ are given in (3.12), $k = 0, \dots, m_\alpha - 1$. Therefore the analytical solution of (1.1)-(1.2) is:

$$u(t,x) = v(t,x) + w(t,x),$$

where $v(t,x)$ and $w(t,x)$ are, respectively, given by (3.13) and (3.1).

Remark 2. In problem (3.2), we had to work with boundary conditions defined on Ω^c and not only on $\partial\Omega$ since the following β-harmonic equation

$$\begin{cases} (-\Delta)^{\beta/2} y(x) = 0, & \text{in } \Omega, \\ y(x) = g_1(x), & \text{in } \partial\Omega. \end{cases}$$

is ill-posed (see [5, 17, 26]. In fact, the probabilistic solution gives the best insight: $(-\Delta)^{\beta/2}$ generates a jump process and this process may jump from Ω into Ω^c, i.e. we need "all of $\mathbb{R}^N$" to understand the corresponding process.

Remark 3. In Theorem 1, the condition on the function g_1 being defined over Ω^c and not on $\partial\Omega$ may seem strange at the first glance. However, this condition was set in order to be compatible with that required (see the remark above) for the existence of (3.2). In fact, assuming some regularity on the set Ω allows us to define g_1 on $\partial\Omega$ and to extend it suitably over Ω^c. For example, if Ω is smooth, convex and bounded, we may choose the same extension as in Step 1 of the proof of Theorem 2 (see Section 4 below)

3.1. Special Cases

The solution of (1.1)-(1.2) can be considered as a generalized solution for a wide variety of evolution equations. However, we only list here two special cases:

Case 1. Take $\alpha = 2\delta$, $m_\alpha = 2$, $m_\delta = 1$, $\beta = 2$, $N = 1$, $\Omega = (0, L)$. Then the eigenvalues and the eigenvectors can be expressed as follows:

$$a_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad \varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right),$$

for all $n \geq 1$. Therefore

$$v_0(t) = 1 - \left(\frac{n\pi}{L}\right)^2 t^{2\delta} E_{(\delta,2\delta),1+2\delta}\left(-2\lambda t^\delta, -\left(\frac{n\pi}{L}\right)^2 t^{2\delta}\right)$$

and

$$v_1(t) = t - 2\lambda t^{1+\delta} E_{(\delta,2\delta),2+\delta}\left(-2\lambda t^\delta, -\left(\frac{n\pi}{L}\right)^2 t^{2\delta}\right) - \left(\frac{n\pi}{L}\right)^2 t^{1+2\delta} E_{(\delta,2\delta),2+2\delta}\left(-2\lambda t^\delta, -\left(\frac{n\pi}{L}\right)^2 t^{2\delta}\right).$$

Moreover the particular solution w of (3.1) can be given by

$$w(t, x) = \begin{cases} g_1(0)g_2(t) + \dfrac{(g_1(L)g_2(t) - g_1(0)g_2(t))x}{L}, & \text{for } x \in [0, L], \\[2mm] g_1(L)g_2(t), & \text{for } x \geq L, \\[2mm] g_1(0)g_2(t), & \text{for } x \leq 0. \end{cases} \tag{3.14}$$

The analytical solution of (1.1)-(1.2) is

$$u(t, x) = v(t, x) + w(t, x),$$

where

$$v(t, x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \left[\int_0^t s^{2\delta-1} E_{(\delta,2\delta),2\delta}\left(-2\lambda s^\delta, -\left(\frac{n\pi}{L}\right)^2 s^{2\delta}\right) \tilde{f}_n(t-s)\, ds + B_n(0)v_0(t) + B'_n(0)v_1(t) \right] \sin\left(\frac{n\pi}{L} x\right),$$

with

$$B_n(0) = \sqrt{\frac{2}{L}} \int_0^L \tilde{\varphi}_0(x) \sin(\frac{n\pi}{L} x)\, dx, \quad B'_n(0) = \sqrt{\frac{2}{L}} \int_0^L \tilde{\varphi}_1(x) \sin(\frac{n\pi}{L} x)\, dx \tag{3.15}$$

and

$$\tilde{f}_n(t) = \sqrt{\frac{2}{L}} \int_0^L \tilde{f}(t, x) \sin(\frac{n\pi}{L} x)\, dx. \tag{3.16}$$

This result is in accord with the result obtained in [9, Section 3].

Case 2. Take $\alpha = 2$, $\delta = 1$, $m_\alpha = 2$, $m_\delta = 1$, $\beta = 2$, $\lambda = 0$, $N = 1$, $\Omega = (0, L)$. As above, one can choose the following expression as the eigenvalues and the eigenvectors respectively:

$$a_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad \varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right),$$

for all $n \geq 1$. Therefore

$$v_0(t) = 1 - \left(\frac{n\pi}{L}\right)^2 t^2 E_{(1,2),3}\left(0, -\left(\frac{n\pi}{L}\right)^2 t^2\right) = 1 - \left(\frac{n\pi}{L}\right)^2 t^2 E_{2,3}\left(-\left(\frac{n\pi}{L}\right)^2 t^2\right) = \cos\left(\frac{n\pi}{L} t\right)$$

and

$$v_1(t) = t - \left(\frac{n\pi}{L}\right)^2 t^3 E_{(1,2),4}\left(0, -\left(\frac{n\pi}{L}\right)^2 t^2\right) = t - \left(\frac{n\pi}{L}\right)^2 t^3 E_{2,4}\left(-\left(\frac{n\pi}{L}\right)^2 t^2\right) = \frac{L}{n\pi} \sin\left(\frac{n\pi}{L} t\right).$$

