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MANAGING FINANCIAL RISKS DUE TO NATURAL CATASTROPHES

Robert Kast
CNRS, IFP Pondicherry, LAMETA Montpellier and IDEP Marseille.

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March 2010

Preface and abstract: this paper was prepared for the First Journées d'Economie et Econométrie de l'Asssurance, in Rennes, October 22nd and 23rd, 2009. It's a melting pot of several papers that I have written with my co-authors, plus a very short summary of Graciella Chichilnisky enlightening results. But, before proposing partial answers to the problem of modelling catastrophes in such a way as to be able to propose how to manage them, I try and grasp what we mean by catastrophe and what are the main problems, in the introduction.

JEL classification n°: G1, G22, H43, K32, N22,Q51, Q54
Keywords: catastrophe, risk, hedging instruments, cat-bonds, ambiguity, long term

Introduction to Catastrophes

Whatever the way you characterize a catastrophic event, you end up with
+ very high negative potential losses,
+ and a controversial (if not completely unknown) supposedly infinitesimal measure of their likelihood
+ in an indeterminate future.

Otherwise stated: catastrophe risks, are not "risks" in the sense of Ramsey (1926), but are of the domain of what Knight (1921) labelled "uncertainty". The problem we are facing is then: how do we make decisions in front of such catastrophe risks? Can we use our usual managing tool kit? Do we have to reset the problems at stake, seek new theories, build new instruments? As I shall try to show in this presentation, a little of each can be useful. The first difficult lies in our culture of risks defined by a known, objective, reliable probability distribution (Ramsey). Instead, we are facing a problem where the outcomes are uncertain (Knight).

Even though the problem of making decisions under uncertainty has been tackled mathematically (de Finetti: subjective probabilities, Savage : Bayesian decision theory) and economically (Arrow: contingent contracts, Arrow and Debreu: general equilibrium under uncertainty, Merton: continuous time equilibrium), the results are not well known and recent theories (such as Rank Dependant Expected Utility and Choquet Expected Utility in decision theory, or generalised "real investments" theory in Finance) are not well spread. However, these new developments are founded on assumptions that are more apt to grasp some aspects of catastrophe risks than the classical ones, notably those found in textbooks on the economics of insurance, for instance.

Indeed, all we can say is: there is little we know in economics about how to deal with catastrophe risks, and econometrics is out of the picture if we don't agree on a probability space of events! However, as I shall try to show and even though we're taking some risks doing so, there are bridges that can rely what we know how to do and what could be done if we knew how to.

The theory (theories, in fact) of financial markets used statistics to value contingent contracts with payoffs that are uncertain. Statistics are done on prices of traded contracts and not on the events on which the financial contracts are contingent. So, beware: this theory measures risks that are uncertain payoffs or payoff streams. But it measures a risk seen as a whole (a security, a call, etc.) by the contract's price assumed to grasp (yield a measure of) the risk represented by the traded contract. In some good cases (notably complete markets[1]) prices yield a measure on events (as formalized by the underlying model) that can be used to value any contingent contracts by the Fundamental Finance Formula:

The price of a contract is the integral of its discounted payoffs with respect to this measure

(the equivalent measure, or risk adjusted measure, often misleadingly called the "risk neutral measure" that would assume prices to be expected utilities). This measure is not the objective measure used to describe risks as random variables (e.g. in the CAPM) but it is "equivalent" as it puts zero probability on the same events and then it is defined by a Radon-Nikodym derivative with respect to the objective distribution.

[1] Markets are complete if any asset can be replicated by a portfolio of other traded assets.

In real[2] investments (real options) theory, this measure is deducted from traded assets prices that are assumed to be related to the real investment to be valued (for instance copper mines payoffs are related to futures on copper, see section 2).
An extension of this theory that we developed some years ago with my colleagues (Kast et al. 2002? 2003, 2004) consists of using the same principle for assets that have no clear economical relationships with traded assets but on which we have observations on past payoffs. In order to replicate such a "real" investment's payoffs, we construct an ad-hoc portfolio of traded assets so as to fit payoffs data, in such a way that the portfolio's payoffs "imitate" the payoffs stream of the investment to be priced. This is close to what has been done in "real options" pricing applications, but what we propose goes further as it is based on pure statistical correlation, when available, or stronger relations such as "comonotonicity". This statistical property is what makes it interesting for risks with non traded damages (payoffs) that can hardly be approximated (nor hedged) by tradable assets and commodities. It can be justified a contrario with the usual incentive to open new hedging instruments for catastrophes: traders are attracted to these instruments because they are not correlated to other traded assets. On the one hand, this is only true if a (dynamical) portfolio of traded assets can replicate (aproximatively) the risk to be hedged[3]. On the other hand it gives an idea of what its price could be if it were introduced on a market where no arbitrage prevails.

Another and more common way to extend the real investment theory to cases where there is no easy link between the asset to be traded and marketed assets is to use an old trick (developed in Macroeconomics and also used in Finance). Namely, the trick consists in representing the market by the behaviour of a "representative agent" whose ask (and bid) prices are the market prices. Following Mossin's (1966) interpretation of the CAPM, IDT can be thought of as a representation of a competitive markets with a representative agent whose valuation would be the equilibrium price. Although very dangerous because the assumptions of the primal theory are easy to forget in this setting and their violations can lead to completely absurd results[4], it has shown fruitful to extend it to the so called "real assets" theory. Accordingly, asset pricing becomes an optimisation problem in individual decision

[2] "Real" means that assets to be priced are not designed theoretically from other traded assets, e.g. an option to sell an underlying security at a given strike price, they are investments in some risky activity, e.g. mining with the real option to stop if a given rate of extraction is not reached.

[3] Markets are not statically complete in general, but they can be dynamically complete: an asset can be replicated by a limited number of traded assets used in dynamic portfolio strategy.

[4] e.g. a number of famous "paradoxes" that gave rise to a huge amount of literature.

theory. In this line, there are two ways that I know of extending results to the case of catastrophe risks.

The first one is to introduce the extension of classical decision theory (EUT) in order to take uncertainty, or at least ambiguity, into account. There are however difficult problems to solve in order to do so: Non additive criteria do not easily respect "dynamic consistency", a necessary condition to apply optimisation methods to cash flows, option pricing, and the like. In the trend of a developing literature we proposed a Dynamically Consistent Choquet Random Walk (Kast and Lapied 2009) in order to have a model close enough to the ones used in finance but taking ambiguity into account.
In a more general and theoretical approach, we have shown (Kast and Lapied 2008) that both uncertainty about the relevant future events and about the time when future information is revealed can be represented by a generalisation of discounted expected utility where discounted factors and/or expectation factors (probabilities or non-additive measures) reflect ambiguity. These extensions are meaningful for catastrophes where both time (of occurrence) and probabilities are undetermined and where flexible decisions based on information arrivals are crucial.

Another path has been opened by Chichilnisky (1998 and 2009), it consists of a generalisation of EUT in order to represent a change in behaviour in front of rare events and in front of long terms. In the latter case, this axiomatic theory (1998) defines a discounted expected utility for short terms combined with a long term valuation of time. In the former case (2009), axioms yield criteria that combine a "classical" expected utility with respect to a known continuous (on real numbers) probability distribution and another integral of payoffs with respect to a countable additive measure on rare events. The difficulty in applying this theory is to understand, and to estimate, what the compound factor that ponders the two criteria is. Another problem with these criteria is that they violate dynamic consistency!
However, the criteria being closer to the ones we know how to deal with (linear), they are attractive because a lot of usual applications can be worked out. But beware of dynamic consistency (and consequentialism) that are indeed crucial to deal with risks for which valuing flexibility of decision processes is relevant.

In order to end on more usual things, even though on a more restricted area, some things can be said to analyse some particularly well known, or close to be well known, common

catastrophes: e.g. floods. I'll present shortly some work that has been achieved on the coverage of floods in agriculture (Kast, Enjolras and Sentis 2009, following a long trend of literature in agricultural economics).

The plan of the paper is as follows:
First, I'll develop some aspects founded on the theory of financial markets, so as to propose some recent instruments designed to hedge catastrophic financial risks.
In section 1, I present my current knowledge about catastrophic assets that have been developed recently, together with a summary of the theory behind the practical tools (the latter are based on Kast and Lapied's 2006 book and the former on two Swiss-Re reports).
In the second section, I suggest a trend of research that extends the "classical" real investment literature to risks that are not naturally linked to traded assets (Kast, Lapied and Pardo, 2002 , 2003 and 2004).
The following sections rely on individual decision theory (IDT). When sticking to discounted expected utility, it reduces most results to local properties based on reliable probabilities at some known in advance dates, so that it can hardly apply to catastrophes. However, it may be extended so as to take into account:
+ ambiguity on the probability distribution (section 3, based on Kast and Lapied 2009)
+ both ambiguity and the future arrival of information (section 4, based on Kast and Lapied 2008)
+ long term discount rates and unusual behaviours in front of rare events (section 5, based on Chichilnisky, 1998 and 2009).
The last section describes the way some "not too catastrophic" risks can be hedged by a combination of existing decision structures (based on Kast, Enjolras and Sentis 2009).

Notice that, except for part 3 (the two papers by Kast and Lapied), all the other approaches rely on econometrics based on observed prices and data. Statistical inference is called for obtaining the relevant estimates of distributions. Obviously, if ambiguity or total uncertainty are to be considered for a catastrophe risk, classical statistical inference doesn't apply, at least directly: There is no prior probability on uncertain states, and even if one is considered, there is none on the set of parameters that determine likelihoods. Generalisations of statistical inference in the case of ambiguity, are in progress, some time will be needed however to have reliable properties enabling us to extend econometric method to the relevant formalisation of uncertainty in catastrophes.

A last point I'd like to make in this introductory note is that, contrary to common thinking, a catastrophe risk is not only something we have to bear, it's a risk that can be taken, as well as any other kind of risk, this is an example:

In *Newsweek*'s September the 14th, 2009 : « Learning to love the bomb », excerpt: 'The logic of nuclear peace rests on a scary bargain : you accept a small chance that something extremely bad will happen [a nuclear war] in exchange for a much bigger chance that something very bad – conventional war - won't happen.'

In terms of risk taking this means: you accept to take a catastrophe risk, in order to hedge a very bad risk.

Conversely, it is more common to consider that in order to hedge a catastrophe risk, one can be led to take a number (a portfolio) of very bad risks (very risky assets) that are available on specialised markets. As we have experienced recently (2008), that should only be done if markets are properly regulated in order to satisfy the theory's assumptions, at least approximatively. Otherwise, ill-defined portfolios of very bad risks, with prices calculated on the basis of theories whose assumptions are not satisfied, can generate a worldwide financial, and then social, catastrophe! The same could be said about the ill-managed industrial bad risks on climate taken over a century, which are generating an earthly catastrophe, notably because the terms of consequences have not been properly measured (discounted) … or taken into account at all.

1. Catastrophic assets

Catastrophes have always (at least since 3000 b.c.) been dealt with through mutual aid, and, when States are able to organise it, by public funds. However, this approach is very costly and, through many efforts and researches, the risks have been dealt with, more and more, by completing the role of public funds with incentives to insure, reinsure, or to rely on local, mutual or not, funds. Several instruments have been developed in order to smooth the losses of public funds and insurance companies, or to hedge reinsurance companies' portfolios.

For example, Swiss-Re has been able to develop specific instruments to help governments and local agencies to manage seismic risks. This is the result of a wide cooperation between the public and the private sector. The first example concerns risks in Mexico, and the second one a group of Mediterranean countries (but not France!).

1.1 Public and private cooperation

Catastrophes such as earthquakes can't be confronted at the individual level. However, individual behaviours do count, that's where Prudence is called for. But Prudence can be imposed by collective rules, public information, and encouraged by subsidies and/or compensations. Up to a certain point, prudent behaviours may induce banks to lend money for building, for example, even though the house may be destroyed by a seism. More typically, even though earthquakes belong to the category of non-insurable risks, insurance contracts can be designed under the condition that individual prudent attitudes are enforced and some prevention devices are built. It is necessary to take into account prudent attitudes guaranteed by public and insurance companies experts, once some collective preventive devices have been set up.

