Decision Problems for Recognizable Languages of Infinite Pictures

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Abstract

Altenbernd, Thomas and Wöhrle have considered in [ATW03] acceptance of languages of infinite two-dimensional words (infinite pictures) by finite tiling systems, with the usual acceptance conditions, such as the Büchi and Muller ones, firstly used for infinite words. Many classical decision problems are studied in formal language theory and in automata theory and arise now naturally about recognizable languages of infinite pictures. We first review in this paper some recent results of [Fin09b] where we gave the exact degree of numerous undecidable problems for Büchi-recognizable languages of infinite pictures, which are actually located at the first or at the second level of the analytical hierarchy, and "highly undecidable". Then we prove here some more (high) undecidability results. We first show that it is Π_2^1-complete to determine whether a given Büchi-recognizable languages of infinite pictures is unambiguous. Then we investigate cardinality problems. Using recent results of [FL09], we prove that it is $D_2(\Sigma_1^1)$-complete to determine whether a given Büchi-recognizable language of infinite pictures is countably infinite, and that it is Σ_1^1-complete to determine whether a given Büchi-recognizable language of infinite pictures is uncountable. Next we consider complements of recognizable languages of infinite pictures. Using some results of Set Theory, we show that the cardinality of the complement of a Büchi-recognizable language of infinite pictures may depend on the model of the axiomatic system **ZFC**. We prove that the problem to determine whether the complement of a given Büchi-recognizable language of infinite pictures is countable (respectively, uncountable) is in the class $\Sigma_3^1 \setminus (\Pi_2^1 \cup \Sigma_2^1)$ (respectively, in the class $\Pi_3^1 \setminus (\Pi_2^1 \cup \Sigma_2^1)$).

Keywords: Languages of infinite pictures; recognizability by tiling systems; decision problems; unambiguity problem; cardinality problems; highly undecidable problems; analytical hierarchy; models of set theory; independence from the axiomatic system **ZFC**.

1 Introduction

Languages of infinite words accepted by finite automata were first studied by Büchi to prove the decidability of the monadic second order theory of one successor over the integers. Since then regular ω-languages have been much studied and many applications have been found for specification and verification of non-terminating systems, see [Tho90, PP04] for many results and references.

Altenbernd, Thomas and Wöhrle have considered in [ATW03] acceptance of languages of infinite two-dimensional words (infinite pictures) by finite tiling systems, with the usual acceptance conditions, such as the Büchi and Muller ones, firstly used for infinite words. This way they extended both the classical theory of ω-regular languages and the classical theory of recognizable languages of finite pictures, [GR97], to the case of infinite pictures.

Many classical decision problems are studied in formal language theory and in automata theory and arise now naturally about recognizable languages of infinite pictures.

In a recent paper, we gave the exact degree of numerous undecidable problems for Büchi-recognizable languages of infinite pictures. In particular, the non-emptiness and the infiniteness problems are Σ^1_1-complete, and the universality problem, the inclusion problem, the equivalence problem, the complementability problem, and the determinizability problem, are all Π^1_2-complete. These decision problems are then located at the first or at the second level of the analytical hierarchy, and "highly undecidable". This gave new natural examples of decision problems located at the first or at the second level of the analytical hierarchy.

Here we first review some of these results, and we study new decision problems, obtaining new results of high undecidability.

We first consider the notion of unambiguous Büchi tiling system, and of unambiguous Büchi-recognizable language of infinite pictures. We show that every language of infinite pictures which is accepted by an unambiguous Büchi tiling system is a Borel set. As a corollary this shows the existence of inherently ambiguous Büchi-recognizable language of infinite pictures. Then we use this result to prove that it is Π_2^1-complete to determine whether a given Büchi-recognizable language of infinite pictures is unambiguous.

Next we study cardinality problems. Using recent results of Finkel and Lecomte in [FL09], we first show that it is $D_2(\Sigma_1^1)$-complete to determine whether a given Büchi-recognizable language of infinite pictures is countably infinite, where $D_2(\Sigma_1^1)$ is the class of 2-differences of Σ_1^1-sets, i.e. the class of sets which are intersections of a Σ_1^1-set and of a Π_1^1-set. And it is Σ_1^1-complete to determine whether a given Büchi-recognizable language of infinite pictures is uncountable.

Then we consider the complements of Büchi-recognizable languages of infinite pictures. By using some results of Set Theory, we show that the cardinality of the complement of a Büchi-recognizable language of infinite pictures may depend on the actual model of the axiomatic system **ZFC**. We prove that one can effectively construct a Büchi tiling system $\mathcal{T}$ accepting a language $L \subseteq \Sigma^{\omega,\omega}$, whose complement is $L^- = \Sigma^{\omega,\omega} - L$, such that:

1. There is a model V_1 of **ZFC** in which L^- is countable.
2. There is a model V_2 of **ZFC** in which L^- has cardinal $2^{\aleph_0}$.
3. There is a model V_3 of **ZFC** in which L^- has cardinal $\aleph_1$ with $\aleph_0 < \aleph_1 < 2^{\aleph_0}$.

Then, using the proof of this result and Schoenfield's Absoluteness Theorem, we prove that the problem to determine whether the complement of a given Büchi-recognizable language of infinite pictures is countable (respectively, uncountable) is in the class $\Sigma_3^1 \setminus (\Pi_2^1 \cup \Sigma_2^1)$ (respectively, in the class $\Pi_3^1 \setminus (\Pi_2^1 \cup \Sigma_2^1)$). This shows that natural cardinality problems are actually located at the **third level** of the analytical hierarchy.

The paper is organized as follows. We recall in Section 2 the notions of tiling systems and of recognizable languages of pictures. In section 3, we recall the definition of the analytical hierarchy on subsets of $\mathbb{N}$. The definitions of the Borel hierarchy and of analytical sets of a Cantor space, along with their effective counterparts, are given in Section 4. Some notions of Set Theory, which are useful in the sequel, are exposed in Section 5. We study decision problems in Section 6, proving new results. Some concluding remarks are given in Section 7.

2 Tiling Systems

We assume the reader to be familiar with the theory of formal (ω)-languages [Tho90, Sta97]. We recall usual notations of formal language theory.

When Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 \ldots a_k$, where $a_i \in \Sigma$ for $i = 1, \ldots, k$, and k is an integer ≥ 1. The *length* of x is k, denoted by $|x|$. The *empty word* has no letter and is denoted by λ; its length is 0. $\Sigma^\star$ is the *set of finite words* (including the empty word) over Σ.
The *first infinite ordinal* is ω. An ω*-word* over Σ is an ω -sequence $a_1 \ldots a_n \ldots$, where for all integers $i \geq 1$, $a_i \in \Sigma$. When σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)\ldots$, where for all i, $\sigma(i) \in \Sigma$, and $\sigma[n] = \sigma(1)\sigma(2)\ldots\sigma(n)$ for all $n \geq 1$ and $\sigma[0] = \lambda$.
The usual concatenation of two finite words u and v is denoted $u.v$ (and sometimes just uv). This product is extended to the product of a finite word u and an ω-word v: the infinite word $u.v$ is then the ω-word such that:
$(u.v)(k) = u(k)$ if $k \leq |u|$, and $(u.v)(k) = v(k - |u|)$ if $k > |u|$.
The *set of* ω*-words* over the alphabet Σ is denoted by Σ^ω. An ω*-language* over an alphabet Σ is a subset of Σ^ω.

We now define two-dimensional words, i.e. pictures.
Let Σ be a finite alphabet, let $\#$ be a letter not in Σ and let $\hat{\Sigma} = \Sigma \cup \{\#\}$. If m and n are two positive integers or if $m = n = 0$, a picture of size (m, n) over Σ is a function p from $\{0, 1, \ldots, m+1\} \times \{0, 1, \ldots, n+1\}$ into $\hat{\Sigma}$ such that $p(i, j) = \#$ if $i \in \{0, m+1\}$ or $j \in \{0, n+1\}$ and $p(i, j) \in \Sigma$ otherwise. The empty picture is the only picture of size $(0, 0)$ and is denoted by λ. Pictures of size $(n, 0)$ or $(0, n)$, for $n > 0$, are not defined. $\Sigma^{\star,\star}$ is the set of pictures over Σ. A picture language L is a subset of $\Sigma^{\star,\star}$. The research on picture languages was firstly motivated by the problems arising in pattern recognition and image processing, a survey on the theory of picture languages may be found in [GR97].

