A Speech Act Calculus
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Building on the work of Peter Hinst and Geo Siegwart, we develop a pragmatised natural deduction calculus, i.e. a natural deduction calculus that incorporates illocutionary operators at the formal level, and prove the equivalence between the consequence relation for the calculus and the classical model-theoretic consequence relation.
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**Introductory Remarks**

In this text\(^1\), we build on the works of Peter Hinst and Geo Siegwart on the pragmatisation of natural deduction calculi\(^2\) and develop a (classical) speech act calculus\(^3\) of natural deduction that has the following properties: (i) Every sentence sequence \(\tilde{s}\), which here means: every sequence of assumption- and inference-sentences, is not a derivation of a proposition (i.e. a closed formula) from a set of propositions or there is exactly one proposition \(\Gamma\) and exactly one set of propositions \(X\) such that \(\tilde{s}\) is a derivation of \(\Gamma\) from \(X\), this being determinable for every sentence sequence without recourse to any metatheoretical means of commentary.\(^4\) (ii) The classical first-order model-theoretic consequence relation is equivalent to the consequence relation for the calculus.

Developing the calculus, we presuppose the grammatical framework of pragmatised first-order languages, which has been developed by Peter Hinst and Geo Siegwart, and supplement it with some additional concepts (1). Then the concept of the availability of propositions is established: In contrast to the calculi developed by Hinst and Siegwart, the formulation of the speech act rules for this calculus does not take recourse to a de-

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\(^1\) This text is basically a translation of our German paper: Ein Redehandlungskalkül. Ein pragmatisierter Kalkül des natürlichen Schließens nebst Metatheorie. Version 2.0. Online available at [http://hal.archives-ouvertes.fr/hal-00532643/en/](http://hal.archives-ouvertes.fr/hal-00532643/en/).

\(^2\) Pragmatised natural deduction calculi are natural deduction calculi that incorporate illocutionary operators at the formal level: For each speech act governed by the calculus (i.e. making an assumption or drawing an inference) there is a specific type of illocutionary operator, called performator, whose application to a proposition yields a sentence (i.e. an assumption or an inference sentence). These performators and the sentences that result from their application to propositions are part of the language of the respective calculus and their use in speech acts is governed by the rules of the respective calculus. Pragmatised calculi thus allow for the formal treatment of the linguistic practice of uttering derivations. More generally, the framework of pragmatised languages developed by Hinst and Siegwart allows for a formal treatment of all kinds of speech acts and linguistic practices. See HINST, P.: *Pragmatische Regeln, Logischer Grundkurs, Logik*, and SIEGWART, G.: *Vorfragen, Denkwerkzeuge* and, in English and most recent, *Alethic Acts*.

\(^3\) Our use of the expression 'speech act calculus' (German: Redehandlungskalkül) to designate pragmatised natural deduction calculi follows Sebastian Paasch.

\(^4\) Note that we regulate the predicate ‘.. is a derivation of .. from ..’ in such a way that the set of propositions mentioned at the third place is identical to the set of assumptions which actually occur in the sentence sequence that is named at the first place and which are not eliminated in that sequence. If one regulates the predicate so that the set of propositions named at the third place has to be a superset of the set of assumptions that actually occur in the respective sentence sequence and are not eliminated there, which is not unusual either, the calculus accordingly ensures that every sentence sequence \(\tilde{s}\) is either not a derivation of a proposition from a set of propositions or that there is a proposition \(\Gamma\) and a set of propositions \(X\), such that for every proposition \(\Delta\) and set of propositions \(Y\) one has: \(\tilde{s}\) is a derivation of \(\Delta\) from \(Y\) iff \(\Delta = \Gamma\) and \(X \subseteq Y\).
pendence relation between sets of propositions and propositions, but to an availability relation between propositions, sequences of sentences and positions (natural numbers in the domain of sequences). The concept of availability is inspired by the idea that all propositions in a subproof except the conclusion of the subproof should not be available after the subproof has been closed, which is implemented, for example, in the KALISH-MONTAGUE calculus.\textsuperscript{5} Here, however, only subproofs that aim at conditional introduction (CdI), negation introduction (NI) or particular-quantifier elimination (PE), are treated in this way and the calculus is established in such a way that neither graphic means nor meta-theoretical commentaries have to be used: Which propositions are available in a given sentence sequence can be unambiguously determined without recourse to any kind of commentary (2).

Next the Speech Act Calculus is established. As is usual for pragmatised natural deduction calculi, the calculus contains a rule of assumption, which allows one to assume any proposition, and two rules for every logical operator, one regulating its introduction and the other one its elimination. Except for the rule of identity introduction (II), which allows the premise-free inference of self-identity propositions, the introduction and elimination rules always demand that suitable premises have already been gained, i.e. are available. So, for example, the rule of conditional elimination (CdE) allows one to infer $\Gamma$ if one has already gained $\Delta$ and $\Delta \rightarrow \Gamma$, i.e. if $\Delta$ and $\Delta \rightarrow \Gamma$ are available. Propositions are gained or made available by being inferred or assumed. One gains a proposition $\Gamma$ departing from an assumption if this assumption is the last one that has been made before gaining $\Gamma$ and that is still available.

Three of the rules, CdI, NI and PE, allow one to discharge assumptions one has made: If one has gained a proposition $\Gamma$ departing from the assumption of a proposition $\Delta$, then one may infer $\Delta \rightarrow \Gamma$ and thus discharge the assumption of $\Delta$ (CdI); if one has gained propositions $\Gamma$ and $\neg \Gamma$ departing from the assumption of a proposition $\Delta$, then one may infer $\neg \neg \Delta$ and thus discharge the assumption of $\Delta$ (NI), if a particular-quantification $\forall \xi \Delta$ is available and one has gained a proposition $\Gamma$ departing from the representative instance assumption $[\beta, \xi, \Delta]$, then one may infer $\Gamma$ and thus discharge the representative

instance assumption (PE). The discharge of the respective initial assumptions is achieved as each application of CdI, NI and PE closes the whole subproof beginning with the respective assumption. One consequence of this is that the respective initial assumptions are not any more available, but it also makes the intermediate conclusions drawn during the subproof unavailable as premises – these intermediate conclusions only served the purpose of preparing the application of the respective rule and have been gained under the respective assumption. If the assumption is not any more available, then neither should any propositions that one was only able to gain under this assumption be available. One may reflect on this using the example of the pair \( \Gamma \) and \( \neg \neg \Gamma \) that has to be gained to prepare the application of NI.

After the establishment of the calculus, a derivation and a consequence concept for the calculus are established. A sequence of sentences \( \mathcal{A} \) will then be a derivation of a proposition \( \Gamma \) from a set of propositions \( X \) if and only if \( \mathcal{A} \) can be uttered in compliance with the rules of the calculus, \( \Gamma \) is the proposition of the last member of \( \mathcal{A} \) and \( X \) is the set of the assumptions available in \( \mathcal{A} \). Accordingly, a proposition \( \Gamma \) will then be a deductive consequence of a set of propositions \( X \) if and only if there is a derivation of \( \Gamma \) from a \( Y \subseteq X \) (3).

The reflexivity, closure under introduction and elimination, transitivity as well as other properties of the deductive consequence relation have to be shown in order to prepare the proof of the adequacy of the then established concept of deductive consequence (4). Subsequently, a version of the classical model-theoretic consequence concept that fits the grammatical framework is established (5). Then the correctness and the completeness of the deductive consequence concept relative to this model-theoretic concept of consequence are shown (6). We conclude with some remarks on ways to elaborate on the approach taken here (7).