Since

$$s E_{(1,2),2}\left(0, -\left(\frac{n\pi}{L}\right)^2 s^2\right) = \sin\left(\frac{n\pi}{L} s\right),$$

the solution is given by

$$u(t, x) = v(t, x) + w(t, x)$$

with

$$v(t, x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \left[\int_0^t \sin\left(\frac{n\pi}{L} s\right) \tilde{f}_n(t-s)\, ds + B_n(0)v_0(t) + B'_n(0)v_1(t) \right] \sin\left(\frac{n\pi}{L} x\right),$$

where $B_n(0)$, $B'_n(0)$ and $\tilde{f}_n(t)$ are, respectively, given by (3.14), (3.15) and (3.16). Indeed, this result is the solution of the wave equation.

4. Proof of Theorem 2

In this section, we derive the analytical solution of problem (1.1) with nonhomogeneous Neumann boundary conditions (1.3).

Proof of Theorem 2. We proceed in four steps.

Step 1 (β-harmonic equation with Neumann conditions). Consider the particular extension ξ of g_1 over Ω^c:

$$\xi(x) = g_1(\overline{x})|x - \overline{x}|, \qquad x \in \Omega^c$$

where $\overline{x} := \text{Pr}_{\overline{\Omega}}(x)$ is the orthogonal projection of x on $\overline{\Omega}$. This function is well-defined by the uniqueness of $\overline{x}$ due to the convexity of Ω. Then, for all $z \in \partial\Omega$, we have

$$
\begin{aligned}
\frac{\partial \xi}{\partial n}(z) &= \lim_{h \to 0} \frac{\xi(z + h\vec{n}) - \xi(z)}{h} \\
&= \lim_{h \to 0} \frac{g_1(\overline{z + h\vec{n}})|(z + h\vec{n}) - (\overline{z + h\vec{n}})| - g_1(\overline{z})|z - \overline{z}|}{h} \\
&= \lim_{h \to 0} \frac{hg_1(z) - 0}{h} \\
&= g_1(z),
\end{aligned}
$$

where we have used the fact that $\overline{z + h\vec{n}} = \overline{z} = z$. Let y be a solution of the following β-harmonic equation with Dirichlet conditions

$$
\begin{cases}
(-\Delta)^{\beta/2} y(x) = 0, & \text{in } \Omega, \\
y(x) = \xi(x), & \text{in } \Omega^c.
\end{cases}
$$

Note that

$$\frac{\partial y}{\partial n}(z) = \frac{\partial \xi}{\partial n}(z) = g_1(z), \qquad z \in \partial\Omega.$$

This yields that y is a solution of the following β-harmonic equation with Neumann conditions

$$
\begin{cases}
(-\Delta)^{\beta/2} y = 0, & \text{in } \Omega, \\
\dfrac{\partial y}{\partial n} = g_1, & \text{on } \partial\Omega.
\end{cases}
\tag{4.1}
$$

Step 2 (Change of variables). We argue in a similar manner as in Step 1 of the proof of Theorem 1.1. In fact, let

$$u(x, t) = v(x, t) + w(x, t),$$

where

$$w(x, t) = y(x)g_2(t) \tag{4.2}$$

with y is a solution of (4.1) and consequently

$$\frac{\partial w}{\partial n}(x, t) = g_1(x)g_2(t), \qquad \text{for all } x \in \partial\Omega, \ t \geq 0.$$

The function $v(x, t)$ is an unknown function satisfying the following problem with homogeneous boundary conditions

$$
\begin{cases}
D_t^\alpha v(x, t) + 2\lambda D_t^\delta v(x, t) = -c^2(-\Delta)^{\beta/2} v(x, t) + \tilde{f}(t, x), & x \in \Omega, \ t > 0, \\
\partial_t^{(k)} v(x, 0) = \tilde{\phi}_k(x), \quad k = 0, \ldots, m_\alpha - 1, & x \in \overline{\Omega}, \\
\dfrac{\partial v}{\partial n}(x, t) = 0, & x \in \partial\Omega, \ t \geq 0,
\end{cases}
\tag{4.3}
$$

where

$$\begin{aligned}
\tilde{f}(t,x) &= -D_t^\alpha w(x,t) - 2\lambda D_t^\delta w(x,t) + f(t,x), \\
\tilde{\varphi}_k(x) &= \varphi_k(x) - \partial_t^{(k)} w(x,0), \quad k = 0, \ldots, m_\alpha - 1.
\end{aligned}$$