An ω-picture over Σ is a function p from $\omega \times \omega$ into $\hat{\Sigma}$ such that $p(i, 0) = p(0, i) = \#$ for all $i \geq 0$ and $p(i, j) \in \Sigma$ for $i, j > 0$. $\Sigma^{\omega,\omega}$ is the set of ω-pictures over Σ. An ω-picture language L is a subset of $\Sigma^{\omega,\omega}$.
For Σ a finite alphabet we call Σ^{ω^2} the set of functions from $\omega \times \omega$ into Σ. So the set $\Sigma^{\omega,\omega}$ of ω-pictures over Σ is a strict subset of $\hat{\Sigma}^{\omega^2}$.

We shall say that, for each integer $j \geq 1$, the j^{th} row of an ω-picture $p \in \Sigma^{\omega,\omega}$ is the infinite word $p(1, j).p(2, j).p(3, j)\ldots$ over Σ and the j^{th} column of p is the infinite word $p(j, 1).p(j, 2).p(j, 3)\ldots$ over Σ.
As usual, one can imagine that, for integers $j > k \geq 1$, the j^{th} column of p is on the right of the k^{th} column of p and that the j^{th} row of p is "above" the k^{th} row of p.

We introduce now (non deterministic) tiling systems as in the paper [ATW03].
A tiling system is a tuple $\mathcal{A}$=(Q, Σ, Δ), where Q is a finite set of states, Σ is a finite alphabet, $\Delta \subseteq (\hat{\Sigma} \times Q)^4$ is a finite set of tiles.

A Büchi tiling system is a pair $(\mathcal{A},F)$ where $\mathcal{A}$=(Q, Σ, Δ) is a tiling system and $F \subseteq Q$ is the set of accepting states.
A Muller tiling system is a pair $(\mathcal{A}, \mathcal{F})$ where $\mathcal{A}$=(Q, Σ, Δ) is a tiling system and $\mathcal{F}$$\subseteq 2^Q$ is the set of accepting sets of states.

Tiles are denoted by $\begin{pmatrix} (a_3, q_3) & (a_4, q_4) \\ (a_1, q_1) & (a_2, q_2) \end{pmatrix}$ with $a_i \in \hat{\Sigma}$ and $q_i \in Q$,

and in general, over an alphabet Γ, by $\begin{pmatrix} b_3 & b_4 \\ b_1 & b_2 \end{pmatrix}$ with $b_i \in \Gamma$.
A combination of tiles is defined by:

$$\begin{pmatrix} b_3 & b_4 \\ b_1 & b_2 \end{pmatrix} \circ \begin{pmatrix} b'_3 & b'_4 \\ b'_1 & b'_2 \end{pmatrix} = \begin{pmatrix} (b_3, b'_3) & (b_4, b'_4) \\ (b_1, b'_1) & (b_2, b'_2) \end{pmatrix}$$

A run of a tiling system $\mathcal{A}$=(Q, Σ, Δ) over a (finite) picture p of size (m, n) over Σ is a mapping ρ from $\{0, 1, \ldots, m + 1\} \times \{0, 1, \ldots, n + 1\}$ into Q such that for all $(i, j) \in \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$ with $p(i, j) = a_{i,j}$ and $\rho(i, j) = q_{i,j}$ we have

$$\begin{pmatrix} a_{i,j+1} & a_{i+1,j+1} \\ a_{i,j} & a_{i+1,j} \end{pmatrix} \circ \begin{pmatrix} q_{i,j+1} & q_{i+1,j+1} \\ q_{i,j} & q_{i+1,j} \end{pmatrix} \in \Delta.$$

A run of a tiling system $\mathcal{A}$=(Q, Σ, Δ) over an ω-picture $p \in \Sigma^{\omega,\omega}$ is a mapping ρ from $\omega \times \omega$ into Q such that for all $(i, j) \in \omega \times \omega$ with $p(i, j) = a_{i,j}$ and $\rho(i, j) = q_{i,j}$ we have

$$\begin{pmatrix} a_{i,j+1} & a_{i+1,j+1} \\ a_{i,j} & a_{i+1,j} \end{pmatrix} \circ \begin{pmatrix} q_{i,j+1} & q_{i+1,j+1} \\ q_{i,j} & q_{i+1,j} \end{pmatrix} \in \Delta.$$

We now recall acceptance of finite or infinite pictures by tiling systems:

Definition 2.1 *Let $\mathcal{A}$=(Q, Σ, Δ) be a tiling system, $F \subseteq Q$ and $\mathcal{F}$$\subseteq 2^Q$.*

- *The picture language recognized by $\mathcal{A}$ is the set of pictures $p \in \Sigma^{\star,\star}$ such that there is some run ρ of $\mathcal{A}$ on p.*
- *The ω-picture language Büchi-recognized by $(\mathcal{A},F)$ is the set of ω-pictures $p \in \Sigma^{\omega,\omega}$ such that there is some run ρ of $\mathcal{A}$ on p and $\rho(v) \in F$ for infinitely many $v \in \omega^2$. It is denoted by $L^B((\mathcal{A},F))$.*
- *The ω-picture language Muller-recognized by $(\mathcal{A}, \mathcal{F})$ is the set of ω-pictures $p \in \Sigma^{\omega,\omega}$ such that there is some run ρ of $\mathcal{A}$ on p and $Inf(\rho) \in \mathcal{F}$ where $Inf(\rho)$ is the set of states occurring infinitely often in ρ. It is denoted by $L^M((\mathcal{A},\mathcal{F}))$.*

Notice that an ω-picture language $L \subseteq \Sigma^{\omega,\omega}$ is recognized by a Büchi tiling system if and only if it is recognized by a Muller tiling system, [ATW03].
We shall denote $TS(\Sigma^{\omega,\omega})$ the class of languages $L \subseteq \Sigma^{\omega,\omega}$ which are recognized by some Büchi (or Muller) tiling system.

3 Recall of Known Basic Notions

3.1 The Analytical Hierarchy

The set of natural numbers is denoted by $\mathbb{N}$ and the set of all mappings from $\mathbb{N}$ into $\mathbb{N}$ will be denoted by $\mathcal{F}$.

We assume the reader to be familiar with the arithmetical hierarchy on subsets of $\mathbb{N}$. We now recall the notions of analytical hierarchy and of complete sets for classes of this hierarchy which may be found in [Rog67].

Definition 3.1 *Let $k, l > 0$ be some integers. Φ is a partial recursive function of k function variables and l number variables if there exists $z \in \mathbb{N}$ such that for any $(f_1, \ldots, f_k, x_1, \ldots, x_l) \in \mathcal{F}^k \times \mathbb{N}^l$, we have*

$$\Phi(f_1, \ldots, f_k, x_1, \ldots, x_l) = \tau_z^{f_1,\ldots,f_k}(x_1, \ldots, x_l),$$

where the right hand side is the output of the Turing machine with index z and oracles $f_1, \ldots, f_k$ over the input $(x_1, \ldots, x_l)$. For $k > 0$ and $l = 0$, Φ is a partial recursive function if, for some z,

$$\Phi(f_1, \ldots, f_k) = \tau_z^{f_1,\ldots,f_k}(0).$$

The value z is called the Gödel number or index for Φ.

Definition 3.2 *Let $k, l > 0$ be some integers and $R \subseteq \mathcal{F}^k \times \mathbb{N}^l$. The relation R is said to be a recursive relation of k function variables and l number variables if its characteristic function is recursive.*

We now define analytical subsets of $\mathbb{N}^l$.

Definition 3.3 *A subset R of $\mathbb{N}^l$ is analytical if it is recursive or if there exists a recursive set $S \subseteq \mathcal{F}^m \times \mathbb{N}^n$, with $m \geq 0$ and $n \geq l$, such that*

$$R = \{(x_1, \ldots, x_l) \mid (Q_1 s_1)(Q_2 s_2) \ldots (Q_{m+n-l} s_{m+n-l}) S(f_1, \ldots, f_m, x_1, \ldots, x_n)\},$$

where Q_i is either $\forall$ or $\exists$ for $1 \leq i \leq m + n - l$, and where $s_1, \ldots, s_{m+n-l}$ are $f_1, \ldots, f_m, x_{l+1}, \ldots, x_n$ in some order.

