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Abstract

We study the average complexity of certain numerical algorithms when adapted to solving systems of multivariate polynomial equations whose coefficients belong to some fixed proper real subspace of the space of systems with complex coefficients. A particular motivation is the study of the case of systems of polynomial equations with real coefficients. Along these pages, we accept methods that compute either real or complex solutions of these input systems. This study leads to interesting problems in Integral Geometry: the question of giving estimates on the average of the normalized condition number along great circles that belong to a Schubert subvariety of the Grassmannian of great circles on a sphere. We prove that this average equals a closed formula in terms of the spherical Radon transform of the condition number along a totally geodesic submanifold of the sphere.

Keywords: Approximate zero theory, Smale's 17th Problem, Computational complexity, Probabilistic polynomial time.

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1. Introduction

1.1. The context of our new results

The main result of these pages is motivated by the study of the real version of Smale's 17th Problem. In [42], S. Smale proposed the following problem:

Problem 1 (Smale's 17th Problem)**.** *"Can a zero of n complex polynomial equations in n unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?"*

This problem was answered affirmatively in [13]: The authors exhibited a **ZPP** (Las Vegas) algorithm that solves systems of complex multivariate polynomial equations in average time $O(N^3)$, where N is the input length for dense encoding of multivariate polynomials (cf. also [12] for a survey on the topic). Another **ZPP** algorithm solving the same problem in average time $O(N^2)$ was shown in [14].

There is, however, much room for improvement and further research. Some open questions follow:

- Find a deterministic *average polynomial time* algorithm that solves systems of multivariate complex polynomial equations. Some deep advances in this direction have been made in [20]. These authors use the powerful "smoothed analysis", by Cheng and Spielman, to exhibit a deterministic algorithm with sub-exponential average time with a small exponent of order $O(\log_2 \log_2 N)$. But the problem of a deterministic average polynomial time algorithm remains open.

- Find an algorithm (either deterministic or probabilistic) with polynomial complexity on average that solves systems of multivariate polynomial equations when the inputs are given by encoding alternatives to dense encoding: sparse/fewnomials systems, straight-line program encoding, etc... To our knowledge, no meaningful advance has been made to date in this direction.

In his original statement of Problem 17th, S. Smale also addressed the question about real solving:

Problem 2 (Smale's 17th Problem, real case)**.** *"...Similar, more difficult, problems may be raised for real polynomial systems (and even with inequalities)."*

Namely, try to solve real systems in average polynomial time. In these pages we focus on this real case of Smale's problem. To date, real solving of systems of polynomial equations with real coefficients has shown strong resistance to be solved in polynomial time on average.

There are two main approaches dealing with this kind of problems: Symbolic/Geometric and Numerical Solving. We are not concerned here with Symbolic/Geometric methods. The reader interested in this approach may follow [3, 4, 5, 6, 7] and references therein.

In this article, we are concerned with the numerical approach. A serious attempt to solve numerically systems of polynomial equations with real coefficients was made in the series [23, 24, 25]. Their proposal is based on the study of the probability distribution of a real condition number and then apply exhaustive search. The complexity has not been shown to be tractable.

Other studies of the properties of real systems on average have been made in [19, 21, 33] and references therein. Other attempts to use search algorithms (in this case, using exclusion methods) may be found in [27] and references therein.

On a completely different basis, a very positive experiment, using evolutive algorithms, is exhibited in [18]: The experiment shows excellent performance and a high probability of success to find an approximate zero for real zeros of real systems of multivariate polynomial equations. However, these experiments lack appropriate mathematical foundations.

Nevertheless, search is not necessarily the unique approach to numerical solving of real systems. Firstly, because we may not be interested in computing *all* solutions (which certainly forces an exponential running time) but computing *one* solution (see [11] for a discussion between universal and non-universal solving in numerical analysis). As in the methods shown to be efficient in the complex case, one may try to use an *homotopic deformation* technique approach (also called path following methods or continuation methods) to compute just one (real or complex) solution of systems of real polynomial equations. See, for instance, the books [1, 16, 34, 43] or surveys like [12, 32] and references therein for different statements of the algorithmic scheme of continuation methods.

The main drawback to the use of an homotopic deformation technique for systems with real coefficients is the codimension of the discriminant variety $\Sigma^{\mathbb{R}}$ in the space of polynomial equations with real coefficients $\mathcal{H}_{(d)}^{\mathbb{R}}$. Since the codimension of $\Sigma^{\mathbb{R}}$ in $\mathcal{H}_{(d)}^{\mathbb{R}}$ is one and since the number of real solutions (in $\mathbb{P}_n(\mathbb{R})$) is constant along each connected component of $\mathcal{H}_{(d)}^{\mathbb{R}}$, we conclude:

- The number of connected components outside the discriminant variety is exponential in the number $n+1$ of variables.
- The probability that for any two randomly chosen systems $f, g \in \mathcal{H}_{(d)}^{\mathbb{R}}$, every continuous path joining them in $\mathcal{H}_{(d)}^{\mathbb{R}}$ intersects the discriminant variety $\Sigma^{\mathbb{R}}$ is greater than the probability that they have a different number of real solutions (in $\mathbb{P}_n(\mathbb{R})$). To our knowledge there is no precise estimate for this quantity. See some related estimates in [2, 19, 40, 41, 44] and references therein.
- In the case of linear deformations, for any two randomly chosen systems $f, g \in \mathcal{H}_{(d)}^{\mathbb{R}}$ of norm 1, the expected number of points in the intersection between the great circle joining f and g and the discriminant variety equals the (codimension one) volume of the projection of the discriminant variety onto the sphere in $\mathcal{H}_{(d)}^{\mathbb{R}}$ of radius one. This is a mere consequence of Crofton-Poincaré's formula.

These facts cause some troubles for the standard method based on a lifting of these paths (through a covering map) and force the search for alternatives. One could be the proposal in [15]: follow a path inside the solution variety. This method has the inconvenience that there is no known method to construct the path to be followed without prior knowledge of the zero to be computed. This could be, perhaps, improved if we were able to compute geodesics with respect to the non-linear condition number metric (cf. the excellent manuscript [10], for instance). But, for the moment, there is no efficient method to compute them. Another proposal for real systems of equations could be that of [8], which traces real curves connecting the solutions of one system of equations to those of another but, in this case, no estimate of the number of steps is provided and, hence, no complexity estimate is known.

A different proposal is the one we suggest in these pages. First we choose to follow simplest paths as in the complex case: great circles on spheres. Then, instead of trying to solve real systems of multivariate polynomial equations by homotopic deformation that follows a path that goes from real systems to real systems, we propose to open up the space and *apply an homotopic deformation by following paths that begin in a complex (not real) initial system of equations and ends in a real system of equations.* This may be modeled in a simple saying:

Apply the (complex) algorithm described in [14] to real systems of polynomial equations and study its average complexity.

Certainly this approach is not expected to provide only real solutions of real systems: we just want to know if there is a low average complexity algorithm that computes approximate zeros of a single solution of systems of equations with real coefficients, accepting both real and complex solutions without establishing any preference among them.

This study leads to interesting problems in Integral Geometry, some of which are solved here. In principle, studying the average complexity of this kind of algorithm leads to the question of giving estimates on the average behavior of condition number along great circles that belong to certain Schubert subvariety of the Grassmannian of great circles on a sphere. We prove that this average equals a closed formula in terms of the spherical Radon transform of the condition number along an N-dimensional totally geodesic submanifold of the sphere of systems of polynomial equations with complex coefficients. This is the main result in these pages.

1.2. Statement of the main results

The first result explains the behavior of the expected value of an integrable function in certain Schubert subvarieties of real Grassmannians given as the set of great circles that intersect a given vector subspace. In order to state it we need to introduce some notation.

Let $S^n \subseteq \mathbb{R}^{n+1}$ be the real hypersphere of radius one, centered at the origin. For a real vector subspace $M \subseteq \mathbb{R}^{n+1}$ we denote by $S(M) \subseteq S^n$ the hypersphere defined by M. From now on, we assume that the codimension of M in $\mathbb{R}^{n+1}$ is greater than 2. We assume that S^n is endowed with the standard Riemannian structure and we denote by $\mathrm{d}S^n$ its canonical volume form. We denote by d_R the Riemannian distance in S^n and by $\mathrm{d}_{\mathbb{P}}$ the "projective" distance (i.e. $\mathrm{d}_{\mathbb{P}}(f,g) = \sin \mathrm{d}_R(f,g)$, for all $f, g \in S^n$). As the total volume of S^n is finite, we may define a probability distribution on S^n in the canonical way. Similarly, we may define in $S(M)$ and $S^n \times S(M)$ their canonical probability distributions. Given a point $(g,f) \in S^n \times S(M)$, we denote by $L_{(g,f)}$ the great circle in S^n passing through f and g. We may assume on

$L_{(g,f)}$ the standard volume form $\mathrm{d}L_{(g,f)}$ (the standard length). We begin by recalling the definition of spherical Radon Transform from [36].

Definition 1 ([36]). With the same notation, let $\varphi : S^n \longrightarrow \mathbb{R}_+$ be an integrable function, and let $k = n - p$ be the codimension of M in $\mathbb{R}^{n+1}$. The spherical Radon transform of φ with respect to $S(M)$ of order α is defined in the following terms:

$$\mathbf{R}^\alpha \varphi(S(M)) = \rho_{n,p}(\alpha) \int_{S^n} \frac{\varphi(g)}{\mathrm{d}_{\mathbb{P}}(g, S(M))^{n-p-\alpha}} \, \mathrm{d}S^n,$$

where

$$\rho_{n,p}(\alpha) = \frac{\mathrm{B}(\frac{n-p-\alpha+1}{2}, \frac{\alpha+p-1}{2})}{\nu_n},$$

ν_n is the standard volume of the unit sphere S^n and B is the usual Beta function.

Remark 3. Our normalization constant $\rho_{n,p}(\alpha)$ differs slightly from $\gamma_{n,p}(\alpha)$, the one used in [36]. Multiplying by $\rho_{n,p}(\alpha)^{-1}\gamma_{n,p}(\alpha)$, we obviously obtain the original definition of B. Rubin.

Then, we prove:

Theorem 4. *With the same notation as above, for every integrable function $\varphi : S^n \longrightarrow \mathbb{R}_+$, let E be the expectation given by the following identity:*

$$E = E_{(g,f)\in S^n \times S(M)} \left[\int_{L_{(g,f)}} \varphi(h) \, \mathrm{d}L_{(g,f)}(h) \right].$$

Moreover, for every n, p and i, let us define the constants:

$$C(n,p,i) := 2\binom{\frac{n-p}{2} - 1}{i} \frac{\mathrm{B}(\frac{n+2}{2}, \frac{1}{2})}{\mathrm{B}(\frac{p-1}{2}, \frac{1}{2})} \text{ and } B_0(n,p,i) := 2\binom{\frac{n-p-3}{2}}{i} \frac{\mathrm{B}(\frac{n+2}{2}, \frac{1}{2})}{\mathrm{B}(\frac{p-1}{2}, \frac{1}{2})}.$$

In terms of the value of the codimension $k = n - p$, the following equalities and inequalities hold:

1. *If $k = 1$, then*

$$\frac{4\sqrt{2\pi}}{(n+\sqrt{3})^{1/2}} E_{S^n}[\varphi] \leq E \leq \frac{\sqrt{(n-2)\pi}}{2} \mathbf{R}^0 \varphi(S(M));$$

2. *If* $k \in 2\mathbb{N}^*$, *then*

$$E = \sum_{i=0}^{\frac{n-p-2}{2}} C(n,p,i)\mathbf{R}^{n-p-2i-1}\varphi(S(M));$$

3. *If* $k \in (2\mathbb{N}^* + 1)$, *then*

$$E \geq \sum_{i=0}^{\frac{n-p-3}{2}} \frac{(n-2)B_0(n,p,i)}{\sqrt{i+\sqrt{3/2}}}\mathbf{R}^{n-p-2i-2}\varphi(S(M)),$$

$$E \leq \sum_{i=0}^{\frac{n-p-3}{2}} \frac{8B_0(n,p,i)}{(2i+1)(n-2)}\mathbf{R}^{n-p-2i-2}\varphi(S(M)).$$

Remark 5. Note that using Gautschi's [30] and Kershaw's [31] inequalities we also have the following sharp bounds of our coefficients:

$$2\binom{\frac{n-p}{2}-1}{i}\sqrt{\frac{p-\frac{3}{2}}{n+\sqrt{3}}} \leq C(n,p,i) \leq 2\binom{\frac{n-p}{2}-1}{i}\sqrt{\frac{p-3+\sqrt{3}}{n+\frac{3}{2}}},$$

$$\frac{(n-2)B_0(n,p,i)}{\sqrt{i+\sqrt{3/2}}} \geq 2\binom{\frac{n-p-3}{2}}{i}(n-2)\sqrt{\frac{(2p-3)}{(2i+\sqrt{3})(n+\sqrt{3})}},$$

and

$$\frac{8B_0(n,p,i)}{(2i+1)(n-2)} \leq 16\binom{\frac{n-p-3}{2}}{i}\frac{\sqrt{2(p-3+\sqrt{3})}}{(2i+1)(n-2)\sqrt{2n+3}}.$$

Note that the largest integral terms in identities (2) and (3) of Theorem 4 correspond to the case $i = 0$. Some less sharp, but illustrative, upper and lower bounds are exhibited in the following corollary.

Corollary 6. *With the same notation as above, for* $k = n - p \geq 2$, E *is bounded as follows:*

$$2\sqrt{\frac{p+\frac{1}{2}}{n+\sqrt{3}}}\mathbf{R}^1\varphi(S(M)) \leq E \leq 2\sqrt{\frac{p-1+\sqrt{3}}{n+\frac{3}{2}}}\frac{1}{\mathrm{B}(\frac{n-p}{2},\frac{p}{2})}\mathbf{R}^1\varphi(S(M)).$$

Note that the upper bound satisfies:

$$\frac{1}{\mathrm{B}(\frac{n-p}{2}, \frac{p}{2})}\mathbf{R}^1\varphi(S(M)) = E_{S^n}\left[\frac{\varphi(g)}{\mathrm{d}_{\mathbb{P}}(g, S(M))^{n-p-1}}\right],$$

where E_{S^n} means expectation.

In the path to the proof of this statement, we also prove the following integral formula in some incidence subvariety of the Grassmann manifold:

Let $\mathcal{L}$ be the Grassmannian given as the set of great circles in S^n and denote by $\mathcal{L}_M$ the semialgebraic subset defined as those great circles $L \in \mathcal{L}$ such that $L \cap M \neq \emptyset$.

We shall see that $\mathcal{L}_M$ may be decomposed as a union of two real manifolds $\mathfrak{C}_M \cup G_{2,p+1}(\mathbb{R})$, where $\mathfrak{C}_M$ is the manifold of all great circles $L \in \mathcal{L}$ that intersect $S(M)$ in exactly 2 points and, $G_{2,p+1}(\mathbb{R})$ is the Grassmannian of great circles in $S(M)$. In fact, $\mathfrak{C}_M$ is formed by smooth regular points of maximal dimension in $\mathcal{L}_M$ and is a dense semialgebraic subset of $\mathcal{L}_M$.

The Riemann manifold $\mathfrak{C}_M$ is endowed with a natural volume form that we denote by $\mathrm{d}\nu_M$. This volume form extends to its closure $\mathcal{L}_M$ as a measure in the obvious way. For every function $\varphi : \mathcal{L}_M \longrightarrow \mathbb{R}$ we denote by

$$\int_{\mathcal{L}_M} \varphi \ \mathrm{d}\nu_M,$$

the integral of the restriction of φ to $\mathfrak{C}_M$ with respect to $\mathrm{d}\nu_M$ and for every subset $F \subseteq \mathcal{L}_M$ we denote by $\nu_M[F]$ the volume of the intersection $F \cap \mathfrak{C}_M$. We will prove that the volume $\nu_M[\mathcal{L}_M]$ is finite and, hence, this induces a natural probability distribution in $\mathcal{L}_M$.

Next, for every $L \in \mathcal{L}_M$, we have a function $\mathrm{d}_M : L \longrightarrow \mathbb{R}_+$ given by $\mathrm{d}_M(h) = \mathrm{d}_{\mathbb{P}}(h, S(M))^{\mathrm{codim}_{\mathbb{R}^{n+1}}(M)-1}$. We may define a measure on every line $L \in \mathcal{L}_M$ that we denote $\mathrm{d}\nu_M$ given by

$$E_{L_M}[\varphi] = \frac{1}{\mathrm{vol}_M[L]}\int_L \varphi \ \mathrm{d}_M(x) \ \mathrm{d}L,$$

where

$$\mathrm{vol}_M[L] = \int_L \mathrm{d}_M(x) \ \mathrm{d}L = \mathrm{B}\left(\frac{k+2}{2}, \frac{1}{2}\right)\partial_M(L)^{k-1},$$

where $k = \mathrm{codim}_{\mathbb{R}^{n+1}}(M)$ and $\partial_M(L) = \max\{\mathrm{d}_{\mathbb{P}}(h, S(M)), \ h \in L\}$.

