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# A novel stability analysis of linear systems under asynchronous samplings

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## Abstract

This article proposes a novel approach to assess stability of continuous linear systems with sampled-data inputs. The method, which is based on the discrete-time Lyapunov theorem, provides easy tractable stability conditions for the continuous-time model. Sufficient conditions for asymptotic and exponential stability are provided dealing with synchronous and asynchronous samplings and uncertain systems. An additional stability analysis is provided for the cases of multiple sampling periods and packet losses. Several examples show the efficiency of the method.

*Key words:* Sampled-Data systems, Uncertain systems, Lyapunov function, Packet losses.

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## 1 Introduction

In the last decades, a large amount of attention has been devoted to Networked Control Systems (NCS) (see [6, 20]). Such systems are controlled systems containing several distributed plants which are connected through a communication network. In such applications, a heavy temporary load of computation in a processor can corrupt the sampling period of a given controller. Another phenomenon, which has been widely investigated, concerns stability under packet losses in wireless networks, where the communication is not always guaranteed. In such situations, the variations of the sampling period will affect the stability properties. Thus an important issue is the development of robust stability conditions with respect to the variations of the sampling period.

Sampled-data systems have been extensively studied in the literature [1, 3, 4, 21, 22] and the references therein. It is now reasonable to design controllers which guarantee the robustness of the solutions of the closed-loop system under periodic samplings. However the case of asynchronous samplings still leads to several open problems. This corresponds to the realistic situation

where the difference between two successive sampling instants is time-varying. Several articles drive the problem of time-varying periods based on a discrete-time approach [8, 14, 18]. An input delay approach using the Lyapunov-Krasovskii (LK) theorem is provided in [3]. Improvements are provided in [4, 12], using the small gain theorem and in [13] based on the analysis of impulsive systems. Recently [2, 10, 16] refine those approaches and obtain tighter conditions. These approaches are very relevant to the problem considered in this paper, because they cope with time-varying sampling periods as well as with uncertain systems in a simple manner. Nevertheless, these sufficient conditions are still more conservative than discrete-time approaches.

This article proposes a novel framework for the stability analysis of linear sampled-data systems using the discrete-time Lyapunov theorem and the continuous-time model of sampled-data systems. Asymptotic and exponential stability criteria are derived from this method. The criteria, which are expressed in terms of linear matrix inequalities, provide tighter upper-bounds of the maximum allowable sampling period than the existing ones, which are based on the continuous-time modelling. A stability analysis of sampled-data systems under multiple sampling rates is also addressed.

This article is organized as follows. The next section formulates the problem. Sections 3 and 4 expose the main contributions of the paper on asymptotic and exponential stability. Section 5 proposes a stability analysis of systems under multiple sampling rates. Some examples are provided in Section 6.

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**Notation.** Throughout the article, the sets  $\mathbb{N}$ ,  $\mathbb{R}^+$ ,  $\mathbb{R}^n$ ,  $\mathbb{R}^{n \times n}$  and  $\mathbb{S}^n$  denote the sets of nonnegative integers, nonnegative scalars,  $n$ -dimensional vectors,  $n \times n$  matrices and symmetric matrices of  $\mathbb{R}^{n \times n}$ , respectively. Define  $\mathbb{K}$ , as the set of differentiable functions from an interval of the form  $[0, T]$  to  $\mathbb{R}^n$ , where  $T \in \mathbb{R}^+$ . The notation  $|\cdot|$  and the superscript  $T$  stand for the Euclidean norm and for matrix transposition, respectively. The notation  $P > 0$  for  $P \in \mathbb{S}^n$  means that  $P$  is positive definite. For any matrix  $A \in \mathbb{R}^{n \times n}$ , the notation  $\text{He}\{A\} > 0$  refers to  $A + A^T > 0$ . The symbols  $I$  and  $0$  represent the identity and the zero matrices of appropriate dimensions.

## 2 Problem formulation

Let  $\{t_k\}_{k \in \mathbb{N}}$  be an increasing sequence of positive scalars such that  $\bigcup_{k \in \mathbb{N}} [t_k, t_{k+1}) = [0, +\infty)$ , for which there exist two positive scalars  $\mathcal{T}_1 \leq \mathcal{T}_2$  such that

$$\forall k \in \mathbb{N}, \quad T_k = t_{k+1} - t_k \in [\mathcal{T}_1, \mathcal{T}_2]. \quad (1)$$

Consider the following sampled-data system

$$\forall t \in [t_k, t_{k+1}), \quad \dot{x}(t) = Ax(t) + Bu(t_k), \quad (2)$$

where  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$  represent the state and the input vectors. The sequence  $\{t_k\}_{k \in \mathbb{N}}$  represents the sampling instants of the controller. The matrices  $A$  and  $B$  are constant, known and of appropriate dimension. The control law is a linear state feedback,  $u = Kx$  with a given gain  $K \in \mathbb{R}^{m \times n}$ . The system is governed by

$$\forall t \in [t_k, t_{k+1}), \quad \dot{x}(t) = Ax(t) + BKx(t_k). \quad (3)$$

Integrating the previous differential equation, the dynamics of the system satisfy

$$\begin{aligned} \forall t \in [t_k, t_{k+1}), \quad x(t) &= \Gamma(t - t_k)x(t_k), \\ \forall \tau \in [0, T_k], \quad \Gamma(\tau) &= [e^{A\tau} + \int_0^\tau e^{A(\tau-\theta)} d\theta BK]. \end{aligned} \quad (4)$$