In the development of the calculus, we assume an established set or class-set theory, such as ZF or NBG(U). Since we do not want to restrict our meta-theory to a purely set-theoretical framework, we sometimes have to stipulate additional properties – such as, for example, \( X \in \{X\} \) – that are trivial within a pure set theory, but informative within a class-set-theory. The development and meta-theoretical analysis of the Speech Act Calcu-
lus employ common set-theoretical and meta-logical instruments and techniques, which are presented in the works listed in the references.

A note concerning the use of this document: All entries in the table of contents link to the respective chapters and are bookmarked. Moreover, all cross-references as well as all mentions of postulates, definitions, theorems and speech-act rules link to the respective item.

We would like to thank SEBASTIAN PAASCH for pointing out various problems which motivated the development of our calculus, for valuable hints and for his helpful criticism of an earlier version of this text. Also, we would like to thank GEO SIEGWART for valuable hints, patience and an open ear.
1 Grammatical Framework

The Speech Act Calculus and its meta-theory are developed for denumerable pragmatised first-order languages. To simplify the following presentation, we suppress any reference to specific languages, or, more precisely, we assume an arbitrary but fixed language of this kind with a denumerably infinite vocabulary, the language L. First, the vocabulary and syntax of L are to be specified (1.1). Then the substitution operation is to be developed and some theorems on substitution are to be proved (1.2).

1.1 Vocabulary and Syntax

L is supposed to be an arbitrary, but fixed representative of languages of the desired kind with a denumerably infinite non-logical vocabulary. However, the calculus also works for languages with finitely many descriptive constants. Since L is not an actually constructed language, it is now just stipulated that a suitable vocabulary and a suitable concatenation operation for expressions exist. Which vocabulary is chosen in particular cases or how it is constructed (and how it is set-theoretically modelled, e.g. with recourse to subsets of \( \mathbb{N} \) in NBG or ZF, or described, e.g. with recourse to axiomatically characterised (sets of) urelements in NBGU) is left open. The same holds for the concatenation operation for expressions: It is left open how this concatenation operation is established, e.g. with recourse to finite sequences or in some other way. The first postulate demands the existence of suitable sets of basic expressions for the vocabulary of L:

Postulate 1-1. The vocabulary of L (CONST, PAR, VAR, FUNC, PRED, CON, QUANT, PERF, AUX)

The following sets are well-defined, pairwise disjunct and do not have \( \emptyset \) as an element:

(i) The denumerably infinite set \( \text{CONST} = \{c_i \mid i \in \mathbb{N}\} \), where for all \( i, j \in \mathbb{N} \) with \( i \neq j \): \( c_i \neq c_j \) and \( c_i \in \{c_i\} \), (the set of individual constants; metavariables: \( \alpha, \alpha', \alpha^*, \ldots \)),

(ii) The denumerably infinite set \( \text{PAR} = \{x_i \mid i \in \mathbb{N}\} \), where for all \( i, j \in \mathbb{N} \) with \( i \neq j \): \( x_i \neq x_j \) and \( x_i \in \{x_i\} \), (the set of parameters; metavariables: \( \beta, \beta', \beta^*, \ldots \)),

(iii) The denumerably infinite set \( \text{VAR} = \{x_i \mid i \in \mathbb{N}\} \), where for all \( i, j \in \mathbb{N} \) with \( i \neq j \): \( x_i \neq x_j \) and \( x_i \in \{x_i\} \), (the set of variables; metavariables: \( \xi, \zeta, \omega, \xi', \zeta', \omega', \xi^*, \zeta^*, \omega^*, \ldots \)),

(iv) The denumerably infinite set \( \text{FUNC} = \{f_{i,j} \mid i \in \mathbb{N} \setminus \{0\} \text{ and } j \in \mathbb{N}\} \), where for all \( i, k \in \mathbb{N} \setminus \{0\} \) and \( j, l \in \mathbb{N} \) with \( (i, j) \neq (k, l) \): \( f_{i,j} \neq f_{k,l} \) and \( f_{i,j} \in \{f_{i,j}\} \), (the set of function con-

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6 See the literature mentioned in footnote 2. For a rigorous development of the grammatical framework see especially HINST, P.: Logik, ch. 1.
The denumerably infinite set \( \text{PRED} = \{=\} \cup \{P_{i,j} \mid i \in \mathbb{N}\setminus\{0\} \text{ and } j \in \mathbb{N}\} \), where \( \{=\} \subset \{P_{i,j} \mid i \in \mathbb{N}\setminus\{0\} \text{ and } j \in \mathbb{N}\} \) and for all \( i, k \in \mathbb{N}\setminus\{0\} \text{ and } j, l \in \mathbb{N} \) with \((i, j) \neq (k, l)\):
\[ P_{i,j} \neq P_{k,l} \text{ and } P_{i,j} \in \{P_{i,j}\}, \]

The 5-element set \( \text{CON} = \{\neg, \rightarrow, \leftrightarrow, \land, \lor\} \) (the set of connectives; metavariables: \( \psi, \psi', \psi^*, \ldots \)),

The 2-element set \( \text{QUANT} = \{\exists, \forall\} \) (the set of quantificators; metavariables: \( \Pi, \Pi', \Pi^*, \ldots \)),

The 2-element set \( \text{PERF} = \{\text{Suppose, Therefore}\} \) (the set of performators; metavariables: \( \Xi, \Xi', \Xi^*, \ldots \)), and

The 3-element set \( \text{AUX} = \{\{\} \cup \{\} \cup \{}\} \) (the set of auxiliary symbols).

The meta-theoretical expressions by which the elements of the sets PERF and AUX are designated will also be used as meta-theoretical performators and auxiliary symbols, the same holds for the identity predicate. To avoid confusion and to enhance intuitive readability, we will therefore use quasi-quotation marks ("\( ^\prime \), "\( ^\ast \)) if object-language expressions are to be designated. \( \mu, \tau, \mu', \tau', \mu^*, \tau^*, \ldots \) serve as general metavariables for object-language expressions. The vocabulary of \( L \) is now simply defined as the set of the sets postulated in Postulate 1-1:

**Definition 1-1.** The vocabulary of \( L \) (VOC)

\[ \text{VOC} = \{\text{CONST, PAR, VAR, FUNC, PRED, CON, QUANT, PERF, AUX}\}. \]

The syntax of \( L \) contains the categories of terms, quantifiers, formulas and sentences according to the definitions found below. First, however, the set of basic expressions is established:

**Definition 1-2.** The set of basic expressions (BEXP)

\[ \text{BEXP} = \text{UVOC}. \]

Now, we demand the existence of a suitable operation with which we can concatenate expressions to form larger expressions. As already remarked above, the way in which this operation is constructed in particular cases is left open. To do this, we first regulate the concatenation of basic expressions, and then, after defining the set of expressions and the expression length function, we regulate the general concatenation of arbitrary expressions.
Postulate 1-2. Concatenation of basic expressions\(^7\)

The concatenation of expressions expressed by juxtaposition is well-defined and it holds that:

(i) For all \(k, j \in \mathbb{N}\setminus\{0\}\): If \(\{\mu_0, \ldots, \mu_{k-1}\} \subseteq \text{BEXP}\) and \(\{\mu'_0, \ldots, \mu'_{j-1}\} \subseteq \text{BEXP}\), then: \(\mu_0 \ldots \mu_{k-1} = \mu'_0 \ldots \mu'_{j-1}\) iff \(j = k\) and for all \(i < k\): \(\mu_i = \mu'_i\),

(ii) If \(\mu \in \text{BEXP}\), then there is no \(k \in \mathbb{N}\setminus\{0, 1\}\) such that \(\{\mu_0, \ldots, \mu_{k-1}\} \subseteq \text{BEXP}\) and \(\mu = \mu_0 \ldots \mu_{k-1}\), and

(iii) For all \(k \in \mathbb{N}\setminus\{0\}\): If \(\{\mu_0, \ldots, \mu_{k-1}\} \subseteq \text{BEXP}\), then \(\mu_0 \ldots \mu_{k-1} \neq \emptyset\) and \(\mu_0 \ldots \mu_{k-1} \in \{\mu_0 \ldots \mu_{k-1}\}\).