In the path to prove the main result (Theorem 4) we also prove the following statement:

Proposition 7. *With the same notation as above, for every integrable function $\varphi : S^n \longrightarrow \mathbb{R}_+$ the following equality holds :*

$$E_{\mathcal{L}_M}[E_{L_M}[\varphi]] = E_{S^n}[\varphi].$$

In particular, we have

$$\mathrm{vol}[\mathcal{L}_M] = \frac{\mathrm{vol}[S^n]\,\mathrm{vol}[S(M)]}{\mathrm{B}(\frac{k+1}{2}, \frac{1}{2})},$$

where k is the codimension of M in $\mathbb{R}^{n+1}$.

1.3. The case of polynomial equations

As said before, the motivation of this study is the analysis of the average complexity of homotopic deformation algorithms for polynomial system solving. Here we will state some corollaries of Theorem 4 and of Proposition 7 above. We need some additional notation to state these corollaries.

For every positive integral number $d \in \mathbb{N}$, let H_d be the complex vector space spanned by the homogeneous polynomials $f \in \mathbb{C}[X_0, \ldots, X_n]$ of degree d. The complex space H_d is naturally endowed with a unitarily-invariant Hermitian inner product, known as Bombieri's Hermitian product (other authors use the terms Bombieri-Weyl's or even Kostlan's norm for the associated norm, cf. [16] for details). For every degree list $(d) = (d_1, \ldots, d_n)$ of positive integer numbers, we denote by $\mathcal{H}_{(d)}$ the complex vector space given as the product $\mathcal{H}_{(d)} = \prod_{i=1}^n H_{d_i}$. Note that if for every i, $1 \leq i \leq n$, $f_i \in \mathbb{C}[X_0, \ldots, X_n]$ is homogeneous of degree d_i, then $\mathcal{H}_{(d)}$ may be seen as the vector space of homogeneous systems of equations $f = (f_1, \ldots, f_n)$. The complex space $\mathcal{H}_{(d)}$ is endowed with the unitarily-invariant Hermitian product $\langle \cdot, \cdot \rangle_\Delta$ defined as the cartesian product of Bombieri's Hermitian products in H_{d_i}.

Let us denote by $N+1$ the complex dimension of $\mathcal{H}_{(d)}$ and by $\mathcal{D} = \prod_{i=1}^n d_i$ the *Bézout number* associated to the list $(d) = (d_1, \ldots, d_n)$.

Let $\| \cdot \|_\Delta$ be the norm associated to $\langle \cdot, \cdot \rangle_\Delta$ and let us denote by $\mathbb{S}^{2N+1} = \mathbb{S}(\mathcal{H}_{(d)})$ the unit sphere in $\mathcal{H}_{(d)}$ with respect to the norm $\| \cdot \|_\Delta$.

For every systems of equations $f = (f_1, \ldots, f_n) \in \mathcal{H}_{(d)}$, we denote by $V_{\mathbb{P}}(f) \subseteq \mathbb{P}_n(\mathbb{C})$ the complex projective algebraic variety of their common zeros. Namely,

$$V_{\mathbb{P}}(f) = \{\zeta \in \mathbb{P}_n(\mathbb{C}),\ f_i(\zeta) = 0,\ 1 \leq i \leq n\}.$$

Given $f \in \mathcal{H}_{(d)}$ and given $\zeta \in V_{\mathbb{P}}(f)$, we denote by $\mu_{\text{norm}}(f, \zeta)$ the normalized condition number of f at ζ (as introduced in [39]) and for every positive real $\alpha \in \mathbb{R}$, we will denote by $\mu_{\text{av}}^{\alpha}(f)$ the average of the αth power of condition number of f along its complex zeros. Namely,

$$\mu_{\text{av}}^{\alpha}(f) = \frac{1}{\sharp(V_{\mathbb{P}}(f))} \sum_{\zeta \in V_{\mathbb{P}}(f)} \mu_{\text{norm}}^{\alpha}(f, \zeta).$$

Studies of the average values of $\mu_{\text{av}}(f)^{\alpha}$, for $1 \leq \alpha < 4$ are exhibited in [14].

From [38] (and the explicit descriptions of the constants in [9, 20, 26]) the number of deformation homotopy steps along a great circle path performed by Newton's method from an initial system g with initial zero $\zeta \in V_{\mathbb{P}}(g)$ and target system f is bounded by the quantity:

$$\mathcal{C}(f, g, \zeta) = \int_L \mu_{\text{norm}}(h, \zeta_h)^2 \, \mathrm{d}L,$$

where L is the great circle containing g and f (which is assumed not to intersect the discriminant variety $\Sigma \subseteq \mathbb{S}(\mathcal{H}_{(d)})$.

Now we consider a probabilistic (We see it is **Z**ero-Error **P**robability or, in fact, Las Vegas in our case) algorithm based on the one introduced in [14], with set of initial pairs $\mathcal{G}_{(d)}$ that we call BP in the sequel. We also consider $M \subseteq \mathcal{H}_{(d)}$ a real vector subspace of the space of complex systems. For instance, M can be the real vector subspace $\mathcal{H}_{(d)}^{\mathbb{R}}$ of $\mathcal{H}_{(d)}$ of systems of equations with real coefficients. Another example could be the sparse case defined by the real vector space of polynomials with coefficients in a given polytope.

We denote by $\mathbb{S}(M) \subseteq \mathbb{S}^{2N+1}$ the sphere of radius 1 given by points in M with respect to Bombieri-Weyl's norm.

Our goal is the design of algorithms adapted to M as input space. Our proposal here will be the following variation of BP:

INPUT: A system $f \in M$

guess at random $(g, \zeta) \in \mathcal{G}_{(d)}$

Apply deformation homotopy with initial pair (g, ζ) and target f.

OUTPUT:

– Either FAILURE

– or an approximate zero $z \in \mathbb{P}_n(\mathbb{C})$ of f with associated zero $\zeta \in \mathbb{P}_n(\mathbb{C})$.

The first obvious consequence of our study is the following one:

Corollary 8. *Let* $\Sigma \subseteq \mathbb{S}^{2N+1}$ *be the discriminant variety (as defined in [16, 39]). Assume that* $\dim(\Sigma \cap \mathbb{S}(M)) < \dim \mathbb{S}(M)$*. Then, the probability that the algorithm above outputs* Failure *is* 0*. Namely, the probability that the algorithm outputs an approximate zero associated to some input system* $f \in M = \mathcal{H}_{(d)}^{\mathbb{R}}$ *is* 1*.*

Nevertheless, the problem is not the soundness of the algorithm, but the average complexity. The usual upper bound for the average complexity of such an algorithm (assuming Gaussian distribution on M) will be the expected value

$$E_M = E_M[\mathrm{Time}] = E_{\mathcal{G}_{(d)} \times \mathbb{S}(M)}[\mathcal{C}(f, g, \zeta)].$$

The following statements are different estimates for this quantity E.

As in the previous subsection, we will denote by $\mathcal{L}$ the Grassmannian of real great circles in $\mathbb{S}^{2N+1}$ and by $\mathcal{L}_M$ the great circles in $\mathcal{L}$ that intersect $\mathbb{S}(M)$.

From Theorem 4 we also obtain the following consequence:

Corollary 9. *With the same notation as above, assume* $\dim(M) = p + 1$ *and let* $C(2N+1, p, i), B_0(2N+1, p, i)$ *be the same constants as defined in Theorem 4. Let* $k = 2N - p + 1$ *be the codimension, then, we have:*

1. *If* $k = 1$*, then:*

$$\frac{4\sqrt{\pi}}{(N+2)^{1/2}} E_{\mathbb{S}^{2N+1}}[\mu_{\mathrm{av}}^2] \leq E \leq \sqrt{\frac{\left(N - \frac{1}{2}\right)\pi}{2}} \mathbf{R}^0[\mu_{\mathrm{av}}^2](\mathbb{S}(M));$$

2. *If* $k \in 2\mathbb{N}^*$*, then the average estimate of the complexity based on the condition number* E_M *satisfies:*

$$E_M = \sum_{i=0}^{\frac{2N-p-1}{2}} C(2N+1, p, i) \mathbf{R}^{2(N-i)-p}[\mu_{\mathrm{av}}^2](\mathbb{S}(M));$$

3. *If* $k \in (2\mathbb{N}^* + 1)$*, then*

$$E_M \geq \sum_{i=0}^{\frac{2N-p-2}{2}} \frac{(2N-1)B_0(2N+1,p,i)}{\sqrt{i+\sqrt{3/2}}} \mathbf{R}^{2(N-i)-p-1}[\mu_{\mathrm{av}}^2](\mathbb{S}(M)),$$

$$E_M \leq \sum_{i=0}^{\frac{2N-p-2}{2}} \frac{8B_0(2N+1,p,i)}{(2i+1)(2N-1)} \mathbf{R}^{2(N-i)-p-1}[\mu_{\mathrm{av}}^2](\mathbb{S}(M)).$$

We also have:

Corollary 10. *With the same notation as above, the following inequalities hold:*

$$\sqrt{\frac{4p+2}{2N+1+\sqrt{3}}}\mathbf{R}^1[\mu_{\mathrm{av}}^2](\mathbb{S}(M)) \leq E_M \leq 2\sqrt{\frac{p-1+\sqrt{3}}{2N+\frac{5}{2}}}\left(\frac{\mathbf{R}^1[\mu_{\mathrm{av}}^2](\mathbb{S}(M))}{\mathrm{B}(N+\frac{1-p}{2},\frac{p}{2})}\right).$$

Or, equivalently,

$$\sqrt{\frac{4p+2}{2N+1+\sqrt{3}}}\mathbf{R}^1[\mu_{\mathrm{av}}^2](\mathbb{S}(M)) \leq E_M,$$

$$E_M \leq 2\sqrt{\frac{p-1+\sqrt{3}}{2N+\frac{5}{2}}} E_{\mathbb{S}^{2N+1}}\left[\frac{\mu_{\mathrm{av}}^2(g)}{\mathrm{d}_{\mathbb{P}}(g,\mathbb{S}(M))^{2N-p}}\right].$$

Corollary 11. *With the same notation as above, let* $p+1$ *be the dimension of* M *and* $k = 2N-p+1$ *be the codimension of* M *in* $\mathcal{H}_{(d)}$*. Then the following equality holds:*

$$E_M = T(N,p)E_{\mathcal{L}_M}\left[\frac{1}{\partial_M(L)}\int_L \mu_{\mathrm{av}}^2(h)\ \mathrm{d}L\right],$$

where

$$T(N,p) = \frac{2\mathrm{B}(N+\frac{3}{2},\frac{1}{2})}{\mathrm{B}(N+1-\frac{p}{2},\frac{1}{2})}.$$

Note that, according to Gautschi's and Kershaw's bounds, $T(N,p)$ is asymptotically in $\Theta\left(\left(1-\frac{p}{2N}\right)^{1/2}\right)$.

Now we are in conditions to exhibit some average complexity upper bounds for the application of the algorithm in [14] to systems with real coefficients. This is resumed in the following corollary.

Corollary 12. *Assume now that M is the real vector subspace of systems with real coefficients (i.e. $M = \mathcal{H}_{(d)}^{\mathbb{R}}$). Denote by $E_{\mathbb{R}}$ the expected number of steps of the underlying homotopy of [38] (i.e. $E_{\mathbb{R}} = E_M$ under our hypothesis). As $\dim_{\mathbb{R}} M = p = N+1$ and $\dim_{\mathbb{R}} \mathcal{H}_{(d)} = 2N+2$, then the codimension k of M is $N+1$ and the following holds:*

1. *If the codimension $(N+1) \in 2\mathbb{N}^*$, then $E_{\mathbb{R}}$ satisfies:*

$$E_{\mathbb{R}} = \sum_{i=0}^{\frac{N-1}{2}} C(2N+1, N, i)\mathbf{R}^{N-2i}[\mu_{\text{av}}^2](\mathbb{S}(\mathcal{H}_{(d)}^{\mathbb{R}}));$$

2. *If the codimension $(N+1) \in (2\mathbb{N}^*+1)$, then $E_{\mathbb{R}}$ satisfies the following inequalities:*

$$\sqrt{\frac{4N+2}{2N+1+\sqrt{3}}}\mathbf{R}^1[\mu_{\text{av}}^2](\mathbb{S}^N) \le E_{\mathbb{R}},$$

$$E_{\mathbb{R}} \le 2\sqrt{\frac{N-1+\sqrt{3}}{2N+\frac{5}{2}}} E_{\mathbb{S}^{2N+1}}\left[\frac{\mu_{\text{av}}^2(g)}{\mathrm{d}_{\mathbb{P}}(g, \mathbb{S}^N)^N}\right],$$

and therefore

$$\mathbf{R}^1[\mu_{\text{av}}^2](\mathbb{S}^N) \le E_{\mathbb{R}} \le \sqrt{2}\left(\frac{\mathbf{R}^1[\mu_{\text{av}}^2](\mathbb{S}^N)}{\mathrm{B}(\frac{N+1}{2}, \frac{N}{2})}\right) = \sqrt{2}E_{\mathbb{S}^{2N+1}}\left[\frac{\mu_{\text{av}}^2(g)}{\mathrm{d}_{\mathbb{P}}(g, \mathbb{S}^N)^N}\right],$$

where $\mathbb{S}^N = \mathbb{S}(\mathcal{H}_{(d)}^{\mathbb{R}})$ and $\mathbb{S}^{2N+1} = \mathbb{S}(\mathcal{H}_{(d)})$.

The manuscript is structured as follows. In Section 2 we establish some basic facts about the underlying geometry of $\mathcal{L}_M$ as semialgebraic set and we also describe the Riemannian structure at regular points. In Section 3 we prove some technical results from Integral Geometry (mostly computing some normal Jacobians and basic integrals). In Section 4 we prove Theorem 4, Corollary 6 and Proposition 7 (the results stated in Section 1.2 above). In Section 5 we prove the corollaries stated in Section 1.3 above.

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2. The underlying geometry

The aim of this section is to prove the following statement concerning the geometry of the Schubert variety (as semialgebraic set) $\mathcal{L}_M$. We have not found an appropriate reference where both the algebraic geometry and the Riemannian metric statements (including an explicit description of the tangent spaces to the smooth points of $\mathcal{L}_M$) of the following lemma are stated. As we need both of them to prove our Theorem 4, we decided to include a self-contained proof.

Lemma 13. *Let $M \subseteq \mathbb{R}^{n+1}$ be a proper vector subspace of dimension $p+1$ and codimension $k = n - p > 0$. Let $\mathcal{L}_M$ be the set of great circles in $\mathcal{L}$ such that $L \cap S(M) \neq \emptyset$. Then, the following properties hold:*

1. *The semialgebraic set $\mathcal{L}_M$ decomposes as the union of two Riemannian manifolds $\mathfrak{C}_M \cup G_{2,p+1}(M)$, where*
   - *$\mathfrak{C}_M$ is the set of great circles $L \in \mathcal{L}$ such that L intersects $S(M)$ in exactly two points (i.e. $\sharp(L \cap S(M)) = 2$),*
   - *$G_{2,p+1}(M)$ may be identified with the Grassmannian of great circles in $S(M)$.*
2. *Manifold $\mathfrak{C}_M$ is made of smooth regular points of maximal dimension in $\mathcal{L}_M$ and it is a dense subset of $\mathcal{L}_M$ with respect to the topology induced in $\mathcal{L}_M$ by the Riemannian metric of $\mathcal{L}$.*
3. *The dimension of $\mathfrak{C}_M$ equals the dimension of $\mathcal{L}_M$ and satisfies:*

$$\dim_{\mathbb{R}} \mathfrak{C}_M = \dim_{\mathbb{R}} \mathcal{L}_M = n + p - 1.$$

4. *For every great circle $L \in \mathfrak{C}_M$ given as the intersection with $S(M)$ of a real plane spanned by a matrix A in the Stiefel manifold $ST_{2,n+1}(\mathbb{R})$, the tangent space $T_L\mathfrak{C}_M$ can be isometrically identified with*

$$T_L\mathfrak{C}_M = \left\{B \in T_L G_{2,n+1}(\mathbb{R}), \exists \eta \in T_f S^p, (\mathrm{Id}_{n+1} - A^T A)\eta^T = B^T A f^T\right\},$$

   where $\{\pm f\} = L \cap S(M)$, $G_{2,n+1}(\mathbb{R})$ is the Grassmannian of great circles in S^n, A^T, η^T, B^T, f^T are respectively the transposed matrices of A, η, B, f and Id_{n+1} is the $(n+1) \times (n+1)$ identity matrix.