Prevention is adapted to situation of scientific certainty, hence, all efforts are made to get hazards measured and to estimate probability measures and the damages' probability distribution if possible. If all this is sufficiently reliable, then insurance contracts can be proposed to individuals.

However, because risks can't be reduced enough through the two principles on which the insurance industry is built (pooling and the law of large numbers), it is still necessary that some public funds (e.g. the Cat-Nat fund in France) warrants damages that are above the expected ones after preventive measures and regulations have been implemented. The problem that remains is to guarantee, or insure, or cover, this fund in case a catastrophe makes it default.

This is because we are in situations of scientific uncertainty, even though not in a situation of complete ignorance. The measurement of uncertain losses is left to the dealers of financial instruments, through the prices they set, in a way that satisfies both the demand (public funds, banks and insurance companies linked to them) and the supply: mainly financial investors looking for high return rates, disconnected from financial markets volatility.

1.2 The Parametric Earthquake Catastrophe Bond (from Swiss-Re Media information)

In Mexico, negotiations were initiated between some States and the Federal Government on the one side, and banks and insurance companies on the other side. The purpose was to obtain loans and insurance for these loans that would make them manageable by both parties. First of

all, physical hazards were to be studied with precision that was needed for the financial side to consider the kind of contracts and adapted premiums. Once the physical data were sufficiently reliable for insurance and banks, the organization of some pooling of regional risks (in order to lower premiums) and some public funds system had to be developed. The public funds had to deal with insurance companies and be hedged by some financial instruments: notably specific cat-bonds.

A cat-bond is a bond (in this case a portfolio of government bonds), but including a default possibility that is linked to the occurrence of a given catastrophe (see section 1.3). The problem is to define an index that will measure this catastrophe and be related to the default it may induce. Several indexes can be used. In case of a bond from a private company, the index is generally a market index for shares (S&P 500, for example). In case of a government bond, it depends on the rate attributed to the State[5]. For the case of a cat-bond, the index must be related to the catastrophic losses that may induce the default of the public fund. In the case of a cat-bond on seism risk for Mexico, the default has been related directly to the seism magnitude on the Richter scale[6]. In fact, only one trigger point has been chosen: magnitude 8.0. But depending on the epicentre place relative to damages, the defaults have been measured according to expected losses.

Magnitude	**7.0**	**7.5**	**8.0**
% bond loss inner zone	40	100	-
% bond loss outer zone	20	60	100

Hence, the returns of the bonds, according to their ratings that go from BB to B- can be fixed so as to represent the price of risks. For instance if the issuer has a portfolio of riskless government bonds at 5% interest rate, plus a public fund risky bond, the total interest rate can be, say: 5+4 = 9% where 4% is the risk premium paid by the public fund. Such working out the design of the bond has been done through cooperation between public fund managers, the

5 The rate of a State (from AAA or A+ to C) is determined by its expected ability to reimburse. For instance, lately, several European countries have been rated B instead of A

6 Obviously, a statistical index makes sense only if there are enough studies that rely the index to amounts of damages.

central Bank of Mexico, local banks, insurance companies and Swis-Re for the re-insurance and financial engineering technology.

1.3 Hedging catastrophe risks (from Kast and Lapied, 2006)

Catastrophe-bonds, or cat-bonds for short, are the most famous among the hedging instruments. As we shall see, they have some features in common with bonds, justifying their appellation, but these assets can be contingent on three types of random variables: Whether a damage index (insured casualties), or indexes that are specific to particular insurance companies, or/and a parameter index, based on (statistically measured) catastrophe characteristics. The choice between such indexes as a reference to define a cat-bond, requires to take two risks into account: Moral hazard, due to insured agents and/or insurance societies behaviours on the one hand. And, on the other hand, basis risks, due to the imperfect correlation between the insured risks and the insurance claims. Securities defined on an insurer's specific risk have no basis risk but make the investors to face likely moral hazards[7]. Conversely, securities founded on industrial damage indexes, sometimes completely eliminate moral hazards, but the hedger is running basis risks.

Cat-bonds are issued whether to address specific risks, or for well delimited geographical zones, for a fixed time horizon in both cases.

Cat-bonds are called this way because, indeed, they are bonds: i.e. tradable debts based on market exchange index such as the Euribor or the Libor, which is calculated in the London market in US $ or in UK £ or other currencies. Expiration dates are often between 3 and 10 years and these securities are contingent on conditions: If no catastrophes occurs, or more exactly if no damage claims are above the determined in advance level, investors perceive the due payments integrally. The returns on investments are above the Treasury Bonds ones (so called riskless, i.e. the price of time), often by more than 300 points. On the contrary, if the claims exceed the fixed level, then coupons and/or the principals are reduced so as to reimburse the concerned insurance companies.

Because of regulation and tax concerns, a specific offshore structure called a "Special Purpose Vehicule" (SPV) proceeds to the bonds' emission. The SPV offers a reinsurance contract to

[7] When perfectly hedged, an insurance company may be less meticulous with respect to the damage claims it insures.

the insurance companies seeking for one, at a cost. The total costs' amount is invested into Treasury Bills at a riskless rate, for a part, and into a short term securities' portfolio, for the remaining part. The riskless rate investments are meant to warrant the pledge on the investors, the short term high risky rate portfolio aims at hedging potential reinsured damage claims.

Catastrophe risks can also be hedged by so called "derivatives". The futures market is open on the CBOT since 1992. From 1995 on, claims are based on a group of indexes given by an official organism: The Property Claims Services (PCS). There are nine of such index-based insured damage claims, which are estimated from enquiries among insurance societies and other available information sources. Such indexes are revised on a day by day basis, they are relevant to determined geographical zones and for fixed expiration dates (trimesters or years). One index concerns the US territory as a whole and five indexes are specialised in States that are running particular risks (California, Florida and Texas, notably).

Call options on catastrophe exist as well, they are called cat-options: In exchange of a subscription that is paid in advance (premium), call options give the right, but not the obligation, to buy the PCS index at a fixed in advance price (exercise price) at a fixed expiration date. Only call options are traded on the CBOT.

Cat-spreads can be found. A spread is a combination of buying a call for a given exercise price and selling another one for a different exercise price, both calls with the same expiration date. Buying a cat-spread is a way to hedge an insurer's portfolio of claims, as an alternative to buying traditional insurance with deductible, or stop-loss insurance contracts.

Comparisons between the different types of instruments can be based on: transaction costs, market liquidity and moral hazards.

- Cat-options can be traded at little cost (bid-ask spreads). Conversely, cat-bonds incur important transaction costs due to the organisational complexity and to the analysis of the underlying risk.
- Markets for cat-options are bound to be easily cleared because of the participants' anonymity and if claims have a standardised form. Conversely the cat-bond market is not very active up to now and trades may wait to be cleared due to a lack of agreements on some standard forms for contingencies.
- The main advantage of cat-bonds over cat-options is that the basis risk that an investor is facing is much lower.

Seen from the insurer's point of view, the basis risk value is a fundamental argument in favour of bonds. On the other side of the market, investors are mainly interested by claims contingent

on catastrophes because they offer an alternative to traditional securities for diversifying their portfolios. Researches have shown that the returns of securities contingent on catastrophes have zero, or close to zero, correlation with other major traditional assets such that stocks and bonds. Estimates of the assets' returns correlations in the US are given in the following tableau.

	Cat-bonds	**S&P 500**	**Treasury Bills**
Cat-bonds	-	-	-
S&P 5000	– 0.13	-	-
Treasury Bills	– 0.07	0.40	-

Cat-bonds (*Source*: Froot, Murphy, Stern and Usher (1995): "The emerging asset class: Insurance risk", Special report from Guy Carpenter and Company, Inc.)

This new class of securities attractiveness can also be measured by the Sharpe ratio between excess return with respect to the riskless rate and the risk's standard error.

1.4 Swiss-Re cat-bonds for Mediterranean seisms (from Swiss-re Media)

On June 1st 2007, Swiss-Re obtained 100 Million US$ protection against earthquake risk in Turkey, Greece, Israel, Portugal and Cyprus.

The problem has been solved on a very similar basis than the one of Mexico, however it was complicated by the reference to different countries, regulations and hazards. The special sponsor vehicle to issue the cat-bonds, is MedQuake ltd. The real issuer is Swiss-Re with a retrocession agreement between the two companies. MedQuake issued notes that cover severe earthquake risk (measured by a parametric trigger) in the countries at stake, from May 2007 to May 2010. There were two classes of issues, with different ratings (depending on two different risky parts in the portfolio of the issuer) for the same redemption date (June 2010).

Class	**Rating**	**Size in M US$**	**Coupon (spread in basis points to LIBOR 3month rate)**
A	BB-	50	355
B	B	50	510

2.Statistically replicating non tradable assets
(from Kast, Lapied and Pardo, 2002, 2003, 2004)

Here, the idea is to extend the so called « real investments » theory (Dixit and Pindyck, 1994) to the case where investments, or projects, or well defined financial assets, are not linked to some easily related traded assets. This is in contrast with the literature where many applications to real investments where done in the case of copper mines (Brennan et Schwartz, 1985, Cortazar et al.,1998, Slade, 2001) for which related traded assets (futures on copper) are easy to find.

The two main methods that have been developed to relate traded asset prices to a given payoff process (to be valued) are well known in Finance : estimating the ß of real investment with a market portfolio (Sharpe and CAPM), or several ß's and related marketed assets (Ross' APT). But, in both cases, the idiosyncratic risks can't be taken into account, which may be highly misleading if the asset to be valued is clearly disconnected from the market (think of an environmental asset).

An alternative to the ß's method is to construct an ad hoc portfolio in such a way that it has a sufficiently close to 1 correlation coefficient (functional correlation; Kast et al. (2001) and (2002) and Pardo (2002)). The idea is to construct a portfolio of traded assets so as to maximise its functional correlation with the asset to be valued. As a result, we construct a tradable asset with payoffs as close as possible (given actual data) to the asset we want to price. The price process of the portfolio of traded assets yields the equivalent martingale measure from which any asset linearly dependent from the portfolio can be priced.

The following is an illustration of this method, it is based on purely financial data which makes it kind of "too easy"!

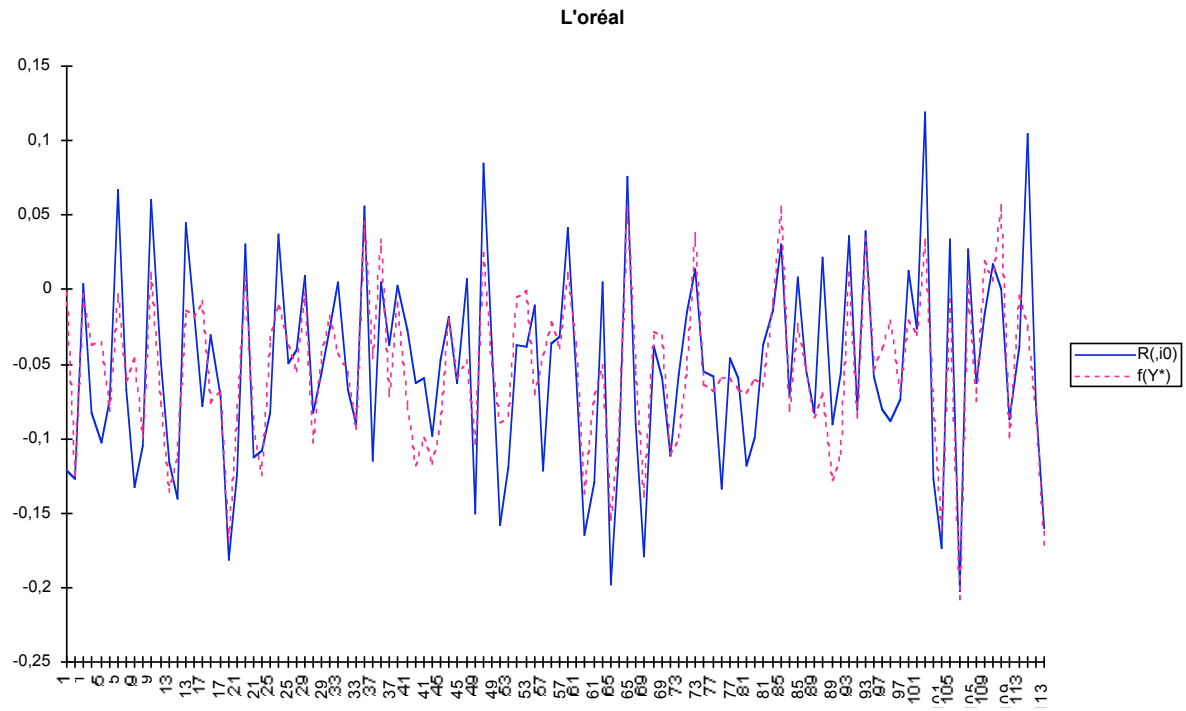
2.1 A statistical illustration

En plus, it's in French!