The expression of the concatenation operation by juxtaposition already presupposes the associativity of the concatenation operation. This property can thus be regarded as implicitly stipulated. Now, the set of all expressions, i.e. all concatenations of basic expressions, will be defined. This set will be a superset of all grammatical categories that are to be defined. Then a function that assigns each expression its length will be defined:

**Definition 1-3. The set of expressions (\(\text{EXP};\) metavariables: \(\mu, \tau, \mu', \tau', \mu^*, \tau^*, \ldots\))**

\[
\text{EXP} = \{\mu_0 \ldots \mu_{k-1} \mid k \in \mathbb{N}\setminus\{0\} \text{ and } \{\mu_0, \ldots, \mu_{k-1}\} \subseteq \text{BEXP}\}.
\]

**Definition 1-4. Length of an expression (\(\text{EXPL}\))**

\[
\text{EXPL} = \{(\mu, k) \mid \mu \in \text{EXP}, k \in \mathbb{N}\setminus\{0\} \text{ and there is } \{\mu_0, \ldots, \mu_{k-1}\} \subseteq \text{BEXP} \text{ with } \mu = \mu_0 \ldots \mu_{k-1}\}.
\]

**Theorem 1-1. \(\text{EXPL}\) is a function on \(\text{EXP}\)**

(i) \(\text{Dom}(\text{EXPL}) = \text{EXP}\) and

(ii) For all \(\mu \in \text{EXP}, k, l \in \mathbb{N}\): If \((\mu, k), (\mu, l) \in \text{EXPL}\), then \(k = l\).

**Proof:** (i) follows directly from Definition 1-3 and Definition 1-4. *Ad (ii):* Let \(\mu \in \text{EXP}, k, l \in \mathbb{N}\) and \((\mu, k), (\mu, l) \in \text{EXPL}. Then there is \(\{\mu_0, \ldots, \mu_{k-1}\} \subseteq \text{BEXP}\) with \(\mu = \mu_0 \ldots \mu_{k-1}\) and there is \(\{\mu'_0, \ldots, \mu'_{l-1}\} \subseteq \text{BEXP}\) with \(\mu = \mu'_0 \ldots \mu'_{l-1}\). According to Postulate 1-2-(i), it then holds that \(k = l\). ■

---

\(^7\) Here and in the following, we assume: If \(k \in \mathbb{N}\setminus\{0\}\) and \(\{a_0, \ldots, a_{k-1}\} \subseteq X\), where \(X \in \{X\}\), then for all \(i < k\): \(a_i \in \{a_0, \ldots, a_{k-1}\}\).
Theorem 1-2. Expressions are concatenations of basic expressions

If \( \mu \in \text{EXP} \), then there is \( \{ \mu_0, \ldots, \mu_{\text{EXP}(\mu)-1} \} \subseteq \text{BEXP} \) such that \( \mu = \mu_0 \cdots \mu_{\text{EXP}(\mu)-1} \).

Proof: Follows directly from Definition 1-3 and Definition 1-4. \( \blacksquare \)

Theorem 1-3. Identification of concatenation members

If \( k \in \mathbb{N}\setminus\{0\} \) and for all \( i < k \): \( \mu_i \in \text{EXP} \), then for all \( s < \sum_{j=0}^{k-1} \text{EXPL}(\mu_j) \):

(i) \( s < \text{EXPL}(\mu_0) \)

or

(ii) \( \text{EXPL}(\mu_0) \leq s \) and there are \( l, r \) such that

a) \( 0 < l < k \) and \( r < \text{EXPL}(\mu_l) \) and \( s = (\sum_{n=0}^{l-1} \text{EXPL}(\mu_n)) + r \), and

b) For all \( l', r' \): If \( 0 < l' < k \) and \( r' < \text{EXPL}(\mu_l) \) and \( s = (\sum_{n=0}^{l-1} \text{EXPL}(\mu_n)) + r' \), then \( l' = l \) and \( r' = r \).

Proof: Suppose \( k \in \mathbb{N}\setminus\{0\} \) and that for all \( i < k \): \( \mu_i \in \text{EXP} \). Now, suppose \( s < \sum_{j=0}^{k-1} \text{EXPL}(\mu_j) \). We have that \( s < \text{EXPL}(\mu_0) \) or \( \text{EXPL}(\mu_0) \leq s \). In the first case, the theorem holds. Now, suppose \( \text{EXPL}(\mu_0) \leq s \). Then we have that \( 1 < k \), because otherwise we would have \( 1 = k \) and thus \( \text{EXPL}(\mu_0) = \sum_{j=0}^{l-1} \text{EXPL}(\mu_j) > s \). Thus, there is at least one \( i \), namely 1, such that \( 0 < i < k \) and \( \sum_{n=0}^{i-1} \text{EXPL}(\mu_n) \leq s \). Now, let \( l = \max(\{ i \mid 0 < i < k \) and \( \sum_{n=0}^{i-1} \text{EXPL}(\mu_n) \leq s \}) \). Then we have \( 0 < l < k \) and \( \sum_{n=0}^{l-1} \text{EXPL}(\mu_n) \leq s \). Then there is an \( r \) such that \( (\sum_{n=0}^{l-1} \text{EXPL}(\mu_n)) + r = s \). Suppose for contradiction that \( \text{EXPL}(\mu_l) \leq r \). We have that \( l < k-1 \) or \( l = k-1 \). Suppose \( l < k-1 \). Then we have \( l+1 < k \). Then we would have \( \sum_{n=0}^{l} \text{EXPL}(\mu_n) = (\sum_{n=0}^{l-1} \text{EXPL}(\mu_n)) + \text{EXPL}(\mu_l) \leq (\sum_{n=0}^{l-1} \text{EXPL}(\mu_n)) + r = s \), which contradicts the maximality of \( l \). Suppose \( l = k-1 \). Then we would have \( l-1 = k-2 \). Thus we would have \( \sum_{n=0}^{l-1} \text{EXPL}(\mu_n) = (\sum_{n=0}^{l-2} \text{EXPL}(\mu_n)) + \text{EXPL}(\mu_{l-1}) \leq (\sum_{n=0}^{l-2} \text{EXPL}(\mu_n)) + r = s \), which contradicts the assumption about \( s \). Thus, the assumption that \( \text{EXPL}(\mu_l) \leq r \) leads to a contradiction in both cases. Therefore we have \( r < \text{EXPL}(\mu_l) \). Hence we have \( 0 < l < k \) and \( r < \text{EXPL}(\mu_l) \) and \( s = (\sum_{n=0}^{l-1} \text{EXPL}(\mu_n)) + r \).