2.1. Some known facts about Grassmannian, Schubert and incidence varieties

We have not found any appropriate reference for the details of this statement, hence we prove it here. Firstly, we just identify $M = \mathbb{R}^{p+1}$ and $S(M) = S^p$ and prove the lemma for this particular case.

We denote by $\mathcal{L}_n = G_{2,n+1}(\mathbb{R})$ (or simply $\mathcal{L}$ when no confusion arises) the Grassmannian of great circles in S^n. Recall that the Stiefel manifold $ST_{2,n+1}(\mathbb{R})$ is the real manifold of dimension $2n-1$ whose points are orthonormal bases of planes in $\mathbb{R}^{n+1}$ (written as $2 \times (n+1)$ matrices). For every matrix $A \in ST_{2,n+1}(\mathbb{R})$ the tangent space $T_A ST_{2,n+1}(\mathbb{R})$ is given by the following identity:

$$T_A ST_{2,n+1}(\mathbb{R}) := \{B \in \mathcal{M}_{2,n+1}(\mathbb{R}) \; BA^T + AB^T = 0\}.$$

where A^T still means transpose. For the remainder of this section we simplify notation by writing $ST(\mathbb{R}) = ST_{2,n+1}(\mathbb{R})$.

There is a natural left action defined by $O(2)$ over $ST(\mathbb{R})$ and $\mathcal{L}$ is the orbit manifold defined by this left action and the Riemannian structure of $\mathcal{L}$ is defined through the Riemannian structure of $ST(\mathbb{R})$.

We denote by $[A]$ the $O(2)$-orbit defined by $A \in ST(\mathbb{R})$ and we denote by $\mathcal{S}\mathrm{pan}(A) \subseteq \mathbb{R}^n$ the vector subspace of dimension 2 spanned by the rows of A.

Lemma 14. *Let $\pi : ST(\mathbb{R}) \longrightarrow \mathcal{L}$ be the canonical projection onto the orbit space. Then, for every $A \in ST(\mathbb{R})$, the tangent mapping $T_A\pi : T_A ST(\mathbb{R}) \longrightarrow T_{[A]}\mathcal{L}$ is given by the following identity:*

$$T_A\pi(B) = B(\mathrm{Id}_{n+1} - A^T A).$$

Proof. Note that the tangent space to the orbit $T_A[A] \subseteq T_A ST(\mathbb{R})$ is identified with the vector space of anti-symmetric matrices $T_{\mathrm{Id}_2}O(2)$ by the isomorphism $\psi : T_{\mathrm{Id}_2}O(2) \longrightarrow T_A[A]$, given by $\psi(N) = NA$. Note that for every $A \in ST(\mathbb{R})$ the inverse mapping ψ^{-1} is given by $\psi^{-1}(B) = BA^T$.

As $T_{[A]}\mathcal{L} \simeq T_A ST(\mathbb{R})/T_A[A]$, the orthogonal complement of $T_A[A]$ in $T_A ST(\mathbb{R})$ can be isometrically identified with $T_{[A]}\mathcal{L}$. Thus, the mapping $T_A\pi : T_A ST(\mathbb{R}) \longrightarrow T_{[A]}\mathcal{L}$ can be isometrically identified with the orthogonal projection onto the orthogonal complement of $T_A[A]$ in $T_A ST(\mathbb{R})$. Now, for every matrix $B \in T_A ST(\mathbb{R})$, the following decomposition holds: $T_A ST(\mathbb{R}) = T_A[A] \oplus^{\perp} T_A[A]^{\perp}$:

$$B = BA^T A + B(\mathrm{Id}_{n+1} - A^T A).$$

This orthogonal projection then satisfies $T_A\pi(B) = B(\mathrm{Id}_{n+1} - A^TA)$, as claimed. $\square$

Then, we conclude:

Proposition 15. *The Grassmannian $\mathcal{L}$ is a Riemannian manifold whose dimension satisfies:*

$$\dim \mathcal{L} = \dim ST(\mathbb{R}) - \dim O(2) = 2(n-1).$$

Moreover, for every $L = [A] \in \mathcal{L}$, the tangent space $T_L\mathcal{L}$ is given by the following equality:

$$T_L\mathcal{L} \cong \left\{ B \in \mathcal{M}_{2,n+1}(\mathbb{R}),\ AB^T = BA^T = 0 \right\},$$

where the metric is the one induced by Frobenius metric in $T_AST(\mathbb{R})$. That is,

$$\langle B_1, B_2 \rangle_F = \mathrm{Tr}(B_1B_2^T).$$

Proof. Since $AA^T = \mathrm{Id}_2$, for every $B \in T_AST_{2,n+1}(\mathbb{R})$, the following equalities hold:

$$B(\mathrm{Id}_{n+1} - A^TA)A^T = B(A^T - A^T) = 0.$$

Analogously, we have $A(\mathrm{Id}_{n+1} - A^TA)B^T = 0$. This proves that the image of $T_A\pi$ is contained in $\{B \in \mathcal{M}_{2,n+1}(\mathbb{R}),\ BA^T = AB^T = 0\}$. Comparing their dimensions we conclude that they are the same vector subspace of dimension $2(n-1)$. $\square$

We now consider the *incidence manifold* $\mathcal{I}(\mathbb{R})$ given by the following equality:

$$\mathcal{I}(\mathbb{R}) = \left\{ ([A], f) \in \mathcal{L} \times S^n,\ f \in \mathcal{S}\mathrm{pan}(A) \right\}.$$

The following are also well-known facts:

Proposition 16. *The incidence manifold $\mathcal{I}(\mathbb{R})$ is a compact Riemannian manifold whose dimension satisfies:*

$$\dim \mathcal{I}(\mathbb{R}) = \dim \mathcal{L} + 1 = 2n - 1.$$

For every $([A], f) \in \mathcal{I}(\mathbb{R})$, the tangent space $T_{([A],f)}\mathcal{I}(\mathbb{R})$ is given by the following equality:

$$T_{([A],f)}\mathcal{I}(\mathbb{R}) = \left\{ (B, \eta) \in T_{[A]}\mathcal{L} \times T_fS^n,\ \left(\mathrm{Id}_n - A^TA\right)\eta^T = B^TAf^T \right\},$$

and the metric structure in $T_{([A],f)}\mathcal{I}(\mathbb{R})$ is the one induced by those of $T_{[A]}\mathcal{L}$ and T_fS^n.

Proof. (Sketch) Let $(A(t), f(t))$ be a lifting to $ST(\mathbb{R}) \times S^n$ of a smooth curve inside $\mathcal{I}(\mathbb{R})$, such that $(A(0), f(0)) = (A, f)$. The fact that $f(t)$ belongs to the vector subspace V_t spanned by the rows of $A(t)$ may also be written as the fact that the orthogonal projection of $f(t)$ onto V_t equals $f(t)$. This yields the equation:

$$f^T(t) := A^T(t)A(t)f^T(t).$$

Differentiating at $t = 0$, we obtain:

$$\dot{f}^T = \dot{A}^T A f^T + A^T \dot{A} f^T + A^T A \dot{f}^T.$$

Thus, $(B, \eta) \in T_{[A]}\mathcal{L} \times T_f S^n$ is in $T_{([A],f)}\mathcal{I}(\mathbb{R})$ if, and only if, the following equality is satisfied:

$$(\mathrm{Id}_{n+1} - A^T A)\eta^T = (B^T A + A^T B) f^T.$$

Now, as f is in the vector space V_0 spanned by the rows of A and, as $B \in T_{[A]}\mathcal{L}$, we conclude that $AB^T = BA^T = 0$. We thus conclude that the rows of B are orthogonal to any vector in V_0 and, in particular, to f. This yields $Bf^T = 0$ and, hence, $A^T B f^T = 0$. $\square$

Let π_1, π_2 be the restrictions to $\mathcal{I}(\mathbb{R})$ of the two canonical projections from $\mathcal{L} \times S^n$. Namely, we consider the mappings:

$$\pi_1 : \mathcal{I}(\mathbb{R}) \longrightarrow \mathcal{L}, \ \pi_2 : \mathcal{I}(\mathbb{R}) \longrightarrow S^n,$$

given by

$$\pi_1([A], f) = [A], \ \pi_2([A], f) = f.$$

Proposition 17. *With this notation, π_1 and π_2 are submersions. In particular, for every $p < n$, the inverse image $\mathcal{I}(S^p) = \pi_2^{-1}(S^p)$ is a Riemannian submanifold of $\mathcal{I}(\mathbb{R})$ whose dimension satisfies:*

$$\dim \mathcal{I}(S^p) = n + p - 1.$$

Moreover, for every $([A], f) \in \mathcal{I}(S^p)$, the tangent space $T_{([A],f)}(\mathcal{I}(S^p))$ satisfies:

$$T_{([A],f)}(\mathcal{I}(S^p)) = T_{([A],f)}\pi_2^{-1}(T_f S^p).$$

Namely, the following equality holds:

$$T_{([A],f)}\mathcal{I}(S^p) = \left\{(B, \eta) \in T_{[A]}\mathcal{L} \times T_f S^p, \ \left(\mathrm{Id}_{n+1} - A^T A\right)\eta^T = B^T A f^T\right\},$$

and the Riemannian metric is the one induced as subspace of $T_{[A]}\mathcal{L} \times T_f S^p$.

Proof. It follows from standard arguments from the fact that π_2 is a submersion. The reader may follow them in [28, Chapter III], for instance. $\square$

2.2. The Schubert variety $\mathcal{L}_M$: Proof of Lemma 13

Definition 2. We define the *Schubert variety $\mathcal{L}_M$* as

$$\mathcal{L}_M = \pi_1\left(\mathcal{I}(S(M))\right) = \pi_1\left(\pi_2^{-1}\left(S(M)\right)\right),$$

where we have identified $S(M)$ as a submanifold of S^n. Namely, $\mathcal{L}_M$ is the semialgebraic set of all great circles in S^n that intersect $S(M)$.

Without loss of generality we may assume $M = \mathbb{R}^{p+1}$, where $\mathbb{R}^{p+1}$ is identified with the vector subspace of $\mathbb{R}^{n+1}$ whose last $n-p$ coordinates are zero. Accordingly, $S(M)$ is identified with S^p.

Let us also define the mapping $\pi_1^{(2)} : \mathcal{I}(S^p) \longrightarrow \mathcal{L}$ as the restriction

$$\pi_1^{(2)} = \pi_1 \mid_{\pi_2^{-1}(S^p)} .$$

Let $\mathfrak{C}_M$ be the set of points $[A] \in \mathcal{L}_M$ such that $\sharp\left(\mathcal{S}\mathrm{pan}(A) \cap S^p\right) = 2$. In other words, $\mathfrak{C}_M$ is the set of great circles in S^n such that their intersection with S^p consists of exactly two points $\pm f$. Note that $\mathcal{L}_M \setminus \mathfrak{C}_M$ is the set of great circles in S^n which are completely embedded in S^p. In particular, $\mathcal{L}_M \setminus \mathfrak{C}_M = G_{2,p+1}(\mathbb{R})$ is the Grassmannian of great circles in S^p. The following proposition implies Lemma 13.

Proposition 18. *With this notation, the following properties hold:*

1. *For every $([A], f) \in \mathcal{I}(S^p)$, the tangent mapping $T_{([A],f)}\pi_1^{(2)}$ is injective if, and only if, $[A] \in \mathfrak{C}_M$. In particular, $\pi_1^{(2)} : \mathcal{I}(S^p) \longrightarrow \mathcal{L}_M$ is an immersion at every $([A], f) \in \mathcal{I}(S^p)$ such that $[A] \in \mathfrak{C}_M$;*
2. *For every $[A] \in \mathfrak{C}_M$ and $([A], f) \in \mathcal{I}(S^p)$ the following properties hold:*
   - *The point $[A]$ is a regular point of maximal dimension in $\mathcal{L}_M$;*
   - *The mapping $\pi_1^{(2)} : \mathcal{I}(S^p) \longrightarrow \mathcal{L}_M$ is a 2-fold smooth covering map and a submersion in a neighborhood of $([A], f)$;*
   - *The following equality holds:*

$$\dim_{([A],f)} \mathcal{I}(S^p) = \dim_{[A]} \mathcal{L}_M = \dim \mathcal{L}_M = n + p - 1.$$

In particular, for every $[A] \in \mathfrak{C}_M$ the tangent spaces satisfy:

$$T_{[A]}\mathcal{L}_M = T_{([A],f)}\pi_1^{(2)}\left(T_{([A],f)}\mathcal{I}(S^p)\right).$$

Namely,

$$T_{[A]}\mathcal{L}_M = \left\{B \in T_{[A]}\mathcal{L},\ \exists \eta \in T_f S^p, (\mathrm{Id}_{n+1} - A^T A)\eta^T = B^T A f^T\right\},$$

where $\mathcal{S}\mathrm{pan}(A) \cap S^p = \{\pm f\}$.

Proof. First of all the following inequalities obviously hold.

$$\dim_{[A]} \mathcal{L}_M \leq \dim \mathcal{L}_M \leq \dim_{([A],f)} \mathcal{I}(S^p) = n + p - 1.$$

There is a natural isometric action of the orthogonal group $O(n+1)$ on the compact Stiefel manifold $ST(\mathbb{R})$ which may be translated to the Grassmannian $\mathcal{L}$ and, then, to the incidence variety $\mathcal{I}(\mathbb{R})$ as follows:

$$\begin{aligned} O(n+1) \times \mathcal{I}(\mathbb{R}) &\longrightarrow \mathcal{I}(\mathbb{R}) \\ (U, ([A], f)) &\longmapsto ([AU], fU). \end{aligned}$$

Let us now consider the Lie subgroup $\mathfrak{O}(p+1, n-p) = O(p+1) \times O(n-p)$ of $O(n+1)$. This group acts isometrically both on $\mathcal{I}(S^p)$ and $\mathcal{L}_M$. Up to some isometry defined by some orthogonal matrix $U \in \mathfrak{O}(p+1, n-p)$, we may assume

$$([A], f) = \left(\left[\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r & 0 & \cdots & 0 & s \end{pmatrix}\right], (1, 0, 0, \ldots, 0, 0)\right),$$

where $r^2 + s^2 = 1$, and $s \neq 0$ if, and only if, $[A] \in \mathfrak{C}_M$.

Now we prove that $T_{([A],f)}\pi_1^{(2)}$ is a monomorphism if and only if $s \neq 0$. Note that for every $(B, \eta) \in T_{([A],f)}\left(\mathcal{I}(S^p)\right)$ the following properties hold:

$$BA^T = 0,\ \langle \eta, f\rangle = 0,\ \eta = (x_1, \ldots, x_{p+1}, 0, \ldots, 0) \in T_f S^n,$$

and

$$\left(\mathrm{Id}_{n+1} - A^T A\right)\eta^T = B^T A f^T.$$

Let $(B, \eta) \in T_{([A],f)}\left(\mathcal{I}(S^p)\right)$ be in the kernel of $T_{([A],f)}\pi_1^{(2)}$. Then,

$$T_{([A],f)}\pi_1^{(2)}(B, \eta) = B = 0$$

and we have:

$$\eta = (0, x_2, \ldots, x_{p+1}, 0, \ldots, 0),\ \left(\mathrm{Id}_{n+1} - A^T A\right)\eta^T = 0.$$

As $s^2 + r^2 = 1$, we also have

$$(\mathrm{Id}_{n+1} - A^T A) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & s^2 & \cdots & -rs \\ \vdots & \vdots & \mathrm{Id}_{n-2} & \vdots \\ 0 & -rs & \cdots & r^2 \end{pmatrix}.$$

Hence,

$$0 = (\mathrm{Id}_{n+1} - A^T A) \begin{pmatrix} 0 \\ x_2 \\ x_3 \\ \vdots \\ x_{p+1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ s^2 x_2 \\ x_3 \\ \vdots \\ x_{p+1} \\ 0 \\ \vdots \\ 0 \\ -rsx_2 \end{pmatrix}.$$

Thus, if $s \neq 0$, we conclude $\eta = 0$ and $T_{([A],f)}\pi_1^{(2)}$ is a linear monomorphism. Otherwise, if $s = 0$, $([0], (0, 1, 0, \ldots, 0))$ would be a non-zero element in the kernel of $T_{([A],f)}\pi_1^{(2)}$. This proves claim (1) of the proposition.