Ces données sont constituées des rentabilités mensuelles de 30 actifs financiers appartenant au CAC 40, observées au premier jour de chaque mois, pour lesquelles des séries complètes sont disponibles. La période d'observation est de 01/01/1989 au 01/09/1998, correspondant à une taille $T = 117$ des échantillons.

Prenons, par exemple et pour vérification de la méthode, le titre L'Oréal qui fait partie de la cohorte, comme risque à couvrir. Nous faisons donc comme si nous ne connaissions pas la valeur de ce titre pour construire un portefeuille formé des autres titres et qui réplique les paiements de L'Oréal. La valeur optimale du coefficient de corrélation fonctionnelle empirique obtenue est de 0.86, relativement proche de 1.
Dans la figure suivante sont superposés le graphique de la rentabilité du portefeuille optimal (en pointillé) et celui de la rentabilité de L'Oréal (en trait plein), ceci illustre le degré d'identification entre les processus de rentabilités obtenus.

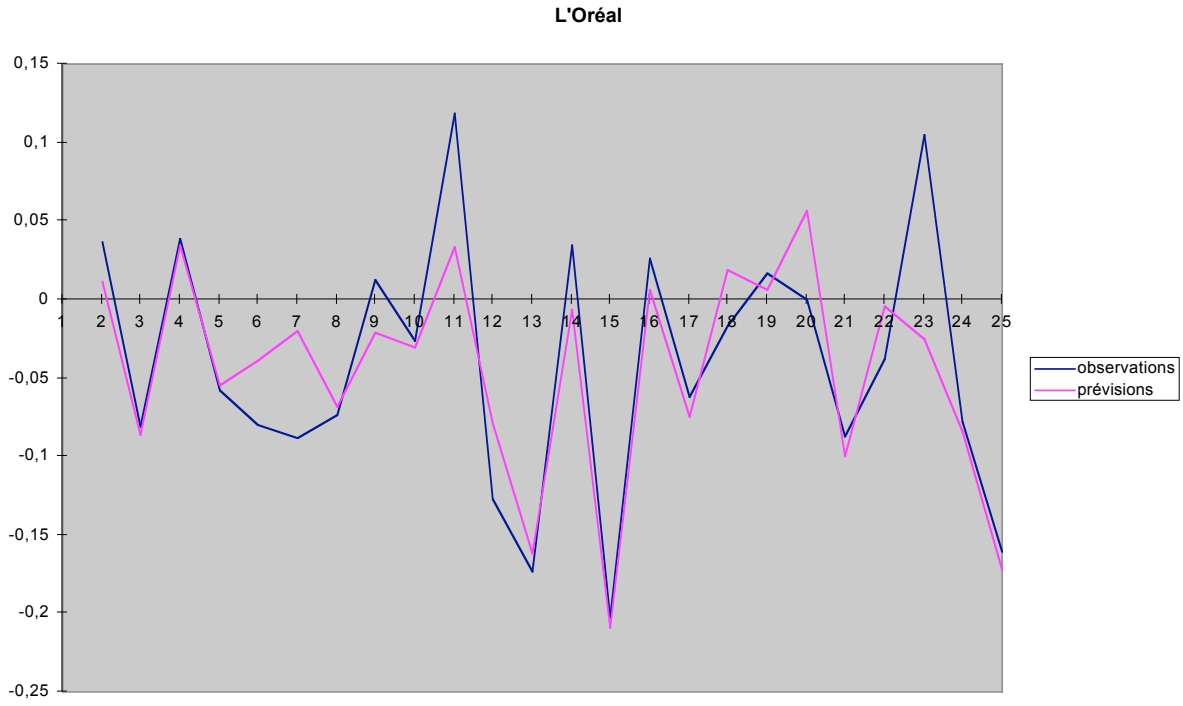
Figure 1 : *Comparaison entre le portefeuille virtuel et L'Oreal*



Nous pouvons vérifier le pouvoir prédictif du portefeuille de réplication, sur des données hors échantillon. La période initiale d'observation $\{1,...,T\}$ a été divisée en deux sous-périodes, une première période pour l'estimation et une seconde pour la prévision. La méthode a été

appliquée sur les données correspondant à la période $\{1,...,T-T_0\}$, avec $T_0 = \{ 6, 12, 24 \}$. Ainsi, un nombre T_0 de mois a été éliminé dans chaque échantillon pour calculer le portefeuille optimal. Les prévisions pour les taux de rendement du portefeuille sur la période$\{T-T_0 +1,..,T\}$ ont été ensuite formées en utilisant les coefficients calculés et les valeurs des taux de rendement observées pour chaque actif. La comparaison entre les prévisions obtenues à l'horizon de 24 mois et les valeurs observées correspondantes est illustrée par la figure suivante.

Figure 2 : *Comparaison des prévisions*



Dans le cadre de données simulées i.i.d., on peut également constater que le portefeuille optimal construit par la maximisation du coefficient de corrélation fonctionnelle est très proche de l'actif à répliquer, ce qui conforte la validité de cette méthode.

2.2 Looking for comonotonicity

When data are scarce, or if the risk is better represented by a finite set of values, correlation coefficients are not very relevant. A stronger notion of dependence can be looked for: two

random variables having increments always of the same sign are said to be comonotonic. Obviously two comonotonic variables cannot hedge each other. An equivalent definition will explain why we may look for a comonotonic relationship between a portfolio of marketed assets and the risk to be valued.

Comonotonicity: *Two random variables X_1 and X_2 are comonotonic, if and only if there exists a third random variable, X_3 and two non decreasing functions g_1 and g_2 such that:*
$X_1 = g_1(X_3)$ *and* $X_2 = g_2(X_3)$.

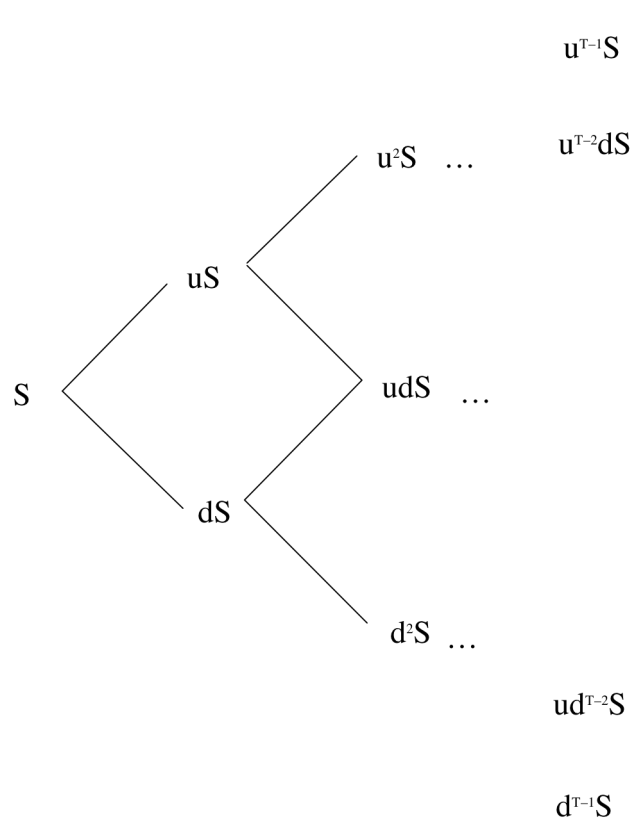
Otherwise stated, if g_1^{-1} exists: $X_2 = g_2 o g_1^{-1}(X_1)$ so that X_2 can be considered as a derivative asset of X_1.

This method is adapted to the case where the risk to be valued is well represented by a discrete variable for instance if its rates of returns can be represented by a binomial or Bernoullian tree. No probability distribution is needed at this stage, binomial tree representation only means that there is a finite number n of outcomes and that they can be obtained from a single one by multiplicative increments. If S is the found portfolio of marketed assets with rates that replicate those of X and S is comonotonic with X, then the same binomial tree applies to S and to X rates of returns. As is well known since Cox, Ross, Rubinstein (1979) the binomial formula yields a risk adjusted probability and doesn't refer to a known probability distribution for the risk to be valued (in consistence with Arrow's assets equilibrium model). This is an important good point in a context of scientific uncertainty about the outcomes.

Comonotonicity can be tested by the Kendall coefficient being equal to 1. The method consists in choosing a portfolio of marketed assets such that it minimises the difference between 1 and its Kendall coefficient with the risk (otherwise stated it maximizes the Kendall index). A difficulty arises because the Kendall coefficient uses the observed values directly and the portfolio's coefficients do not intervene in the formula. The programming method must then make a detour by an approximation methods such as genetic algorithms[8].

Without loss of generality, let us rank the T final values of the risk in increasing order: $C_1 < ... < C_T$. The portfolio that maximises Kendall's index is assumed to be strictly comonotonic

8 See Kast, Lapied and Pardo (2004).



to the risk to be valued in the sequel. From the portfolio's observed values, we can construct the relevant binomial tree with T final outcomes (i.e. over $T - 1$ periods), where $u > d$ without loss of generality as in figure 12.4.

Estimation of u and d parameters can be done according to two methods:

- Directly by estimation on the sample of a discrete distribution:

$(u^{T-1}S, u^{T-2}dS, \ldots, ud^{T-2}S, d^{T-1}S)$;

- By estimation of the volatility and then by discretization of the continuous process:

$$m = \frac{\sum_{t=1}^{T-1} \ln[\frac{S(t+1)}{S(t)}]}{T-1}, \; s^2 = \frac{\sum_{t=1}^{T-1} \{\ln[\frac{S(t+1)}{S(t)}] - \mu\}^2}{T-2} .$$

Once these parameters are determined, the risk adjusted probability is given by:

$P = (p, 1-p), p = \frac{(1+R)-d}{u-d}$, where R is the riskless interest rate.

Because C and S are comonotonic, C's payoffs can be ranked in the same order than those of S, then they can be placed on the final nodes of the binomial tree designed for S and assign them the corresponding risk adjusted probabilities (table above).

$u^{T-1}S$	→	C_T	→	p^{T-1}
$u^{T-2}dS$	→	C_{T-1}	→	$p^{T-2}(1-p)$
…		…		…

$ud^{T-2}S$	→	C_2	→	$p\,(1-p)^{T-2}$
$d^{T-1}S$	→	C_1	→	$(1-p)^{T-1}$

Then we can apply the FFF: the value of the risk is its discounted payoffs expectation with respect to the risk adjusted probability. The present value of C is then:

$$C(0) = \frac{1}{(1+R)^{T-1}} \sum_{i=0}^{i=T-1} \frac{(T-i)!}{i!(T-1-i)!} p^i (1-p)^{T-1-i} C_{i+1} .$$

Concerning controversial risks, this method is appealing because it doesn't require to use empirical probabilities on future payoffs: such an empirical distribution is typically a source of questions and/or of controversies among experts.