This equality leads naturally to the introduction of the following notation. For any integer  $k \in \mathbb{N}$ , define the function  $\chi_k : \mathbb{K}$ , such that, for all  $\tau \in [0, T_k]$

$$\begin{cases} \chi_k(\tau) = x(t_k + \tau) = \Gamma(\tau)\chi_k(0), \\ \dot{\chi}_k(\tau) = \frac{d}{d\tau}\chi_k(\tau) = A\chi_k(\tau) + BK\chi_k(0). \end{cases} \quad (5)$$

The definition of  $\chi_k$  yields  $x(t_{k+1}) = \chi_k(T_k) = \chi_{k+1}(0)$ . If  $A, BK$  are constant and known and  $T_k = \mathcal{T}$ , the dynamics become  $x(t_{k+1}) = \Gamma(\mathcal{T})x(t_k)$ . The system is thus asymptotically stable if and only if  $\Gamma(\mathcal{T})$  has all eigenvalues inside the unit circle. If  $T_k$  is time-varying, this does not hold anymore. Relevant stability analysis based on uncertain representations of  $\Gamma(T_k)$  have been already investigated for instance in [7, 14, 18]. However

extensions to uncertain systems lead to additional difficulties induced by the definition of  $\Gamma$ .

Concerning the analysis using continuous-time models, the input delay approach provided in [3] and refined in [2, 12, 13, 16], leads to relevant stability criteria since it is able to take into account uncertain systems. However this approach is still conservative in comparison to discrete-time ones. The present article establishes a novel framework for the stability analysis of sampled-data systems.

## 3 Asymptotic stability analysis

The following theorem shows an equivalence between the discrete-time and the continuous-time approaches.

**Theorem 1** *Let  $0 < \mathcal{T}_1 \leq \mathcal{T}_2$  be two positive scalars and  $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$  be a differentiable function for which there exist positive scalars  $\mu_1 < \mu_2$  and  $p$  such that*

$$\forall x \in \mathbb{R}^n, \quad \mu_1|x|^p \leq V(x) \leq \mu_2|x|^p. \quad (6)$$

*Then the two following statements are equivalent.*

(i) *The increment of the Lyapunov function is strictly negative for all  $k \in \mathbb{N}$  and  $T_k \in [\mathcal{T}_1, \mathcal{T}_2]$ , i.e.,*

$$\Delta_0 V(k) = V(\chi_k(T_k)) - V(\chi_k(0)) < 0;$$

(ii) *There exists a continuous and differentiable functional  $\mathcal{V}_0 : [0, \mathcal{T}_2] \times \mathbb{K} \rightarrow \mathbb{R}$  which satisfies for all  $z \in \mathbb{K}$*

$$\forall T \in [\mathcal{T}_1, \mathcal{T}_2] \quad \mathcal{V}_0(T, z(\cdot)) = \mathcal{V}_0(0, z(\cdot)), \quad (7)$$

*and such that, for all  $(k, T_k, \tau) \in \mathbb{N} \times [\mathcal{T}_1, \mathcal{T}_2] \times [0, T_k]$ ,*

$$\dot{\mathcal{W}}_0(\tau, \chi_k) = \frac{d}{d\tau}[V(\chi_k(\tau)) + \mathcal{V}_0(\tau, \chi_k)] < 0. \quad (8)$$

*Moreover, if one of these two statements is satisfied, then the solutions of the system (3) are asymptotically stable.*

**Proof.** Let  $k \in \mathbb{N}$ ,  $T_k \in [\mathcal{T}_1, \mathcal{T}_2]$  and  $\tau \in [0, T_k]$ . Assume that (ii) is satisfied. Integrating  $\dot{\mathcal{W}}_0$  with respect to  $\tau$  over  $[0, T_k]$  and assuming that (7) holds, this leads to  $\int_0^{T_k} \dot{\mathcal{W}}_0(\tau, \chi_k) d\tau = \Delta_0 V(k)$ . Then  $\Delta_0 V(k)$  is strictly negative since  $\dot{\mathcal{W}}_0$  is negative over  $[0, T_k]$ .

Assume that (i) is satisfied. Introduce the functional  $\mathcal{V}_0(\tau, \chi_k) = -V(\chi_k(\tau)) + \tau/T_k \Delta_0 V(k)$ , as in Lemma 2 in [15]. By simple computations, it is easy to see that it satisfies (7) and  $\dot{\mathcal{W}}_0(\tau, \chi_k) = \Delta_0 V(k)/T_k$ . This proves the equivalence between (i) and (ii).

The function  $\Gamma(\cdot)$  is continuous and consequently bounded over  $[0, \mathcal{T}_2]$ . Then Equation (4) proves that  $x(t)$  and the continuous Lyapunov function uniformly and asymptotically tend to zero. ■

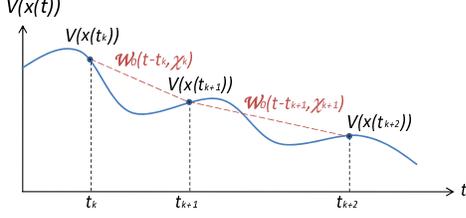


Fig. 1. Illustration of Theorem 1 with  $\mathcal{V}_0(T_k, \chi_k) = \mathcal{V}_0(0, \chi_k) = 0$ .