Now, we still have to show b), i.e. that \( l \) and \( r \) are uniquely determined. For this, suppose \( 0 < l' < k \) and \( r' < \text{EXPL}(\mu_l) \) and \( s = (\sum_{n=0}^{l'-1} \text{EXPL}(\mu_n)) + r' \). Then it holds that \( \sum_{n=0}^{l'-1} \text{EXPL}(\mu_n) \leq s \). From the maximality of \( l \), it then follows that \( l' \leq l \). Now, suppose for contradiction that \( l' < l \). Then we would have \( l' \leq l-1 \). Thus we would have \( (\sum_{n=0}^{l'-1} \text{EXPL}(\mu_n)) + \text{EXPL}(\mu_l) = \sum_{n=0}^{l'-1} \text{EXPL}(\mu_n) \leq \sum_{n=0}^{l-1} \text{EXPL}(\mu_n) \leq s = \sum_{n=0}^{l'-1} \text{EXPL}(\mu_n) + r' \).

\( \blacksquare \)
Postulate 1-3. Concatenation of expressions
If \( k \in \mathbb{N} \setminus \{0\} \) and if for all \( i < k \): \( \mu_i \in \text{EXP} \) and \( \mu_i = \tau_{\mu_i^0 \ldots \mu_i^{\text{EXPL}(\mu_i)-1}} \), where \( \{\mu_i^0, \ldots, \mu_i^{\text{EXPL}(\mu_i)-1}\} \subseteq \text{BEXP} \), then there are \( m \in \mathbb{N} \setminus \{0\} \) and \( \{\mu_s^0, \ldots, \mu_s^{m-1}\} \subseteq \text{BEXP} \) such that for all \( i < k \):

\[
\tau_{\mu_0 \ldots \mu_{k-1}} = \tau_{\mu_0 \ldots \mu_s^0 \ldots \mu_{k-1}} \mu_{i+1} \ldots \mu_{k-1},
\]

where

\[
\begin{align*}
a) & \quad m = \sum_{j=0}^{k-1} \text{EXPL}(\mu_j), \text{ and} \\
b) & \quad \text{For all } s < m: \\
& \quad \mu_s^* = \mu_s^0, \text{ if } s < \text{EXPL}(\mu_0), \text{ and} \\
& \quad \mu_s^* = \mu_s^r \text{ for the uniquely determined } l, r \text{ for which } 0 < l < k \text{ and } r < \text{EXPL}(\mu_l) \text{ and } s = (\sum_{n=0}^{l-1} \text{EXPL}(\mu_n)) + r, \text{ if } \text{EXPL}(\mu_0) \leq s.
\end{align*}
\]

As an immediate consequence of Postulate 1-3, we have that every concatenation of expressions is identical to a concatenation of basic expressions and thus itself an expression.

Now, we will prove some general theorems on expressions and their concatenations (Theorem 1-4 to Theorem 1-8). Then, we will define the arity of operators and subsequently the categories of terms, quantifiers and formulas.

**Theorem 1-4. On the identity of concatenations of expressions (a)**

If \( k \in \mathbb{N} \setminus \{0\} \), for all \( i < k \): \( \mu_i \in \text{EXP} \) and \( \mu_i = \tau_{\mu_i^0 \ldots \mu_i^{\text{EXPL}(\mu_i)-1}} \), where \( \{\mu_i^0, \ldots, \mu_i^{\text{EXPL}(\mu_i)-1}\} \subseteq \text{BEXP} \), then:

\[
\begin{align*}
(i) \quad & \tau_{\mu_0 \ldots \mu_{k-1}} = \tau_{\mu_0^0 \ldots \mu_{k-1}^0 \ldots \mu_0^{\text{EXPL}(\mu_0)-1} \ldots \mu_{k-1}^{\text{EXPL}(\mu_{k-1})-1}}, \\
(ii) \quad & \text{EXPL}(\tau_{\mu_0 \ldots \mu_{k-1}}) = \sum_{j=0}^{k-1} \text{EXPL}(\mu_j), \text{ and}
\end{align*}
\]
(iii) If \( m \in \mathbb{N}\setminus\{0\} \) and \( \{\mu_0', \ldots, \mu_{m-1}'\} \subseteq \text{BEXP} \), then:

\[
\begin{align*}
\prod_{i=0}^{m-1} \mu_i^{\text{EXPL}(\mu_i)-1} &= \\
\text{iff} \quad m &= \sum_{i=0}^{k-1} \text{EXPL}(\mu_i) + r, \text{ if } \text{EXPL}(\mu_0) \leq s.
\end{align*}
\]

Proof: Suppose \( k \in \mathbb{N}\setminus\{0\} \), for all \( i < k \): \( \mu_i \in \text{EXP} \) and \( \mu_i = \prod_{j=0}^{i-1} \text{EXPL}(\mu_j) \), where \( \{\mu_0', \ldots, \mu_{i-1}'\} \subseteq \text{BEXP} \).

Ad (i): First, we show, by induction on \( i \), that for all \( i < k \):

\[
\begin{align*}
\prod_{i=0}^{k-1} \mu_i &= \\
\text{iff} \quad m &= \sum_{i=0}^{k-1} \text{EXPL}(\mu_i) + r.
\end{align*}
\]

Then, this statement also holds for \( i = k-1 \), and thus we have (i). Now, suppose the statement holds for all \( l < i \). Suppose \( i < k \). Then we have that \( i = 0 \) or \( 0 < i \). Suppose \( i = 0 \). Because of \( \mu_0 = \prod_{j=0}^{0} \text{EXPL}(\mu_j) \), we then have, with Postulate 1-3:

\[
\begin{align*}
\prod_{i=0}^{k-1} \mu_i &= \\
\text{iff} \quad m &= \sum_{i=0}^{k-1} \text{EXPL}(\mu_i) + r.
\end{align*}
\]

Now, suppose \( 0 < i \). Then it holds for all \( l < i \) that \( l < k \) and thus, according to the I.H., that

\[
\begin{align*}
\prod_{i=0}^{k-1} \mu_i &= \\
\text{iff} \quad m &= \sum_{i=0}^{k-1} \text{EXPL}(\mu_i) + r.
\end{align*}
\]

Since \( i-1 < i \), we thus have

\[
\begin{align*}
\prod_{i=0}^{k-1} \mu_i &= \\
\text{iff} \quad m &= \sum_{i=0}^{k-1} \text{EXPL}(\mu_i) + r.
\end{align*}
\]

Because of \( \mu_i = \prod_{j=0}^{i-1} \text{EXPL}(\mu_j) \), we then have, with Postulate 1-3:
Hence we have

\[
\begin{align*}
\mu_0 \cdots \mu_{k-1} &= \\
\mu_0 \cdots \mu_{k-1} &< \mu_0 \cdots \mu_{m-1}.
\end{align*}
\]

**Ad (ii) and (iii):** With Postulate 1-3, there are \( m^* \in \mathbb{N}\setminus\{0\} \) and \( \{\mu_0^*, \ldots, \mu_{m^*-1}^*\} \subseteq \text{BEXP} \) such that \( \mu_0 \cdots \mu_{k-1}^* = \mu_0^* \cdots \mu_{m^*-1}^* \) and \( m^* = \sum_{j=0}^{k-1} \text{EXPL}(\mu_j) \) and for all \( s < m^*: \mu_s^* = \mu_s^0 \) if \( s < \text{EXPL}(\mu_0) \), and \( \mu_s^* = \mu_s^v \) for the uniquely determined \( l, r \) for which \( 0 < l < k, r < \text{EXPL}(\mu_l) \) and \( s = \left( \sum_{j=0}^{l-1} \text{EXPL}(\mu_j) \right) + r \), if \( \text{EXPL}(\mu_0) \leq s \). Then we have \( \sum_{j=0}^{k-1} \text{EXPL}(\mu_j) = m^* = \text{EXPL}(\mu_0^* \cdots \mu_{m^*-1}^*) = \text{EXPL}(\mu_0 \cdots \mu_{k-1}^*) \). Thus we have (ii).