Step 3 (Separation of variables). Let $v(t,x) = X(x)T(t)$ (supposing that $\tilde{f}(t,x) = 0$) and substitute it in (4.3), we get the following fractional differential equation with Neumann boundary conditions:

$$\begin{cases} (-\Delta)^{\beta/2} X = \dfrac{a}{c^2} X, & \text{in } \Omega, \\ \dfrac{\partial X}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Applying Theorem 6 with $A = -\Delta$, $\alpha = \beta/2$, $H = L^2(\Omega)$ and $D(A) = \{u \in H^2(\Omega); \ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$, there exists $\{\lambda_n^{\beta/2}\}_{n\geq 1}$, $\{\varphi_n\}_{n\geq 1}$ eigenvalues and eigenvectors of (4.4) where $\{\lambda_n\}_{n\geq 1}$, $\{\varphi_n\}_{n\geq 1}$ are the eigenvalues and eigenvectors of (4.4) with $\beta = 2$ (see Theorem 5). Note that for $a = a_n$, $n \geq 1$, we have $X = X_n = \varphi_n$ and $\frac{a_n}{c^2} = \lambda_n^{\beta/2}$. Hence, the solution of (4.3) has the form

$$v(t,x) = \sum_{n=1}^\infty B_n(t)\varphi_n(x).$$

By repeating the same argument as in Step 2 of the proof of Theorem 1.1, we conclude that

$$\begin{cases} D_t^\alpha B_n(t) + 2\lambda D_t^\delta B_n(t) + a_n B_n(t) = \tilde{f}_n(t) \\ \partial_t^{(k)} B_n(0) = \displaystyle\int_\Omega \tilde{\varphi}_k(x)\varphi_n(x)\,dx =: c_{k,n}, \quad n \geq 1, \ k = 0, \ldots, m_\alpha - 1. \end{cases}$$

Step 4 (Analytical solution). The analytical solution of (1.1)-(1.3) is

$$u(t,x) = v(t,x) + w(t,x),$$

where $v(t,x)$ and $w(t,x)$ are given by (3.13) and (4.2) respectively.

4.1. A Special Case

We give a special case of the solution of (1.1)-(1.3). Take $\alpha = 2\delta$, $m_\alpha = 2$, $m_\delta = 1$, $\beta = 2$, $N = 1$, $\Omega = (0,L)$ and $g_1 = C$ is a constant function. Then the eigenvalues and the eigenvectors of problem (4.4) can be expressed as follows:

$$a_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad \varphi_n(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi}{L}x\right),$$

for all $n \geq 1$. Therefore

$$v_0(t) = 1 - \left(\frac{n\pi}{L}\right)^2 t^{2\delta} E_{(\delta,2\delta),1+2\delta}\left(-2\lambda t^\delta, -\left(\frac{n\pi}{L}\right)^2 t^{2\delta}\right)$$

and

$$v_1(t) = t - 2\lambda t^{1+\delta} E_{(\delta,2\delta),2+\delta}\left(-2\lambda t^\delta, -\left(\frac{n\pi}{L}\right)^2 t^{2\delta}\right) - \left(\frac{n\pi}{L}\right)^2 t^{1+2\delta} E_{(\delta,2\delta),2+2\delta}\left(-2\lambda t^\delta, -\left(\frac{n\pi}{L}\right)^2 t^{2\delta}\right).$$

Moreover the particular solution w of (3.1) is given by

$$w(t,x) = Cxg_2(t) \qquad x \in [0,L], \ t \geq 0. \tag{4.5}$$

The analytical solution is

$$u(t,x) = v(t,x) + w(t,x),$$

where

$$v(t,x) = \sqrt{\frac{2}{L}} \sum_{n=1}^\infty \left[\int_0^t s^{2\delta-1} E_{(\delta,2\delta),2\delta}\left(-2\lambda s^\delta, -\left(\frac{n\pi}{L}\right)^2 s^{2\delta}\right) \tilde{f}_n(t-s)\,ds + B_n(0)v_0(t) + B_n'(0)v_1(t) \right] \cos\left(\frac{n\pi}{L}x\right),$$

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with

$$B_n(0) = \sqrt{\frac{2}{L}} \int_0^L \tilde{\varphi}_0(x) \cos(\frac{n\pi}{L}x)\, dx, \quad B_n'(0) = \sqrt{\frac{2}{L}} \int_0^L \tilde{\varphi}_1(x) \cos(\frac{n\pi}{L}x)\, dx$$

and

$$\tilde{f}_n(t) = \sqrt{\frac{2}{L}} \int_0^L \tilde{f}(t,x) \cos(\frac{n\pi}{L}x)\, dx.$$

This result is in accord with the result obtained in [9, Section 4].

5. Conclusions

In this paper we have considered the nonhomogeneous space-time fractional telegraph equation (1.1) under Dirichlet and Neumann boundary conditions (1.2) and (1.3), respectively. We have derived the analytical solutions of problems (1.1)-(1.2) and (1.1)-(1.3) using the separation of variables method. The time fractional derivative is considered in the Caputo sense and the space fractional derivative in the sense of fractional Laplacian. The general solutions, which are given in the form of the multivariate Mittag-Leffler function, reduce to those of the classical telegraph equation and corresponding diffusion and wave equations.

Acknowledgments

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