Recall now that the real Grassmannian $\mathcal{L}$ may be viewed as an affine semialgebraic set (cf. [17], for instance). Then, $\mathcal{L}_M$ may also be viewed as a semialgebraic subset of the Grassmannian. As $\pi_1^{(2)}$ is an immersion at $([A], f)$, there is some semialgebraic subset V of $\mathcal{L}_M$ containing $[A]$ and such that V is diffeomorphic to some open neighborhood of $([A], f)$ in $\mathcal{I}(S^p)$. In particular, we have

$$\begin{aligned} n + p - 1 = \dim_{[A]} V = \dim_{[A]} \mathcal{I}(S^p) &\leq \dim_{[A]} \mathcal{L}_M \\ &\leq \dim \mathcal{L}_M \leq n + p - 1, \end{aligned}$$

for all $[A] \in \mathfrak{C}_M$ and the last statement of claim (2) holds.

Moreover, for every $[A] \in \mathfrak{C}_M$ and for every f such that $([A], f) \in \mathcal{I}(S^p)$, there is a compact neighborhood of $([A], f)$ in $\mathcal{I}(S^p)$ such that the restriction of $\pi_1^{(2)}$ to its interior is injective and, hence, a proper embedding. In particular, $[A]$ is a smooth regular point of $\mathcal{L}_M$ of maximal dimension and $\pi_1^{(2)}$ is a 2-fold covering map in a neighborhood of $[A]$. This proves the other two statements of claim (2). The last claim of the proposition immediately follows from these facts and the previously proved statements. □

3. Some geometric integration tools

In this section we prove the following statements concerning normal Jacobians of certain mappings we define.

With the same notation as in Section 2 above, let $M \subseteq \mathbb{R}^{n+1}$ be a real vector subspace of dimension $p+1$ and codimension $k = n-p$ and let $\Phi : S^n \times S(M) \setminus \mathrm{Diag} \longrightarrow \mathcal{L}_M$ be the mapping given by:

$$\Phi(g,f) = L_{(g,f)}, \ \forall (g,f) \in S^n \times S(M) \setminus \mathrm{Diag},$$

where $\mathrm{Diag} = \{(g,f), \ g = \pm f\}$ and $L_{(g,f)}$ is the great circle containing g and f. In terms of classes $[A]$ modulo $O(2)$ of matrices A in the Stiefel manifold, the mapping Φ is given by the following rule:

$$\Phi(g,f) = \begin{bmatrix} f \\ \frac{GS_f(g)}{(1-\langle f,g\rangle^2)^{1/2}} \end{bmatrix},$$

where $GS_f(g) = g - \langle f, g\rangle f$.

Proposition 19. *With this notation, for every $g \in S^n \setminus S(M)$ and $f \in S(M)$, the normal Jacobian of Φ satisfies:*

$$NJ_{(g,f)}\Phi = \frac{\partial_M \left(\Phi(g,f)\right)^n}{\mathrm{d}_{\mathbb{P}}(g, S(M))^{n-1}},$$

where $\partial_M(\Phi(g,f)) = \partial_M(L_{(g,f)}) = \max\left\{\mathrm{d}_{\mathbb{P}}(h, S(M)), \ h \in L_{(g,f)}\right\}$.

With the same notation we define the following incidence variety:

$$\begin{aligned} \mathcal{IC}(M) &= \pi_1^{-1}\left(\pi_1\left(\pi_2^{-1}\left(S(M)\right)\right)\right) = \pi_1^{-1}(\mathcal{L}_M) \\ &= \left\{([A], g) \in \mathcal{I}(\mathbb{R}), \ [A] \in \mathcal{L}_M\right\}. \end{aligned}$$

We have two canonical projections:

$$p_1 = \pi_1 \mid_{\mathcal{IC}(M)} : \mathcal{IC}(M) \longrightarrow \mathcal{L}_M,$$

and

$$p_2 = \pi_2 \mid_{\mathcal{IC}(M)} : \mathcal{IC}(M) \longrightarrow S^n.$$

Observe that p_1 is onto and that $\dim p_1^{-1}(L) = 1$. Thus,

$$\dim \mathcal{IC}(M) = n + p - 1 + 1 = n + p.$$

The following property holds:

Proposition 20. *With the same notation as above, given $([A], g) \in \mathcal{IC}(M)$, such that $g \in S^n \setminus S(M)$. Then $([A], g)$ is a smooth regular point in $\mathcal{IC}(M)$, p_1 and p_2 are submersions at $([A], g)$ and, if $\mathcal{S}\mathrm{pan}(A) \cap S(M) = \{\pm f\}$, the quotient of the normal Jacobians of p_1 and p_2 satisfies the following equality:*

$$\frac{NJ_{([A],g)}p_1}{NJ_{([A],g)}p_2} = \left(\frac{1}{\|g - \langle f, g\rangle f\|}\right)^{k-1} = \left(\frac{\partial_M([A])}{\mathrm{d}_{\mathbb{P}}(g, S(M))}\right)^{k-1},$$

where k is the codimension of M in $\mathbb{R}^{n+1}$.

With the same notation, for every $g \in S^n$, we denote by $\mathcal{IC}(M)_g$ the fiber by projection p_2 over g. Namely, $\mathcal{IC}(M)_g = p_2^{-1}(\{g\})$. We also prove the following statement.

Proposition 21. *With the same notation, let $I(g)$ be the following quantity:*

$$I(g) = \int_{(L,g)\in\mathcal{IC}(M)_g} \frac{1}{\partial_M(L)} \frac{NJ_{(L,g)}p_1}{NJ_{(L,g)}p_2} \, \mathrm{d}\mathcal{IC}(M)_g.$$

Following the values of the codimension $k = n - p$, we have

1. *If $k = 1$:*
$$I(g) = 2\nu_{p-1} \int_0^1 \frac{(1-t^2)^{\frac{p}{2}-1}}{(1 - r^2(1-t^2))^{1/2}} \, \mathrm{d}t,$$
where $r^2 = 1 - \mathrm{d}_{\mathbb{P}}(g, S^p)^2$. In particular, we have
$$\nu_{p-1}\mathrm{B}\left(\frac{1}{2}, \frac{p}{2}\right) \le I(g) \le \frac{\nu_{p-1}\mathrm{B}(\frac{1}{2}, \frac{p}{2})}{\mathrm{d}_{\mathbb{P}}(g, S^p)};$$

2. *If $k \in 2\mathbb{N}^*$, then*
$$I(g) = \sum_{i=0}^{\frac{k}{2}-1} \frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g, S^p)^{2i+1}} \frac{\mathrm{B}(i + \frac{1}{2}, \frac{n}{2} - i - 1)}{k\mathrm{B}(\frac{k}{2} - i, i + 1)};$$

3. *If $k \in (2\mathbb{N}^* + 1)$, then*
$$I(g) = \sum_{i=0}^{\infty} \frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g, S^p)^{2i+1}} \frac{\mathrm{B}(i + \frac{1}{2}, \frac{n}{2} - i - 1)}{k\mathrm{B}(\frac{k}{2} - i, i + 1)}.$$

In the latter case, we may also exhibit the following upper and lower bounds given by finite sums:

$$I(g) \le \sum_{i=0}^{\frac{k-3}{2}} \frac{4\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{2i+2}} \frac{\mathrm{B}(i+\frac{1}{2},\frac{n}{2}-i-\frac{3}{2})}{(k-1)\mathrm{B}(\frac{k-1}{2}-i,i+1)},$$

and

$$I(g) \ge \sum_{i=0}^{\frac{k-3}{2}} \frac{4\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{2i+2}} \frac{\mathrm{B}(i+1,\frac{n}{2}-i-\frac{3}{2})}{(k-1)\mathrm{B}(\frac{k-1}{2}-i,i+1)}.$$

Remark 22. Let $s = \mathrm{d}_{\mathbb{P}}(g, S(M))$ and r be such that $r^2 + s^2 = 1$ and let F be the following function

$$\begin{aligned} F(r,s) &= \int_0^{1/s} \left(1+r^2z^2\right)^{\frac{n-p-2}{2}} (1-s^2z^2)^{\frac{p-2}{2}}\, \mathrm{d}z \\ &= \frac{1}{\mathrm{d}_{\mathbb{P}}(g,S(M))} F_1\left(\frac{1}{2}, \frac{p+2-n}{2}, \frac{2-p}{2}, \frac{3}{2}; -\cot(\mathrm{d}_R(g,S^p)), 1\right), \end{aligned}$$

where F_1 is Appell's hypergeometric funtion and $\cot(\mathrm{d}_R(g, S(M)))$ is the cotangent of the Riemannian distance of g to $S(M)$. Then, quantity $I(g)$ can be rewritten

$$I(g) = 2\nu_{p-1} F(r,s).$$

Remark 23. Whenever the codimension is greater than 2, the following bounds hold:

$$\frac{\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{k-1}} \mathrm{B}\left(\frac{k-1}{2}, \frac{p}{2}\right) \le I(g) \le \frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{k-1}},$$

Here we follow the same notation as in Section 2 above. In Section 3.3 we prove Proposition 19, in Section 3.4 we prove Proposition 20 and in Section 3.5 we prove Proposition 21.

We assume $M = \mathbb{R}^{p+1}$ as real vector subspace of $\mathbb{R}^{n+1}$, S^p is the sphere $S(M)$ as Riemannian submanifold of S^n. We denote by $\mathcal{L}$ the Grassmannian of great circles in S^n and by $\mathcal{L}_M$ the semialgebraic subset of $\mathcal{L}$ given as the lines $L \in \mathcal{L}$ that intersect $S(M)$. Finally, $\mathfrak{C}_M$ is the manifold given as the subset of $\mathcal{L}_M$ such that $\sharp(L \cap S(M))) = 2$. Before getting into the proofs of these two propositions, we need to establish some basic facts.

3.1. Normal Jacobians and the Co-area formula

Our first statement is a classical formula discovered by Federer that can be found in many places in the literature. Some classic references are [29, 35, 37]. Our formulation below has been taken from [16, p. 241].

Let X and Y be Riemannian manifolds, and let $F : X \longrightarrow Y$ be a C^1 surjective map. Let $p = \dim(Y)$ be the real dimension of Y. For every point $x \in X$ such that the tangent mapping T_xF is surjective, let $\left(v_1^x, \ldots, v_p^x\right)$ be an orthonormal basis of $\ker(T_xF)^\perp$. Then, we define the normal Jacobian of F at x, NJ_xF, as the volume in $T_{F(x)}Y$ of the parallelepiped spanned by $\left(T_xF(v_1^x), \ldots, T_xF(v_p^x)\right)$. In the case that T_xF is not surjective, we define NJ_xF as 0.

Note that, in particular, normal Jacobians remain equal under the action of Riemannian isometries. Namely, the following statement holds:

Proposition 24. *Let X, Y be two Riemannian manifolds, and let $F : X \longrightarrow Y$ be a C^1 map. Let $x_1, x_2 \in X$ be two points. Assume that there exist isometries $\varphi_X : X \longrightarrow X$ and $\varphi_Y : Y \longrightarrow Y$ such that $\varphi_X(x_1) = x_2$, and*

$$F \circ \varphi_X = \varphi_Y \circ F.$$

Then, the following equality holds:

$$NJ_{x_1}F = NJ_{x_2}F.$$

Moreover, if there exists an inverse $G : Y \longrightarrow X$, then

$$NJ_xF = \frac{1}{NJ_{F(x)}G}.$$

Theorem 25 (Co-area formula)**.** *Consider a differentiable map $F : X \longrightarrow Y$, where X and Y are Riemannian manifolds of respective real dimensions $n \geq p$. Consider a measurable function $f : X \longrightarrow \mathbb{R}$, such that f is integrable. Then, for every $y \in Y$ except in a zero-measure set, $F^{-1}(y)$ is empty or a real submanifold of X of real dimension $n - p$. Moreover, the following equality holds (and the integrals appearing on it are well-defined):*

$$\int_X f NJ_xF \, \mathrm{d}X = \int_{y \in Y} \left(\int_{x \in F^{-1}(y)} f(x) \, \mathrm{d}F^{-1}(y) \right) \mathrm{d}Y,$$

where NJ_xF is the normal Jacobian of F in x.

3.2. Distances in $\mathcal{L}_M$: Some technical results

We denote by $\mathrm{d}_{\mathbb{P}} : (S^n)^2 \longrightarrow \mathbb{R}_+$ the "projective" distance on the sphere as in [16] (i.e. $\mathrm{d}_{\mathbb{P}}(f,g) = \sin \mathrm{d}_R(f,g)$, where $\mathrm{d}_R(f,g)$ is the standard Riemannian (arclength) distance in S^n.

Let $L = [A] \in \mathfrak{C}_M$ be a great circle that intersects S^p in exactly two points. Assume $\mathcal{S}\mathrm{pan}(A) \cap S^p = \{\pm f\}$. Up to some isometry in $O(p+1) \times O(n-p)$ we may assume that $f = (1,0,\ldots,0)$ and that

$$L = [A] = \left[\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r & 0 & \cdots & 0 & s \end{pmatrix}\right],$$

where $r^2 + s^2 = 1$. Moreover, the following mapping is an isometry between L and S^1:

$$\begin{aligned} \varphi : S^1 &\longrightarrow L, \\ (\lambda, \mu) &\longmapsto (\lambda, \mu r, 0, \ldots, 0, \mu s). \end{aligned}$$

Lemma 26. *With this notation, let $g = \varphi(\lambda,\mu)$ be any point in L, then the following properties hold:*

- $\mathrm{d}_{\mathbb{P}}(g, S^p) = |\mu s|$,
- $\partial_M(L) = \max\{\mathrm{d}_{\mathbb{P}}(g, S^p),\ g \in L\} = |s|$,
- $\frac{\mathrm{d}_{\mathbb{P}}(g,S^p)}{\partial_M(L)} = |\mu| = \|g - \langle f, g\rangle f\| = (1 - \langle f, g\rangle^2)^{1/2}$.

The proof comes from simple calculations. The following statement also holds:

Lemma 27. *For every $L \in \mathfrak{C}_M$, the following equality holds for every positive integer $r \in \mathbb{N}$, $r \geq 2$:*

$$I_r(L) = \int_L \mathrm{d}_{\mathbb{P}}(x, S^p)^r \ \mathrm{d}L = \frac{\nu_{r+2}}{\nu_{r+1}} \partial_M(L)^r = \mathrm{B}\left(\frac{r+3}{2}, \frac{1}{2}\right) \partial_M(L)^r,$$

where ν_r is the volume of the rth dimensional sphere, namely

$$\nu_r = \mathrm{vol}[S^r] = \frac{\pi^{r/2}}{\Gamma(\frac{r}{2}+1)}.$$

Proof. Using the isometry φ above, we have $\mathrm{d}_{\mathbb{P}}(g, S^p) = |s\mu| = \partial_M(L)|\mu|$ and hence, we have:

$$I_r(L) = \partial_M(L)^r \int_{S^1} |\mu|^r \, \mathrm{d}\nu_{S^1}.$$

Now, we project $\pi : S^1 \longrightarrow [-1, 1]$, where $\pi(\lambda, \mu) = \mu$. The normal Jacobian $NJ_x\pi$ equals $(1 - |\pi(x)|^2)^{1/2}$ (cf. [16, p. 206], for instance) and we use the Co-area formula to conclude:

$$I_r(L) = \partial_M(L)^r \int_{-1}^{1} \frac{|\mu|^r}{(1-\mu^2)^{1/2}} \, \mathrm{d}\mu = 2\partial_M(L)^r \int_0^1 \frac{\mu^r}{(1-\mu^2)^{1/2}} \, \mathrm{d}\mu.$$

The following equality is classical (cf. [22], for instance) and finishes the proof:

$$2\int_0^1 \frac{\mu^r}{(1-\mu^2)^{1/2}} \, \mathrm{d}\mu = \frac{\nu_{r+2}}{\nu_{r+1}}. \qquad \square$$

We may define a density function on every great circle $L \in \mathfrak{C}_M$. We denote $\mathrm{d}L^{(M)}$ the probability distribution defined in the following terms. For every integrable function $\Phi : S^n \longrightarrow \mathbb{R}_+$, we define:

$$E_{L^{(M)}}[\Phi] = \int_L \Phi \, \mathrm{d}L^{(M)} = \frac{\nu_k}{\nu_{k+1}\partial_M(L)} \int_L \Phi(x) \, \mathrm{d}_{\mathbb{P}}(x, S^p)^{k-1} \, \mathrm{d}L,$$

where $k = n - p$ is the codimension of M in $\mathbb{R}^{n+1}$.