3. Real investments with ambiguity (Kast and Lapied, 2009)

3.1 The model and references

The real investments (or real options) literature (Pindyck and Dixit,1994) assumes that uncertainty is described by known stochastic processes (e.g. Brownian motions), as does most of the literature in finance (derivative asset pricing, optimal portfolio choice, CAPM etc.). In order to enlarge the scope of applications to uncertainty situations described by controversial probability distributions, or ambiguity about them, a way is to refer to new results in decision theory such as the multi-prior model of Gilboa and Schmeidler (1989) or the Choquet expected utility model of Schmeidler (1989) that yield non-linear criteria. Non additivity of the criteria expresses ambiguity of the decision maker about what is the relevant representation of uncertainty: the set of priors, or a Choquet capacity that is the envelope of probabilities inside its core. For instance, a convex capacity: i.e. for two disjoints events A and B, a capacity μ such that $\mu(AUB) \geq \mu(A) + \mu(B)$, expresses aversion to ambiguity. However many problems with non-linear models are still open. Notably, there is not a unique way to update non-additive measures such as Choquet capacities (see e.g. a survey in Kast, Lapied and Toqueboeuf, 2008). Furthermore, most models collapse to a linear one when dynamic consistency is required (Sarin and Wakker, 1998). Dynamics is central to investment problems, and the consistency of decisions with future decisions based on information arrivals

is at the core of real options theory. Epstein and Schneider (2003) propose a recursive multi-prior model that opens the way to applications: First, the multi-priors model is very close to linear ones, so that the usual mathematics are (relatively) easy to adapt; Second, a recursive model, although not really dynamical (it only looks one period ahead), is sufficient to address most issues. However, because of its similarity with classical models it may be the case that this recursive multi-prior model restricts the kind of ambiguity that one wants to address. In our approach, we propose a particular family of probability distributions that is summarized by a non-additive measure: a Choquet capacity. Our approach is axiomatic and subjective (the measure derives from the decision maker's preferences) there is no reference to an objective probability distribution that would be subjectively distorted (although it could be an interpretation). Furthermore we consider a discrete time dynamics that we make converge toward a continuous time model, instead of the converse as in Epstein and Schneider (2003). We address both the optimal portfolio choice of traded assets and the investment choice in a new non-traded asset (real investment).

In our model uncertainty is measured by the decision maker's subjective representation of preferences. Dynamics is described by a discrete time Brownian motion in which probability ½ is replaced by a constant *c* (the ambiguous weight that the decision maker is putting both on the event « up » and the event « down » instead of the unambiguous ½). At the limit, we obtain a deformed Brownian motion where both the drift and the volatility are changed, in contrast with the Chen and Epstein (2002) model where only the drift is affected.

Preferences on future payoffs are represented by V, a discounted Choquet expected utility of assets' cash flows. This is implied by 6 axioms (3 for Choquet expected utility and 3 for discounting) plus one to justify why we sum over uncertain states before summing over future dates (the converse would yield a different criterion given the non additivity of the Choquet integral).

Information is formalised by $Y: S \times T \rightarrow I$, where I is assumed finite, and the Y_t's define a filtration $(F_t)_{t \in T}$ on S.

We require a weak form of dynamic consistency, as expressed for example by Nishimura and Osaki (2003), so that the axiom of Dynamic Consistency will be stated as:

$$\forall X, X', [\forall i \in I, V^{[Y=i]}(X) \geq V^{[Y=i]}(X')] \Rightarrow V(X) \geq V(X'),$$ where $V^{[Y=i]}(X)$ is the criterion conditional on information given by the random variable Y, if it takes the value i.

We require furthermore that preferences conditional on information satisfy the same 8 axioms.

Then we can state the following condition expressing consistency between conditional and unconditional expectations.

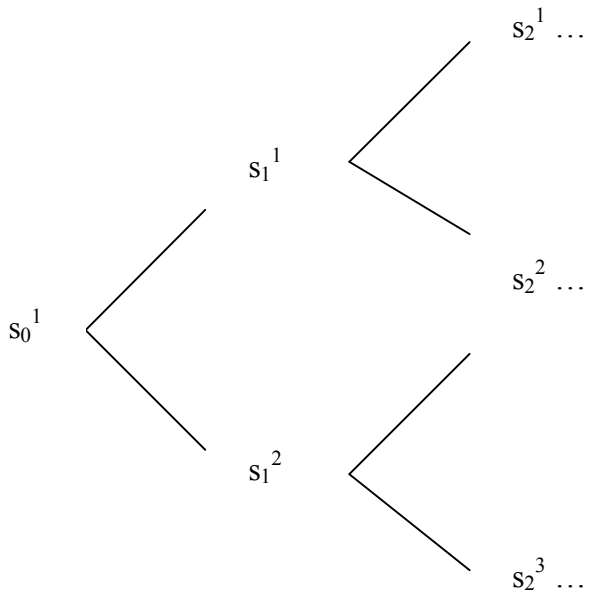
Proposition 3.1: Under the representation of preferences satisfying our nine axioms, for any $X \in R^{S \times T}$, $\forall \tau \in T$, $\forall i \in I$, $\forall [Y_\tau = i] \subset F_\tau$, $\forall t \geq \tau$, $E_\nu(X_t) = E_\nu[E_\nu^{[Y_\tau = i]}(X_t)]$.

3.2 ***Dynamically Consistent Choquet Random Walk (DCCRW)***

Consider an investment with payoffs contingent on future states according to a binomial type tree (no probabilities for up and down movements are needed).

Time is defined by: $t = 0, 1, \ldots, T$.

Uncertainty is described by a binomial tree so that the uncertain states, $s_1, \ldots, s_n$ in S are trajectories, i.e. sequences of nodes in the tree: for $i = 1, \ldots N$, $s_i = (s_0, s_1^{i_1}, \ldots, s_T^{i_T})$ with $i_t = 1, \ldots, t+1$:



A each t, the possible nodes are in $S_t = \{s_t^1, \ldots, s_t^{t+1}\}$. The information process is such that the DM knows, at time t, the state that is realised at this date. The set of parts of S_t is A_t.

The preferences of the DM over payoff processes such that: $X = (X_0,\ldots, X_T)$ are represented by a discounted (by a discount factor π) Choquet expectation with respect to a capacity ν on $(S, 2^S)$ so that the certainty equivalent of the process is:

$$DE(X) = \sum_{t=0}^{T} \pi(t) E_\nu(X_t),$$

where: $E_\nu(X_t) = \sum_{s_t \in S_t} X_t(s_t)\Delta\nu(s_t)$, with the usual notation for a Choquet integral for which, if, for instance, $X_t(s_1) \leq \ldots \leq X_t(s_N)$, $\Delta\nu(s_n) = \nu(\{s_n,\ldots, s_N\}) - \nu(\{s_{n+1}, \ldots, s_N\})$, with $\{s_{N+1}\} = \varnothing$, for notational convenience.

In order to characterise a Choquet Random Walk, we impose that, for any node s_t at date t ($0 \leq t < T$), if s^u_{t+1} and s^d_{t+1} are the two possible successors of s_t at date t+1 (for, respectively, an "up" or a "down" movement in the binomial tree), the conditional capacity is a constant: $\nu(s^u_{t+1}/s_t) = \nu(s^d_{t+1}/s_t) = c$, with $0 < c < 1$. The common value c expresses the DM's ambiguity about the likelihood of the states to come.

The conditional capacities are normalized in the following way:

$$\nu(\varnothing/s_t) = 0, \nu(\{s^u_{t+1}, s^d_{t+1}\}/s_t) = 1, \forall B \in A_{t+1}, \nu(B/s_t) = \nu(B \cap \{s^u_{t+1}, s^d_{t+1}\}/s_t).$$

From proposition 2.2.1, dynamic consistency implies:

$$\forall\ \tau = 1,\ldots,T-1,\ \forall\ t = \tau,\ldots,T,\ \sum_{s_\tau \in S_\tau} [\ \sum_{s_t \in S_t} X_t(s_t)\Delta\nu(s_t/s_\tau)]\Delta\nu(s_\tau) = \sum_{s_t \in S_t} X_t(s_t)\Delta\nu(s_t) \qquad (1.1)$$

Now that a Choquet random walk is characterised, we show that preferences that satisfy the dynamic consistency axiom don't leave much choice for the subjective capacity that represents them.

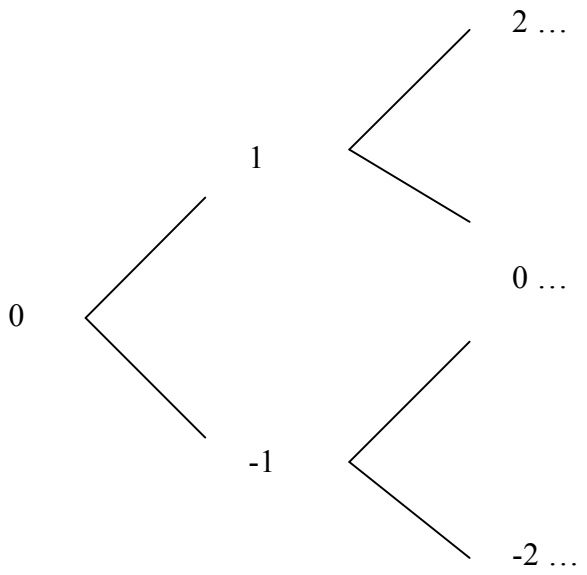
Proposition 3.2.1: A Dynamically Consistent Choquet Random Walk satisfying relation (1.1) is completely defined by a unique capacity ν satisfying: $\nu(s^u_{t+1}/s_t) = \nu(s^d_{t+1}/s_t) = c$.

Proposition 3.2.2: In a Dynamically Consistent Choquet Random Walk the capacity ν is sub-linear[9] if and only if $c \leq ½$. Moreover it does not reduce to a probability if and only if $c \neq ½$.

3.3 Symmetric Random Walk

[9] A sub- linear (or convex) capacity characterizes aversion to ambiguity (Gilboa and Schmeidler 1993).

We call Symmetric Random Walk a binomial process for which the "up" and the "down" movements correspond to the same magnitude. Without loss of generality, we take this increment to be the unity, and the departure point to be zero. In the probabilistic model, i.e. the case where $c = ½$, this process is a discrete time Brownian motion.



To compute the Choquet expectation of such a process we need to characterize the decumulative distribution function of capacity ν.

Proposition 3.3.1: The decumulative function of capacity ν is obtained by iteration from:

$$\forall\, t = 2,\ldots,T,\ \forall\, n = 1,\ldots,t,\ \nu(s_t^1,\ldots,s_t^n) = c\nu(s_{t-1}^1,\ldots,s_{t-1}^n) + (1-c)\nu(s_{t-1}^1,\ldots,s_{t-1}^{n-1}) \qquad (3.1)$$

and $\nu(s_1^1) = c$.

The closed form of the decumulative function is:

$$\forall\, t = 1,\ldots,T,\ \forall\, n = 1,\ldots,t-1, \qquad \nu(s_t^1,\ldots s_t^n) = c^{t-n+1}\sum_{j=0}^{n-1}\binom{j}{t-n+j}(1-c)^j \qquad (3.2)$$

Proposition 3.3.2: The Choquet Expectation of the payoffs at date t of a Symmetrical Choquet Random Walk is: $\forall\, t = 0,\ldots,T, \quad E(X_t) = t\,(2c-1)$ (3.3).

Remark 1: $c < ½ \Rightarrow E(X_t) < 0$. This is consistent with the DM's aversion to ambiguity that makes her give a negative value to a fair game.

Remark 2: Other Symmetric Random Walks can be obtained from this one by a positive affine transformation. The Choquet integral is linear with respect to this transformation.

For $\forall\ t = 0,\ldots, T$, $Y_t = a\,X_t + b$, $a > 0$, we have: $E(Y_t) = a\,t\,(2c - 1) + b$.

As a result we can address the cases where the mean is non zero and we can make volatility vary.