A graphical illustration of the proof of Theorem 1 is shown in Figure 1. The main contribution of the theorem is the introduction of a new kind of Lyapunov functionals for sampled-data systems. In [2, 13, 16], the authors developed stability criteria from the LK theorem with the same type of functionals. However no direct relation between the discrete-time Lyapunov theorem and the LK theorem was provided. Theorem 1 proves that they are equivalent and allows relaxing the constraint on the positivity of the functional. Asymptotic stability of linear sampled-data systems under asynchronous samplings is provided in the sequel. The objective is to design functionals satisfying the statements of Theorem 1.

**Theorem 2** *Let  $0 < \mathcal{T}_1 \leq \mathcal{T}_2$  be two positive scalars. Assume that there exist  $P > 0$ ,  $R > 0$ ,  $S_1$  and  $X \in \mathbb{S}^n$ ,  $S_2 \in \mathbb{R}^{n \times n}$  and  $N \in \mathbb{R}^{2n \times n}$  that satisfy*

$$\begin{aligned} \forall i = 1, 2, \quad \Psi_0^1(\mathcal{T}_i) &= \Pi_1 + \mathcal{T}_i(\Pi_2 + \Pi_3) < 0, \\ \Psi_0^2(\mathcal{T}_i) &= \begin{bmatrix} \Pi_1 - \mathcal{T}_i \Pi_3 & \mathcal{T}_i N \\ * & -\mathcal{T}_i R \end{bmatrix} < 0, \end{aligned} \quad (9)$$

where

$$\begin{aligned} \Pi_1 &= \text{He}\{M_1^T P M_0 - M_{12}^T (\frac{1}{2} S_1 M_{12} + S_2 M_2 + N^T)\} \\ \Pi_2 &= M_0^T R M_0 + \text{He}\{M_0^T (S_1 M_{12} + S_2 M_2)\}, \\ \Pi_3 &= M_2^T X M_2, \end{aligned} \quad (10)$$

with  $M_0 = [A \ BK]$ ,  $M_1 = [I \ 0]$ ,  $M_2 = [0 \ I]$  and  $M_{12} = M_1 - M_2$ . Then the system (3) is asymptotically stable for any asynchronous sampling satisfying (1) and for all  $T$  in  $[\mathcal{T}_1, \mathcal{T}_2]$ ,  $\Gamma^T(T) P \Gamma(T) - P < 0$  holds.

**Proof.** Introduce a quadratic Lyapunov function defined by  $V(x) = x^T P x$  for all  $x \in \mathbb{R}^n$ . This candidate clearly satisfies (6). A candidate for  $\mathcal{V}_0$  is defined for all  $k \in \mathbb{N}$ ,  $T_k \in [\mathcal{T}_1, \mathcal{T}_2]$  and  $\tau \in [0, T_k]$  by

$$\begin{aligned} \mathcal{V}_0(\tau, \chi_k) &= (T_k - \tau) \left\{ \zeta_k^T(\tau) [S_1 \zeta_k(\tau) + 2S_2 \chi_k(0)] \right. \\ &\quad \left. + \int_0^\tau \dot{\chi}_k^T(\theta) R \dot{\chi}_k(\theta) d\theta + \tau \chi_k^T(0) X \chi_k(0) \right\}, \end{aligned}$$

where  $\zeta_k(\tau) = \chi_k(\tau) - \chi_k(0)$ . The functional  $\mathcal{V}_0$  satisfies condition (7) since  $\mathcal{V}_0(0, \chi_k) = \mathcal{V}_0(T_k, \chi_k) = 0$ . The two

first terms of the functional  $\mathcal{V}_0$  were already introduced in [13]. The integral term was also considered in [2, 16]. A first contribution of this article is the introduction of the last term of  $\mathcal{V}_0$ , which depends on the matrix  $X$ . As suggested from Theorem 1, no positivity constraints of the functional are required. The differentiation of  $\mathcal{W}_0$  yields, for all  $\tau \in [0, T_k]$

$$\begin{aligned} \dot{\mathcal{W}}_0(\tau, \chi_k) &\leq \xi_k^T(\tau) [\Pi_1 + (T_k - \tau) \Pi_2 \\ &\quad + \tau N R^{-1} N^T + (T_k - 2\tau) \Pi_3] \xi_k(\tau). \end{aligned} \quad (11)$$

where  $\xi_k(\tau) = [\chi_k^T(\tau) \ \chi_k^T(0)]^T$ . This upper-bound is provided using the following inequality

$$\begin{aligned} \int_0^\tau \dot{\chi}_k^T(\theta) R \dot{\chi}_k(\theta) d\theta - 2\xi_k^T(\tau) N (\chi_k(\tau) - \chi_k(0)) \\ + \tau \xi_k^T(\tau) N R^{-1} N^T \xi_k(\tau) \geq 0, \end{aligned}$$

To prove that  $\dot{\mathcal{W}}_0$  is negative definite for all  $\tau$ , a convexity property is employed. Since the previous inequality is linear with respect to  $\tau$ , it is necessary and sufficient to ensure the negativity at the edges. This leads to  $\Psi_0^1(T_k) < 0$  and  $\Psi_0^2(T_k) < 0$ . The application of the same convexity argument on  $T_k$  leads to (9). By virtue of Theorem 1, the previous conditions are equivalent to  $\Gamma^T(T) P \Gamma(T) - P < 0$  for all  $T$  in  $[\mathcal{T}_1, \mathcal{T}_2]$  and the solutions of the system (3) are asymptotically stable. ■

In practice, it may happen that there is no lower bound of the sampling period. Consider the particular case of Theorem 2 where  $\mathcal{T}_1$  tends to zero. The following corollary is derived

**Corollary 1** *Let  $\mathcal{T}_2 > 0$ . Assume that there exist the same matrix variables as in Theorem 2 that satisfy*

$$P(A + BK) + (A + BK)^T P < 0, \quad (12)$$

$\Psi_0^1(\mathcal{T}_2) < 0$  and  $\Psi_0^2(\mathcal{T}_2) < 0$ . Then the system (3) is asymptotically stable for any asynchronous sampling in  $(0, \mathcal{T}_2]$ . Moreover, the following inequality  $\Gamma^T(T) P \Gamma(T) - P < 0$  holds for all  $T \in (0, \mathcal{T}_2]$ .