Now, for (iii), suppose \( m \in \mathbb{N}\setminus\{0\} \) and \( \{\mu_0^0, \ldots, \mu_{m-1}^0\} \subseteq \text{BEXP} \). (L-R): Suppose \( \mu_0^0 \cdots \mu_1^0 \cdots \mu_{k-1}^0 = \mu_0^0 \cdots \mu_{m-1}^0 \) and \( m^* = \sum_{j=0}^{k-1} \text{EXPL}(\mu_j) \) and for all \( s < m: \mu_s^0 = \mu_s^0 \) if \( s < \text{EXPL}(\mu_0) \), and \( \mu_s^0 = \mu_s^v \) for the uniquely determined \( l, r \) for which \( 0 < l < k, r < \text{EXPL}(\mu_l) \) and \( s = \left( \sum_{j=0}^{l-1} \text{EXPL}(\mu_j) \right) + r \), if \( \text{EXPL}(\mu_0) \leq s \).

(R-L): Suppose \( m = \sum_{j=0}^{k-1} \text{EXPL}(\mu_j) \) and that it hold for all \( s < m \) that \( \mu_s^0 = \mu_s^0 \) if \( s < \text{EXPL}(\mu_0) \), and \( \mu_s^0 = \mu_s^v \) for the uniquely determined \( l, r \) for which \( 0 < l < k, r < \text{EXPL}(\mu_l) \) and \( s = \left( \sum_{j=0}^{l-1} \text{EXPL}(\mu_j) \right) + r \), if \( \text{EXPL}(\mu_0) \leq s \). Then it holds that \( m^* = m \) and that for all \( s < m: \mu_s^* = \mu_s^* \). With Postulate 1-2(i), we then have \( \mu_0^0 \cdots \mu_{m-1}^0 = \mu_0^* \cdots \mu_{m-1}^* \). With (i), we then have \( \mu_0^0 \cdots \mu_{k-1}^0 = \mu_0^0 \cdots \mu_{k-1}^\prime = \mu_0^* \cdots \mu_{m-1}^* = \mu_0^0 \cdots \mu_{m-1}^0 \).

**Theorem 1-5.** *On the identity of concatenations of expressions (b)***

If \( k, k' \in \mathbb{N}\setminus\{0\} \) and for all \( i < k: \mu_i \in \text{EXP} \) and \( \mu_i = \mu_i^0 \cdots \mu_i^{\text{EXPL}(\mu_i)-1} \), where \( \{\mu_0^0, \ldots, \mu_k^{\text{EXPL}(\mu_k)-1}\} \subseteq \text{BEXP} \), and for all \( i < k': \mu'_i \in \text{EXP} \) and \( \mu'_i = \mu_i' \cdots \mu_i'^{\text{EXPL}(\mu_i')-1} \), where \( \{\mu_0'^0, \ldots, \mu_k'^{\text{EXPL}(\mu_k')-1}\} \subseteq \text{BEXP} \), and if \( \mu_0^0 \cdots \mu_{k-1}^0 = \mu_0' \cdots \mu_{k'-1}' \), then:

(i) \( \mu_0 \cdots \mu_{k-1}^0 = \mu_0' \cdots \mu_{k'-1}' \)

\[
\begin{align*}
\mu_0^0 \cdots \mu_{k-1}^0 &= \mu_0'^0 \cdots \mu_{k'-1}' \\
\mu_0^0 \cdots \mu_{k-1}^0 &= \mu_0'^0 \cdots \mu_{k'-1}' \text{EXPL}(\mu_{k-1})^{-1} \\
\mu_0^0 \cdots \mu_{k-1}^0 &= \mu_0'^0 \cdots \mu_{k'-1}' \text{EXPL}(\mu_{k'-1})^{-1}.
\end{align*}
\]
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\[ (\mu_0 \ldots \mu_{k-1}) \]

(ii) \( \text{EXPL}(\mu_0 \ldots \mu_{k-1}) = \sum_{j=0}^{k-1} \text{EXPL}(\mu_j) = \sum_{j=0}^{k-1} \text{EXPL}(\mu_j^*) = \text{EXPL}(\mu_0^* \ldots \mu_{k-1}^*) \), and

(iii) For all \( i < k, k' \): If \( \text{EXPL}(\mu_i) = \text{EXPL}(\mu_i^*) \) for all \( j \leq i \), then:

\[
\begin{align*}
\text{a) } & \quad \gamma_{\mu_0 \ldots \mu_i^*} = \\
& = \gamma_{\mu_0^* \ldots \mu_i^*} = \\
& = \gamma_{\mu_0^* \ldots \mu_i^*} = \\
& = \gamma_{\mu_0^* \ldots \mu_i^*}
\end{align*}
\]

\[
\begin{align*}
\text{b) } & \quad \text{For all } j \leq i: \mu_j = \mu_j^*.
\end{align*}
\]

Proof: Suppose \( k, k' \in \mathbb{N} \setminus \{0\} \) and for all \( i < k \): \( \mu_i \in \text{EXP} \) and \( \mu_i^* = \gamma_{\mu_0^* \ldots \mu_i^*} \), where \( \{ \mu_0^*, \ldots, \mu_{k-1}^* \} \subseteq \text{BEXP} \), and for all \( i < k' \): \( \mu_i \in \text{EXP} \) and \( \mu_i^* = \gamma_{\mu_0^* \ldots \mu_i^*} \), where \( \{ \mu_0^*, \ldots, \mu_{k-1}^* \} \subseteq \text{BEXP} \), and suppose \( \gamma_{\mu_0 \ldots \mu_{k-1}^*} = \gamma_{\mu_0^* \ldots \mu_{k-1}^*} \). Then clauses (i) and (ii) follow with Theorem 1-4-(i) and -(ii).

Now, for (iii), suppose \( i < k, k' \) and suppose \( \text{EXPL}(\mu_i) = \text{EXPL}(\mu_i^*) \) for all \( j \leq i \). First, with Postulate 1-3, we have that there are \( m^* \in \mathbb{N} \setminus \{0\} \) and \( \{ \mu_0^*, \ldots, \mu_{m-1}^* \} \subseteq \text{BEXP} \) such that \( \gamma_{\mu_0 \ldots \mu_{m-1}^*} = \gamma_{\mu_0^* \ldots \mu_{m-1}^*} \) and \( m = \sum_{n=0}^{k-1} \text{EXPL}(\mu_n) \) and for all \( s < m \): \( \mu_s = \mu_s^* \), if \( s < \text{EXPL}(\mu_0) \), and \( \mu_s^* = \mu_s^* \), for the uniquely determined \( l, r \) for which \( 0 < l < k, r < \text{EXPL}(\mu_i) \) and \( s = (\sum_{n=0}^{l-1} \text{EXPL}(\mu_n)) + r \), if \( \text{EXPL}(\mu_0) \leq s \); and that there are \( m' \in \mathbb{N} \setminus \{0\} \) and \( \{ \mu_0^*, \ldots, \mu_{m'-1}^* \} \subseteq \text{BEXP} \) such that \( \gamma_{\mu_0^* \ldots \mu_{k-1}^*} = \gamma_{\mu_0^* \ldots \mu_{m'-1}^*} \) and \( m' = \sum_{n=0}^{k'-1} \text{EXPL}(\mu_n) \) and for all \( s < m' \): \( \mu_s = \mu_s^* \), if \( s < \text{EXPL}(\mu_0) \), and \( \mu_s^* = \mu_s^* \), for the uniquely determined \( l', r' \) for which \( 0 < l' < k', r' < \text{EXPL}(\mu_i) \) and \( s = (\sum_{n=0}^{l'-1} \text{EXPL}(\mu_n)) + r' \), if \( \text{EXPL}(\mu_0) \leq s \). With (ii), we then have \( m = m' \). Furthermore, we have, with (i):