3.4 Independence and convergence toward a Brownian Motion

In order to study the convergence of the binomial random walk, we need to define independence and "independent random variables" have to be defined in the context of non-additive measures. We do it in the following, using a slightly more general definition than the one usually adopted (e.g. Marinacci, 1997). In the sequel, we omit the time index, which is useless because we assumed dynamic consistency.

For a real interval $[a, b]$, which doesn't contain all these values but at least one of them, let D be the information of the decision maker: $D = \{s \in S \,/\, Y(s) \in [a, b]\}$. Similarly, let us define: $B = \{s \in S \,/\, X(s) \in [a, b]\}$.

In this framework, relation (1.1) becomes:

$$\sum_{i=D,D^C}[\sum_{s\in S} X(s)\Delta\nu^i(s)]\Delta\nu(i) = \sum_{s\in S} X(s)\Delta\nu(s) \qquad (4.1)$$

As in section 3.1, the conditional capacities are normalised in the following way:

$\forall\ D \subset S$, $\nu(\varnothing/D) = 0$, $\nu(S/D) = 1$, $\forall\ B \subset S$, $\nu(B/D) = \nu(B\cap D/D)$.

Whatever the updating rule used to define the conditional capacity, independence between two random variables expresses the idea that conditioning their joint distribution by any one of them yields the other's marginal measure:

Definition 3.4.1: The random variables X and Y are independent if and only if:

$\forall\ D = \{s \in S \,/\, Y(s) \in [a, b]\} \subset S$, $\forall\ B = \{s \in S \,/\, X(s) \in [a, b]\} \subset S$: $\quad \nu(B/D) = \nu(B)$.

Notice that, most papers following Marinacci (1997) give a definition that is a particular case of this one: $\nu(B\cap D) = \nu(B)\ \nu(D)$. Indeed, <u>whatever the updating rule</u> we have:

Proposition 3.4.1: Under relation (4.1), if the random variables X and Y are independent:
$\forall D \subset S, \forall B \subset S, \nu(B \cap D) = \nu(B).\nu(D)$.

Proposition 3.4.2: $\forall t = 1,\ldots, T, \forall n = 1,\ldots, t+1$,

$$\Delta\nu_t^n \equiv \nu(s_t^1,\ldots s_t^n) - \nu(s_t^1,\ldots s_t^{n-1}) = \binom{n-1}{t} c^{t-n+1}(1-c)^{n-1} \quad (5.1)$$ where we set: $\nu(s_t^0) = 0$.

Proposition 3.4.3: When the time interval converges toward 0, the Symmetric Random Walk defined by (5.1) converges towards a general Wiener process with mean $m = 2c-1$ and variance $s^2 = 4c(1-c)$.

Notice that if $c < ½$, then $m < 0$ and $s^2 < 1$: both the mean and the variance are lower than in the probabilistic model. Indeed, ambiguity aversion yields lower weights on the ups and downs ($c < ½$) for given values (+1, -1), hence the variance is lower.

3.5. Applications

We can find possible applications of this approach to any model with ambiguity, for example macroeconomics "uncertainty models" where central banks are not aware of the "true" economic model. Another famous application would be a generalisation of the portfolio optimal choice under risk (Merton 1969, 1971, 1973). Here, let us consider a generalisation of the so called real option theory, i.e. a basic problem of optimal investment (cf. Dixit and Pindyck (1994)).

A firm can use a patent to invest and, after that, develop a production. The investment is irreversible and the corresponding cost is totally sunk. Therefore, we face an optimal stopping problem: The firm has to choose the optimal date to exerts its option to invest (if it is worth it).

In the basic model, the profit obtained when the patent is used follows a geometric Brownian motion $(\pi_t)_{0 \le t \le T}$ (where T is the expiration date of the patent after which there is no profits) such that:

(5.1) $d\pi_t = \mu\, \pi_t\, dt + \sigma\, \pi_t\, dB_t$, with $\pi_0 > 0$,

where B_t is the standard Brownian motion and μ and σ are some real numbers, with $\sigma > 0$ and $\mu < \rho$, where ρ, $\rho > 0$, is the firm discount rate.

With the Choquet Random Walk, the profit is:

$d\pi_t = \mu\, \pi_t\, dt + \sigma\, \pi_t\, dW_t$, where W_t is given by Proposition 5.2:

$dW_t = m\, dt + s\, dB_t$, with: $m = 2\, c - 1$, and $s^2 = 4\, c\, (1 - c)$, and then:

$d\pi_t = (\mu + m\, \sigma)\, \pi_t\, dt + s\, \sigma\, \pi_t\, dB_t$.

This relation is of the same type as (5.1), with $\mu' = \mu + m\, \sigma$ and $\sigma' = s\, \sigma$ in place of μ and σ. We can see that $0 < c < ½$ implies $-1 < m < 0$ and $0 < s < 1$, and then $\mu - \sigma < \mu' < \mu$ and $0 < \sigma' < \sigma$, a reduction of the instantaneous mean. But we also have the reduction of the volatility: an unexpected result!

If we suppose that the horizon T is infinite, the value of the utilized patent $V(\pi_t, t)$ does not depend on t directly and the model becomes stationary. The well known solution of the optimal stopping problem is then to invest at date t if and only if the value $V(\pi_t)$ is larger than a reservation value V^* such that:

$$(6.5) \quad V^* = \frac{\alpha' I}{\alpha' - 1},$$

where I is the cost of the investment and the constant α' is given by:[10]

$$(6.6) \quad \alpha' = \frac{-(\mu' - \frac{1}{2}\sigma'^2) + \sqrt{(\mu' - \frac{1}{2}\sigma'^2)^2 + 2\rho\sigma'^2}}{\sigma'^2}.$$

The effect of the Choquet distortion on the standard solution is equivocal, because it reduces at the same time the instantaneous mean and the volatility. It follows that the comparison between α' and α (with parameters μ and σ) is a matter of empirical data.

Consider now a stationary version of the Intertemporal Capital Asset Pricing Model (Merton (1969, 1971, 1973).

Let $(k_t)_{0 \le t \le T}$, the capital of the investor, which has to be allocated between a riskless asset with rate of return r, $r > 0$, and a risky asset the price of whom follows a geometric Brownian motion:

$$dP_t = \mu\, P_t\, dt + \sigma\, P_t\, dB_t \qquad (6.7)$$

with $P_0 > 0$, where B_t is the standard Brownian motion and μ and σ are some real numbers, with $\mu > 0$ and $\sigma > 0$.

If (x_t) is the part of the capital invested in the risky asset at date t, the program of the agent for a time horizon T is:

[10] It is easy to check that $\mu < \mu' < \rho$ implies $\alpha > 1$ and then (6.5) is a solution of the problem.

$$\underset{(x_t)_{0\le t\le T}}{Max}\ E_t[u(k_T)],\ k_0 > 0 \qquad (6.8)$$

where $u(.)$ is an increasing and concave utility function.

In the iso-elastic case:

$$u(k) = \frac{k^{1-\alpha}}{1-\alpha},\ \alpha > 0,\ \alpha \neq 1,$$

the optimal solution is a constant:

$$x^*(t,k) = x = \frac{1}{\alpha}\left(\frac{\mu - r}{\sigma^2}\right) \qquad (6.9)$$

With the Random Choquet Walk, this solution becomes:

$$x^*(t,k) = x' = \frac{1}{\alpha}\left(\frac{\mu' - r}{\sigma'^2}\right) \qquad (6.10)$$

with $\mu' = \mu + m\ \sigma$, $\sigma' = s\ \sigma$, $m = 2\ c - 1$, and $s^2 = 4\ c\ (1 - c)$.

If $0 < c < ½$, we have $\mu - \sigma < \mu' < \mu$ and $0 < \sigma' < \sigma$. It is easy to check that:

$$x' < x \Leftrightarrow \lambda \equiv \frac{\mu - r}{\sigma} < -\frac{m}{1-s^2} = \frac{1}{1-2c}.$$

The value of the market price of risk λ relatively to the ambiguity parameter c, gives the hierarchy between investment in the risky asset under ambiguity and under risk.

Similarly:

$$\frac{\partial x'}{\partial c} > 0 \Leftrightarrow \lambda \equiv \frac{\mu - r}{\sigma} < \frac{1 - 2c + 2c^2}{1 - 2c}.$$

The fact that investment in the risky asset is increasing with the reduction of ambiguity (when c increase towards ½, the ambiguity decreases) depends on the value of the market price of risk.

In both applications, the effect of ambiguity is ambiguous! Our results differ from the ones obtained from applications of Epstein and Schneider (2003)'s recursive multi-priors model. As we shall see in the next section, results in both applications to real options (Nishimura and Osaki 2007) and to continuous time CAPM (Chen and Epstein 2002) are not modified because the effect of ambiguity is not straightforward. This is due to the deformations of both the mean and the variance in our model of Choquet expectations.

4. The future as a product space of uncertain events and dates: the role of information (Kast, Lapied 2008)

We consider that the future payoffs of an asset have an uncertain component represented by a set of states and a time component represented by a set of dates (both finite in the paper so as to concentrate on concepts instead of mathematical subtleties).

The results are founded on the Ghirardato-Fubini theorem (Ghirardato 1997) that allows to analyse the problem of discounting future uncertain payoffs. Results are that we concentrate on two special cases: The first one assumes separability in time and then yields a (linear) discounted Choquet expectation expressing ambiguity on uncertain payoffs. The second one yields a non linear discounted (a Choquet expectation with respect to a bounded measure on dates) of (linear) expectations on uncertain payoffs. Both criteria satisfy dynamic consistency (by construction) but slack consequentialism (things that will not happen after information is released are still taken into account before information). The criteria yield uptaded measures of time and of uncertainty after information is released. Thanks to this, options value can be taken into account into the present valuation on the basis of which decisions are taken.

Let me just show some results that may be more puzzling than reassuring as far as tackling catastrophe risks are concerned but, I think, open some fruitful ways of research on understanding the stakes of "new" risks and catastrophe risks.

4.1. The model

We consider that a payoff is a measurable function $X: S \times T \rightarrow R_+$ where $S = \{s_1, \ldots, s_N\}$ represents the set of uncertain states to whom the payoffs are contingent and $T = \{1, \ldots, T\}$ the set of future dates, both with the sets of parts, 2^S and 2^T, as algebras.

Given we consider finite spaces, we can refer to a simple representation of preferences model, namely the generalisation (for finite sets) of de Finetti's (1930) axioms by Diecidue and Wakker (2002)[11]. Notice that in these models the DM's attitude toward the future payoffs is

[11] In a more general setting, we could refer to Chateauneuf's (1991) model, for instance.

completely grasped by the measure, and its decision criterion is defined by a cash amount (present certain value) such that the DM is indifferent between this present cash amount and the cash flow.

The certainty equivalent of uncertain payoff (*E* for expected) is:

$\forall X_t : R^S \rightarrow R$ is $E(X_t) = \int_S X_t \, d\nu$ if ν is a capacity, $E(X_t) = \int_S X_t \, d\mu$ if μ is additive.

The present equivalent of date contingent payoffs (*D* for discounted) is:

$\forall X_s : R^T \rightarrow R_+$, , $D(X_s) = \int_T X_s \, d\rho$ if ρ is a capacity $D(X_s) = \int_T X_s \, d\pi$ if π is additive.

In most economic models these two representations are assumed to be known and the problem is to define a representation of preferences over $R^{S \times T}$ that is consistent with the previous ones. Two obvious candidates are:

$\forall X: S \times T \rightarrow R_+$, $DE(X) = D[E(X)] = \int_T [\int_S X(s,t) \, d\nu(s)] \, d\rho \, (t)$ (Discounted Expectation) and:

$ED(X) = E[D(X)] = \int_S [\int_T X(s,t) \, d\rho(t)] \, d\nu(s)$ (Expected Discounting).

In the special case where de Finetti's coherence axiom (axiom 3') is satisfied, the cash flows' valuation representing the DM's preferences is unambiguously the (subjective) present certainty equivalent. Indeed, in this case we have:

$V(X) = D[E(X)] = E[D(X)]$.