**Proof.** When  $\mathcal{T}_1$  tends to zero, both LMIs (9) tend to the ‘limit’ LMI  $\Pi_1 < 0$ . By virtue of the Finsler lemma [17], the matrix  $N$  can be removed. An equivalent LMI is

$$M_{12}^{\perp T} [\text{He}\{M_1^T P M_0 - M_{12}^T (\frac{1}{2} S_1 M_{12} + S_2 M_2)\}] M_{12}^{\perp} < 0$$

where  $M_{12}^{\perp T} = [I \ I]^T$ . Then the condition (12) is retrieved. It thus ensures that the system is asymptotically stable for arbitrarily small sampling periods. ■ The following theorem introduces additional matrix variables to relax the conditions from Theorem 2.

**Theorem 3** Consider two positive scalars  $0 < \mathcal{T}_1 < \mathcal{T}_2$  and  $\epsilon \in \mathbb{R}$ . If there exist  $P > 0 \in \mathbb{S}^n$ , some matrices  $X^i$ ,  $R^i$  and  $S_1^i \in \mathbb{S}^n$ ,  $S_2^i \in \mathbb{R}^{n \times n}$  and  $N^i \in \mathbb{R}^{2n \times n}$  that satisfy

$$\forall i = 1, 2, \Theta_1^i(\mathcal{T}_i) = \Pi_1 + \mathcal{T}_i \Pi_2^i + \mathcal{T}_i \Pi_3^i < 0, \quad (13)$$

$$\Theta_2^i(\mathcal{T}_i) = \begin{bmatrix} \Pi_1 - \mathcal{T}_i \Pi_3^i & \mathcal{T}_i N^i \\ * & -\mathcal{T}_i R^i \end{bmatrix} < 0,$$

where  $S_1^2 = \epsilon S_1^1$ ,  $S_2^2 = \epsilon S_2^1$ ,  $N^2 = \epsilon N^1$  and  $\Pi_1$ ,  $\Pi_2^i$  and  $\Pi_3^i$  as in (10) but with the matrices  $S_1^i$ ,  $S_2^i$ ,  $N^i$ ,  $X^i$  and  $R^i$ . Then the system (3) is asymptotically stable for any asynchronous sampling in  $[\mathcal{T}_1, \mathcal{T}_2]$  and for all  $T \in [\mathcal{T}_1, \mathcal{T}_2]$ ,  $\Gamma^T(T)P\Gamma(T) - P < 0$  holds.

**Proof.** Consider  $k \in \mathbb{N}$  and the associated sampling period  $T_k \in [\mathcal{T}_1, \mathcal{T}_2]$ . Then there exists  $\gamma_k \in [0, 1]$  such that  $T_k = \gamma_k \mathcal{T}_1 + (1 - \gamma_k) \mathcal{T}_2$ . Introduce, for  $j = 1, 2$ ,

$$\Theta_j(T_k) = \gamma_k \Theta_j^1(\mathcal{T}_1) + (1 - \gamma_k) \Theta_j^2(\mathcal{T}_2).$$

If the conditions of Theorem 3 hold, then it implies that  $\Theta_j(T_k) < 0$ , for  $j = 1, 2$  and all  $T_k \in [\mathcal{T}_1, \mathcal{T}_2]$ . Define the new variables  $U(k) = (\gamma_k + \epsilon(1 - \gamma_k))U^1$ , where  $U(k)$  stands for the matrix variables  $S_1(k)$ ,  $S_2(k)$ ,  $N(k)$  and  $\tilde{U}(k) = (\gamma_k \tilde{U}^1 + (1 - \gamma_k) \tilde{U}^2)/T_k$  where  $\tilde{U}(k)$  stands for  $R(k)$  and  $X(k)$ . This choice makes  $\Theta_j(T_k) = \Psi^j(T_k) < 0$  as in Theorem 2, which ensures asymptotic stability of the closed-loop system. ■

Consider now that the  $A$  and  $B$  are uncertain but are included in a polytope defined by some positive scalars  $\lambda_j$ 's such that  $\sum_{j=1}^M \lambda_j = 1$  and  $M_0 = [A \ B \ K] = \sum_{j=1}^M \lambda_j [A_j \ B_j \ K] = \sum_{j=1}^M \lambda_j M_{0j}$ . An extension of Theorem 2 to this class of uncertain systems is proposed is straightforward.

## 4 Exponential stability analysis

In the sequel an extension of Theorem 1 including performance properties is proposed. The objective is to guarantee exponential stability with a guaranteed decay rate.

**Theorem 4** Consider positive scalars  $\alpha$ ,  $0 < \mathcal{T}_1 \leq \mathcal{T}_2$  and a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$ , which satisfies (6). The two following statements are equivalent.