\[
\begin{align*}
\gamma_{\mu_0^* \ldots \mu_{m'-1}^*} = \\
& = \gamma_{\mu_0^* \ldots \mu_{m'-1}^*} = \\
& = \gamma_{\mu_0^* \ldots \mu_{m'-1}^*} = \\
& = \gamma_{\mu_0^* \ldots \mu_{m'-1}^*}.
\end{align*}
\]
With Postulate 1-2-(i), we then have for all \( s < m = m' \): \( \mu^*_s = \mu^{i*}_s \). We have that \( i = 0 \) or \( 0 < i \). First, suppose \( i = 0 \). By hypothesis, we have \( \text{EXPL}(\mu_0) = \text{EXPL}(\mu_0') \). Now, suppose \( s < \text{EXPL}(\mu_0) \). Then we have \( s < \text{EXPL}(\mu_0') \) and \( s < m = m' \). Then we have \( \mu^*_s = \mu^{i0}_s \) and \( \mu^*_s = \mu^{i0'}_s \). Then we have \( \mu^{i0}_s = \mu^{i0'}_s \). Thus we have for all \( s < \text{EXPL}(\mu_0) = \text{EXPL}(\mu_0') \) that \( \mu^{i0}_s = \mu^{i0'}_s \). Thus we have, with Postulate 1-2-(i), that \( \mu_0 = \mu^{i0}_0 \ldots \mu^{i0}_{\text{EXPL}(\mu_0)-1} = \mu^{i0'}_0 \ldots \mu^{i0'}_{\text{EXPL}(\mu_0)-1} = \mu_0' \). Thus a) holds for \( i = 0 \). Also, if \( i = 0 \), we have for all \( j \leq i \) that \( j = i = 0 \) and thus b) holds as well for \( i = 0 \).

Now, suppose \( 0 < i \). By hypothesis, we have \( \text{EXPL}(\mu_j) = \text{EXPL}(\mu'_j) \) for all \( j \leq i \). From this, we get: \( \sum_{i=0}^{n-1} \text{EXPL}(\mu_n) = \sum_{i=0}^{j} \text{EXPL}(\mu'_n) \). With Postulate 1-3, we have that there are \( t \in \mathbb{N} \setminus \{0\} \) and \( \{\mu^+_0, \ldots, \mu^+_t, \} \subseteq \text{BEXP} \) such that \( \gamma \mu_0 \ldots \mu^+_t \gamma = \gamma \mu^+_0 \ldots \mu^+_t \) and \( t = \sum_{i=0}^{j} \text{EXPL}(\mu_i) \) and for all \( s < t \): \( \mu^+_s = \mu^{i0}_s \) if \( s \leq \text{EXPL}(\mu_i) \), and \( \mu^+_s = \mu^{i0'}_s \) for the uniquely determined \( l^o \), \( r^o \) for which \( 0 < l^o < i+1 \), \( r^o < \text{EXPL}(\mu_j) \) and \( s = (\sum_{n=0}^{t-1} \text{EXPL}(\mu_n)) + r^o \), if \( \text{EXPL}(\mu_i) \leq s \). Then we have \( t = \sum_{i=0}^{n-1} \text{EXPL}(\mu_n) = \sum_{i=0}^{j} \text{EXPL}(\mu'_n) = t' \). Because of \( \sum_{i=0}^{n-1} \text{EXPL}(\mu_n) \leq \sum_{i=0}^{n-1} \text{EXPL}(\mu'_n) \), we also have \( t \leq m = m' \).

Now, suppose \( s < t \). Then we have \( s < t' \) and \( s < m = m' \). We have that \( s < \text{EXPL}(\mu_0) \) or \( \text{EXPL}(\mu_0) \leq s \). Suppose \( s < \text{EXPL}(\mu_0) \). Since \( 0 < i \), we have, by hypothesis, that \( \text{EXPL}(\mu_0) = \text{EXPL}(\mu'_0) \), and thus also that \( s < \text{EXPL}(\mu'_0) \). Then we have \( \mu^*_s = \mu^{i0'}_s = \mu^*_s \) und \( \mu^*_s = \mu^{i0'}_s = \mu^*_s \). Because of \( \mu^*_s = \mu^*_s \), we thus have \( \mu^*_s = \mu^*_s \).

Now, suppose \( \text{EXPL}(\mu_0) = \text{EXPL}(\mu'_0) \leq s \). Then it holds that

\[
\begin{align*}
\mu^*_s &= \mu^{i0'}_s \text{ for the uniquely determined } l, r \text{ for which } 0 < l < k, r < \text{EXPL}(\mu_i) \text{ and } s = (\sum_{n=0}^{t-1} \text{EXPL}(\mu_n)) + r + \gamma \\
\text{and } \mu^*_{s'} &= \mu^{i0'}_{s'} \text{ for the uniquely determined } l', r' \text{ for which } 0 < l' < k', r' < \text{EXPL}(\mu_i') \text{ and } s = (\sum_{n=0}^{t-1} \text{EXPL}(\mu_n')) + r' + \gamma \\
\text{and } \mu^*_{s} &= \mu^{i0'}_{s} \text{ for the uniquely determined } l^o, r^o \text{ for which } 0 < l^o < i+1, r^o < \text{EXPL}(\mu_j) \text{ and } s = (\sum_{n=0}^{t-1} \text{EXPL}(\mu_n)) + r^o + \gamma \\
\text{and } \mu^*_{s} &= \mu^{i0'}_{s} \text{ for the uniquely determined } l^o, r^o \text{ for which } 0 < l^o < i+1, r^o < \text{EXPL}(\mu_j') \text{ and } s = (\sum_{n=0}^{t-1} \text{EXPL}(\mu_n')) + r^o + \gamma.
\end{align*}
\]
With \( l^0, l^0 < i+1 \), we then have \( l^0, l^0 \leq i \). By hypothesis, we thus have that \( \text{EXPL}(\mu^r) = \sum_{n=0}^{r-1} \text{EXPL}(\mu_n) \). Then we have \( 0 < l^0 < i+1 \) and \( r^0 < \text{EXPL}(\mu^r) \) and \( s = (\sum_{n=0}^{r-1} \text{EXPL}(\mu_n)) + r^0 \). By Theorem 1-3, we then have \( l^0 = l^0 \) und \( r^0 = r^0 \). Now, suppose for contradiction that \( i+1 \leq l \). Then we would have \( i \leq l \). But then we would have \( t = \sum_{n=0}^{l-1} \text{EXPL}(\mu_n) \leq \sum_{n=0}^{l-1} \text{EXPL}(\mu_n) \leq s \). Contradiction! Thus we have \( l < i+1 \). From this, we get \( l = l^0 \) und \( r = r^0 \). In the same way, we get \( r = l^0 \) and \( r^0 = r^0 \). Thus we have \( l = l^0 = l^0 \) und \( r = r^0 = r^0 \). With this, we have \( \mu^s = \mu^r = \mu^s \) and \( \mu^s = \mu^r = \mu^s \). Since \( \mu^s = \mu^s \), we thus have \( \mu^s = \mu^s \). Thus it holds for all \( s < t = l^0 \) that \( \mu^s = \mu^s \) and thus, with Postulate 1-2(i), that \( \mu^s = \mu^s \) and \( \mu^s = \mu^s \). Moreover, we have, with Theorem 1-4-(i), that \( \mu^s = \mu^s \) and \( \mu^s = \mu^s \). Hence a) also holds for \( 0 < i \).