The equalities are obtained because Fubini's theorem applies to Lebesgue integrals with respect to additive measures.

However, it is not the case that the two candidates yield the same result if the measures are not additive because Fubini's theorem doesn't apply. This why, in section 2.3, we shall invoke the Ghirardato-Fubini theorem that will allow us to construct *V* as whether *DE* or *ED* and investigate separately the effect of information arrivals on *E* and on *D*.

Integrating informational values in the linear valuation of a cash flow is straightforward: If some information arrives at some date τ, it is valued at that date by the conditional valuation, say V^τ, and the original cash flow $X = (X_1, \ldots X_T)$ is indifferent to the cash flow $(X_1, \ldots, X_{\tau-1}, V^\tau(X), 0, \ldots, 0)$. Then, the later cash flow can be discounted under the usual conditions.

The aim of this paper is to extend this result, as far as it can be done, to non-linear valuations.

We need two more axioms in order to insure that conditional values are well defined:

Model consistency imposes that future preferences be represented by the same type of criterion (here a Choquet integral).

Dynamic consistency expresses consistence between conditional preferences and present ones: if in all state of information an asset X is preferred to Y, then it should be preferred before information. In order to address the problem of consistently conditioning V, D and E when $V = DE$ or $V = ED$, we need an extension of Fubini's theorem, it was given by Ghirardato (1997):

Definition (Slice comonotonicity):

- $X \in R^{S \times T}$ is T-slice (resp. S-slice) comonotonic, if for all t in T, its t-sections on R^S (resp. for all s in S, its s-sections on R^T) are comonotonic.
- $X \in R^{S \times T}$ is slice comonotonic, if all its t-sections and its s-sections are comonotonic.
- A set $F \subset R^{S \times T} = R$ is said to be comonotonic if all the s-sections of its characteristic function 1_F are comonotonic, which is equivalent to: all its t-sections are comonotonic, and then to: 1_F is slice comonotonic.

The relevance of this definition for the problem of valuing an investment is related to the notion of hedging future variations, and hence to preferences showing more or less variation aversion. T-slice comonotonicity excludes the possibility that uncertain variations are hedged as time passes. We proved that provided another axiom (comontonic consistency) is satisfied, the Fubini property is also satisfied:

For any comonotonic subset F of $2^{S \times T}$,

$$\int_{S \times T} 1_F(s,t)\, d\psi(s,t) = \int_T d\rho(t) \int_S 1_F(s,t)\, d\nu(s) = \int_S d\nu(s) \int_T 1_F(s,t)\, d\rho(t).$$

Now, Ghirardato's theorem (his lemma 3 in our simple model) yields the following decomposition of preferences on $R^{S \times T}$ and preferences on R^S and on R^T.

Ghirardato-Fubini theorem:

1- If X in $R^{S \times T}$ is T-slice comonotonic, then: $V(X) = E[D(X)]$.

Furthermore, for any comonotonic class C_k in R^S containing all the comonotonic t-sections of X, there exists a probability distribution μ_k defining an additive representation E_k of preferences on C_k such that for any X' with all its comonotonic t-sections in C_k:

$V(X') = E_k[D(X')]$.

2- If X in $R^{S\times T}$ is S-slice comonotonic, then: $V(X) = D[E(X)]$.

Furthermore, for any comonotonic class C_h containing all the comonotonic s-sections of X, there exists a probability distribution π_h defining an additive representation D_h of preferences on C_h such that for any X' with all its comonotonic s-sections in C_h:

$V(X') = D_h[E(X')]$.

3- If X in $R^{S\times T}$ is slice comonotonic, then: $V(X) = E[D(X)] = D[E(X)]$.

Conditional valuation of S-slice comonotonic cash payoffs

From the previous theorem, apllied to charateristic functions we derive :

Proposition 4.1: Under relation (3.1), for any $i \in I_\tau$, the conditional capacity of a set $A \in F_t$, $t > \tau$, is given by:

(i) If $A \subset i$, $\quad \nu^i(A) = \dfrac{\nu(A \cap i)}{\nu(i)}$ (Bayes updating rule).

(ii) If $A^C \subset i$, $\quad \nu^i(A) = \dfrac{\nu(A \cup i^C) - \nu(i^C)}{1 - \nu(i^C)}$ (Dempster-Schafer updating rule).

The two rules we obtain result from the ranking of values after information obtains, and it depends on the type of information (comonotonic or antimonotonic with payoffs). The type of information can be interpreted as a “good” or “bad” news (with respect to what was expected). The fact that these rules integrate values that couldn’t not obtain after information is in contradiction with consequentialism as we confirm below.

Another familiar consistency condition known as consequentialism (Hammond (1989)) is usually imposed as an axiom on preferences. It is well known however (see, for instance Sarin and Wakker (1998), Machina (1998), Karni and Schmeidler (1991), Ghirardato (2002) and Lapied and Toquebeuf (2007)) that Model consistency, Dynamic consistency and Consequentialism imply additive (or quasi always additive) models. Ours is not, under the two first assumptions, hence it must be that Consequentialism is not satisfied, as we show in the paper.

Conditional valuation of T-slice comonotonic cash payoffs

If updating means that we modify the measure of uncertainty according to information at some date, then we dubb "upstating" the fact that we modify the measure of time according to

information (on the set of states) at the date at which it is obtained. Given that the DM's preferences satisfy axiom 5 (Dynamic Consistency) and hence relation (4.1) we have:

Proposition 4.2: Under relation (4.1), for $F \subset T$, with $\tau^- = \{0,\ldots,\tau\}$, and $\tau^+ = \{\tau,\ldots,T\}$, the "upstated" discount factors are given by:

(i) If $\rho(F) \geq \rho[(F \cap \tau^-) \cup \{\tau\}]$:

$$\rho^\tau(F \cap \tau^+) = \frac{\rho(F) - \rho[(F \cap \tau^-) \cup \{\tau\}] + \rho(\{\tau\})}{\rho(\{\tau\})}.$$

(ii) If $\rho(F) \leq \rho[(F \cap \tau^-) \cup \{\tau\}]$:

$$\rho^\tau(F \cap \tau^+) = \frac{\rho(F) - \rho(F \cap \tau^-)}{\rho[(F \cap \tau^-) \cup \{\tau\}] - \rho(F \cap \tau^-)}.$$

In the more familiar case where $F = \{0,\ldots,T\}$ we have the following:

Corollary 4.2: Under relation (4.1), for $F = \{0,\ldots,T\}$, the "upstated" discount factors are given by:

$$\rho^\tau(\{\tau,\ldots,T\}) = \frac{\rho(\{0,\ldots,T\}) - \rho(\{0,\ldots,\tau\}) + \rho(\{\tau\})}{\rho(\{\tau\})}.$$

As in the previous section, the different formulas come from the ranking of payoffs, but this time it's the payoffs before information obtains that make the difference.

The interpretations of these "upstating" formula are not straightforward. We can however propose the following: Given we deal with T-slice comonotonic payoffs, the important variations for the DM are the ones due to time. Hence, the timing of decisions is the most relevant feature for the valuation problem (and not the subsets of states that may be obtained after information). The weights given to the payoffs after information is obtained depend on the weights given in the past because these enter into the payoffs' ranking. As a result, aversion to time variations, say, may be modified depending on the relative importance of future vs past payoffs. The important point to note, is that the value of the past does count. This is in contrast with the additive case where the usual compound discount factors formula would yield: $\pi^\tau(\{\tau,\ldots,T\}) = \frac{\pi(\{\tau,\ldots,T\})}{\pi(\{\tau\})}$.

Notice that, here again, consequentialism is not satisfied.

Conclusion to part 4

Time and uncertainty are certainly the two most usual factors that have been taken into account in economics and we have available models to represent them. We based the intuition on Ghirardato's theorem and we have restricted the models to two cases: one were time didn't play a role because of comonotonicity of trajectories, and another one where it is uncertainty that can't be hedged because all random variables in the process are comonotonic. These restrictions greatly simplify the analysis because, in fact, one of the components is simplified away in the formulas. We have furthermore restricted the information process to a unique random variable occurring at a given date for reasons of tractability. This is obviously a first analytic step and it already yields some insights into the difficulties we'll have to face when we consider an information process. Still, the formulas we obtain for conditioning are meaningful and unusual.

From there, many things are left to analysis in order to retrieve the generality of a global vision of the future a decision maker is facing (and may be more than what we formalized as dates and states). More than that, even if we keep the two components decomposition of the future, some results would be useful when Ghirardato's theorem doesn't apply and both measures are non additive. Applications to problems in economics should lead us to seek results that put forward different methods to valuate the future, depending on whether the decision problem is more related to the timing of decisions than on the information bearing on the resolution of uncertainty, even though both account and interfere for the variability of payoffs to which the DM may be averse.

5. Separating what we know about how to deal with, and what is more scary
(from Chichilnisky, 1996, 1998, 2009)

It is necessary at this point to recognise the *clairvoyance* of G. Chischilnisky and her co-authors, mainly G. Heal and Andrea Beltratti (1998), about environmental and catastrophe risks. In papers and books, she relied pioneering works by Arrow, Debreu, Malinvaud, Drèze,

Radner, Genakopoulos, Grandmont, and many others on markets under uncertainty (perfect foresight, rational expectations, incomplete, and temporary equilibria) and decision theory (though ignoring the so called "non expected utility", in fact a generalisation of EU, approaches) with the problem we are facing in environmental (climate change for instance) and catastrophe risks. Furthermore, she addressed first and in an axiomatic form the problem of long term discount rates (the Chichilnisky criterion).

What makes her work and results attractive is that, instead of breaking away from classical approaches (i.e. discounted expected utility theory), she expands this criterion to take care of the problem at hand without depriving us from our usual (and may be wrong but efficient) way to solve them.
The first Chichilnisky criterion, addresses the problem of long vs short term discount rates.
The topology of fears, as she calls it, addresses the problem of the usual probabilistic approach when confronted with particular events (catastrophes) that can't be measured the same way as most insurable damages.

5.1 The Chichilnisky (1998) criterion for short and long term discounting in order to obtain the 'Green Golden Rule'.

For one part, there is a usual expected utility, however the discount factor (expressed as an absolutely continuous measure with respect to the uniform one on R with density f must be such that the discount rate $-f'/f$ converges toward zero as time goes to infinity. This excludes discount factors such as $\exp -(\partial t)$.
Preferences on consequences are a two factors: consumption and external state (ecological, for example). The criterion is of the form:

$$\alpha \int_0^{\infty} u(c_t, s_t) f(t) dt + (1-\alpha) \lim_{t \to \infty} (c_t, s_t)$$

Actually, the result holds for u additive (separable in c and in s).

It's a relevant criterion and not so different from a linear one, so that many usual techniques can be applied. It presents however one handicap (together with most criteria that are non additive, e.g. Choquet expectations) : it doesn't satisfy dynamic consistency (Beltratti et al. 1998). As a consequence, problem of the value of future options and the like are out of reach.

5.2 Chichilnisky's « topology of fear » (2009) and a criterion for rare events

Axiom 1: The ranking W : $L_\infty \rightarrow R$ is linear and continuous on lotteries

Axiom 2: The ranking W : $L_\infty \rightarrow R$ is sensitive to rare events

Axiom 3: The ranking W : $L_\infty \rightarrow R$ is sensitive to frequent events

The expected utility of a lottery f is a ranking denoted by $W(f) = \int f(x)\, d\mu(x)$

where is μ a measure with an integrable density function φ such that for all event A, $\mu(A)=\int_A \varphi(x)dx)$ where dx is the standard Lebesgue measure on R.

Expected utility satisfies Axioms 1 and 3, but not Axiom 2: Expected utility is insensitive to rare events.

Expected utility is derived from Debreu's monotone continuity axiom :

Theorem 1: A ranking of lotteries W : $L_\infty \rightarrow R$ satisfies the Monotone Continuity (MC) Axiom if and only if it is insensitive to rare events.