The expression $(Q_1 s_1)(Q_2 s_2)\dots(Q_{m+n-l} s_{m+n-l})S(f_1,\dots,f_m,x_1,\dots,x_n)$ *is called a predicate form for* R*. A quantifier applying over a function variable is of type* 1*, otherwise it is of type* 0*. In a predicate form the (possibly empty) sequence of quantifiers, indexed by their type, is called the prefix of the form. The reduced prefix is the sequence of quantifiers obtained by suppressing the quantifiers of type* 0 *from the prefix.*

The levels of the analytical hierarchy are distinguished by considering the number of alternations in the reduced prefix.

Definition 3.4 *For* $n > 0$*, a* Σ^1_n*-prefix is one whose reduced prefix begins with* $\exists^1$ *and has* $n-1$ *alternations of quantifiers. A* Σ^1_0*-prefix is one whose reduced prefix is empty. For* $n > 0$*, a* Π^1_n*-prefix is one whose reduced prefix begins with* $\forall^1$ *and has* $n-1$ *alternations of quantifiers. A* Π^1_0*-prefix is one whose reduced prefix is empty.*
A predicate form is a Σ^1_n *(*Π^1_n*)-form if it has a* Σ^1_n *(*Π^1_n*)-prefix. The class of sets in some* $\mathbb{N}^l$ *which can be expressed in* Σ^1_n*-form (respectively,* Π^1_n*-form) is denoted by* Σ^1_n *(respectively,* Π^1_n*).*
The class $\Sigma^1_0 = \Pi^1_0$ *is the class of arithmetical sets.*

We now recall some well known results about the analytical hierarchy.

Proposition 3.5 *Let* $R \subseteq \mathbb{N}^l$ *for some integer* l*. Then* R *is an analytical set iff there is some integer* $n \geq 0$ *such that* $R \in \Sigma^1_n$ *or* $R \in \Pi^1_n$*.*

Theorem 3.6 *For each integer* $n \geq 1$*,*

(a) $\Sigma^1_n \cup \Pi^1_n \subsetneq \Sigma^1_{n+1} \cap \Pi^1_{n+1}$.

(b) A set $R \subseteq \mathbb{N}^l$ *is in the class* Σ^1_n *iff its complement is in the class* Π^1_n*.*

(c) $\Sigma^1_n - \Pi^1_n \neq \emptyset$ *and* $\Pi^1_n - \Sigma^1_n \neq \emptyset$*.*

Transformations of prefixes are often used, following the rules given by the next theorem.

Theorem 3.7 *For any predicate form with the given prefix, an equivalent predicate form with the new one can be obtained, following the allowed prefix transformations given below :*

(a) $\dots\exists^0\exists^0\dots \rightarrow \dots\exists^0\dots, \qquad \dots\forall^0\forall^0\dots \rightarrow \dots\forall^0\dots;$

(b) $\dots\exists^1\exists^1\dots \rightarrow \dots\exists^1\dots, \qquad \dots\forall^1\forall^1\dots \rightarrow \dots\forall^1\dots;$

(c) $\ldots\exists^0 \quad \ldots \rightarrow \ldots\exists^1\ldots, \qquad \ldots\forall^0 \quad \ldots \rightarrow \ldots\forall^1\ldots;$

(d) $\ldots\exists^0\forall^1\ldots \rightarrow \ldots\forall^1\exists^0\ldots, \qquad \ldots\forall^0\exists^1\ldots \rightarrow \ldots\exists^1\forall^0\ldots;$

We can now define the notion of 1-reduction and of Σ^1_n-complete (respectively, Π^1_n-complete) sets. Notice that we give the definition for subsets of $\mathbb{N}$ but one can easily extend this definition to the case of subsets of $\mathbb{N}^l$ for some integer l.

Definition 3.8 *Given two sets $A, B \subseteq \mathbb{N}$ we say A is 1-reducible to B and write $A \leq_1 B$ if there exists a total computable injective function f from $\mathbb{N}$ to $\mathbb{N}$ such that $A = f^{-1}[B]$.*

Definition 3.9 *A set $A \subseteq \mathbb{N}$ is said to be Σ^1_n-complete (respectively, Π^1_n-complete) iff A is a Σ^1_n-set (respectively, Π^1_n-set) and for each Σ^1_n-set (respectively, Π^1_n-set) $B \subseteq \mathbb{N}$ it holds that $B \leq_1 A$.*

For each integer $n \geq 1$ there exists some Σ^1_n-complete set $E_n \subseteq \mathbb{N}$. The complement $E^-_n = \mathbb{N} - E_n$ is a Π^1_n-complete set. These sets are precisely defined in [Rog67] or [CC89].

3.2 Borel Hierarchy and Analytic Sets

We assume now the reader to be familiar with basic notions of topology which may be found in [Mos80, LT94, Kec95, Sta97, PP04].

There is a natural metric on the set Σ^ω of infinite words over a finite alphabet Σ containing at least two letters which is called the *prefix metric* and defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $\delta(u, v) = 2^{-l_{\mathrm{pref}(u,v)}}$ where $l_{\mathrm{pref}(u,v)}$ is the first integer n such that the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v. This metric induces on Σ^ω the usual Cantor topology for which *open subsets* of Σ^ω are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$. A set $L \subseteq \Sigma^\omega$ is a *closed set* iff its complement $\Sigma^\omega - L$ is an open set. Now let define the *Borel Hierarchy* of subsets of Σ^ω:

Definition 3.10 *For a non-null countable ordinal α, the classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ of the Borel Hierarchy on the topological space Σ^ω are defined as follows:*
$\mathbf{\Sigma}^0_1$ is the class of open subsets of Σ^ω, $\mathbf{\Pi}^0_1$ is the class of closed subsets of Σ^ω, and for any countable ordinal $\alpha \geq 2$:
$\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Pi}^0_\gamma$.
$\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Sigma}^0_\gamma$.

For a countable ordinal α, a subset of Σ^ω is a Borel set of *rank* α iff it is in $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$ but not in $\bigcup_{\gamma<\alpha}(\mathbf{\Sigma}^0_\gamma \cup \mathbf{\Pi}^0_\gamma)$.

There are also some subsets of Σ^ω which are not Borel. Indeed there exists another hierarchy beyond the Borel hierarchy, which is called the projective hierarchy and which is obtained from the Borel hierarchy by successive applications of operations of projection and complementation. The first level of the projective hierarchy is formed by the class of *analytic sets* and the class of *co-analytic sets* which are complements of analytic sets. In particular the class of Borel subsets of Σ^ω is strictly included into the class $\mathbf{\Sigma}^1_1$ of *analytic sets* which are obtained by projections of Borel sets.

Definition 3.11 *A subset A of Σ^ω is in the class $\mathbf{\Sigma}^1_1$ of* **analytic** *sets iff there exist a finite set Y and a Borel subset B of $(\Sigma \times Y)^\omega$ such that $[x \in A \leftrightarrow \exists y \in Y^\omega$ $(x, y) \in B]$, where (x, y) is the infinite word over the alphabet $\Sigma \times Y$ such that $(x, y)(i) = (x(i), y(i))$ for each integer $i \geq 1$.*

We now define completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, a set $F \subseteq \Sigma^\omega$ is said to be a $\mathbf{\Sigma}^0_\alpha$ (respectively, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^1_1$)-*complete set* iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \mathbf{\Sigma}^0_\alpha$ (respectively, $E \in \mathbf{\Pi}^0_\alpha$, $E \in \mathbf{\Sigma}^1_1$) iff there exists a continuous function $f : Y^\omega \to \Sigma^\omega$ such that $E = f^{-1}(F)$. $\mathbf{\Sigma}^0_n$ (respectively $\mathbf{\Pi}^0_n$)-complete sets, with n an integer ≥ 1, are thoroughly characterized in [Sta86].

In particular $\mathcal{R} = (0^\star.1)^\omega$ is a well known example of a $\mathbf{\Pi}^0_2$-complete subset of $\{0, 1\}^\omega$. It is the set of ω-words over $\{0, 1\}$ having infinitely many occurrences of the letter 1. Its complement $\{0, 1\}^\omega - (0^\star.1)^\omega$ is a $\mathbf{\Sigma}^0_2$-complete subset of $\{0, 1\}^\omega$.