3.3. Normal Jacobians I: Proof of Proposition 19

We follow the same notation as in previous sections and subsections.

As the normal Jacobian is invariant under the action of isometries (Proposition 24 above), we may assume that

$$f = (1, 0, \ldots, 0) \in S^p, g = (\lambda, \mu r, 0, \ldots, 0, \mu s) \in S^n,$$

where $r^2 + s^2 = 1$ and $\lambda^2 + \mu^2 = 1$. Hence,

$$\Phi(g, f) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r & 0 & \cdots & 0 & s \end{bmatrix}.$$

We may decompose $\Phi = \pi \circ \varphi$ as the composition of the following two mappings:

- A first mapping into the Stiefel manifold:

$$\begin{aligned}\varphi : S^n \times S^p \setminus \mathrm{Diag} &\longrightarrow ST(\mathbb{R})\\ (h_1,h_2) &\longmapsto \begin{pmatrix} h_2 \\ \frac{GS_{h_2}(h_1)}{(1-\langle h_1,h_2\rangle^2)^{1/2}}\end{pmatrix},\end{aligned}$$

  where $GS_{h_2}(h_1) = h_1 - \langle h_1, h_2\rangle h_2$ was defined above.

- The canonical projection $\pi : ST(\mathbb{R}) \longrightarrow ST(\mathbb{R})/O(2) = \mathcal{L}$. In this case the tangent mapping $T_A\pi$ is the orthogonal projection of Lemma 14 above, and it is given by the following matrix:

$$\mathrm{Id}_{n+1} - A^T A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & s^2 & \cdots & -rs \\ \vdots & \vdots & \mathrm{Id}_{n-2} & \vdots \\ 0 & -rs & \cdots & r^2 \end{pmatrix}.$$

Then, for every $(\dot g, \dot f) \in T_g S^n \times T_f S^p$, the following equality holds:

$$T_{(g,f)}\Phi(\dot g, \dot f) = T_A\pi\left(T_{(g,f)}\varphi(\dot g, \dot f)\right),$$

where $A = \varphi(g,f)$.

We start by computing the tangent mapping $T_{(g,f)}\varphi$, which is given by the following identities:

$$\begin{aligned} T_{(g,f)}\varphi : T_gS^n \times T_fS^p &\longrightarrow T_{\varphi(g,f)}ST(\mathbb{R}),\\ T_{(g,f)}\varphi(\dot g, \dot f) &\longmapsto \begin{pmatrix} \dot f \\ \left(\dot{\frac{GS_f(g)}{(1-\langle f,g\rangle^2)^{1/2}}}\right)\end{pmatrix},\end{aligned}$$

where

$$\left(\dot{\frac{GS_f(g)}{(1-\langle f,g\rangle^2)^{1/2}}}\right) = \frac{(1-\langle f,g\rangle^2)^{1/2}\rho_{\dot f,\dot g}(f,g) + (1-\langle f,g\rangle^2)^{-1/2}\tau_{\dot f,\dot g}(f,g)}{(1-\langle f,g\rangle^2)},$$

and

$$\rho_{\dot f,\dot g}(f,g) = GS_f(\dot g) - (\langle g,\dot f\rangle f + \langle g,f\rangle \dot f) = \dot g - \langle \dot g, f\rangle f - (\langle g,\dot f\rangle f + \langle g,f\rangle \dot f),$$

$$\tau_{\dot f,\dot g}(f,g) = \langle f,g\rangle \left[\langle g,\dot f\rangle + \langle \dot g, f\rangle\right] GS_f(g).$$

Now we consider the following orthonormal bases of the tangent spaces T_fS^p and T_gS^n:

- $T_f S^p$ is generated by the list of tangent vectors $\{\dot{f}_2, \dots, \dot{f}_{p+1}\}$ where $\dot{f}_i$ is the vector whose coordinates are all zero excepting the ith coordinate which is 1. Therefore $f = f_1$.
- $T_g S^n$ is generated by the list of tangent vectors $\{\dot{g}_1, \dots, \dot{g}_n\}$, where
  - $\dot{g}_1 = (-\mu, \lambda r, 0, \dots, 0, \lambda s)$,
  - $\dot{g}_2 = (0, s, 0, \dots, 0, -r)$,
  - and for every i, $3 \leq i \leq n$, $\dot{g}_i$ is the vector whose coordinates are all zero excepting the ith coordinate which is 1.

Now some calculations would yield

- For every i, $3 \leq i \leq p+1$, we have
$$T_{(g,f)}\varphi(0, \dot{f}_i) = \begin{pmatrix} \dot{f}_i \\ -\frac{\langle f,g\rangle}{(1-\langle f,g\rangle^2)^{1/2}} \dot{f}_i \end{pmatrix} = \begin{pmatrix} \dot{f}_i \\ -\frac{\lambda}{\mu} \dot{f}_i \end{pmatrix}.$$
- As for the case $i = 2$ we have:
$$T_{(g,f)}\varphi(0, \dot{f}_i) = \begin{pmatrix} \dot{f}_i \\ u_1 \end{pmatrix},$$
where
$$u_1 = \left(-r, -\frac{\lambda}{\mu} s^2, 0, \dots, 0, \frac{\lambda}{\mu} rs \right).$$
- For every j, $3 \leq j \leq n$, we have
$$T_{(g,f)}\varphi(\dot{g}_j, 0) = \begin{pmatrix} 0 \\ \frac{1}{\mu} \dot{g}_j \end{pmatrix}.$$
- For $j = 1$ we have
$$T_{(g,f)}\varphi(\dot{g}_1, 0) = \begin{pmatrix} 0 \\ u_2 \end{pmatrix},$$
where
$$u_2 = \lambda\mu(0, r, 0, \dots, 0, s).$$

- Finally, for $j = 2$, we have

$$T_{(g,f)}\varphi(\dot{g}_2, 0) = \begin{pmatrix} 0 \\ \frac{1}{\mu}\dot{g}_3 \end{pmatrix}.$$

Now we consider the following matrices in $T_{\varphi(g,f)}ST(\mathbb{R})$ which are part of an orthonormal basis with respect to Frobenius inner product. In fact, all of them belong to $T_{\Phi(g,f)}\mathfrak{C}_M$ and also to $T_{\Phi(g,f)}L$.

- The matrix $E_{1,2}$ given by:

$$E_{1,2} = \begin{pmatrix} 0 & s & 0 & \cdots & 0 & -r \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

- For every i, $3 \leq i \leq p+1$, let $E_{1,i}$ be the matrix given as:

$$E_{1,i} = \begin{pmatrix} \dot{f}_i \\ 0 \end{pmatrix}.$$

- The matrix $E_{2,2}$ given by

$$E_{2,2} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s & 0 & \cdots & 0 & -r \end{pmatrix}.$$

- For every j, $3 \leq j \leq n$, let $E_{2,j}$ be the matrix given as:

$$E_{2,j} = \begin{pmatrix} 0 \\ \dot{g}_j \end{pmatrix}.$$

Now, we have:

- For every i, $3 \leq i \leq p+1$,

$$T_{(g,f)}\Phi(0, \dot{f}_i) = T_A\pi\left(T_{(g,f)}\varphi\left(0, \dot{f}_i\right)\right) = T_{(g,f)}\varphi\left(0, \dot{f}_i\right) = E_{1,i} - \frac{\lambda}{\mu}E_{2,i}.$$

- For $i = 2$,

$$\begin{aligned} T_{(g,f)}\Phi(0, \dot{f}_2) &= T_A\pi\left(T_{(g,f)}\varphi\left(0, \dot{f}_2\right)\right) = T_{(g,f)}\varphi\left(0, \dot{f}_2\right)(\mathrm{Id}_{n+1} - A^T A) \\ &= sE_{1,2} + \frac{\lambda s}{\mu}E_{2,2}. \end{aligned}$$

- For every j, $3 \leq j \leq n$,

$$T_{(g,f)}\Phi(\dot{g}_j, 0) = T_A\pi\left(T_{(g,f)}\varphi\left(\dot{g}_j, 0\right)\right) = T_{(g,f)}\varphi\left(\dot{g}_j, 0\right) = \frac{1}{\mu}E_{2,j}.$$

- For $j = 1$,

$$T_{(g,f)}\Phi(\dot{g}_2, 0) = T_A\pi\left(T_{(g,f)}\varphi\left(\dot{g}_2, 0\right)\right) = T_{(g,f)}\varphi\left(\dot{g}_2, 0\right)(\mathrm{Id}_{n+1} - A^T A) = 0.$$

- Finally, for $j = 2$,

$$T_{(g,f)}\Phi(\dot{g}_2, 0) = T_A\pi\left(T_{(g,f)}\varphi\left(\dot{g}_2, 0\right)\right) = T_{(g,f)}\varphi\left(\dot{g}_2, 0\right) = \frac{1}{\mu}E_{2,2}.$$

In particular, we conclude that the kernel of $T_{(g,f)}\Phi$ is the vector subspace generated by $(\dot{g}_2, 0) \in T_g S^n \times T_f S^p$. The restriction of $T_{(g,f)}\Phi$ to the orthogonal complement of its kernel, taking orthonormal basis, is given by a triangular matrix of the following form:

$$\begin{pmatrix} s & * & * \\ 0 & \mathrm{Id}_{p-1} & * \\ 0 & 0 & \frac{1}{\mu}\mathrm{Id}_{n-1} \end{pmatrix}.$$

Then, the normal Jacobian satisfies

$$NJ_{(g,f)}\Phi = \frac{|s|}{\mu^{n-1}} = \frac{\partial_M(L)^n}{\mathrm{d}_{\mathbb{P}}(g, S^p)^{n-1}},$$

as wanted. □

3.4. Normal Jacobians II: Proof of Proposition 20

Once again we follow the same notation as above.

First of all, observe that if $([A], g) \in \mathcal{IC}(M)$, then $[A] \in \mathfrak{C}_M$ and this is a smooth point of maximal dimension in $\mathcal{L}_M$. Now, we proceed by computing the tangent space $T_{([A],g)}\mathcal{IC}(M)$. Again, due to the right action of $O(p+1) \times O(n-p)$ on $\mathcal{I}(S^p)$ and $\mathcal{I}(\mathbb{R})$. Since Proposition 24 about the invariance of normal Jacobians holds, we may assume:

$$([A], g) = \left(\left[\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r & 0 & \cdots & 0 & s \end{pmatrix}\right], (\lambda, \mu r, 0, \ldots, 0, \mu s)\right),$$

where $r^2+s^2=1$, $\lambda^2+\mu^2=1$, $\mu\neq 0$ (since $g\notin S^p$) and (then) $s\neq 0$. Let us also write $f=(1,0,0,\dots,0)\in S^p\cap\mathcal{S}\mathrm{pan}(A)$. Observe that $\|g-\langle f,g\rangle f\| = (1-\langle f,g\rangle^2)^{1/2}=|\mu|$. For sake of simplicity, assume $\mu\geq 0$ from now on.

We need to compute an orthonormal basis of $T_{([A],g)}\mathcal{IC}(M)$ and then its images under the two projections $T_{([A],g)}p_1$ and $T_{([A],g)}p_2$. This is done in the following technical lemma:

Lemma 28. *Let $v_1=(0,-s,0,\dots,0,r)$, $v_2=(\mu,-\lambda r,0,\dots,0,-\lambda s)$ and $(e_1,\dots,e_{n+1})$ be the canonical orthonormal basis of $\mathbb{R}^{n+1}$. Let $(\omega_1,\dots,\omega_{n+1})$ and $(\omega'_1,\omega'_3,\dots,\omega'_{p+1})$ be defined as follows:*

- $\omega_1=\left(\begin{pmatrix}0 & -s\lambda & 0 & \cdots & 0 & r\lambda\\ 0 & -s\mu & 0 & \cdots & 0 & r\mu\end{pmatrix},v_1\right)$,
- $\omega_2=\left(\begin{pmatrix}0 & \cdots & 0\\ 0 & \cdots & 0\end{pmatrix},v_2\right)$,
- $\omega_i=\left(\begin{pmatrix}0 & \cdots & 0 & \lambda & 0 & \cdots & 0\\ 0 & \cdots & 0 & \mu & 0 & \cdots & 0\end{pmatrix},e_i\right)$, *for* $3\leq i\leq p+1$,
- $\omega_j=\left(\begin{pmatrix}0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & \cdots & 0 & \mu^{-1} & 0 & \cdots & 0\end{pmatrix},e_j\right)$, *for* $p+2\leq j\leq n$,
- $\omega'_1=\left(\begin{pmatrix}0 & s\mu & 0 & \cdots & 0 & -r\mu\\ 0 & -s\lambda & 0 & \cdots & 0 & r\lambda\end{pmatrix},0\right)$,
- $\omega'_i=\left(\begin{pmatrix}0 & \cdots & 0 & -\mu & 0 & \cdots & 0\\ 0 & \cdots & 0 & \lambda & 0 & \cdots & 0\end{pmatrix},0\right)$, *for* $3\leq i\leq p+1$.

Then, the following family is an orthonormal basis of $T_{([A],g)}\mathcal{IC}(M)$:

$$\mathrm{B}=\left\{\frac{1}{\sqrt{2}}\omega_1,\omega_2,\frac{1}{\sqrt{2}}\omega_3,\dots,\frac{1}{\sqrt{2}}\omega_{p+1},\frac{1}{\sqrt{1+\mu^{-2}}}\omega_{p+2},\dots,\frac{1}{\sqrt{1+\mu^{-2}}}\omega_n\right\}\cup\left\{\omega'_1,\omega'_3,\dots,\omega'_{p+1}\right\}.$$

Proof. From Section 2 we have the following description of $T_{([A],g)}\mathcal{IC}(M)$:

A pair $(B,\eta)\in T_{[A]}\mathcal{L}\times T_gS^n$ is in the tangent space $T_{([A],g)}\mathcal{IC}(M)$ if, and only if, the following properties hold:

1. $BA^T=0$, since $B\in T_{[A]}\mathcal{L}$;

2. $\langle \eta, g\rangle = 0$, since $\eta \in T_g S^n$,
3. $(\mathrm{Id}_{n+1} - A^T A)\eta^T = B^T A g^T$, since $(B, \eta) \in T_{([A],g)}\mathcal{I}(\mathbb{R})$;
4. There exists $\nu \in T_f S^p$, such that $B = T_{([A],f)}\pi_1^{(2)}(B, \nu)$ As, B already satisfies property (1) above, this may be rewritten as:

$$\exists \nu \in T_f S^p,\ (\mathrm{Id}_{n+1} - A^T A)\nu^T = B^T A f^T.$$

Let us rewrite these properties in terms of matrices and coordinates to prove that β is an orthonormal basis of $T_{([A],g)}\mathcal{IC}(M)$.