(i) The function  $V$  satisfies for all  $(k, T_k) \in \mathbb{N} \times [\mathcal{T}_1, \mathcal{T}_2]$

$$\Delta_\alpha V(k) = e^{2\alpha T_k} V(\chi_k(T_k)) - V(\chi_k(0)) < 0.$$

(ii) There exists  $\mathcal{V}_\alpha : [0, \mathcal{T}_2] \times \mathbb{K} \rightarrow \mathbb{R}$  satisfying

$$\forall (T, z) \in [\mathcal{T}_1, \mathcal{T}_2] \times \mathbb{K}, \quad e^{2\alpha T} \mathcal{V}_\alpha(T, z(\cdot)) = \mathcal{V}_\alpha(0, z(\cdot)), \quad (14)$$

such that the functional  $\mathcal{W}_\alpha(\tau, \chi_k) = e^{2\alpha\tau} [V(\chi_k(\tau)) + \mathcal{V}_\alpha(\tau, \chi_k)]$  satisfies

$$\forall (k, T_k, \tau) \in \mathbb{N} \times [\mathcal{T}_1, \mathcal{T}_2] \times [0, T_k], \quad \dot{\mathcal{W}}_\alpha(\tau, \chi_k) < 0.$$

Moreover, if one of these statements holds, the solutions of the system (3) are exponentially stable with the rate  $\alpha$ .

**Proof.** Consider a given  $\alpha > 0$  and a positive integer  $k \in \mathbb{N}$ . Following Theorem 1, (ii) implies (i). Assume now that (i) holds. Consider the functional  $\mathcal{V}_\alpha(\tau, \chi_k) = -V(\chi_k(\tau)) + \Delta_\alpha V(k)/(e^{2\alpha T_k} - 1)$ . This functional satisfies (14) and  $\dot{\mathcal{W}}_\alpha(\tau, \chi_k) = \frac{2\alpha T_k}{e^{2\alpha T_k} - 1} \frac{e^{2\alpha\tau}}{T_k} \Delta_\alpha V(k)$ . Since  $(e^{2\alpha T_k} - 1)/2\alpha T_k$  is positive, for all  $\alpha$ ,  $\dot{\mathcal{W}}_\alpha$  has the same sign as  $\Delta_\alpha V(k)$ . This proves the equivalence between (i) and (ii). The proof is concluded as in Theorem 1. ■

If  $\alpha < 0$ , Theorem 4 still is valid. It means that the solutions of the system can be unstable but the divergence rate of the solutions is not greater than  $\alpha$ . In the sequel, a system is said  $\alpha$ -stable if there is an  $\alpha \in \mathbb{R}$  such that Theorem 4 is verified. A study of  $\alpha$ -stability of the solutions of sampled-data systems under a constant period is provided ( $T_k = T$ , for all  $k \in \mathbb{N}$ ).

**Theorem 5** For a given  $\alpha \neq 0$  and a positive scalar  $\mathcal{T}$ , assume that there exist the same variables as in Theorem 2, that satisfy

$$\Psi_\alpha^1(\mathcal{T}) = h_\alpha^1(0)\Pi_{1\alpha} + h_\alpha^2(\mathcal{T}, 0)\Pi_2 + h_\alpha^4(\mathcal{T}, 0)\Pi_3 < 0,$$

$$\Psi_\alpha^2(\mathcal{T}) = \begin{bmatrix} h_\alpha^1(\mathcal{T})\Pi_{1\alpha} + h_\alpha^4(\mathcal{T}, \mathcal{T})\Pi_3 & h_\alpha^3(\mathcal{T}, \mathcal{T})N \\ * & -h_\alpha^3(\mathcal{T}, \mathcal{T})R \end{bmatrix} < 0, \quad (15)$$

where  $\Pi_{1\alpha} = \Pi_1 + 2\alpha M_1^T P M_1$ ,  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$  are given in Theorem 2 and where the functions  $h_\alpha^i$  are defined for all  $\tau \in [0, \mathcal{T}]$  by

$$h_\alpha^1(\tau) = e^{2\alpha\tau}, \quad h_\alpha^2(\mathcal{T}, \tau) = (e^{2\alpha\mathcal{T}} - e^{2\alpha\tau})/2\alpha,$$

$$h_\alpha^3(\mathcal{T}, \tau) = \begin{cases} e^{2\alpha\mathcal{T}}(e^{2\alpha\tau} - 1)/2\alpha, & \text{if } \alpha > 0, \\ (e^{2\alpha\tau} - 1)/2\alpha, & \text{if } \alpha < 0, \end{cases}$$

$$h_\alpha^4(\mathcal{T}, \tau) = \frac{1}{\alpha} \left[ \frac{e^{2\alpha\mathcal{T}} - 1}{2\alpha\mathcal{T}} - e^{2\alpha\tau} \right].$$

Then the system (3) is  $\alpha$ -stable for the sampling period  $\mathcal{T}$  and  $\Gamma^T(\mathcal{T})P\Gamma(\mathcal{T}) - e^{-2\alpha\mathcal{T}}P < 0$ .