We recall now the definition of the arithmetical hierarchy of ω-languages which form the effective analogue to the hierarchy of Borel sets of finite ranks.
Let X be a finite alphabet. An ω-language $L \subseteq X^\omega$ belongs to the class Σ_n if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N})^{n-1} \times X^\star$ such that

$$L = \{\sigma \in X^\omega \mid \exists a_1 \ldots Q_n a_n \quad (a_1, \ldots, a_{n-1}, \sigma[a_n + 1]) \in R_L\}$$

where Q_i is one of the quantifiers $\forall$ or $\exists$ (not necessarily in an alternating order). An ω-language $L \subseteq X^\omega$ belongs to the class Π_n if and only if its complement $X^\omega - L$ belongs to the class Σ_n. The inclusion relations that hold between the classes Σ_n and Π_n are the same as for the corresponding classes of the Borel hierarchy. The classes Σ_n and Π_n are included in the respective classes $\mathbf{\Sigma^0_n}$ and $\mathbf{\Sigma^0_n}$ of the Borel hierarchy, and cardinality arguments suffice to show that these inclusions are strict.

As in the case of the Borel hierarchy, projections of arithmetical sets (of the second Π-class) lead beyond the arithmetical hierarchy, to the analytical hierarchy of ω-languages. The first class of this hierarchy is the (lightface) class Σ_1^1 of *effective analytic sets* which are obtained by projection of arithmetical sets. An ω-language $L \subseteq X^\omega$ belongs to the class Σ_1^1 if and only if there exists a recursive relation $R_L \subseteq \mathbb{N} \times \{0,1\}^\star \times X^\star$ such that:

$$L = \{\sigma \in X^\omega \mid \exists\tau(\tau \in \{0,1\}^\omega \wedge \forall n \exists m((n, \tau[m], \sigma[m]) \in R_L))\}$$

Then an ω-language $L \subseteq X^\omega$ is in the class Σ_1^1 iff it is the projection of an ω-language over the alphabet $X \times \{0,1\}$ which is in the class Π_2. The (lightface) class Π_1^1 of *effective co-analytic sets* is simply the class of complements of effective analytic sets. We denote as usual $\Delta_1^1 = \Sigma_1^1 \cap \Pi_1^1$.
Recall that an ω-language $L \subseteq X^\omega$ is in the class Σ_1^1 iff it is accepted by a non deterministic Turing machine (reading ω-words) with a Büchi or Muller acceptance condition [CG78, Sta97].

For Γ a finite alphabet having at least two letters, the set $\Gamma^{\omega\times\omega}$ of functions from $\omega\times\omega$ into Γ is usually equipped with the product topology of the discrete topology on Γ. This topology may be defined by the following distance d. Let x and y in $\Gamma^{\omega\times\omega}$ such that $x \neq y$, then

$$d(x, y) = \frac{1}{2^n} \qquad \text{where}$$

$$n = min\{p \geq 0 \mid \exists(i, j) \;\; x(i, j) \neq y(i, j) \text{ and } i + j = p\}.$$

Then the topological space $\Gamma^{\omega\times\omega}$ is homeomorphic to the topological space Γ^ω, equipped with the Cantor topology. Borel subsets of $\Gamma^{\omega\times\omega}$ are defined from open subsets as in the case of the topological space Γ^ω. Analytic subsets of $\Gamma^{\omega\times\omega}$ are obtained as projections on $\Gamma^{\omega\times\omega}$ of Borel subsets of the product space $\Gamma^{\omega\times\omega} \times \Gamma^\omega$.
The set $\Sigma^{\omega,\omega}$ of ω-pictures over Σ, considered a topological subspace of $\hat{\Sigma}^{\omega\times\omega}$, is easily seen to be homeomorphic to the topological space $\Sigma^{\omega\times\omega}$, via the mapping $\varphi : \Sigma^{\omega,\omega} \to \Sigma^{\omega\times\omega}$ defined by $\varphi(p)(i, j) = p(i + 1, j + 1)$ for all $p \in \Sigma^{\omega,\omega}$ and $i, j \in \omega$.

3.3 Some Results of Set Theory

We now recall some basic notions of set theory which will be useful in the sequel, and which are exposed in any textbook on set theory, like [Jec02].

The usual axiomatic system **ZFC** is Zermelo-Fraenkel system **ZF** plus the axiom of choice **AC**. A model (**V**, $\in$) of the axiomatic system **ZFC** is a collection **V** of sets, equipped with the membership relation $\in$, where "$x \in y$" means that the set x is an element of the set y, which satisfies the axioms of **ZFC**. We shall often say " the model **V**" instead of "the model (**V**, $\in$)".

The axioms of **ZFC** express some natural facts that we consider to hold in the universe of sets. For instance a natural fact is that two sets x and y are equal iff they have the same elements. This is expressed by the *Axiom of Extensionality*. Another natural axiom is the *Pairing Axiom* which states that for all sets x and y there exists a set $z = \{x, y\}$ whose elements are x and y. Similarly the *Powerset Axiom* states the existence of the set of subsets of a set x. We refer the reader to any textbook on set theory, like [Jec02], for an exposition of the other axioms of **ZFC**.

The infinite cardinals are usually denoted by $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$ The cardinal $\aleph_\alpha$ is also denoted by ω_α, as usual when it is considered an ordinal.

The continuum hypothesis **CH** says that the first uncountable cardinal $\aleph_1$ is equal to $2^{\aleph_0}$ which is the cardinal of the continuum. Gödel and Cohen proved that the continuum hypothesis **CH** is independent from the axiomatic system **ZFC**: providing **ZFC** is consistent, there exist some models of **ZFC + CH** and also some models of **ZFC +** $\neg$ **CH**, where $\neg$ **CH** denotes the negation of the continuum hypothesis, [Jec02].

Let $\mathbf{ON}$ be the class of all ordinals. Recall that an ordinal α is said to be a successor ordinal iff there exists an ordinal β such that $\alpha = \beta + 1$; otherwise the ordinal α is said to be a limit ordinal and in that case $\alpha = \sup\{\beta \in \mathbf{ON} \mid \beta < \alpha\}$.

The class **L** of *constructible sets* in a model **V** of **ZF** is defined by

$$\mathbf{L} = \bigcup_{\alpha \in \mathbf{ON}} \mathbf{L}(\alpha)$$

where the sets $\mathbf{L}(\alpha)$ are constructed by induction as follows:

1. $\mathbf{L}(0) = \emptyset$
2. $\mathbf{L}(\alpha) = \bigcup_{\beta < \alpha} \mathbf{L}(\beta)$, for α a limit ordinal, and
3. $\mathbf{L}(\alpha + 1)$ is the set of subsets of $\mathbf{L}(\alpha)$ which are definable from a finite number of elements of $\mathbf{L}(\alpha)$ by a first-order formula relativized to $\mathbf{L}(\alpha)$.

If **V** is a model of **ZF** and **L** is the class of *constructible sets* of **V**, then the class **L** forms a model of **ZFC + CH**. Notice that the axiom (**V=L**) means "every set is constructible" and that it is consistent with **ZFC**.

Consider now a model **V** of the axiomatic system **ZFC** and the class of constructible sets $\mathbf{L} \subseteq \mathbf{V}$ which forms another model of **ZFC**. It is known that the ordinals of **L** are also the ordinals of **V**. But the cardinals in **V** may be different from the cardinals in **L**.

In particular, the first uncountable cardinal in **L** is denoted $\aleph_1^{\mathbf{L}}$. It is in fact an ordinal of **V** which is denoted $\omega_1^{\mathbf{L}}$. It is known that this ordinal satisfies the inequality $\omega_1^{\mathbf{L}} \leq \omega_1$. In a model **V** of the axiomatic system **ZFC + V=L** the equality $\omega_1^{\mathbf{L}} = \omega_1$ holds. But in some other models of **ZFC** the inequality may be strict and then $\omega_1^{\mathbf{L}} < \omega_1$. This is explained in [Jec02, page 202]: one can start from a model **V** of **ZFC + V=L** and construct by forcing a generic extension **V[G]** in which the cardinals ω and ω_1 are collapsed; in this extension the inequality $\omega_1^{\mathbf{L}} < \omega_1$ holds.

We now recall the notion of a perfect set.

Definition 3.12 *Let $P \subseteq \Sigma^\omega$, where Σ is a finite alphabet having at least two letters. The set P is said to be a perfect subset of Σ^ω if and only if :*
(1) P is a non-empty closed set, and
(2) for every $x \in P$ and every open set U containing x there is an element $y \in P \cap U$ such that $x \neq y$.