The condition $BA^T = 0$ implies that we may assume

$$B = \begin{pmatrix} 0 & -sx_2 & b_{1,3} & \cdots & rx_2 \\ 0 & -sy_2 & b_{2,3} & \cdots & ry_2 \end{pmatrix}.$$

Let $e_i,\ 1 \leq i \leq n+1$ be the canonical (usual) orthonormal basis of $\mathbb{R}^{n+1}$ and let $v_1 = (0, -s, \ldots, r)$ and $v_2 = (\mu, -\lambda r, 0, \ldots, -\lambda s)$. The following family is an orthonormal basis of $T_g S^n$:

$$\beta = \{v_1, v_2, e_3, \ldots, e_n\}.$$

As $Ag^T = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$, we conclude

$$B^T A g^T = \begin{pmatrix} 0 \\ (-s)(\lambda x_2 + \mu y_2) \\ \lambda b_{1,3} + \mu b_{2,3} \\ \vdots \\ \lambda b_{1,n} + \mu b_{2,n} \\ (r)(\lambda x_2 + \mu y_2) \end{pmatrix}.$$

Hence, property (3) may be rewritten as:

$$(\mathrm{Id}_{n+1} - A^T A)\eta^T = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & s^2 & \cdots & -rs \\ \vdots & \vdots & \mathrm{Id}_{n-2} & \vdots \\ 0 & -rs & \cdots & r^2 \end{pmatrix} \eta^T = \begin{pmatrix} 0 \\ (-s)(\lambda x_2 + \mu y_2) \\ \lambda b_{1,3} + \mu b_{2,3} \\ \vdots \\ \lambda b_{1,n} + \mu b_{2,n} \\ (r)(\lambda x_2 + \mu y_2) \end{pmatrix}.$$

Observe that $(\mathrm{Id}_{n+1} - A^T A)v_1^T = v_1^T$ and $(\mathrm{Id}_{n+1} - A^T A)v_2^T = 0$. Hence, assuming that $\eta = z_1 v_1 + z_2 v_2 + \sum_{i=3}^n z_i e_i$, property (3) becomes:

$$\begin{pmatrix} 0 \\ (-s)z_1 \\ z_3 \\ \vdots \\ z_n \\ rz_1 \end{pmatrix} = \begin{pmatrix} 0 \\ (-s)(\lambda x_2 + \mu y_2) \\ \lambda b_{1,3} + \mu b_{2,3} \\ \vdots \\ \lambda b_{1,n} + \mu b_{2,n} \\ (r)(\lambda x_2 + \mu y_2) \end{pmatrix}.$$

Now we consider property (4). Since $\nu \in T_f S^p$, we may assume that

$$\nu = (0, u_2, \ldots, u_{p+1}, 0, \ldots, 0) \in \mathbb{R}^{n+1}.$$

As $Af^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, property (4) may be rewritten as:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & s^2 & \cdots & -rs \\ \vdots & \vdots & \mathrm{Id}_{n-2} & \vdots \\ 0 & -rs & \cdots & r^2 \end{pmatrix} \begin{pmatrix} 0 \\ u_2 \\ u_3 \\ \vdots \\ u_{p+1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ s^2 u_2 \\ u_3 \\ \vdots \\ u_{p+1} \\ 0 \\ \vdots \\ 0 \\ -rsu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -sx_2 \\ b_{1,3} \\ \vdots \\ b_{1,n} \\ rx_2 \end{pmatrix}.$$

This yields theses equalities

$$\begin{aligned} -su_2 &= x_2, \\ b_{1,j} &= 0, \quad p+2 \le j \le n. \end{aligned}$$

Putting all these properties together, we get the following characterization of tangent space $T_{([A],g)}\mathcal{IC}(M)$:

$$\left(\begin{pmatrix} 0 & -sx_2 & b_{1,3} & \cdots & rx_2 \\ 0 & -sy_2 & b_{2,3} & \cdots & ry_2 \end{pmatrix}, \eta \right) \in T_{([A],g)}\mathcal{IC}(M)$$

if, and only if, the following properties hold:

- $b_{1,j} = 0, p + 2 \leq j \leq n,$
- $\eta = z_1 v_1 + z_2 v_2 + \sum_{i=3}^{n} z_i e_i,$
- $\lambda x_2 + \mu y_2 = z_1,$
- $\lambda b_{1,i} + \mu b_{2,i} = z_i,\ 3 \leq i \leq p + 1,$
- $\mu b_{2,j} = z_j,\ p + 2 \leq j \leq n.$

The collection of vectors in β described in the statement of the lemma satisfies these properties, they are linearly independent and a family of orthonormal vectors with the accurate number of elements (equal to the dimension of $T_{([A],g)}\mathcal{IC}(M)$) as wanted. $\square$

Then, note that $\ker(T_{([A],g)}p_1) = \mathcal{S}\mathrm{pan}(\{\omega_2\})$ and $T_{([A],g)}p_1(B, \eta) = B$. Then, using this orthonormal basis, we immediately compute the list of vectors in $T_{([A],g)}p_1(\beta)$. They are mutually orthogonal and we may compute the normal Jacobian as the product of their norms, yielding the following equality:

$$NJ_{([A],g)}p_1 = \left(\frac{1}{\sqrt{2}}\right)^p \left(\frac{\mu^{-1}}{\sqrt{1+\mu^{-2}}}\right)^{n-p-1} = \left(\frac{1}{\sqrt{2}}\right)^p \left(\frac{1}{\sqrt{1+\mu^{2}}}\right)^{n-p-1}.$$

On the other hand,

$$\ker(T_{([A],g)}p_2) = \mathcal{S}\mathrm{pan}(\{\omega_1', \omega_3', \ldots, \omega_{p+1}'\}),\ \text{and}\ T_{([A],g)}p_2(B, \eta) = \eta.$$

Again, we may compute the list of vectors in $T_{([A],g)}p_2$(B) and then compute the corresponding normal Jacobian, obtaining :

$$NJ_{([A],g)}p_2 = \left(\frac{1}{\sqrt{2}}\right)^p \left(\frac{1}{\sqrt{1+\mu^{-2}}}\right)^{n-p-1}.$$

Then, the quotient satisfies:

$$\frac{NJ_{([A],g)}p_1}{NJ_{([A],g)}p_2} = \frac{\left(\frac{1}{\sqrt{2}}\right)^p \left(\frac{\mu^{-1}}{\sqrt{1+\mu^{-2}}}\right)^{n-p-1}}{\left(\frac{1}{\sqrt{2}}\right)^p \left(\frac{1}{\sqrt{1+\mu^{-2}}}\right)^{n-p-1}} = \left(\frac{1}{\mu}\right)^{n-p-1},$$

which proves Proposition 20 as wanted. $\square$

3.5. Fibers over "complex" points: Proof of Proposition 21

We begin with the following statement.

Proposition 29. *With the same notation as above, for every $g \in S^n \setminus S^p$, there is an isometry*

$$\Psi_g : S^p \longrightarrow \mathcal{IC}(M)_g.$$

In particular, the volume of the fiber $\mathcal{IC}(M)_g$ is constant and independent of g. In fact,

$$\mathrm{vol}[\mathcal{IC}(M)_g] = \nu_p = \mathrm{vol}[S^p].$$

Proof. Simply observe that the following mapping is an isometry, an immersion and its image is the fiber $\mathcal{IC}(M)_g$, where $g = (0, r, 0, \ldots, 0, s)$, $r^2+s^2 = 1$, $s \neq 0$:

$$\Psi_g : S^p \longrightarrow \mathcal{I}(\mathbb{R}),$$

given by

$$\Psi_g(x_1, \ldots, x_{p+1}) = \left(\begin{bmatrix} x_1 & -sx_2 & x_3 & \cdots & x_{p+1} & 0 & \cdots & 0 & rx_2 \\ 0 & r & 0 & \cdots & 0 & 0 & \cdots & 0 & s \end{bmatrix}, g \right).$$

First of all, it is clear that $\Psi_g(x) \in \mathcal{IC}(M)_g$ for all $x \in S^p$. The matrix

$$\psi(x) = \begin{pmatrix} x_1 & -sx_2 & x_3 & \cdots & x_{p+1} & 0 & \cdots & 0 & rx_2 \\ 0 & r & 0 & \cdots & 0 & 0 & \cdots & 0 & s \end{pmatrix}$$

is in the Stiefel manifold $ST(\mathbb{R})$ and so the orbit $\Psi(x) = [\psi(x)]$ is in the Grassmannian $\mathcal{L}$. But observe that

$$\left((x_1, -sx_2, x_3, \ldots, x_{p+1}, 0, \ldots, 0, rx_2) - \frac{rx_2}{s} g\right) \in \mathcal{S}\mathrm{pan}(\psi(x)) \cap \mathbb{R}^{p+1} \neq \emptyset.$$

Thus $\Psi_g(x) \in \mathcal{IC}(M) \cap p_2^{-1}(g)$ as wanted.

Additionally, observe that the tangent mapping is given by

$$T_x\Psi_g(\eta) = \left(\begin{pmatrix} \eta_1 & -s\eta_2 & \eta_3 & \cdots & \eta_{p+1} & 0 & \cdots & 0 & r\eta_2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, 0 \right),$$

where $\eta = (\eta_1, \ldots, \eta_{p+1}) \in x^\perp = T_xS^p$ is orthogonal to x. Moreover, for $\eta, \eta' \in T_xS^p$ we have:

$$\langle T_x\Psi_g(\eta), T_x\Psi_g(\eta') \rangle = \eta_1\eta_1' + s^2\eta_2\eta_2' + \sum_{i=3}^{p+1} \eta_i\eta_i' + r^2\eta_2\eta_2' = \langle \eta, \eta' \rangle,$$

and Ψ_g is an isometry. Then, its normal Jacobian is 1 and the equality between the corresponding volumes holds. □

Corollary 30. *For every point $g \in S^n \setminus S^p$ and for every couple $([A], g) \in \mathcal{IC}(M)$, the quotient of normal Jacobians satisfies*

$$\frac{NJ_{([A],g)}p_1}{NJ_{([A],g)}p_2} = \left(\frac{1}{s^{n-p-1}}\right)\left(s^2 + r^2 x_2^2\right)^{\frac{n-p-1}{2}},$$

where $x = (x_1, x_2, x_3, \ldots, x_{p+1}) \in S^p$ is such that $\Psi_g(x) = ([A], g)$, $s^2 = \mathrm{d}_{\mathbb{P}}(g, S^p)^2$ and $r^2 + s^2 = 1$.

Proof. According to Proposition 20, the quotient of normal Jacobians satisfies:

$$\frac{NJ_{([A],g)}p_1}{NJ_{([A],g)}p_2} = \left(\frac{1}{\|g - \langle f, g\rangle f\|}\right)^{n-p-1} = \left(\frac{1}{1 - \langle f, g\rangle^2}\right)^{\frac{n-p-1}{2}},$$

where $\mathcal{S}\mathrm{pan}(A) \cap S^p = \{\pm f\}$. With the same notation as in the proof of the previous proposition, we may assume $g = (0, r, 0, \ldots, 0, s)$, $r^2 + s^2 = 1$, $s \neq 0$, and $\Psi_g(x) = ([A], g)$. Thus, we have seen that

$$v = \left((x_1, -sx_2, x_3, \ldots, x_{p+1}, 0, \ldots, 0, rx_2) - \frac{rx_2}{s} g\right) \in \mathcal{S}\mathrm{pan}(A) \cap \mathbb{R}^{p+1},$$

and, hence we may choose

$$f = \frac{v}{\|v\|},$$

to compute the normal Jacobian. Observe that

$$v = \left(x_1, -\frac{x_2}{s}, x_3, \ldots, x_{p+1}, 0, \ldots, 0\right),$$

and

$$\|v\|^2 = 1 + \left(\frac{1}{s^2} - 1\right) x_2^2 = 1 + \frac{r^2 x_2^2}{s^2},$$

whereas

$$\langle v, g\rangle^2 = \frac{r^2 x_2^2}{s^2}.$$

Hence

$$1 - \langle f, g\rangle^2 = 1 - \frac{\langle v, g\rangle^2}{\|v\|^2} = 1 - \frac{\frac{r^2 x_2^2}{s^2}}{1 + \frac{r^2 x_2^2}{s^2}} = \frac{s^2}{s^2 + r^2 x_2^2}.$$

Finally, we conclude:

$$\frac{NJ_{([A],g)}p_1}{NJ_{([A],g)}p_2} = \left(\frac{1}{1 - \langle f, g\rangle^2}\right)^{\frac{n-p-1}{2}} = \left(\frac{s^2 + r^2 x_2^2}{s^2}\right)^{\frac{n-p-1}{2}},$$

as wanted. □

3.5.1. Proof of Proposition 21

As in the proof of Proposition 29, assuming that $x = (x_1, x_2, \dots, x_m)$ and $g = (0, r, 0, \dots, 0, s)$, $r^2 + s^2 = 1$, $s \neq 0$, we have

$$I(g) = \frac{1}{\mathrm{d}_{\mathbb{P}}(g, S^p)} \left(\int_{x \in S^p} \left(\frac{s^2 + r^2 x_2^2}{s^2} \right)^{\frac{n-p-2}{2}} \mathrm{d}S^p \right).$$

Integrating in polar coordinates we get:

$$I(g) = \frac{1}{\mathrm{d}_{\mathbb{P}}(g, S^p)} \int_{-1}^{1} \left(\int_{S^{p-1}_{\sqrt{1-t^2}}} \mathrm{d}S^{p-1} \right) \left(\frac{s^2 + r^2 t^2}{s^2} \right)^{\frac{n-p-2}{2}} (1 - t^2)^{-1/2} \, \mathrm{d}t.$$

Then,

$$I(g) = \frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g, S^p)} \int_{0}^{1} \left(\frac{s^2 + r^2 t^2}{s^2} \right)^{\frac{n-p-2}{2}} (1 - t^2)^{\frac{p-2}{2}} \, \mathrm{d}t. \tag{1}$$

In other words.

$$I(g) = \frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g, S^p)} \int_{0}^{1} \left((1 - t^2) + \frac{t^2}{s^2} \right)^{\frac{k}{2}-1} (1 - t^2)^{\frac{p}{2}-1} \, \mathrm{d}t, \tag{2}$$

where $k = n - p$ is the codimension.

In the case of codimension 1, this equation becomes:

$$I(g) = 2\nu_{p-1} \int_{0}^{1} \frac{(1 - t^2)^{\frac{p}{2}-1}}{(1 - r^2(1 - t^2))^{1/2}} \, \mathrm{d}t,$$

as wanted. In particular, the upper and lower bounds are given by

$$2\nu_{p-1} \int_{0}^{1} (1 - t^2)^{\frac{p}{2}-1} \, \mathrm{d}t \leq I(g) \leq \frac{2\nu_{p-1}}{(1 - r^2)^{1/2}} \int_{0}^{1} (1 - t^2)^{\frac{p}{2}-1} \, \mathrm{d}t,$$

which yields

$$\nu_{p-1} \mathrm{B}\left(\frac{1}{2}, \frac{p}{2} \right) \leq I(g) \leq \frac{\nu_{p-1} \mathrm{B}(\frac{1}{2}, \frac{p}{2})}{\mathrm{d}_{\mathbb{P}}(g, S^p)},$$

as wanted.

In the case of even codimension $k=n-p=2\tau$, with $\tau\in\mathbb{N}^*$, equation (2) yields:

$$I(g)=\frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)}\sum_{i=0}^{\tau-1}\int_0^1\binom{\tau-1}{i}\left(\frac{t^2}{s^2}\right)^i(1-t^2)^{\tau-i+\frac{p}{2}-2}\ \mathrm{d}t.$$

Then,

$$I(g)=\sum_{i=0}^{\tau-1}\binom{\tau-1}{i}\frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{2i+1}}\int_0^1 t^{2i}(1-t^2)^{\frac{n}{2}-i-2}\ \mathrm{d}t,$$

and

$$I(g)=\sum_{i=0}^{\tau-1}\frac{\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{2i+1}}\frac{\mathrm{B}(i+\frac{1}{2},\frac{n}{2}-i-1)}{\tau\mathrm{B}(\tau-i,i+1)}.$$

In the case of odd codimension $k=n-p=2\tau+1$, with $\tau\in\mathbb{N}^*$, equation (2) yields:

$$I(g)=\frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)}\int_0^1\left((1-t^2)+\frac{t^2}{s^2}\right)^{\tau-\frac{1}{2}}(1-t^2)^{\frac{p}{2}-1}\ \mathrm{d}t.$$

Observing that

$$\frac{t}{s}\leq\left((1-t^2)+\frac{t^2}{s^2}\right)^{1/2}=\left(\frac{s^2+r^2t^2}{s^2}\right)^{1/2}\leq\frac{1}{s},\tag{3}$$

equation (2) becomes:

$$I(g)=\frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)}\int_0^1\left((1-t^2)+\frac{t^2}{s^2}\right)^{(\tau-1)+\frac{1}{2}}(1-t^2)^{\frac{p}{2}-1}\ \mathrm{d}t.$$

Then, expanding $\left((1-t^2)+\frac{t^2}{s^2}\right)^{\tau-1}$ yields

$$I(g)=\sum_{i=0}^{\tau-1}\frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{2i+1}}\binom{\tau-1}{i}\int_0^1 t^{2i}(1-t^2)^{\frac{n}{2}-i-\frac{5}{2}}\left((1-t^2)+\frac{t^2}{s^2}\right)^{1/2}\ \mathrm{d}t,$$

where

$$\binom{\tau-\frac{1}{2}}{i}=\frac{(\tau-\frac{1}{2})_i}{i!}=\frac{1}{(\tau+\frac{1}{2})\mathrm{B}(\tau-i+\frac{1}{2},i+1)}.$$

and $(\tau - \frac{1}{2})_i$ is Pochhammer symbol:

$$\left(\tau - \frac{1}{2}\right)_i = \frac{\Gamma(\tau + \frac{1}{2})}{\Gamma(\tau - i + \frac{1}{2})}.$$

Thus,

$$I(g) \leq \sum_{i=0}^{\tau-1} \frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g, S^p)^{2i+2}} \frac{\int_0^1 t^{2i}(1-t^2)^{\frac{n}{2}-i-\frac{5}{2}} \,\mathrm{d}t}{\tau \mathrm{B}(\tau - i, i+1)},$$

and

$$I(g) \geq \sum_{i=0}^{\tau-1} \frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g, S^p)^{2i+2}} \frac{\int_0^1 t^{2i+1}(1-t^2)^{\frac{n}{2}-i-\frac{5}{2}} \,\mathrm{d}t}{\tau \mathrm{B}(\tau - i, i+1)}.$$