Now, suppose, for b), that \( j < i \). For \( j = 0 \), we have already shown above that \( \mu_j = \mu_j \).

Suppose \( 0 < j \leq i \). Now, suppose \( r < \text{EXPL}(\mu) = \text{EXPL}(\mu^r) \). Then we have \( (\sum_{n=0}^{j-1} \text{EXPL}(\mu_n)) + r = \sum_{n=0}^{j-1} \text{EXPL}(\mu_n) + r < t = t^0 \). With \( s = (\sum_{n=0}^{j-1} \text{EXPL}(\mu_n)) + r \), then it holds that \( \mu^s = \mu^s \) and \( \mu^s = \mu^s \). Since \( s < t = t^0 \), we then have, as we have just shown, that \( \mu^s = \mu^s \) and thus that \( \mu^s = \mu^s \). Thus it holds for all \( r < \text{EXPL}(\mu) = \text{EXPL}(\mu^r) \) that \( \mu^r = \mu^r \). Then it holds, with Postulate 1-2-(i), that \( \mu_j = \mu^r \) and \( \mu_j = \mu^r \). Hence b) also holds for \( 0 < i \). ■

**Theorem 1-6. On the identity of concatenations of expressions (c)***

If \( k, s \in \mathbb{N} \setminus \{0\} \) and \( \{\mu_0, ..., \mu_{k-1}\} \subseteq \text{EXP} \) and \( \{\mu_0, ..., \mu_{s-1}\} \subseteq \text{EXP} \) and \( j < k \) and \( \mu_j = \mu^r \) and \( \mu_j = \mu^r \), then \( \mu_0...\mu_{k-1} = \mu_0...\mu_{j-1}\mu_{j+1}...\mu_{k-1} \).

**Proof:** Suppose \( k, s \in \mathbb{N} \setminus \{0\} \) and \( \{\mu_0, ..., \mu_{k-1}\} \subseteq \text{EXP} \) and \( \{\mu_0, ..., \mu_{s-1}\} \subseteq \text{EXP} \) and \( j < k \) and \( \mu_j = \mu^r \) and \( \mu_j = \mu^r \). With \( \{\mu_0, ..., \mu_{s-1}\} \subseteq \text{EXP} \) and Theorem 1-2, it then holds for all \( i < s \) that there is \( \{\mu^r, ..., \mu^r\} \subseteq \text{EXP} \) such that \( \mu^r = \mu^r \) and \( \mu^r = \mu^r \).

With Theorem 1-4-(i), we then have \( \mu_j = \mu^r \) and \( \mu_j = \mu^r \) and \( \mu_j = \mu^r \) and \( \mu_j = \mu^r \). With Postulate 1-3, we then have \( \mu_0...\mu_{s-1} = \mu^r \) and \( \mu_0...\mu_{s-1} = \mu^r \). Now, we first show by induction on \( i \) that for all \( i < s \):

\[
\mu_0...\mu_j\mu^r...\mu^r...\mu^r...\mu^r...\mu^r...\mu^r
\]
Then we have that
\[ m_{0...i} \in [0, s - 1] \]
Hence the statement holds for all \( i \). Since
\[ 0 < \theta \]
we then have, with Postulate 1-3:
\[ m_{0...i} = \mu_{0...i} \exp(\theta) \]
Then the theorem holds. Now, suppose the statement holds for all \( l < i \). Then, this also holds for \( i = s - 1 \) and thus we get
\[ m_{0...s-1} = \mu_{0...s-1} \exp(\theta) = \mu_{0...s-1} \exp(\theta) = \mu_{0...s-1} \exp(\theta) \]

Then the theorem holds. Now, suppose the statement holds for all \( l < i \). Suppose \( i < s \). Then we have that \( i = 0 \) or \( 0 < i \). Suppose \( i = 0 \). Because of \( m_0 = \mu_{0...0} \exp(\theta) \), we then have, with Postulate 1-3:
\[ m_{0...0} = \mu_{0...0} \exp(\theta) = \mu_{0...0} \exp(\theta) = \mu_{0...0} \exp(\theta) \]

Now, suppose \( 0 < i \). Then it holds for all \( l < i \) that \( l < s \) and thus, according to the I.H.:
\[ m_{0...i} = \mu_{0...i} \exp(\theta) = \mu_{0...i} \exp(\theta) = \mu_{0...i} \exp(\theta) \]

Since with \( 0 < i \), we have \( i - 1 < i \), we thus have
\[ m_{0...i-1} = \mu_{0...i-1} \exp(\theta) = \mu_{0...i-1} \exp(\theta) = \mu_{0...i-1} \exp(\theta) \]

Since \( \mu_i = \mu_{0...i} \exp(\theta) \), we then have, with Postulate 1-3:
\[ m_{0...i} = \mu_{0...i} \exp(\theta) = \mu_{0...i} \exp(\theta) = \mu_{0...i} \exp(\theta) \]

Hence the statement holds for all \( i < s \) and the theorem follows as indicated above. \( \blacksquare \)
Theorem 1-7. Unique initial and end expressions
If \( \mu, \mu' \in \text{EXP} \), then:

(i) \( \mu' \neq \mu\mu^* \),
(ii) \( \mu' \neq \mu^*\mu^* \),
(iii) \( \mu' = \mu^*\mu^* \).

Proof: Suppose \( \mu, \mu' \in \text{EXP} \). Then there are \( i \in \mathbb{N}\setminus\{0\} \) such that \( \{\mu_0, ..., \mu_{i-1}\} \subseteq \text{BEXP} \) and \( \mu = \mu_0...\mu_i \), and \( j \in \mathbb{N}\setminus\{0\} \) such that \( \{\mu^*_{-i}, ..., \mu^*_{-1}\} \subseteq \text{BEXP} \) and \( \mu^* = \mu_{-i}...\mu_{-1} \). Assume for contradiction that \( \mu' = \mu\mu^* \) or \( \mu' = \mu^*\mu^* \). As with Theorem 1-5-(i), we would then have \( \mu' = \mu\mu^* \) or \( \mu' = \mu^*\mu^* \). Therefore \( \mu' \neq \mu\mu^* \) and \( \mu' \neq \mu^*\mu^* \).

Theorem 1-8. No expression properly contains itself
If \( \mu' \in \text{EXP} \), then:

(i) \( \mu' \neq \mu\mu^* \),
(ii) \( \mu' \neq \mu^*\mu^* \),
(iii) \( \mu' \neq \mu^*\mu^* \).