**Proof.** Consider the same quadratic function as in Theorem 2. A candidate for  $\mathcal{V}_\alpha$  is defined for all  $\tau \in [0, \mathcal{T}]$  by

$$\mathcal{V}_\alpha(\tau, \chi_k) = f_\alpha(\cdot, \tau) \zeta_k^T(\tau) [S_1 \zeta_k(\tau) + 2S_2 \chi_k(0)] + f_\alpha(\mathcal{T}, \tau) \int_0^\tau \dot{\chi}_k^T(\theta) R \dot{\chi}_k(\theta) d\theta + H_\alpha^4(\mathcal{T}, \tau) \chi_k^T(0) X \chi_k(0), \quad (16)$$

where for all  $\tau \in [0, T]$ ,  $f_\alpha(\mathcal{T}, \tau) = (e^{2\alpha(\mathcal{T}-\tau)} - 1) / 2\alpha$ ,  $H_\alpha^4(\mathcal{T}, \tau) = \frac{e^{-2\alpha\tau}}{\alpha} \left[ \frac{\tau}{\mathcal{T}} \frac{e^{2\alpha\mathcal{T}} - 1}{2\alpha} - \frac{e^{2\alpha\tau} - 1}{2\alpha} \right]$ . The functional  $\mathcal{V}_\alpha$  satisfies (14) since  $\mathcal{V}_\alpha(\mathcal{T}, \chi_k) = \mathcal{V}_\alpha(0, \chi_k) = 0$ . Then noting that  $\frac{d}{d\tau}(e^{2\alpha\tau} f_\alpha) = -e^{2\alpha\tau}$  and that  $\frac{d}{d\tau}(e^{2\alpha\tau} H_\alpha^4) = h_\alpha^4$ , it yields

$$\begin{aligned} \dot{W}_\alpha(\tau, \chi_k) &\leq \xi_k^T(\tau)[h_\alpha^1(\tau)\Pi_{1\alpha} + h_\alpha^2(\mathcal{T}, \tau)\Pi_2 \\ &\quad + \tau h_\alpha^1(\tau)NR^{-1}N^T + h_\alpha^4(\mathcal{T}, \tau)\Pi_3]\xi_k(\tau). \end{aligned} \quad (17)$$

In order to keep the same formulation as in Theorem 2, the convexity property of the exponential function is employed. Noting that  $e^{2\alpha\tau} \geq 1 + 2\alpha\tau$  if  $\alpha > 0$  and  $e^{-2\alpha\tau} \geq 1 - 2\alpha\tau$  if  $\alpha < 0$ , it yields, for all  $\alpha \neq 0, \tau h_\alpha^1(\tau) \leq h_\alpha^3(\mathcal{T}, \tau)$ . Since  $R$  and  $R^{-1}$ , are positive definite, it yields

$$\begin{aligned} \dot{W}_\alpha(\tau, \chi_k) &\leq \xi_k^T(\tau)[h_\alpha^1(\tau)\Pi_1 + h_\alpha^2(\mathcal{T}, \tau)\Pi_2 \\ &\quad + h_\alpha^3(\mathcal{T}, \tau)NR^{-1}N^T + h_\alpha^4(\mathcal{T}, \tau)\Pi_3]\xi_k(\tau). \end{aligned}$$

The previous inequality is linear with respect to  $e^{2\alpha\tau}$ . It thus suffices to ensure the negativity at the vertices, which leads to (15). By virtue of Theorem 4, the solutions of the system (3) are  $\alpha$ -stable. ■

The stability conditions of Theorem 2 are retrieved from Theorem 5 when  $\alpha$  tends to 0. Since the conditions of the previous theorem do not depend linearly on  $e^{2\alpha\mathcal{T}}$  because of the expression of  $h_\alpha^4$ , an extension of Theorem 5 to the case of asynchronous sampling is not straightforward. The objective in the sequel is to provide  $\alpha$ -stability for a system with an asynchronous sampling.

**Corollary 2** For a given  $\alpha \in \mathbb{R}$  and  $\mathcal{T}_2 > 0$ , assume that there exist the same matrices as in Theorem 5 that satisfy  $\Psi_\alpha^1(\mathcal{T}_2) < 0$ ,  $\Psi_\alpha^2(\mathcal{T}_2) < 0$  with  $X = 0$ . Then the system (3) is  $\alpha$ -stable for all asynchronous samplings in  $(0, \mathcal{T}_2]$  and it yields  $\Gamma^T(T)P\Gamma(T) - e^{-2\alpha T}P < 0$ , for all  $T \in (0, \mathcal{T}_2]$ .

**Proof.** When  $\mathcal{T}_1$  tends to zero, both LMIs (15) tends to the ‘limit’ LMI  $\Pi_{1\alpha} < 0$ . Since  $X = 0$ , this condition is already included in  $\Psi_\alpha^2(\mathcal{T}_2) < 0$ . If  $\alpha$  is negative, then  $\Psi_\alpha^1(\mathcal{T}_k)$  and  $\Psi_\alpha^2(\mathcal{T}_k)$  are linear with respect to  $e^{2\alpha\mathcal{T}_k}$  for any  $\mathcal{T}_k \in (0, \mathcal{T}_2]$ . If  $\alpha$  is positive, it is easy to see that  $h_\alpha^3(\mathcal{T}_k, \mathcal{T}_k) \leq h_\alpha^3(\mathcal{T}_2, \mathcal{T}_k)$  which becomes linear with respect to  $e^{2\alpha\mathcal{T}_k}$ . Then a convexity property allows to conclude the proof. ■

## 5 Multiple sampling periods

In this section, the system is subject to sampling periods  $\mathcal{T}_k$  taking values in a finite set  $\{\mathcal{T}_1, \dots, \mathcal{T}_L\}$  of positive interger, where  $L$  is a positive integer. This problem has been already pointed out in [22]. The objective is here

to prove that the system (3) under several sampling periods is stable even if one of them is greater than the maximum allowable sampling period. Exponential stability conditions from Theorem 5 allow quantifying the convergence and divergence rates of the solutions within each sampling period. As suggested in [19, 23], the combination of these convergence and divergence rates can lead to refined stability conditions.