So a perfect subset of Σ^ω is a non-empty closed set which has no isolated points. It is well known that a perfect subset of Σ^ω has cardinality $2^{\aleph_0}$, see [Mos80, page 66].

We now recall the notion of thin subset of Σ^ω.

Definition 3.13 *A set $X \subseteq \Sigma^\omega$ is said to be thin iff it contains no perfect subset.*

The following important result was proved by Kechris [Kec75] and independently by Guaspari [Gua73] and Sacks [Sac76].

Theorem 3.14 (see [Mos80] page 247) *(**ZFC**) Let Σ be a finite alphabet having at least two letters. There exists a thin Π_1^1-set $\mathcal{C}_1(\Sigma^\omega) \subseteq \Sigma^\omega$ which contains every thin, Π_1^1-subset of Σ^ω. It is called the largest thin Π_1^1-set in Σ^ω.*

An important fact is that the cardinality of the largest thin Π_1^1-set in Σ^ω depends on the model of **ZFC**. The following result on the cardinality of $\mathcal{C}_1(\Sigma^\omega)$, was proved by Kechris and independently by Guaspari and Sacks, see also [Kan97, page 171].

Theorem 3.15 *(**ZFC**) The cardinal of the largest thin Π_1^1-set in Σ^ω is equal to the cardinal of $\omega_1^{\mathbf{L}}$.*

This means that in a given model $\mathbf{V}$ of **ZFC** the cardinal of the largest thin Π_1^1-set in Σ^ω is equal to the cardinal *in* $\mathbf{V}$ of the ordinal $\omega_1^{\mathbf{L}}$ which plays the role of the cardinal $\aleph_1$ in the inner model $\mathbf{L}$ of constructible sets of $\mathbf{V}$.

We can now state the following result which will be useful in the sequel.

Corollary 3.16

(a) There is a model $\mathbf{V}_1$ of **ZFC** *in which the largest thin Π_1^1-set in Σ^ω has cardinal $\aleph_1$, where $\aleph_1 = 2^{\aleph_0}$.*

(b) There is a model $\mathbf{V}_2$ of **ZFC** *in which the largest thin Π_1^1-set in Σ^ω has cardinal $\aleph_0$, i.e. is countably infinite.*

(c) There is a model $\mathbf{V}_3$ of **ZFC** *in which the largest thin Π_1^1-set in Σ^ω has cardinal $\aleph_1$, where $\aleph_0 < \aleph_1 < 2^{\aleph_0}$.*

Proof. (a). In the model $\mathbf{L}$, the cardinal of the largest thin Π_1^1-set in Σ^ω is equal to the cardinal of ω_1. Moreover the continuum hypothesis is satisfied thus $2^{\aleph_0} = \aleph_1$. Thus the largest thin Π_1^1-set in Σ^ω has the cardinality $2^{\aleph_0} = \aleph_1$.

(b). Let $\mathbf{V}$ be a model of (**ZFC** + $\omega_1^{\mathbf{L}} < \omega_1$). In this model ω_1 is the first uncountable ordinal. Thus $\omega_1^{\mathbf{L}} < \omega_1$ implies that $\omega_1^{\mathbf{L}}$ is a countable ordinal in $\mathbf{V}$. Its cardinal is $\aleph_0$ and it is also the cardinal of the largest thin Π_1^1-set in Σ^ω.

(c). It suffices to show that there is a model $\mathbf{V}_3$ of **ZFC** in which $\omega_1^{\mathbf{L}} = \omega_1$ and $\aleph_1 < 2^{\aleph_0}$. Such a model can be constructed by Cohen's forcing. We can start from a model $\mathbf{V}$ of **ZFC + V=L** (in which $\omega_1^{\mathbf{L}} = \omega_1$) and construct by forcing a generic extension **V[G]** in which $\aleph_2$ subsets of ω are added. Notice that the cardinals are preserved under this extension (see [Jec02, page 219]) and that the constructible sets of **V[G]** are also the constructible sets of $\mathbf{V}$. Thus in the new model **V[G]** we still have $\omega_1^{\mathbf{L}} = \omega_1$ but now $\aleph_1 < 2^{\aleph_0}$. □

4 Decision Problems

We now study decision problems for recognizable languages of infinite pictures. We gave in [Fin09b] the exact degree of several natural decision problems. We first recall some of these results.

Castro and Cucker proved in [CC89] that the non-emptiness problem and the infiniteness problem for ω-languages of Turing machines are both Σ_1^1-complete. We easily inferred from this result a similar result for recognizable languages of infinite pictures.

From now on we shall denote by $\mathcal{T}_z$ the non deterministic tiling system of index z, (accepting pictures over $\Sigma = \{a, b\}$), equipped with a Büchi acceptance condition.

Theorem 4.1 ([Fin09b]) *The non-emptiness problem and the infiniteness problem for Büchi-recognizable languages of infinite pictures are Σ_1^1-complete, i.e. :*

1. $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z) \neq \emptyset\}$ *is* Σ_1^1*-complete.*
2. $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ *is infinite* $\}$ *is* Σ_1^1*-complete.*

In a similar way, the universality problem and the inclusion and the equivalence problems, for ω-languages of Turing machines, have been proved to be Π_2^1-complete by Castro and Cucker in [CC89], and we used these results to prove the following results in [Fin09b].

Theorem 4.2 ([Fin09b]) *The universality problem for Büchi-recognizable languages of infinite pictures is Π_2^1-complete, i.e. :*

$$\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z) = \Sigma^{\omega,\omega}\} \text{ is } \Pi_2^1\text{-complete.}$$

Theorem 4.3 ([Fin09b]) *The inclusion and the equivalence problems for Büchi-recognizable languages of infinite pictures are Π_2^1-complete, i.e. :*

1. $\{(y, z) \in \mathbb{N}^2 \mid L^B(\mathcal{T}_y) \subseteq L^B(\mathcal{T}_z)\}$ *is* Π_2^1*-complete.*
2. $\{(y, z) \in \mathbb{N}^2 \mid L^B(\mathcal{T}_y) = L^B(\mathcal{T}_z)\}$ *is* Π_2^1*-complete.*

The class of Büchi-recognizable languages of infinite pictures is not closed under complement [ATW03]. Thus the following question naturally arises: "can we decide whether the complement of a Büchi-recognizable language of infinite pictures is Büchi-recognizable?". And what is the exact complexity of this decision problem, called the complementability problem.

Another classical problem is the determinizability problem: "can we decide whether a given recognizable language of infinite pictures is recognized by a deterministic tiling system?".
Recall that a tiling system is called deterministic if on any picture it allows at most one tile covering the origin, the state assigned to position $(i+1, j+1)$ is uniquely determined by the states at positions $(i, j), (i+1, j), (i, j+1)$ and the states at the border positions $(0, j+1)$ and $(i+1, 0)$ are determined by the state $(0, j)$, respectively $(i, 0)$, [ATW03].
As remarked in [ATW03], the hierarchy proofs of the classical Landweber hierarchy defined using deterministic ω-automata "carry over without essential changes

to pictures". In particular, a language of ω-pictures which is Büchi-recognized by a deterministic tiling system is a $\mathbf{\Pi}_2^0$-set and a language of ω-pictures which is Muller-recognized by a deterministic tiling system is a boolean combination of $\mathbf{\Pi}_2^0$-sets, hence a $\mathbf{\Delta}_3^0$-set.

These topological properties have been used in [Fin09b], along with a dichotomy property, to prove the following results.

Theorem 4.4 ([Fin09b]) *The determinizability problem and the complementability problem for Büchi-recognizable languages of infinite pictures are Π_2^1-complete, i.e. :*

1. $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ *is Büchi-recognizable by a deterministic tiling system*$\}$ *is* Π_2^1*-complete.*
2. $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ *is Muller-recognizable by a deterministic tiling system* $\}$ *is* Π_2^1*-complete.*
3. $\{z \in \mathbb{N} \mid \exists y \ \ \Sigma^{\omega,\omega} - L^B(\mathcal{T}_z) = L^B(\mathcal{T}_y)\}$ *is* Π_2^1*-complete.*

We already mentioned that we used some results of Castro and Cucker in the proof of the above results. Castro and Cucker studied degrees of decision problems for ω-languages accepted by Turing machines and proved that many of them are highly undecidable, [CC89]. We are going to use again some of their results to prove here new results about Büchi-recognizable languages of infinite pictures.