Theorem 2 : A ranking of lotteries W : $L_\infty \rightarrow R$ satisfies **non** MC if and only if there exist two continuous linear functions on L1, and L2 and a real number ; $0 < \lambda < 1$; such that:

$$W(f) = \lambda \int_{x \in R} f(x)\phi_1(x)dx + (1-\lambda) < f, \phi_2 >$$

Where : $\int \phi_1(x)dx = 1$ while ϕ_2 is a purely finitely additive measure.

This criterion is very convenient and relevant for the problem of representing particular behaviours in front of rare events, many applications are proposed in Chichilnisky's paper.

6. Less catastrophe risks
(from Enjolras, Kast and Sentis, 2009)

The problem tackled here is that of insuring several crops in a given area. Previous analyses have focused on a Linear Additive Model (LAM), which has recently received theoretical

foundations. This formulation has been developed for a single crop; instead, we investigate the case where there is a portfolio of crops. In order to formalize the problem, we develop a generalization of the Linear Additive Model: the Multi-LAM, and its implications. An illustration is given for wheat and sugar beet producers in France using the bootstrap technique. We prove that, in most cases, the use of the Multi-LAM generates a significant decrease in the area-yield basis risk. Implications for crop insurance are emphasized.

6.1. The Linear Additive Model and its extensions

The aim of the Linear Additive Model (LAM) is to describe a relationship between individual yield and the mean yields in a given area for a given crop. In this section, we present the potential extensions of the LAM when considering the full crop portfolio of a farm.

The LAM considers only one crop yield for a given farmer *i*. It can be written as follows:

$$\tilde{y}_i = E(\tilde{y}_i) + \beta_i (\tilde{y} - E(\tilde{y})) + \tilde{\varepsilon}_i$$

Where y_i is the individual yield, y is the area yield, yj, is the area yield of crop j, β_i is the sensitivity of the individual crop yield to the movements of the crop area-yield and εi is the residual of the model. E(.) denotes the expectation of a random variable.

6.2. The Multi-Linear Additive Model

Generalizing the LAM results to a Multi-LAM implies to preserve the linear structure of the model. This property allows to determine with precision some essential properties. Moreover, a linear model helps to provide a direct interpretation of the estimated coefficients. In this section, we look for the essential properties that a Multi-LAM must satisfy. In particular, we take care to verify and extend the theoretical results of the LAM (Rawaswami and Roe, 2004) when there are many crops.

In accordance with these authors and to preserve the generality power of the model, we choose to define a structural model which defines the individual yield y_{ij} of each crop *j* for farm *i* as a function *f* of an individual component e_{ij} and a systematic component θ_j. This structural form is written as follows:

$$y_{ij} = f_{ij}\left(e_{ij}, \theta_j\right)$$

At the farm's scale, we can then define:

$$y_i = \sum_j x_{ij} y_{ij} = \sum_j x_{ij} f_{ij}\left(e_{ij}, \theta_j\right)$$

In practice, function f can take several forms. We present the specifications used by Ramaswami and Roe (2004) and their extension to a crop portfolio in the following Table 2.

•	• **1 crop**	• **j crops**
• **Basic Model**	• $y_i = E\left(y_i\right) + \beta_i\left(y - E\left(y\right)\right) + \varepsilon_i$	• $y_i = \sum_j x_{ij} E\left(y_{ij}\right) + \sum_j x_{ij}\beta_{ij}\left(y_j - E\left(y_j\right)\right) + \varepsilon$
• **MRAC**	• $y_i = E\left(y_i\right)\left[\alpha e_i + \gamma\theta\right]$	• $y_i = \sum_j x_{ij} E\left(y_{ij}\right)\left[\alpha e_{ij} + \gamma\theta_j\right]$
• **ARAC**	• $y_i = E\left(y_i\right) + e_i + \theta$	• $y_i = \sum_j x_{ij}\left[E\left(y_{ij}\right) + e_{ij} + \theta_j\right]$
• **JPAC**	• $y_i = E\left(y_i\right) + \sigma_i\left(e_i + \theta\right)$	• $y_i = \sum_j x_{ij}\left[E\left(y_{ij}\right) + \sigma_{ij}\left(e_{ij} + \theta_j\right)\right]$
• **MRMC**	• $y_i = E\left(y_i\right) e_i\theta$	• $y_i = \sum_j x_{ij}\left[E\left(y_{ij}\right) e_{ij}\theta_j\right]$

Legend: MRAC designs a model with Multiplicative Risks and Additive Components. ARAC designs a model with Additive Risks and Additive Components. JPAC is the Just-Pope (1979) model with Additive Components. MRMC designs a model with Multiplicative Risks and Multiplicative Components.

Table 2. Popular yield specifications generalized to the multi-crop analysis

Starting from these specifications, we must answer three main questions: Which properties the Multi-LAM model must satisfy? Conversely, which classes of models imply a Multi-LAM? What is the validity of the Multi-LAM when yield aggregation is performed at a small scale?

6.4. A general structural model for the Multi-LAM

The purpose of this section is to define the class of models which imply a Multi-LAM. Conversely, it does so by looking for model characteristics that do not imply a Multi-LAM.

The structural form of production of crop *j* in farm *i* is given by the following equation:

$$(5) \qquad y_{ij} = f_{ij}\left(\theta_j, e_{ij}, E\left(y_{ij}\right)\right) \equiv f_{ij}\left(\theta_j, e_{ij}\right)$$

where θ_j and e_{ij} are respectively the random realizations of systematic and individual shocks. μ_{ij} is a vector of realized yields, which implies that we can omit it from our notations in the next steps.

Proposition 2: If the relationship between individual and area yields is described by a Multi-LAM as in (1), then the structural model necessarily satisfies: (a) $y_{ij} = \sum_j x_{ij} h_{ij}\left(e_{ij}\right) + \sum_j x_{ij} g_{ij}\left(\theta_j\right)$, *where* h_{ij} *and* g_{ij} *are functions which characterize respectively the impact of individual and systematic shocks on individual yields. (b)* $\forall i$ *, there is a function l(θ) and a parameter* λ_i *such that:* $g_{ij}\left(\theta_j\right) = \lambda_{ij} l\left(\theta_j\right) + c_{ij}$ *,where* c_{ij} *is a constant of integration.*

Proposition 3: Structural model (5) implies a multi-LAM if: (a) The weighted average of the risks can be replaced by the average of a large population. (b) The structural model satisfies:

(6) $y_i = \sum_j x_{ij} a_{ij} + \sum_j x_{ij} b_{ij} l_j\left(\theta_j\right) + \sum_j x_{ij} h_{ij}\left(e_{ij}\right)$, *where* $l_j\left(\theta_j\right)$ *and* $h_{ij}\left(e_{ij}\right)$ *are monotonic functions and* a_{ij} *and* b_{ij} *are parameters which can vary with i and j.*

Proposition 4: The parameters of the general structural model, which is equivalent to a Multi-LAM, satisfy (a) $\beta_{ij} = \frac{b_{ij}}{b_j}$ *and (b)* $\varepsilon_i = \sum_j x_{ij}\left[h_{ij}\left(e_{ij}\right) - E\left(h_{ij}\left(e_{ij}\right)\right)\right]$

Assuming the independence of the $h_{ij}\left(e_{ij}\right)$ terms, part (b) implies that:

$$Var\left(\varepsilon_i\right) = Var\sum_j x_{ij} h_{ij}\left(e_{ij}\right) = \sum_j x_{ij}^2 Var\left[h_{ij}\left(e_{ij}\right)\right]$$

6.5. Application to French agricultural data

The study uses a survey of French farmers who are members of the Farm Accountancy Data Network (FADN). Data are accounted for each year from a representative sample of farms of Northern France[12], whose size can be considered to be commercial. We selected a set of farms whose accounting data were available from 1990 to 2006 and which cultivated at least two crops, *i.e.* 1,732 farms. The regional data come from the *AGRESTE* database, which contains aggregate indicators for each crop and for each administrative region. In line with Mahul *et al.* (2000), this eliminated the need to compute yield expectations with our sample. Moreover, it prevented the problem of small aggregations.

For each region, we computed the correlation coefficients between crop yields using available historical data. This allowed us defining classes of crops.

In order to verify if our main assumption is satisfied, we need to estimate the econometric models for Additive LAM, Farm LAM and Multi-LAM. As our dataset is available at most for 12 years, we decided to resample it with the purpose of estimating linear regressions. In practice and for samples of farm coming from Group 1 and Group 2, we bootstrap the original data. Originally proposed by Efron (1979), this is a computation-intensive method for estimating the distribution of a test statistic or a parameter estimator by resampling the data[13]. Although it retains correlation, the bootstrap is particularly useful in cases where the asymptotic distribution is difficult to obtain, or simply unknown. In addition, this method often generates higher-order accurate estimates of the distribution which improve upon the usual asymptotic approximations (Chou and Zhou, 2006). Because of these advantages, it is not surprising to find applications of the bootstrap method in finance, in particular for estimating regression coefficients (see *e.g.* Balduzzi and Robotti, 2005). Of the different methodologies, the one developed by Hall (1994) provides the most relevant and accurate estimates, as their standard deviation is reduced.

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12 Ile-de-France, Nord-Pas-de-Calais and Picardie.

13 For each farm, we start from available data and create 1,000 new samples, each one containing 1,000 observations. These additional datasets are then used to estimate precisely regression coefficients and residuals for Additive LAM, Farm LAM and Multi-LAM.

In order to perform a direct comparison between the different theoretical approaches, we use each bootstrap resample to estimate the different models (Additive LAM, Farm LAM and Multi-LAM). Then we look at the validity of these three different approaches through the residuals in the models[14]. The residuals help determining the "individual risk" also known as "area yield basis risk" because they are not explained by area yield indices. As stated before, the variance of this risk should be lower using a Multi-LAM approach.

6.6. Perspectives for insurance policies

The estimated parameters of the Multi-LAM offer many implications for crop insurance regarding the traditional uses of the *beta* coefficients in finance. In our model, these parameters estimate the sensibility of an individual yield to an area yield while taking into account the crop portfolio structure. As observed by Ramaswami *et al.* (2004), there exists an analogy between the formulations of the Linear Additive Model (LAM) and of the Sharpe's single-index model. However, Sharpe's model is used for pricing which is not the aim of the LAM. Similarly, when extending the LAM to a Multi-LAM, one can find similarities between this new formulation and a multifactor model (Ross, 1976).

Previous studies emphasized the equivalence between *beta* coefficients and hedge positions with financial policies (see for instance Miranda, 1991, Smith *et al.*, 1994, Mahul, 1999, Chambers and Quiggin, 2002, Rejesus *et al.,* 2006 or Deng *et al.,* 2007). This comes from the financial signification of the *beta* coefficient, which is the sensitivity of the producer's yield to the movements of the area yields of crop *j*. In fact, the literature focused on models with a single variable and a single *beta*. In a mean-variance framework, Miranda (1991) found that optimal farmer's behavior is to take out an insurance contract with a coverage level that equals his positive individual *beta*. Mahul and Vermersch (2000) derived hedge ratios from their estimated *beta*. Then they proposed a set of financial contracts including futures and options in order to hedge crop risk.

However, these studies are constrained to the use of a single beta or a set of beta estimated separately for each crop. The Multi-LAM takes into account farm diversification when

[14] Multi-LAM and Farm LAM directly provides the regression residuals while it is necessary to compute them for Additive LAM. In that case, we had to subtract the estimated values from the original values, with respect to the weight of each crop in the considered farms.

differentiating the individual and systematic components of a given risk. It thus reinforces the results of Barnett *et al.* (2005) in two ways:

The specific risk – or area yield basis risk – is the result of coverage inefficiency. We proved it was significantly reduced with a Multi-LAM, either with uncorrelated crops or groups of uncorrelated crops. Moreover, this kind of risk was independent between the different farmers in a given area because it mainly depended on the structure of the crop portfolio. We thus guess that it can be hedged more easily with private insurance. Therefore, standard policies may be used, or even participating policies (Enjolras and Kast, 2008), in order to provide a fully integrated coverage.