Namely,

$$I(g) \leq \sum_{i=0}^{\tau-1} \frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g, S^p)^{2i+2}} \frac{\mathrm{B}(i + \frac{1}{2}, \frac{n}{2} - i - \frac{3}{2})}{\tau \mathrm{B}(\tau - i, i+1)},$$

and

$$I(g) \geq \sum_{i=0}^{\tau-1} \frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g, S^p)^{2i+2}} \frac{\mathrm{B}(i + 1, \frac{n}{2} - i - \frac{3}{2})}{\tau \mathrm{B}(\tau - i, i+1)}. \qquad \square$$

Remark 31. One may want a close formula for the latter case. In that case, we have to be careful when expanding equation (2) as we have to distinguish both cases when $1 - t^2 \geq \frac{t^2}{s^2}$ and when $1 - t^2 \leq \frac{t^2}{s^2}$. Hence

$$I(g) = \frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g, S^p)} \sum_{i=0}^{\infty} \binom{\tau - \frac{1}{2}}{i} \left(\int_0^{\frac{1}{\sqrt{1+s^2}}} \left(\frac{t^2}{s^2}\right)^i (1-t^2)^{\frac{n}{2}-i-2} \,\mathrm{d}t \right.$$
$$\left. + \int_{\frac{1}{\sqrt{1+s^2}}}^1 \left(\frac{t^2}{s^2}\right)^{\frac{n-p}{2}-i-1} \left(1-t^2\right)^{\frac{p}{2}+i-1} \,\mathrm{d}t \right)$$

This yields

$$I(g) = 2\nu_{p-1} \sum_{i=0}^{\infty} \binom{\tau - \frac{1}{2}}{i} \left(\int_0^{\frac{1}{\sqrt{1+s^2}}} \frac{t^{2i}(1-t^2)^{\frac{n}{2}-i-2} \,\mathrm{d}t}{\mathrm{d}_{\mathbb{P}}(g, S^p)^{2i+1}} \right.$$
$$\left. + \int_{\frac{1}{\sqrt{1+s^2}}}^1 \frac{t^{n-p-2i-2}(1-t^2)^{\frac{p}{2}+i-1} \,\mathrm{d}t}{\mathrm{d}_{\mathbb{P}}(g, S^p)^{n-p-2i-1}} \right),$$

and, hence,

$$I(g) = \frac{\nu_{p-1}}{\tau+\frac{1}{2}} \sum_{i=0}^{\infty} \frac{1}{\mathrm{B}(\tau-i+\frac{1}{2}, i+1)} \left(\frac{\mathrm{B}\left(\frac{1}{1+s^2}; i+\frac{1}{2}, \frac{n}{2}-i-1\right)}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{2i+1}} + \frac{\mathrm{B}\left(\frac{s^2}{1+s^2}; \frac{p}{2}+i, \frac{n-p-1}{2}-i\right)}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{n-p-2i-1}} \right),$$

where $\mathrm{B}(x;a,b)$ is the incomplete Beta function:

$$\mathrm{B}(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\ \mathrm{d}t.$$

3.5.2. Proof of Remark 22.

This remark immediately follows from equation (1). Making the obvious change of variable, this equation yields:

$$I(g) = 2\nu_{p-1} \int_0^{1/s} \left(1+r^2z^2\right)^{\frac{n-p-2}{2}} (1-s^2z^2)^{\frac{p-2}{2}}\ \mathrm{d}z.$$

And, by the standard definition of Appell's F_1 hyper-geometric function, we immediately obtain:

$$I(g) = \frac{\nu_{p-1}}{2\mathrm{d}_{\mathbb{P}}(g,S^p)} F_1\left(\frac{1}{2}, \frac{p+2-n}{2}, \frac{2-p}{2}, \frac{3}{2}; -\cot(\mathrm{d}_R(g,S^p)), 1\right),$$

where $\cot(\mathrm{d}_R(g,S^p))$ is the cotangent of the Riemannian distance of g to S^p. $\square$

4. Proof of the main results

4.1. Proof of Theorem 4

As above, we assume $M = \mathbb{R}^{p+1}$, $S(M) = S^p$ and $k = n-p$ the codimension of S^p in S^n. Let $\varphi : S^n \longrightarrow \mathbb{R}_+$ be an integrable function and let I be the quantity:

$$I = \int_{(g,f)\in S^n\times S^p} \left(\int_{L_{(g,f)}} \varphi(h)\ \mathrm{d}L_{(g,f)}(h) \right) \mathrm{d}S^n\ \mathrm{d}S^p,$$

where $L_{(g,f)} \in \mathcal{L}$ is the great circle containing g and f and $\mathrm{d}L_{(g,f)}$ is the standard measure on the great circle.

Let $\Phi : S^n \times S^p \setminus \mathrm{Diag} \longrightarrow \mathcal{L}_M$, be the mapping discussed in Section 3 and given by $\Phi(g,f) = L_{(g,f)} \in \mathcal{L}_M$, where $\mathcal{L}_M$ is the semialgebraic set of great circles in $\mathcal{L}$ that intersect S^p. According to the Co-area formula (Theorem 25) we have:

$$I = \int_{\mathcal{L}_M} \left(\int_{\Phi^{-1}(L)} \frac{\theta(g,f)}{NJ_{(g,f)}\Phi} \, \mathrm{d}\Phi^{-1}(L) \right) \, \mathrm{d}\mathcal{L}_M,$$

where

$$\theta(g,f) = \int_{L_{(g,f)}} \varphi(h) \, \mathrm{d}L_{(g,f)}(h).$$

Note that, for $L \in \mathfrak{C}_M$, if $L \cap S^p = \{\pm f\}$, we have $\Phi^{-1}(L) = L \times \{f\} \cup L \times \{-f\}$ and we conclude:

$$I = 2 \int_{\mathcal{L}_M} \left(\int_L \frac{\theta(g,f)}{NJ_{(g,f)}\Phi} \, \mathrm{d}L \right) \, \mathrm{d}\mathcal{L}_M.$$

Now, from Proposition 19 we conclude:

$$I = 2 \int_{\mathcal{L}_M} \frac{\theta(g,f)}{\partial_M(L)} \left(\int_L \frac{\mathrm{d}_{\mathbb{P}}(g, S(M))^{n-1}}{\partial_M\,(L)^{n-1}} \, \mathrm{d}L \right) \, \mathrm{d}\mathcal{L}_M.$$

Then, from Lemma 27 we conclude that the inner integral is constant and independent of L and, hence, the following holds:

$$I = 2\mathrm{B}\left(\frac{n+2}{2}, \frac{1}{2}\right) \int_{\mathcal{L}_M} \frac{\theta(g,f)}{\partial_M(L)} \, \mathrm{d}\mathcal{L}_M,$$

i.e.

$$I = 2\mathrm{B}\left(\frac{n+2}{2}, \frac{1}{2}\right) \int_{\mathcal{L}_M} \left(\frac{1}{\partial_M(L)} \int_L \varphi(h) \, \mathrm{d}L(h) \right) d\mathcal{L}_M.$$

Now, considering the incidence variety $\mathcal{IC}(M)$ given by

$$\mathcal{IC}(M) = \left\{ ([A], g) \in \mathcal{L} \times S^n, \; g \in \mathcal{S}\mathrm{pan}(A), \; [A] \in \mathcal{L}_M \right\},$$

and the canonical projections $p_1 : \mathcal{IC}(M) \longrightarrow \mathcal{L}_M$ and $p_2 : \mathcal{IC}(M) \longrightarrow S^n$, and applying twice the Co-area formula allows to conclude:

$$I = 2\mathrm{B}\left(\frac{n+2}{2}, \frac{1}{2}\right) \int_{(L,g) \in \mathcal{IC}(M)} \left(\frac{NJ_{(L,g)}p_1}{\partial_M(L)} \varphi(g) \right) \, \mathrm{d}\mathcal{IC}(M),$$

and,

$$I = 2\mathrm{B}\left(\frac{n+2}{2},\frac{1}{2}\right)\int_{S^n}\left(\int_{p_2^{-1}(g)}\frac{1}{\partial_M(L)}\varphi(g)\frac{NJ_{(L,g)}p_1}{NJ_{(L,g)}p_2}\ \mathrm{d}p_2^{-1}(g)(L)\right)\ \mathrm{d}S^n.$$

Namely, we have:

$$I = 2\mathrm{B}\left(\frac{n+2}{2},\frac{1}{2}\right)\int_{S^n}\varphi(g)\left(\int_{\mathcal{IC}(M)_g}\frac{1}{\partial_M(L)}\frac{NJ_{(L,g)}p_1}{NJ_{(L,g)}p_2}\ \mathrm{d}\mathcal{IC}(M)_g(L)\right)\ \mathrm{d}S^n.$$

According to the notation used in Proposition 21, this equality may be rewritten as:

$$I = 2\mathrm{B}\left(\frac{n+2}{2},\frac{1}{2}\right)\int_{S^n}\varphi(g)I(g)\ \mathrm{d}S^n.$$

This proposition implies the following cases according to the codimension $k = n - p$:

- If $k = 1$, the following inequalities result from Proposition 21:

  $$I \geq 2\nu_{p-1}\mathrm{B}\left(\frac{1}{2},\frac{p}{2}\right)\mathrm{B}\left(\frac{n+2}{2},\frac{1}{2}\right)\int_{S^n}\varphi(g)\ \mathrm{d}S^n,$$

  and

  $$I \leq 2\nu_{p-1}\mathrm{B}\left(\frac{1}{2},\frac{p}{2}\right)\mathrm{B}\left(\frac{n+2}{2},\frac{1}{2}\right)\int_{S^n}\frac{\varphi(g)}{\mathrm{d}_{\mathbb{P}}(g,S^p)}\ \mathrm{d}S^n.$$

  As E is an expectation, we have

  $$E = \frac{1}{\nu_n\nu_p}I,$$

  and hence the following two inequalities:

  $$E \geq \frac{2\nu_{p-1}\mathrm{B}(\frac{1}{2},\frac{p}{2})\mathrm{B}(\frac{n+2}{2},\frac{1}{2})}{\nu_p}\frac{1}{\nu_n}\int_{S^n}\varphi(g)\ \mathrm{d}S^n,$$
  $$E \leq \frac{2\nu_{p-1}\mathrm{B}(\frac{1}{2},\frac{p}{2})\mathrm{B}(\frac{n+2}{2},\frac{1}{2})}{\nu_p\mathrm{B}(1,\frac{n-2}{2})}\frac{\mathrm{B}(1,\frac{n-2}{2})}{\nu_n}\int_{S^n}\frac{\varphi(g)}{\mathrm{d}_{\mathbb{P}}(g,S^p)}\ \mathrm{d}S^n.$$

According to Definition 1, these two inequalities may be rewritten as

$$C(n,p)E_{S^n}[\varphi] \leq E \leq D(n,p)\mathbf{R}^{n-p-1}\varphi(S^p),$$

where

$$C(n,p) = \frac{2\nu_{p-1}\mathrm{B}(\frac{1}{2},\frac{p}{2})\mathrm{B}(\frac{n+2}{2},\frac{1}{2})}{\nu_p},$$

and

$$D(n,p) = \frac{C(n,p)}{\mathrm{B}(1,\frac{n-2}{2})} = \frac{n-2}{2}C(n,p).$$

Using Gautschi's [30] and Kershaw's [31] inequalities we conclude:

$$4\sqrt{\frac{\pi(2p+1)}{(p+\sqrt{3}-2)(n+\sqrt{3})}} \leq C(n,p) \leq 4\sqrt{\frac{\pi(p+\sqrt{3}-1)}{(2p-1)(n+3)}},$$

whereas

$$\frac{n-2}{2}\sqrt{\frac{\pi(2p+1)}{(p+\sqrt{3}-2)(n+\sqrt{3})}} \leq D(n,p) \leq \frac{n-2}{2}\sqrt{\frac{\pi(p+\sqrt{3}-1)}{(2p-1)(n+3)}},$$

- If $k \in 2\mathbb{N}^*$ is an even integer number we have:

$$I = \sum_{i=0}^{\frac{k}{2}-1} 4\mathrm{B}\left(\frac{n+2}{2},\frac{1}{2}\right)\nu_{p-1}\frac{\mathrm{B}(i+\frac{1}{2},\frac{n}{2}-i-1)}{k\mathrm{B}(\frac{k}{2}-i,i+1)}\int_{S^n}\frac{\varphi(g)}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{2i+1}}\,\mathrm{d}S^n.$$

Namely, in terms of Definition 1, we have proved

$$I = \sum_{i=0}^{\frac{k}{2}-1}\frac{4\mathrm{B}(\frac{n+2}{2},\frac{1}{2})\nu_{p-1}\mathrm{B}(i+\frac{1}{2},\frac{n}{2}-i-1)\nu_n}{k\mathrm{B}(\frac{k}{2}-i,i+1)}\mathbf{R}^{k-2i-1}\varphi(S^p).$$

Namely, we have

$$E = \frac{1}{\nu_n\nu_p}I = \sum_{i=0}^{\frac{k}{2}-1}C(n,p,i)\mathbf{R}^{k-2i-1}\varphi(S^p),$$

where

$$C(n,p,i) = 2\binom{\frac{n-p}{2}-1}{i}\frac{\mathrm{B}(\frac{n+2}{2},\frac{1}{2})}{\mathrm{B}(\frac{p-1}{2},\frac{1}{2})}.$$

- If $k \in (2\mathbb{N}^* + 1)$ is an odd integer, according to Proposition 21 we may use the finite sum bounds to conclude:

$$E \leq \sum_{i=0}^{\frac{k-3}{2}} \binom{\frac{k-3}{2}}{i} \frac{4\nu_{p-1}\mathrm{B}(\frac{n+2}{2}, \frac{1}{2})\mathrm{B}(i+\frac{1}{2}, \frac{n-2i-3}{2})}{\nu_p \nu_n} \int_{S^n} \frac{\varphi(g)}{\mathrm{d}_{\mathbb{P}}(g, S^p)^{2i+2}} \, \mathrm{d}S^n.$$

On the other hand the same proposition also yields:

$$E \geq \sum_{i=0}^{\frac{k-3}{2}} \binom{\frac{k-3}{2}}{i} \frac{4\nu_{p-1}\mathrm{B}(\frac{n+2}{2}, \frac{1}{2})\mathrm{B}(i+1, \frac{n-2i-3}{2})}{\nu_p \nu_n} \int_{S^n} \frac{\varphi(g)}{\mathrm{d}_{\mathbb{P}}(g, S^p)^{2i+2}} \, \mathrm{d}S^n.$$

Thus, we conclude

$$\sum_{i=0}^{\frac{k-3}{2}} A_1(n,p,i)\mathbf{R}^{k-2i-2}\varphi(S^p) \leq E \leq \sum_{i=0}^{\frac{k-3}{2}} A_2(n,p,i)\mathbf{R}^{k-2i-2}\varphi(S^p),$$

where

$$A_1(n,p,i) = 2\binom{\frac{k-3}{2}}{i} \frac{(n-2)\mathrm{B}(\frac{n+2}{2}, \frac{1}{2})\Gamma(i+1)}{\mathrm{B}(\frac{p-1}{2}, \frac{1}{2})\Gamma(i+\frac{3}{2})},$$

and

$$A_2(n,p,i) = 4\binom{\frac{k-3}{2}}{i} \frac{\mathrm{B}(\frac{n+2}{2}, \frac{1}{2})\Gamma(i+\frac{1}{2})\Gamma(\frac{n}{2}-1)}{\mathrm{B}(\frac{p-1}{2}, \frac{1}{2})\Gamma(i+\frac{3}{2})\Gamma(\frac{n}{2})}.$$

Now, using Gautschi's [30] and Kershaw's [31] inequalities, we conclude:

$$A_1(n,p,i) \geq \binom{\frac{n-p-3}{2}}{i} \frac{\mathrm{B}(\frac{n+2}{2}, \frac{1}{2})}{\mathrm{B}(\frac{p-1}{2}, \frac{1}{2})} \frac{2\sqrt{2}(n-2)}{\sqrt{2i+\sqrt{3}}} = \frac{B_0(n,p,i)(n-2)}{\sqrt{i+\sqrt{3/2}}}.$$

$$\begin{aligned} A_2(n,p,i) &= 16\binom{\frac{n-p-3}{2}}{i} \frac{\mathrm{B}(\frac{n+2}{2}, \frac{1}{2})}{\mathrm{B}(\frac{p-1}{2}, \frac{1}{2})} \frac{1}{(2i+1)(n-2)} \\ &= \frac{8B_0(n,p,i)}{(2i+1)(n-2)}. \quad \square \end{aligned}$$