We firstly recall the notion of acceptance of infinite words by Turing machines considered by Castro and Cucker in [CC89].

Definition 4.5 *A non deterministic Turing machine $\mathcal{M}$ is a 5-tuple $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$, where Q is a finite set of states, Σ is a finite input alphabet, Γ is a finite tape alphabet satisfying $\Sigma \subseteq \Gamma$, q_0 is the initial state, and δ is a mapping from $Q \times \Gamma$ to subsets of $Q \times \Gamma \times \{L, R, S\}$. A configuration of $\mathcal{M}$ is a 3-tuple (q, σ, i), where $q \in Q$, $\sigma \in \Gamma^\omega$ and $i \in \mathbb{N}$. An infinite sequence of configurations $r = (q_i, \alpha_i, j_i)_{i \geq 1}$ is called a run of $\mathcal{M}$ on $w \in \Sigma^\omega$ iff:*

(a) $(q_1, \alpha_1, j_1) = (q_0, w, 1)$, *and*

(b) *for each* $i \geq 1$, $(q_i, \alpha_i, j_i) \vdash (q_{i+1}, \alpha_{i+1}, j_{i+1})$,

where $\vdash$ is the transition relation of $\mathcal{M}$ defined as usual. The run r is said to be complete if the limsup of the head positions is infinity, i.e. if $(\forall n \geq 1)(\exists k \geq 1)(j_k \geq n)$. The run r is said to be oscillating if the liminf of the head positions is bounded, i.e. if $(\exists k \geq 1)(\forall n \geq 1)(\exists m \geq n)(j_m = k)$.

Definition 4.6 *Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$ be a non deterministic Turing machine and $F \subseteq Q$. The ω-language accepted by $(\mathcal{M}, F)$ is the set of ω-words $\sigma \in \Sigma^\omega$ such that there exists a complete non oscillating run $r = (q_i, \alpha_i, j_i)_{i\geq 1}$ of $\mathcal{M}$ on σ such that, for all $i, q_i \in F$.*

The above acceptance condition is denoted $1'$-acceptance in [CG78]. Another usual acceptance condition is the now called Büchi acceptance condition which is also denoted 2-acceptance in [CG78]. We now recall its definition.

Definition 4.7 *Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$ be a non deterministic Turing machine and $F \subseteq Q$. The ω-language Büchi accepted by $(\mathcal{M}, F)$ is the set of ω-words $\sigma \in \Sigma^\omega$ such that there exists a complete non oscillating run $r = (q_i, \alpha_i, j_i)_{i\geq 1}$ of $\mathcal{M}$ on σ and infinitely many integers i such that $q_i \in F$.*

Recall that Cohen and Gold proved in [CG78, Theorem 8.6] that one can effectively construct, from a given non deterministic Turing machine, another equivalent non deterministic Turing machine, equipped with the same kind of acceptance condition, and in which every run is complete non oscillating. Cohen and Gold proved also in [CG78, Theorem 8.2] that an ω-language is accepted by a non deterministic Turing machine with $1'$-acceptance condition iff it is accepted by a non deterministic Turing machine with Büchi acceptance condition.

From now on, we shall denote $\mathcal{M}_z$ the non deterministic Turing machine of index z, (accepting words over $\Sigma = \{a, b\}$), equipped with a $1'$-acceptance condition.

An important notion in automata theory is the notion of ambiguity. It can be defined also in the context of acceptance by tiling systems, see [AGMR06] for the case of finite pictures.

Definition 4.8 *Let $\mathcal{A}$=(Q, Σ, Δ) be a tiling system, and $F \subseteq Q$. The Büchi tiling system $(\mathcal{A}, F)$ is unambiguous iff every ω-picture $p \in \Sigma^{\omega,\omega}$ has at most an accepting run by $(\mathcal{A}, F)$.*

Definition 4.9 *A Büchi recognizable language $L \subseteq \Sigma^{\omega,\omega}$ is unambiguous iff there exists an unambiguous Büchi tiling system $(\mathcal{A}, F)$ such that $L = L(\mathcal{A}, F)$. Otherwise the language L is said to be inherently ambiguous.*

We can now prove the following result, which is very similar to a corresponding result for recognizable tree languages proved in [FS09].

Proposition 4.10 *Let $L \subseteq \Sigma^{\omega,\omega}$ be an unambiguous Büchi recognizable language of infinite pictures. Then L is a Borel subset of $\Sigma^{\omega,\omega}$.*

Proof. Let $L \subseteq \Sigma^{\omega,\omega}$ be a language accepted by an unambiguous Büchi tiling system $(\mathcal{A}, F)$, where $\mathcal{A}$=(Q, Σ, Δ), and let $R \subseteq (\hat{\Sigma} \times Q)^{\omega\times\omega}$ be defined by:

$R = \{(p, \rho) \mid p \in \Sigma^{\omega,\omega}$ and $\rho \in$ is an accepting run of $(\mathcal{A}, F)$ on the picture $p\}$.

The set R is easily seen to be a $\mathbf{\Pi}^0_2$-subset of $(\hat{\Sigma} \times Q)^{\omega\times\omega}$.

Consider now the projection $\mathrm{PROJ}_{\hat{\Sigma}^{\omega\times\omega}} : \hat{\Sigma}^{\omega\times\omega} \times Q^{\omega\times\omega} \to \hat{\Sigma}^{\omega\times\omega}$ defined by $\mathrm{PROJ}_{\hat{\Sigma}^{\omega\times\omega}}((p, \rho)) = p$ for all $(p, \rho) \in \hat{\Sigma}^{\omega\times\omega} \times Q^{\omega\times\omega}$. This projection is a continuous function and it is *injective* on the Borel set R because the Büchi tiling system $(\mathcal{A}, F)$ is unambiguous. Hence, by a Theorem of Lusin and Souslin, see [Kec95, Theorem 15.1 page 89], the injective image of R by the continuous function $\mathrm{PROJ}_{\hat{\Sigma}^{\omega\times\omega}}$ is Borel. Thus the language $L = \mathrm{PROJ}_{\hat{\Sigma}^{\omega\times\omega}}(R)$ is a Borel subset of $\hat{\Sigma}^{\omega\times\omega}$. But $\Sigma^{\omega,\omega}$ is a closed subset of $\hat{\Sigma}^{\omega\times\omega}$ and $L \subseteq \Sigma^{\omega,\omega}$. Thus L is also a Borel subset of $\Sigma^{\omega,\omega}$. □

Corollary 4.11 *There exist some inherently ambiguous Büchi-recognizable languages of infinite pictures.*

Proof. The result follows directly from the above proposition because we know that there exist some Büchi-recognizable languages of infinite pictures which are not Borel sets, see [Fin04, Fin09b]. □

We can now state that the unambiguity problem for recognizable language of infinite pictures is Π^1_2-complete.

Theorem 4.12 *The unambiguity problem for recognizable languages of infinite pictures is Π^1_2-complete, i.e. :*

$$\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z) \text{ is non ambiguous } \} \text{ is } \Pi^1_2\text{-complete.}$$

Proof. To prove that the unambiguity problem for recognizable language of infinite pictures is in the class Π^1_2, we reason as in the case of the unambiguity problem for ω-languages accepted by 1-counter or 2-tape automata, see [Fin09c].

Notice first, as in [Fin09b], that, using a recursive bijection $b : (\mathbb{N} - \{0\})^2 \to \mathbb{N} - \{0\}$, one can associate with each ω-word $\sigma \in \Sigma^\omega$ a unique ω-picture $p^\sigma \in \Sigma^{\omega,\omega}$ which is simply defined by $p^\sigma(i, j) = \sigma(b(i, j))$ for all integers $i, j \geq 1$. And we can identify a run $\rho \in Q^{\omega\times\omega}$ with an element of Q^ω and finally with a coding of this element over the alphabet $\{0, 1\}$. So the run ρ can be identified with its code $\bar{\rho} \in \{0, 1\}^\omega$.

If a tiling system $\mathcal{A}$=(Q, Σ, Δ) is equipped with a set of accepting states $F \subseteq Q$, then for $\sigma \in \Sigma^\omega$ and $\rho \in \{0, 1\}^\omega$, "ρ is a Büchi accepting run of $(\mathcal{A}, F)$ over the

ω-picture p^σ" can be expressed by an arithmetical formula, see [ATW03, Section 2.4] and [Fin09b].