Instead of considering only the case of a deterministic sequence of several samplings, the probabilities  $p_l$  of employing the sampling period  $\mathcal{T}_l$  in the controller are introduced for all  $l = 1, \dots, L$ . Assume that, for all  $\beta_l > 0$ , there exists  $k_0 > 0$  such that the probability  $p_l$  satisfies

$$\forall k > k_0, \quad \forall l = 1, \dots, L, \quad \left| p_l - \frac{K_l(k)}{k} \right| < \beta_l. \quad (18)$$

where  $K_l(k)$  represents the number of times that the sampling period  $\mathcal{T}_l$  is employed before the  $k^{\text{th}}$  sampling instant. This assumption simply means that the ratio between the times that  $\mathcal{T}_l$  is employed and the total number of sampling instants tends to the probability  $p_l$ .

This situation refers to packet losses where the previous control value holds until a new control packet is received. The loss is associated to a probability  $p$  satisfying (18). Then the probability to implement the same control values during  $l\mathcal{T}$  is  $p_l = p^{l-1}(1-p)$ , where  $l$  is a positive integer. To avoid a large number of successive losses, no more than  $L$  successive losses are permitted. Based on these assumptions, the following theorem is proposed.

**Theorem 6** Consider given  $\alpha_l \in \mathbb{R}$ , for  $l = 1, \dots, L$ . Assume that there exist  $p > 0 \in \mathbb{S}^n$  and some matrices  $X^l, R^l$  and  $S_1^l \in \mathbb{S}^n, S_2^l \in \mathbb{R}^{n \times n}$  and  $N^l \in \mathbb{R}^{2n \times n}$  for  $l = 1, \dots, L$ , that satisfy for  $l = 1, \dots, L$ ,  $\Psi_{\alpha_l}^1(\mathcal{T}_l) < 0$  and  $\Psi_{\alpha_l}^2(\mathcal{T}_l) < 0$  and such that

$$c = \sum_{l=1}^L p_l \alpha_l \mathcal{T}_l > 0,$$

where  $\Pi_{1\alpha_l}^l, \Pi_2^l$  and  $\Pi_3^l$  are defined as in (5) but with the matrices  $X^l, R^l, S_1^l, S_2^l$  and  $N^l$ . Then, the system (3) under the sampling periods  $\mathcal{T}_l$  associated with the probability  $p_l$  is stable.

**Proof.** Thanks to Theorem 5, if the inequalities  $\Psi_{\alpha_l}^1(\mathcal{T}_l) < 0$  and  $\Psi_{\alpha_l}^2(\mathcal{T}_l) < 0$  hold for all  $l = 1, \dots, L$ , then it yields  $V(\chi_k(\mathcal{T}_{\sigma(k)})) \leq e^{-2\alpha_{\sigma(k)}\mathcal{T}_{\sigma(k)}} V(\chi_k(0))$ , for all  $k \in \mathbb{N}$ , where  $\sigma(\cdot)$  belongs to  $\{1, \dots, L\}$  and is such that  $\mathcal{T}_{\sigma(k)} = t_{k+1} - t_k$ . The function  $\sigma$  is basically a stochastic process where the realization is fixed. This leads to  $V(\chi_{k+1}(0)) \leq e^{-2\sum_{i=1}^k \alpha_{\sigma(i)}\mathcal{T}_{\sigma(i)}} V(\chi_0(0))$ . Reordering the terms in the exponential function and introducing the probabilities  $p_l$ , it yields

$$\sum_{i=1}^k \alpha_{\sigma(i)}\mathcal{T}_{\sigma(i)} = k \left[ c + \sum_{l=1}^L \left( \frac{K_l(k)}{k} - p_l \right) \alpha_l \mathcal{T}_l \right].$$

Theorems	Ex.1	Ex.2
[3]	(0, 0.869]	(0, 0.99]
[13]	(0, 1.113]	(0, 1.99]
[2]	(0, 1.695]	(0, 2.53]
[10]	(0, 1.695]	(0, 2.53]
Cor.1	(0, 1.723]	(0, 2.62]

Table 1

Interval of allowable asynchronous samplings.

Consider  $\beta = c/(2L \max_{l=1,\dots,L} |\alpha_l \mathcal{T}_l|)$ . From assumption (18), there exists  $k_0$  such that, for all  $k > k_0$ ,  $(K_l(k)/k - p_l) < \beta$ . This ensures that for all  $l = 1, \dots, L$

$$\left| \left( \frac{K_l(k)}{k} - p_l \right) \alpha_l \mathcal{T}_l \right| \leq \left| \frac{K_l(k)}{k} - p_l \right| |\alpha_l \mathcal{T}_l| \leq c/2L.$$

Consequently, this leads to  $V(\chi_{k+1}(0)) \leq V(\chi_0(0))e^{-ck}$ , for all  $k \geq k_0$ , or in other words  $V(x(t_{k+1})) \leq V(x(t_0))e^{-ck}$ . The proof is concluded by noting that when  $k$  tends to  $+\infty$ ,  $V(x(t_{k+1}))$  tends to 0. ■

**Corollary 3** *If the sampling is a repeated sequence of the form  $\{\mathcal{T}_1, \mathcal{T}_2\}$ , then the condition to ensure stability becomes  $c' = \alpha_1 \mathcal{T}_1 + \alpha_2 \mathcal{T}_2 > 0$ .*