In our two examples, the relative importance of the systematic risk was higher estimating a Multi-LAM. Then, the use of the *beta* coefficients estimated with a Multi-LAM function may be more accurate than the coefficients estimated with single LAMs. Moreover, hedging with existing instruments should be facilitated. This should also make crop risk insurable on financial markets.

These interpretations are related to the linear assumption of the Multi-LAM. Its formulation helps to provide precise theoretical propositions and associated results. It is also a way to perform direct comparisons with existing approaches. Moreover, the strong statistical significance of the Multi-LAM shows the relevancy of this choice. The use of nonlinear models is being developed in the literature and it offers some other promising results. It seems that this approach could be an extension of our work. In particular, its implementation should take into account that diversification contributes to smooth yield variations of the crop portfolio.

Conclusion to part 6

This article contributed to restore the role of diversification in crop insurance. Extending previous studies, we proved that use of a linear model is compatible with many specifications of crop yields already stated. We also showed that considering *ex-ante* the diversity of a farm crop portfolio could lead to a significant reduction of the individual component of the risk, which is neglected when subscribing only financial policies.

Estimating in practice the Multi-LAM, which is an econometric model, requires fulfilling some conditions: for instance the definition of homogeneous areas, *i.e.* with the same climate and with a sufficient number of farms, so that the area yield is correctly correlated with the

individual yields. It is also necessary to define crops of reference in order to avoid correlations effects in the model.

Our database at the French level showed that most farms which cultivate at least two crops are potentially concerned by the Multi-LAM. This result opens up numerous perspectives for a commercial application. With a set of adjusted betas, we can presume that the existing models with a single crop can be adapted in order to design optimal hedging strategies, even with existing instruments. Furthermore, the bootstrap technique makes it possible to overcome the lack of historical data. For this reason, this new parameterization of financial instruments could be adapted to the pricing and the extension of the crop insurance market.

General conclusion

In part 1 and part 6, I presented works that are close extensions of known methods and models developed to deal with ordinary risks. They work rather well when catastrophes are not "too catastrophic". By this I mean that, although we acknowledge that the phenomenon provoking the risk at stake is of the catastrophe type, we know enough about it to manage it using usual collective and individual methods, but we recognize that these methods are extended to their limit so that some more instruments have to be implemented to make them more secure. Otherwise stated: we know how to reduce and hedge the risks, but there are some risks left that need to be managed using new and appropriate tools.

An analysis of agricultural risks on crops, notably provoked by floods and draughts, is proposed in section 6. In enhances that, even though we deal with catastrophe events, refining the risk analysis would allow to design insurance and reinsurance contracts, maybe helped by public funds, to insure the risks they cause.

In the same vein for seism risks that are catastrophes provoked by well studied phenomena, already existing tools that have been proposed by the finance industry in collaboration with public institutions are presented in part 1.

In part 2 and part 5, theories are developed in order to grasp the relevant challenges presented by catastrophe risks, they keep close to theories that permitted to develop tools that are well known and handy (real investment theory and individual decision theory).

In contrast, parts 3 and 4, depart from classical approaches and propose to develop new tools in order to represent the type of uncertainty encountered in most catastrophes. Obviously, managing methods are not yet already available as theory has still to be developed. A way to take into account the type of results expected from the developments, is to question methods and instruments founded on more restrictive assumptions that, clearly, exclude some of the particular features of catastrophe risks. Notably, the foundations of statistical inference, and hence of econometrics, and the valuation of damages, insurance or financial contracts assuming known probabilities and discount factors. This is a way to apply the precautionary principle in front of scientific uncertainty, a situation that clearly characterizes catastrophes.

References

Balduzzi P. and Robotti C. (2005), "Mimicking Portfolios, Economic Risk Premia, and Tests of Multi-beta Models". Working paper series, Federal Reserve Bank of Atlanta, 2005-4.

Barnett, B. J., Black J. R., Hu Y. and Skees J.R. (2005). "Is Area Yield Insurance Competitive with Farm Yield Insurance?". *Journal of Agricultural and Resource Economics*, 30: 285-301.

Beltratti, Chichilnisky G., Heal "Uncertainty future preferences and conservation," in *sustainability dynamics and uncertainty*, Chichinisky, Heal and Vercelli eds. Dordrecht (1998)

Brennan M., Schwartz E., Evaluating natural ressource investments, *Journal of Business*, 58, pp135-157, 1985.

Chambers R. G. and Quiggin J. (2002). "Optimal producer behavior in the presence of area-yield crop insurance". American Journal of Agricultural Economics, 84: 320–34.

Chichilnisky G. "Sustainable development: an axiomatic approach", *Social Choice and Welfare*, 13-2, 199-248, 1996.

Chichilnisky G. "The topology of fear", W.P Greqam 2009.

Chou P.H. and Zhou G. (2006). "Using Bootstrap to Test Portfolio Efficiency". *Annals of Economics and Finance*, 7: 217–249.

Cortazar G., Schwartz E.S., Salinas M., Evaluating Environmental Investments: A Real Options Approach, *Management Science*, 44:8, 1059-1070, 1998.

Cox J., Ross S. and Rubinstein M. (1979), "Option pricing: a simplified approach", *Journal of Financial Economics*, 7, p. 229-263.

De Finetti, B. (1930). "Fondamenti logici del raggionamento probabilistico", *Bolletino dell'unione matematica italiana* 9.

Deng X., Barnett B.J. and Vedenov D.V. (2007). "Is There a Viable Market for Area-Based Crop Insurance?". *American Journal of Agricultural Economics*, 89(2): 508-519.

Dieccidue, E., and P. Wakker. (2002). "Dutch Books: Avoiding strategic and dynamic complications and a comonotonic extension", *Mathematical Social Sciences* 43, 135-149.

Dixit A., Pindyck R., *Investment under uncertainty*, Princeton University Press, 1994.

Efron B. (1979). "Bootstrap Methods: Another Look at the Jackknife". *The Annals of Statistics*, **7** (1): 1–26.

Enjolras G. and Kast R. (2008). "Using Participating and Financial Contracts to Insure Catastrophe Risk: Implications for Crop Risk Management". *Lameta Working Paper*, 2008-1.

Epstein, L., (1999), "A definition of uncertainty", *Review of Economic Studies* 66, 579–608.

Epstein, L. and M. LeBreton, (1993), "Dynamically consistent beliefs must be Bayesian", *Journal of Economic Theory*.

Epstein**,** L. and J. Miao, (2003). "A two-person dynamic equilibrium under ambiguity," *Journal of Economic Dynamics and Control*, Elsevier, Volume 27, Number 7, May 2003 , pp. 1253-1288.

Epstein, L. and M. Schneider, (2003), "Recursive Multiple Priors", *Journal of Economic Theory*,113, 1–31.

Epstein, L. and M. Schneider (2006), "Ambiguity, information quality and asset pricing", University of Rochester, Working Paper.

Epstein, L. and T. Wang (1994), "Intertemporal asset pricing under knightian uncertainty", *Econometrica* 62(3), 283–322.

Epstein, L. and S. Zin, (1989), "Substitution, risk aversion and the temporal behavior of consumption and asset returns: a theoretical framework", *Econometrica* 1989

Gilboa, I. and D. Schmeidler, (1989), "Maxmin Expected Utility with a Non-Unique Prior", *Journal of Mathematical Economics*, 18, 141-153.

Gilboa, I., and D. Schmeidler. (1993). "Updating ambiguous beliefs", *Journal of Econonomic Theory* 59, 33-49.

Ghirardato, P. (1997). "On independence for non-additive measures with a Fubini theorem", *Journal of Economic Theory* 73, 261-291.

Ghirardato, P. (2002). "Revisiting Savage in a conditional world", *Economic Theory* 20, 83-92.

Hall P. (1994). "Methodology and Theory for the Bootstrap, Chapter 39" in Engle R. F. and McFadden D. L., eds. Handbook of Econometrics. Vol. IV. New York: Elsevier, 2342-2379.

Just R.E. and Pope R.D. (1979). “Production Function Estimation and Related Risk Considerations”, American Journal of Agricultural Economics, 61: 277-284.

Karni, E., and D. Schmeidler. (1991). "Atemporal Dynamic Consistency and Expected Utility Theory", *Journal of Economic Theory* 54, 401-408.

Kast R., Lapied A., Pardo S., Protopopescu C.: Evaluation de Risques Controversés par la Méthode des Options Réelles, *Economie et Prévision*, n°149, 3/2001.

Kast R., Lapied A. and Pardo S. (2003), “Virtual underlying security”, WP greqam, 2003.

Kast R., Lapied A. and Pardo S. (2004), "Construction d'un portefeuille sous-jacent virtuel", *Revue Economique*, 55.

Lapied, A. and Toqueoeuf, P. (2007): “Consistent dynamic choice and non expected utility preferences”, DT Greqam.

Kast, R., Lapied, A. (2008): “Dynamically consistent Choquet capacities”, ICER WP, Torino, to appear in *Theory and Decisions*.

Knight F. (1921), *Risk, uncertainty and profit*, Houghton Mifflin, Boston.

Machina, M. (1998). "Dynamic consistency and non expected utility models of choice under uncertainty", *Journal of Economic Literature* 22, 1622-1668.

Mahul O. (1999), “Optimum Area Yield Crop Insurance”. *American Journal of Agricultural Economics*, 81(1): 75–82.

Mahul O. and Vermersch D. (2000). “Hedging crop risk with yield insurance futures and Options”. *European Review of Agricultural Economics*, 27(2): 109-126.

Marinacci, M. (1997). “Limit laws for non additive probabilities and their frequentist interpretation”, *Journal of Economic Theory*, 84, 145-195.

Merton, R.C., (1973a), "An intertemporal capital asset pricing model", *Econometrica* 41, 867–887.

Merton, R.C., (1973b), "Theory of rational option pricing", *Bell Journal of Economics and Management Science* 4, 141–183.

Miranda M.J. (1991). “Area-Yield Crop Insurance Reconsidered”. *American Journal of Agricultural Economics*, 73: 233-42.

Mossin J. (1966), "Equilibrium in a capital market", *Econometrica*, 34, p. 768-783.

Nishimura, K., and H. Ozaki. (2003). "A simple axiomatization of iterated Choquet objectives", W. P. CIRJE, Tokyo University.

Pardo S. (2002): Valuation of a contingent claim and surrogate asset limitations: A way out, *mimeo*, 2002.

Pindyck, A., and R. Dixit. (1994). *Investments under uncertainty*, Princeton University Press.

Ramsey F. (1926), "Truth and probability" reprint in Braithwaite R. (ed.), *The foundations of mathematics and other logical essays*, Routledge, London, 1931).

Ramaswami B. and Roe T.L. (2004). "Aggregation in Area-Yield Crop Insurance: The Linear Additive Model". *American Journal of Agricultural Economics*, 86(2): 420-431.

Rejesus R.M., Coble K.H., Knight T. and Yufei J. (2006). "Developing Experience-Based Premium Rate Discounts in Crop Insurance". *American Journal of Agricultural Economics*, 88(2): 409-419.

Ross S.A. (1976). "The arbitrage theory of capital asset pricing". *Journal of Economic Theory*, 13(3): 341-360.

Sarin, R., and P. Wakker. (1998). "Dynamic choice and Non Expected Utility", *Journal of Risk and Uncertainty* 17, 87-119.

Schmeidler, D. (1989). "Subjective probability and expected utility without additivity", *Econometrica* 57, pp.571-587.

Slade M., Valuing managerial flexibility : An application of real options theory to mining investments, *Journal of Environmental Economics and Management*, vol. 41, No 2, pp 193-223, March 2001.

Smith V.H., Chouinard H.H. and Baquet A.E. (1994). "Almost Ideal Area Yield Crop Insurance Contracts". *Agricultural and Resource Economic Review*, 23:75-83.