We can now first express "$\mathcal{T}_z$ is non ambiguous" by :

$$\text{“}\forall \sigma \in \Sigma^\omega \forall \rho, \rho' \in \{0,1\}^\omega [(\rho \text{ and } \rho' \text{ are accepting runs of } \mathcal{T}_z \text{ on } p^\sigma) \rightarrow \rho = \rho']\text{”}$$

which is a Π_1^1-formula. Then " $L^B(\mathcal{T}_z)$ is non ambiguous" can be expressed by the following formula: "$\exists y[L^B(\mathcal{T}_z) = L^B(\mathcal{T}_y)$ and $\mathcal{T}_y$ is non ambiguous]". This is a Π_2^1-formula because $L^B(\mathcal{T}_z) = L^B(\mathcal{T}_y)$ can be expressed by a Π_2^1-formula, and the quantification $\exists y$ is of type 0. Thus the set $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is non ambiguous $\}$ is a Π_2^1-set.

To prove the completeness part of the theorem, we shall use the following dichotomy result proved in [Fin09b, proof of Theorem 5.11].
There exists an injective computable function $H \circ \theta$ from $\mathbb{N}$ into $\mathbb{N}$ such that:
First case: If $L(\mathcal{M}_z) = \Sigma^\omega$ then $L^B(\mathcal{T}_{H\circ\theta(z)}) = \Sigma^{\omega,\omega}$.
Second case: If $L(\mathcal{M}_z) \neq \Sigma^\omega$ then $L^B(\mathcal{T}_{H\circ\theta(z)})$ is not a Borel set.

In the first case $L^B(\mathcal{T}_{H\circ\theta(z)}) = \Sigma^{\omega,\omega}$ is obviously an unambiguous language. And in the second case the language $L^B(\mathcal{T}_{H\circ\theta(z)})$ cannot be unambiguous because it is not a Borel subset of $\Sigma^{\omega,\omega}$. Thus, using the reduction $H \circ \theta$, we see that :

$$\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\} \leq_1 \{z \in \mathbb{N} \mid L^B(\mathcal{T}_z) \text{ is non ambiguous } \}$$

and the result follows from the Π_2^1-completeness of the universality problem for ω-languages of Turing machines proved by Castro and Cucker in [CC89]. $\square$

Notice that the same dichotomy result above with the reduction $H \circ \theta$ was used in [Fin09b] to prove that topological properties of recognizable languages of infinite pictures are actually highly undecidable.

Theorem 4.13 ([Fin09b]) *Let α be a non-null countable ordinal. Then*

1. $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ *is in the Borel class* $\mathbf{\Sigma}_\alpha^0\}$ *is* Π_2^1*-hard.*
2. $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ *is in the Borel class* $\mathbf{\Pi}_\alpha^0\}$ *is* Π_2^1*-hard.*
3. $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ *is a Borel set* $\}$ *is* Π_2^1*-hard.*

A natural question is to study similar problems by replacing Borel classes by the effective classes of the arithmetical hierarchy. This was not studied in [Fin09b], but a similar problem was solved in [Fin09c] for ω-languages accepted by 1-counter or 2-tape Büchi automata. We can reason in a similar way for the case of recognizable languages of infinite pictures, and state the following result.

Theorem 4.14 *Let $n \geq 1$ be an integer. Then*

1. *$\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is in the arithmetical class $\Sigma_n\}$ is Π^1_2-complete.*
2. *$\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is in the arithmetical class $\Pi_n\}$ is Π^1_2-complete.*
3. *$\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is a Δ^1_1-set $\}$ is Π^1_2-complete.*

We do not give the complete proof here. It is actually very similar to the case of ω-languages accepted by 1-counter or 2-tape Büchi automata in [Fin09c]. A key argument, to prove that $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is in the arithmetical class $\Sigma_n\}$ (respectively, $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is in the arithmetical class $\Pi_n\}$) is a Π^1_2-set, is the existence of a universal set $\mathcal{U}_{\Sigma_n} \subseteq \mathbb{N} \times \Sigma^{\omega,\omega}$ (respectively, $\mathcal{U}_{\Pi_n} \subseteq \mathbb{N} \times \Sigma^{\omega,\omega}$) for the class of Σ_n-subsets of $\Sigma^{\omega,\omega}$, (respectively, Π_n-subsets of $\Sigma^{\omega,\omega}$), [Mos80, p. 172]. Notice also that the completeness part follows easily from the dichotomy result obtained with the reduction $H \circ \theta$.

We now come to cardinality problems. We already know that it is Σ^1_1-complete to determine whether a given recognizable language of infinite pictures is empty (respectively, infinite). Recall that every recognizable language of infinite pictures is an analytic set. On the other hand, every analytic set is either countable or has the cardinality $2^{\aleph_0}$ of the continuum. Then some questions naturally arise. What are the complexities of the following decision problems: "Is a given recognizable language of infinite pictures countable? Is it countably infinite? Is it uncountable?". Notice that similar questions were asked by Castro and Cucker in the case of ω-languages of Turing machines and have been solved very recently by Finkel and Lecomte in [FL09]. We can now state the following result for recognizable languages of infinite pictures. Below $D_2(\Sigma^1_1)$ denotes the class of 2-differences of Σ^1_1-sets, i.e. the class of sets which are intersections of a Σ^1_1-set and of a Π^1_1-set.

Theorem 4.15

1. *$\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is countable$\}$ is Π^1_1-complete.*
2. *$\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is uncountable$\}$ is Σ^1_1-complete.*
3. *$\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is countably infinite$\}$ is $D_2(\Sigma^1_1)$-complete.*

Proof. (1). We can first prove that $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is countable$\}$ is in the class Π^1_1 in the same way as in the case of ω-languages of Turing machines in [FL09]. We know that a recognizable language of infinite pictures $L^B(\mathcal{T}_z)$ is a Σ^1_1-subset of $\Sigma^{\omega,\omega}$. But it is known that a Σ^1_1-subset L of $\Sigma^{\omega,\omega}$ is countable if and only if for every $x \in L$ the singleton $\{x\}$ is a Δ^1_1-subset of $\Sigma^{\omega,\omega}$, see [Mos80, page

243]. Then, using a nice coding of Δ_1^1-subsets of $\Sigma^{\omega,\omega}$ given in [HKL90, Theorem 3.3.1], we can prove that $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is countable$\}$ is in the class Π_1^1, see [FL09] for more details.

To prove the completeness part of Item (1), we shall use the following two lemmas proved in previous papers.

For $\sigma \in \Sigma^\omega = \{a, b\}^\omega$ we denote σ^a the ω-picture whose first row is the ω-word σ and whose other rows are labelled with the letter a. For an ω-language $L \subseteq \Sigma^\omega = \{a, b\}^\omega$ we denote L^a the language of infinite pictures $\{\sigma^a \mid \sigma \in L\}$.

Lemma 4.16 ([Fin04]) *If $L \subseteq \Sigma^\omega$ is accepted by some Turing machine (in which every run is complete non oscillating) with a Büchi acceptance condition, then L^a is Büchi recognizable by a finite tiling system.*

Lemma 4.17 ([Fin09b]) *There is an injective computable function K from $\mathbb{N}$ into $\mathbb{N}$ satisfying the following property.*
If $\mathcal{M}_z$ is the non deterministic Turing machine (equipped with a $1'$-acceptance condition) of index z, and if $\mathcal{T}_{K(z)}$ is the tiling system (equipped with a Büchi acceptance condition) of index $K(z)$, then

$$L(\mathcal{M}_z)^a = L^B(\mathcal{T}_{K(z)})$$

On the other hand, we can easily see that the cardinality of the ω-language $L(\mathcal{M}_z)$ is equal to the cardinality of the ω-picture language $L(\mathcal{M}_z)^a$. Thus using the reduction K given in the above lemma we see that:

$$\{z \in \mathbb{N} \mid L(\mathcal{M}_z) \text{ is countable } \} \leq_1 \{z \in \mathbb{N} \mid L^B(\mathcal{T}_z) \text{ is countable } \}$$

Then the completeness part follows from the fact that $\{z \in \mathbb{N} \mid L(\mathcal{M}_z)$ is countable $\}$ is Π_1^1-complete, proved in [FL09].

(2). The proof of Item (2) follows directly from Item (1).