**Remark 1** *Consider two sampling periods  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , their associated convergence rates,  $\alpha_i$  and probabilities,  $p_i$ ,  $i = 1, 2$ . The convergence rate of discrete-time systems is given by  $\lambda_i = e^{-2\alpha_i \mathcal{T}_i}$ . Rewriting the expression of  $c$  in terms of the  $\lambda_i$ 's, one has  $p_1 \ln \lambda_1 + p_2 \ln \lambda_2 = -2c$ . Combining it with  $p_1 + p_2 = 1$  leads to  $p_2/p_1 = (-2c - \ln \lambda_1)/(\ln \lambda_2 + 2c)$ . The condition  $c > 0$  implies the existence of a  $\lambda^* \in [\lambda_1, 1]$  such that  $2c = -\ln \lambda^*$ . Then Theorem 1 from [9] is retrieved.*

## 6 Examples

Consider the system (3) with

$$\text{Ex.1 [13, 22]: } A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, BK = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \begin{bmatrix} 3.75 \\ 11.5 \end{bmatrix}^T;$$

$$\text{Ex.2 [10]: } A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix};$$

$$\text{Ex.3 [5, 11]: } A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};$$

$$\text{Ex.4 [3, 13]: } A = \begin{bmatrix} 1 & 0.5 \\ g_1 & -1 \end{bmatrix}, BK = \begin{bmatrix} 1 + g_2 \\ -1 \end{bmatrix} \begin{bmatrix} -2.688 \\ -0.664 \end{bmatrix}^T;$$

where  $|g_1| \leq 0.1$ , and  $|g_2| \leq 0.3$ ;

**Asymptotic stability under asynchronous samplings:** Table 1 summarizes the results obtained in the literature and in this article for Examples 1 and 2.

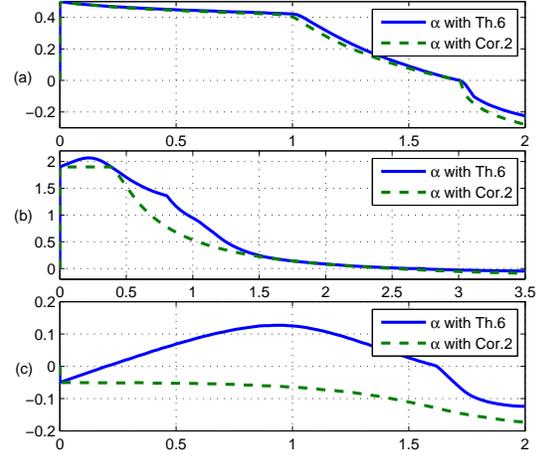


Fig. 2. Relation between the exponential decay rate  $\alpha$  and the sampling period  $\mathcal{T}_2$  with the cases of constant and time-varying periods for Examples 1, (a), 2, (b) and 3, (c).

Corollary 1 is less conservative than the existing ones. Example 3 is well known in the time-delay system theory [5, 11] because the solutions are unstable for a small delay and become stable when the delay is sufficiently large. The same behavior appears with a sampling. The existing conditions from [2, 3, 10, 13] cannot ensure stability of this class of systems. Theorem 2 ensures stability for all constant sampling periods in  $[0.2001, 1.62]$ . In the case of asynchronous sampling, Theorem 3 delivers  $[0.21, 0.43]$ ,  $[0.40, 1.25]$  and  $[1.20, 1.57]$ , with  $\epsilon = 0.8, 0.55$  and  $0.76$ , respectively.

Concerning Example 4, it was proven, in [3], [13] and [16], that this system is stable for any asynchronous sampling smaller than 0.35, 0.4476, 0.602, respectively. Theorem 2 adapted to the case of uncertain systems ensures that the closed-loop system is stable for any asynchronous samplings with periods less than 0.827s. Figure 2 shows the relation between the maximal exponential rate  $\alpha$  and the sampling period  $\mathcal{T}$  based on Theorem 5 and Corollary 2 for Examples 1, 2 and 3. A relevant comment concerns Examples 2 and 3 in Figure 2 (b) and (c). According to Theorem 5, the greatest convergence rate for these two examples is not achieved when the sampling period is zero. This means that, surprisingly, the solutions of a closed-loop system can converge more quickly to the equilibrium with a sampled control input than with a continuous one.

**Stability under two sampling periods:** Consider Example 1 and two sampling periods,  $\mathcal{T}_1 = 1$  and  $\mathcal{T}_2$  which has to be maximized (greater than 1.72). Solving Corollary 3 gives  $\mathcal{T}_2 = 2.01$  with the exponential rates  $\alpha_1 = 0.4121$ ,  $\alpha_2 = -0.204$ , the probabilities  $p_1 = p_2 = 0.5$  and  $c' = 2.10^{-5}$ .

**Stability under packet losses:** Consider Example 1 with a constant sampling  $\mathcal{T}$  with packet losses ( $L = 6$ ). The aim is to find the maximal probability  $p$  for which the system is still stable. Applying Theorem 6, the probabilities  $p = 0.421, 0.378$  and  $0.130$  are obtained

for  $T = 0.8, 1$  and  $1.5s$ , respectively.

## 7 Conclusion

A novel analysis of asynchronous sampled-data systems is provided. It provides the missing link between the discrete-time approaches and the input delay approach. Moreover it allows relaxing the conditions on the functionals traditionally employed in the input delay approach. The examples show its efficiency and the reduction of the conservatism compared to previous literature.

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