4.2. Proof of Corollary 6

With the same notation as above, we make use of inequalities (3) to conclude from equation (2) :

$$\frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{k-1}}\int_0^1 t^{k-2}(1-t^2)^{\frac{p}{2}-1}\,\mathrm{d}t \le I(g) \le \frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{n-p-1}}.$$

Namely,

$$\frac{\nu_{p-1}\mathrm{B}(\frac{n-p}{2},\frac{p}{2})}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{k-1}} \le I(g) \le \frac{2\nu_{p-1}}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{k-1}}.$$

From the proof of Theorem 4 above, we conclude

$$E \ge \frac{2\mathrm{B}(\frac{n+2}{2},\frac{1}{2})\nu_{p-1}\mathrm{B}(\frac{n-p}{2},\frac{p}{2})}{\nu_p\nu_n}\int_{S^n}\frac{\varphi(g)}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{k-1}}\,\mathrm{d}S^n,$$

and

$$E \le \frac{2\mathrm{B}(\frac{n+2}{2},\frac{1}{2})\nu_{p-1}}{\nu_p\nu_n}\int_{S^n}\frac{\varphi(g)}{\mathrm{d}_{\mathbb{P}}(g,S^p)^{k-1}}\,\mathrm{d}S^n.$$

According to Definition 1, this means:

$$E \ge \frac{2\mathrm{B}(\frac{n+2}{2},\frac{1}{2})\nu_{p-1}}{\nu_p}\mathbf{R}^1\varphi(S^p),$$

and

$$E \le \frac{2\mathrm{B}(\frac{n+2}{2},\frac{1}{2})\nu_{p-1}}{\nu_p\mathrm{B}(\frac{n-p}{2},\frac{p}{2})}\mathbf{R}^1\varphi(S^p).$$

Using Gautschi's [30] and Kershaw's [31] inequalities, we finally obtain:

$$\sqrt{\frac{2p+1}{2(n+\sqrt{3})}}\mathbf{R}^1\varphi(S^p) \le E \le 2\sqrt{\frac{2(p+\sqrt{3}-1)}{2n+3}}\frac{1}{\mathrm{B}(\frac{n-p}{2},\frac{p}{2})}\mathbf{R}^1\varphi(S^p),$$

as wanted. $\square$

4.3. Proof of Proposition 7

With the same notation as in the Introduction, according to Lemma 27, for every $L \in \mathfrak{C}_M$, we have:

$$E_{L_M}[\varphi] = \frac{1}{\mathrm{B}(\frac{n-p+2}{2}, \frac{1}{2})\partial_M(L)^{n-p-1}} \int_L \varphi(g)\ \mathrm{d}_{\mathbb{P}}(g, S^p)^{n-p-1}\ \mathrm{d}L.$$

Then, we use the Co-area formula (Theorem 25) as in the proof of Theorem 4 above, to conclude:

$$\int_{\mathcal{L}_M} E_{L_M}[\varphi] =$$

$$\int_{S^n} \frac{\varphi(g)}{\mathrm{B}(\frac{n-p+2}{2}, \frac{1}{2})} \left(\int_{\mathcal{IC}(M)_g} \frac{\mathrm{d}_{\mathbb{P}}(f, S^p)^{n-p-1}}{\partial_M(L)^{n-p-1}} \frac{NJ_{(L,g)}p_1}{NJ_{(L,g)}p_1}\ \mathrm{d}[p_2^{-2}(g)](L) \right)\ \mathrm{d}S^n(g).$$

According to Proposition 20, this yields:

$$\int_{\mathcal{L}_M} E_{L_M}[\varphi] = \int_{S^n} \frac{\varphi(g)}{\mathrm{B}(\frac{n-p+2}{2}, \frac{1}{2})} \left(\int_{\mathcal{IC}(M)_g} \mathrm{d}[p_2^{-2}(g)](L) \right)\ \mathrm{d}S^n(g).$$

Then, applying Proposition 29 we conclude:

$$\int_{\mathcal{L}_M} E_{L_M}[\varphi] = \frac{\nu_p}{\mathrm{B}(\frac{n-p+2}{2}, \frac{1}{2})} \int_{S^n} \varphi(g)\ \mathrm{d}S^n(g) = \frac{\nu_p \nu_n}{\mathrm{B}(\frac{n-p+2}{2}, \frac{1}{2})} E_{S^n}[\varphi].$$

Now, taking $\varphi = 1$, we conclude:

$$\mathrm{vol}\,[\mathcal{L}_M] = \frac{\nu_p \nu_n}{\mathrm{B}(\frac{n-p+2}{2}, \frac{1}{2})} E_{S^n}[1] = \frac{\nu_p \nu_n}{\mathrm{B}(\frac{n-p+2}{2}, \frac{1}{2})},$$

and Proposition 7 follows immediately. □

5. Proof of the statements related to polynomial equation solving

We follow the notation introduced in Section 1.3. We will use the notation $\mathbb{S}^{2N+1}$ to denote $\mathbb{S}(\mathcal{H}_{(d)})$ and $\mathbb{S}^p$ to denote $\mathbb{S}(M)$. As in [39], let $V_{(d)} \subseteq \mathbb{S}^{2N+1} \times \mathbb{P}_n(\mathbb{C})$ be the solution variety. Namely,

$$V_{(d)} = \left\{ (f, \zeta) \in \mathbb{S}^{2N+1} \times \mathbb{P}(\mathbb{C}^{n+1}),\ \zeta \in V(f) \right\}.$$

5.1. Proof of Corollary 8

Let us define $\widetilde{\Sigma} \subseteq \mathcal{L}_M$ as the subset of all great circles $L \in \mathcal{L}_M$ that intersect the discriminant variety Σ. As $\dim(\Sigma \cap \mathbb{S}(M)) < \dim \mathbb{S}(M)$, using the double fibration as in Section 2 above, we may conclude that the codimension of $\widetilde{\Sigma}$ in $\mathcal{L}_M$ is at least 1 and, hence, it is a semialgebraic set of volume zero. Namely,

$$E_{\mathcal{L}_M}[\chi_{\widetilde{\Sigma}}] = 0,$$

where $E_{\mathcal{L}_M}$ means expectation in $\mathcal{L}_M$ and $\chi_{\widetilde{\Sigma}} : \mathcal{L}_M \longrightarrow \{0,1\}$ is the characteristic function defined by $\widetilde{\Sigma}$.

Let us define the mapping $\Theta_{\widetilde{\Sigma}} : V_{(d)} \longrightarrow \mathbb{R}_+$ given by the following identity:

$$\Theta_{\widetilde{\Sigma}}(g,\zeta) = E_{\mathbb{S}^p}[\mathcal{C}(f,g,\zeta)] = \frac{1}{\nu_p}\int_{\mathbb{S}^p} \chi_{\widetilde{\Sigma}}(L_{(g,f)})\ \mathrm{d}\mathbb{S}^p,$$

where $L_{(g,f)}$ is the great circle passing through g and f. Let $\mathcal{G}_{(d)} \subseteq V_{(d)}$ be the strong questor set defined in [14], endowed with its probability distribution. The probability that the algorithm outputs Failure is at most the expectation $E_{\mathcal{G}_{(d)}}[\Theta_{\widetilde{\Sigma}}]$. By [14, Theorem 7], the following equality holds:

$$E_{\mathcal{G}_{(d)}}[\Theta_{\widetilde{\Sigma}}] = \frac{1}{\nu_{2N+1}}\int_{\mathbb{S}^{2N+1}} \frac{1}{\mathcal{D}} \sum_{\zeta \in V_{\mathbb{P}}(g)} \Theta_{\widetilde{\Sigma}}(g,\zeta)\ \mathrm{d}\mathbb{S}^{2N+1}.$$

Namely, this expectation satisfies:

$$E_{\mathcal{G}_{(d)}}[\Theta_{\widetilde{\Sigma}}] = \frac{1}{\nu_{2N+1}\nu_p}\int_{\mathbb{S}^{2N+1}\times\mathbb{S}^p} \frac{1}{\mathcal{D}} \sum_{\zeta \in V_{\mathbb{P}}(g)} \chi_{\widetilde{\Sigma}}(L_{(g,f)})\ \mathrm{d}\mathbb{S}^{2N+1}\ \mathrm{d}\mathbb{S}^p.$$

In other terms,

$$E_{\mathcal{G}_{(d)}}[\Theta_{\widetilde{\Sigma}}] = \frac{1}{\nu_{2N+1}\nu_p}\int_{\mathbb{S}^{2N+1}\times\mathbb{S}^p} \chi_{\widetilde{\Sigma}}(L_{(g,f)})\ \mathrm{d}\mathbb{S}^{2N+1}\ \mathrm{d}\mathbb{S}^p.$$

According to Proposition 19 and the Co-area formula, we have:

$$E_{\mathcal{G}_{(d)}}[\Theta_{\widetilde{\Sigma}}] = \frac{1}{\nu_{2N+1}\nu_p}\int_{\mathfrak{C}_M} \left(\int_{L_{(g,f)}} \chi_{\widetilde{\Sigma}}(L_{(g,f)}) \frac{\mathrm{d}_{\mathbb{P}}(g,\mathbb{S}^p)^{n-1}}{\partial_M\left(L_{(g,f)}\right)^n}\ \mathrm{d}L_{(g,f)}\right)\ \mathrm{d}\mathfrak{C}_M.$$

Finally, as $\mathrm{d}_{\mathbb{P}}(g,\mathbb{S}^p) \leq \partial_M(L_{(g,f)})$ we have

$$0 \le E_{\mathcal{G}_{(d)}}[\Theta_{\widetilde{\Sigma}}] \le \frac{2\pi}{\nu_{2N+1}\nu_p} \int_{\mathcal{L}_M} \chi_{\widetilde{\Sigma}}(L_{(g,f)}) \frac{1}{\partial_M(L_{(g,f)})} \, \mathrm{d}\mathcal{L}_M.$$

As $\widetilde{\Sigma}$ has zero measure in $\mathcal{L}_M$, we conclude $E_{\mathcal{G}_{(d)}}[\Theta_{\widetilde{\Sigma}}] = 0$ and the claim of Corollary 8 follows. □

5.2. Proof of Corollaries 9, 10 and 11

Again we use the same strategy based on [14]. Let us define the mapping $\Theta : V_{(d)} \longrightarrow \mathbb{R}_+$ given by the following identity:

$$\Theta(g, \zeta) = E_{\mathbb{S}^p}[\mathcal{C}(f, g, \zeta)] = \frac{1}{\nu_p} \int_{\mathbb{S}^p} \mathcal{C}(f, g, \zeta) \, \mathrm{d}\mathbb{S}^p,$$

where $\mathrm{d}\mathbb{S}^p$ is the volume form associated to the Riemannian structure of $\mathbb{S}^N$ and ν_p is the volume of $\mathbb{S}^p$.

Let $\mathcal{G}_{(d)} \subseteq V_{(d)}$ be the strong questor set defined in [14], endowed with its probability distribution. By [14, Theorem 7], the following equality holds:

$$E_M[\mathrm{Time}] = E_{\mathcal{G}_{(d)}}[\Theta] = \frac{1}{\nu_{2N+1}} \int_{\mathbb{S}^{2N+1}} \frac{1}{\mathcal{D}} \sum_{\zeta \in V_{\mathbb{P}}(g)} \Theta(g, \zeta) \, \mathrm{d}\mathbb{S}^{2N+1}, \qquad (4)$$

where E denotes expectation, $\mathcal{D} = \prod_{i=1}^n d_i$ is the Bézout number associated to the list $(d) = (d_1, \ldots, d_n)$, $\mathrm{d}\mathbb{S}^{2N+1}$ the volume form in $\mathbb{S}^{2N+1}$ and ν_{2N+1} the volume of this sphere.

Now observe that equation (4) may be rewritten as:

$$E_M = \frac{1}{\nu_{2N+1}\nu_p} \int_{\mathbb{S}^{2N+1} \times \mathbb{S}^p} \frac{1}{\mathcal{D}} \sum_{\zeta \in V_{\mathbb{P}}(g)} \mathcal{C}(f, g, \zeta) \, \mathrm{d}\mathbb{S}^{2N+1} \, \mathrm{d}\mathbb{S}^p.$$

From the definition of $\mu_{\mathrm{av}}^2(g)$, we immediately conclude:

$$E_M = \frac{1}{\nu_{2N+1}\nu_p} \int_{\mathbb{S}^{2N+1} \times \mathbb{S}^p} \left(\int_{L_{(g,f)}} \mu_{\mathrm{av}}^2(h) \, \mathrm{d}L_{(g,f)} \right) \mathrm{d}\mathbb{S}^{2N+1} \, \mathrm{d}\mathbb{S}^p.$$

In other words,

$$E_M = E_{(g,f) \in \mathbb{S}^{2N+1} \times \mathbb{S}^p} \left[\int_{L_{(g,f)}} \mu_{\mathrm{av}}^2(h) \, \mathrm{d}L_{(g,f)}(h) \right].$$

Then, Corollary 9 immediately follows from Theorem 4, whereas Corollary 10 immediately follows from Corollary 6.

As for Corollary 11, we apply the Co-area formula and Proposition 19 to conclude:

$$E_M = \frac{1}{\nu_{2N+1}\nu_p} \int_{\mathcal{L}_M} \left(\int_{(g,f)\in\Phi^{-1}(L)} \frac{C(L_{(g,f)})}{NJ_{(g,f)}\Phi} \, \mathrm{d}\Phi^{-1}(L) \right) \, \mathrm{d}\mathcal{L}_M,$$

where

$$C(L_{(g,f)}) = \int_{L_{(g,f)}} \mu_{\mathrm{av}}^2(h) \, \mathrm{d}L_{(g,f)}.$$

As $L = L_{(g,f)}$, using Proposition 19 we conclude:

$$E_M = \frac{1}{\nu_{2N+1}\nu_p} \int_{\mathcal{L}_M} C(L) \left(\int_{\Phi^{-1}(L)} \frac{\mathrm{d}_{\mathbb{P}}(g, \mathbb{S}(M))^{2N}}{\partial_M(L)^{2N+1}} \, \mathrm{d}\Phi^{-1}(L) \right) \, \mathrm{d}\mathcal{L}_M.$$

Namely,

$$E_M = \frac{1}{\nu_{2N+1}\nu_p} \int_{\mathcal{L}_M} \frac{C(L)}{\partial_M(L)} \left(\int_{\Phi^{-1}(L)} \frac{\mathrm{d}_{\mathbb{P}}(g, \mathbb{S}(M))^{2N}}{\partial_M(L)^{2N}} \, \mathrm{d}\Phi^{-1}(L) \right) \, \mathrm{d}\mathcal{L}_M,$$

For great circles $L \in \mathfrak{C}_M$, this equals:

$$E_M = \frac{2}{\nu_{2N+1}\nu_p} \int_{\mathcal{L}_M} \frac{C(L)}{\partial_M(L)} \left(\int_{L} \frac{\mathrm{d}_{\mathbb{P}}(g, \mathbb{S}(M))^{2N}}{\partial_M(L)^{2N}} \, \mathrm{d}L \right) \, \mathrm{d}\mathcal{L}_M.$$

Then, according to Lemma 27, this yields:

$$E_M = \frac{2\mathrm{B}(\frac{2N+3}{2}, \frac{1}{2})}{\nu_{2N+1}\nu_p} \int_{\mathcal{L}_M} \frac{C(L)}{\partial_M(L)} \, \mathrm{d}\mathcal{L}_M.$$

According to Proposition 7, this equality becomes:

$$E_M = \frac{2\mathrm{B}(N+\frac{3}{2}, \frac{1}{2})}{\mathrm{B}(N+1-\frac{p}{2}, \frac{1}{2})} \frac{1}{\mathrm{vol}[\mathcal{L}_M]} \int_{\mathcal{L}_M} \frac{1}{\partial_M(L)} \int_L \mu_{\mathrm{av}}^2(h) \, \mathrm{d}L \, \mathrm{d}\mathcal{L}_M.$$

Namely, we proved

$$E_M = T(N,p) E_{\mathcal{L}_M} \left[\frac{1}{\partial_M(L)} \int_L \mu_{\mathrm{av}}^2(h) \, \mathrm{d}L \right],$$

where

$$T(N,p) = \frac{2\mathrm{B}(N+\frac{3}{2}, \frac{1}{2})}{\mathrm{B}(N+1-\frac{p}{2}, \frac{1}{2})},$$

and Corollary 11 follows. □

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