(3). We already know that the set $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is infinite$\}$ is in the class Σ_1^1. Thus the set $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is countably infinite$\}$ is the intersection of a Σ_1^1-set and of a Π_1^1-set, i.e. it is in the class $D_2(\Sigma_1^1)$. Using again the reduction K we see that:

$$\{z \in \mathbb{N} \mid L(\mathcal{M}_z) \text{ is countably infinite } \} \leq_1 \{z \in \mathbb{N} \mid L^B(\mathcal{T}_z) \text{ is countably infinite } \}$$

It was proved in [FL09] that $\{z \in \mathbb{N} \mid L(\mathcal{M}_z)$ is countably infinite $\}$ is $D_2(\Sigma_1^1)$-complete. Thus the set $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)$ is countably infinite $\}$ is also $D_2(\Sigma_1^1)$-complete. □

We are now looking at complements of recognizable languages of infinite pictures. We first state the following result which shows that actually the cardinality of the complement of a recognizable language of infinite pictures may depend on the models of set theory. We denote $L^B(\mathcal{T})^-$ the complement $\Sigma^{\omega,\omega} - L^B(\mathcal{T})$ of a Büchi-recognizable language $L^B(\mathcal{T}) \subseteq \Sigma^{\omega,\omega}$.

Theorem 4.18 *The cardinality of the complement of a Büchi-recognizable language of infinite pictures is not determined by the axiomatic system* **ZFC**. *Indeed there is a Büchi tiling system $\mathcal{T}$ such that:*

1. *There is a model V_1 of* **ZFC** *in which $L^B(\mathcal{T})^-$ is countable.*
2. *There is a model V_2 of* **ZFC** *in which $L^B(\mathcal{T})^-$ has cardinal $2^{\aleph_0}$.*
3. *There is a model V_3 of* **ZFC** *in which $L^B(\mathcal{T})^-$ has cardinal $\aleph_1$ with $\aleph_0 < \aleph_1 < 2^{\aleph_0}$.*

Proof. Moschovakis gave in [Mos80, page 248] a Π^1_1-formula ϕ defining the set $\mathcal{C}_1(\Sigma^\omega)$. Thus its complement $\mathcal{C}_1(\Sigma^\omega)^- = \{a, b\}^\omega - \mathcal{C}_1(\Sigma^\omega)$ is a Σ^1_1-set defined by the Σ^1_1-formula $\psi = \neg\phi$.

Recall that one can construct, from the Σ^1_1-formula ψ defining $\mathcal{C}_1(\Sigma^\omega)^-$, a Büchi Turing machine $\mathcal{M}$ accepting the ω-language $\mathcal{C}_1(\Sigma^\omega)^-$.

On the other hand it is easy to see that the language $\Sigma^{\omega,\omega} - (\Sigma^\omega)^a$ of ω-pictures is Büchi recognizable. But the class $TS(\Sigma^{\omega,\omega})$ is closed under finite union, so we get the following result.

Lemma 4.19 ([Fin09b]) *If $L \subseteq \Sigma^\omega$ is accepted by some Turing machine with a Büchi acceptance condition, then $L^a \cup [\Sigma^{\omega,\omega} - (\Sigma^\omega)^a]$ is Büchi recognizable by a finite tiling system.*

Notice that the constructions are effective and that they can be achieved in an injective way. Thus we can construct, from the Büchi Turing machine $\mathcal{M}$ accepting the ω-language $\mathcal{C}_1(\Sigma^\omega)^-$, a Büchi tiling system $\mathcal{T}$ such that

$$L^B(\mathcal{T}) = L(\mathcal{M})^a \cup [\Sigma^{\omega,\omega} - (\Sigma^\omega)^a].$$

It is then easy to see that:

$$L^B(\mathcal{T})^- = (\Sigma^\omega - L(\mathcal{M}))^a = (\mathcal{C}_1(\Sigma^\omega))^a.$$

Thus the cardinality of $L^B(\mathcal{T})^-$ is equal to the cardinality of the ω-language $\mathcal{C}_1(\Sigma^\omega)$, and then we can infer the results of the theorem from previous Corollary 3.16. □

We can now use the proof of the above result to prove the following result which shows that natural cardinality problems are actually located at the third level of the analytical hierarchy.

Theorem 4.20

1. $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)^-$ *is finite* $\}$ *is* Π_2^1*-complete.*
2. $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)^-$ *is countable* $\}$ *is in* $\Sigma_3^1 \setminus (\Pi_2^1 \cup \Sigma_2^1)$.
3. $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)^-$ *is uncountable*$\}$ *is in* $\Pi_3^1 \setminus (\Pi_2^1 \cup \Sigma_2^1)$.

Proof. Item (1) was proved in [Fin09b].

To prove Item (2), we first show that $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)^-$ is countable $\}$ is in the class Σ_3^1.

As in [Fin09b], using a recursive bijection $b : (\mathbb{N} - \{0\})^2 \to \mathbb{N} - \{0\}$, we can consider an infinite word $\sigma \in \Sigma^\omega$ as a countably infinite family of infinite words over Σ : the family of ω-words (σ_i) such that for each $i \geq 1$, σ_i is defined by $\sigma_i(j) = \sigma(b(i,j))$ for each $j \geq 1$. And one can associate with each ω-word $\sigma \in \Sigma^\omega$ a unique ω-picture $p^\sigma \in \Sigma^{\omega,\omega}$ which is simply defined by $p^\sigma(i,j) = \sigma(b(i,j))$ for all integers $i, j \geq 1$.

We can now express "$L^B(\mathcal{T}_z)^-$ is countable " by the formula:

$$\exists \sigma \in \Sigma^\omega \;\; \forall p \in \Sigma^{\omega,\omega} \; [(p \in L^B(\mathcal{T}_z)) \text{ or } (\exists i \in \mathbb{N}\; p = p^{\sigma_i})]$$

This is a Σ_3^1-formula because "$p \in L^B(\mathcal{T}_z)$", and hence also "$[(p \in L^B(\mathcal{T}_z))$ or $(\exists i \in \mathbb{N}\; p = p^{\sigma_i})]$", is expressed by a Σ_1^1-formula.

We can now prove that $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)^-$ is countable $\}$ is neither in the class Σ_2^1 nor in the class Π_2^1, by using Shoenfield's Absoluteness Theorem from Set Theory.

Let $\mathcal{T}$ be the Büchi tiling system obtained in Theorem 4.18 and let z_0 be its index so that $\mathcal{T} = \mathcal{T}_{z_0}$.

Assume now that $\mathbf{V}$ is a model of $(\mathbf{ZFC} + \omega_1^{\mathbf{L}} < \omega_1)$. In the model $\mathbf{V}$, by the proofs of Theorem 4.18 and of Corollary 3.16, the integer z_0 belongs to the set $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)^-$ is countable $\}$.

But, by the proofs of Theorem 4.18 and of Corollary 3.16, in the inner model $\mathbf{L} \subseteq \mathbf{V}$, the language $L^B(\mathcal{T}_{z_0})^-$ has cardinality $2^{\aleph_0}$. Thus the integer z_0 does not belong to the set $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)^-$ is countable $\}$.

On the other hand, Schoenfield's Absoluteness Theorem implies that every Σ^1_2-set (respectively, Π^1_2-set) is absolute for all inner models of (ZFC), see [Jec02, page 490].
In particular, if the set $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)^-$ is countable $\}$ was a Σ^1_2-set or a Π^1_2-set then it could not be a different subset of $\mathbb{N}$ in the models $\mathbf{V}$ and $\mathbf{L}$ considered above. Therefore, the set $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)^-$ is countable $\}$ is neither a Σ^1_2-set nor a Π^1_2-set.

Item (3) follows directly from Item (2). $\square$

5 Concluding Remarks

Using the notion of largest effective coanalytic set, we have proved in another paper that the topological complexity of a recognizable language of infinite pictures is not determined by the axiomatic system **ZFC**. In particular, there is a Büchi tiling system $\mathcal{S}$ and models $\mathbf{V}_1$ and $\mathbf{V}_2$ of **ZFC** such that: the ω-picture language $L(\mathcal{S})$ id Borel in $\mathbf{V}_1$ but not in $\mathbf{V}_2$, [Fin09a].

We have proved in this paper that $\{z \in \mathbb{N} \mid L^B(\mathcal{T}_z)^-$ is countable $\}$ is in $\Sigma^1_3 \setminus (\Pi^1_2 \cup \Sigma^1_2)$. It remains open whether this set is actually Σ^1_3-complete.

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