Proof: Suppose \( \mu', \mu^* \in \text{EXP} \). Then there are \( i \in \mathbb{N}\setminus\{0\} \) such that \( \{\mu_0, ..., \mu_{i-1}\} \subseteq \text{EXP} \) and \( \mu' = \mu_0...\mu_i \), and \( j \in \mathbb{N}\setminus\{0\} \) such that \( \{\mu^*_{-i}, ..., \mu^*_{-1}\} \subseteq \text{EXP} \) and \( \mu^* = \mu_{-i}...\mu_{-1} \). Assume for contradiction that \( \mu' = \mu\mu^* \) or \( \mu' = \mu^*\mu^* \). As with Theorem 1-5-(i), we would then have \( \mu' = \mu\mu^* \) or \( \mu' = \mu^*\mu^* \). Therefore \( \mu' \neq \mu\mu^* \) and \( \mu' \neq \mu^*\mu^* \).

Now, all operators can be assigned an arity, where the category of the operators described in Definition 1-5-(vi) will be defined as the category of quantifiers further below in Definition 1-8. Following the definition of arity, we can also define the categories of terms and formulas and subsequently prove the unique readability for the categories established by then. Afterwards, we will introduce further grammatical concepts up to sentence sequences.
1.1 Vocabulary and Syntax

Definition 1-5. *Arity*

\[ \mu \text{ is } i\text{-ary} \]
iff
\begin{align*}
(i) & \quad \mu \in \text{FUNC} \text{ and there is } j \in \mathbb{N} \text{ such that } \mu = \%f_{ij}^\gamma \text{ or} \\
(ii) & \quad \mu \in \text{PRED} \text{ and there is } j \in \mathbb{N} \text{ such that } \mu = \%P_{ij}^\gamma \text{ or} \\
(iii) & \quad \mu = \%\neg^\gamma \text{ and } i = 2 \text{ or} \\
(iv) & \quad \mu = \%\land^\gamma \text{ and } i = 1 \text{ or} \\
(v) & \quad \mu \in \text{CON}\setminus\{%\neg\}\text{ and } i = 2 \text{ or} \\
(vi) & \quad \text{There are } \Pi \in \text{QUANT} \text{ and } \xi \in \text{VAR} \text{ and } \mu = \%\Pi\xi^\gamma \text{ and } i = 1 \text{ or} \\
(vii) & \quad \mu \in \text{PERF} \text{ and } i = 1.
\end{align*}

Definition 1-6. *The set of terms* (TERM; metavariables: \( \theta, \theta', \theta^*, \ldots \))

\[ \text{TERM} = \bigcap \{ R \mid R \subseteq \text{EXP} \text{ and} \]
\begin{align*}
(i) & \quad \text{CONST } \cup \text{ PAR } \cup \text{ VAR } \subseteq R, \text{ and} \\
(ii) & \quad \text{If } \{ \theta_0, \ldots, \theta_{n-1} \} \subseteq R \text{ and } \varphi \in \text{FUNC } n\text{-ary, then } \%\varphi(\theta_0, \ldots, \theta_{n-1})^\gamma \in R \}.
\end{align*}

*Note:* Here and in the following, blanks only serve the purpose of easing readability, blanks are not a part of the expressions. So, for example, \( \%f_{3,1}(c_0, c_0, c_1)^\gamma \) stands for \( \%f_{3,1}(c_0, c_0, c_1) \).

Definition 1-7. *Atomic and functional terms* (ATERM and FTERM)

(i) \( \text{ATERM} = \text{CONST } \cup \text{ PAR } \cup \text{ VAR} \),
(ii) \( \text{FTERM} = \text{TERM } \setminus \text{ATERM} \).

Definition 1-8. *The set of quantifiers* (QUANTOR)

\[ \text{QUANTOR} = \{ \%\Pi\xi^\gamma \mid \Pi \in \text{QUANT} \text{ and } \xi \in \text{VAR} \}. \]

Definition 1-9. *The set of formulas* (FORM; metavariables: \( A, B, \Gamma, \Delta, A', B', \Gamma', \Delta', A^*, B^*, \Gamma^*, \Delta^*, \ldots \))

\[ \text{FORM} = \bigcap \{ R \mid R \subseteq \text{EXP} \text{ and} \]
\begin{align*}
(i) & \quad \text{If } \{ \theta_0, \ldots, \theta_{n-1} \} \subseteq \text{TERM} \text{ and } \Phi \in \text{PRED } n\text{-ary, then } \%\Phi(\theta_0, \ldots, \theta_{n-1})^\gamma \in R, \\
(ii) & \quad \text{If } \Delta \in R, \text{ then } \%\neg\Delta^\gamma \in R, \\
(iii) & \quad \text{If } \Delta_0, \Delta_1 \in R \text{ and } \psi \in \text{CON}\setminus\{%\neg\}, \text{ then } \%(\Delta_0 \psi \Delta_1)^\gamma \in R, \text{ and} \\
(iv) & \quad \text{If } \Delta \in R \text{ and } \xi \in \text{VAR} \text{ and } \Pi \in \text{QUANT}, \text{ then } \%\Pi\xi\Delta^\gamma \in R \}.
\end{align*}
Definition 1-10. Atomic, connective and quantificational formulas (AFORM, CONFORM, QFORM)

(i) $\text{AFORM} = \{\overline{\Phi(\theta_0, \ldots, \theta_{n-1})} \mid \Phi \in \text{PRED} \text{ n-ary and } \{\theta_0, \ldots, \theta_{n-1}\} \subseteq \text{TERM}\},$

(ii) $\text{CONFORM} = \{\overline{\neg \Delta} \mid \Delta \in \text{FORM}\} \cup \{\overline{(\Delta_0 \psi \Delta_1)} \mid \Delta_0, \Delta_1 \in \text{FORM and } \psi \in \text{CON}\}\{\overline{\neg \gamma}\}\},$

(iii) $\text{QFORM} = \{\overline{\Pi \xi \Delta} \mid \Delta \in \text{FORM and } \Pi \in \text{QUANT und } \xi \in \text{VAR}\}.$

The following theorem leads directly to the theorems on unique readability.

Theorem 1-9. Terms resp. formulas do not have terms resp. formulas as proper initial expressions

(i) If $\theta, \theta' \in \text{TERM}$ and $\mu \in \text{EXP}$, then $\theta' \neq \overline{\theta \mu^n}$, and

(ii) If $\Delta, \Delta' \in \text{FORM}$ and $\mu \in \text{EXP}$, then $\Delta' \neq \overline{\Delta \mu^n}$.

Proof: Ad (i): Suppose $\theta, \theta' \in \text{TERM}$ and $\mu \in \text{EXP}$. The proof is carried out by induction on $\text{EXPL}(\theta')$. For this, suppose the statement holds for all $\theta^* \in \text{TERM}$ with $\text{EXPL}(\theta^*) < \text{EXPL}(\theta')$. For $\text{EXPL}(\theta') = 1$, and thus $\theta' \in \text{ATERM}$, the statement holds trivially, because, according to Postulate 1-2-(ii), there are no $\theta, \mu \in \text{EXP}$ such that $\theta' = \overline{\theta \mu^n}$. Now, suppose $1 < \text{EXPL}(\theta')$. Then $\theta' \notin \text{ATERM}$ and therefore $\theta' \in \text{FTERM}$. Then there are $n' \in \mathbb{N}\setminus\{0\}$ and $\varphi' \in \text{FUNC}, \varphi'$ $n'$-ary, and $\{\theta'_0, \ldots, \theta'_{n'-1}\} \subseteq \text{TERM}$ such that $\theta' = \overline{\varphi'(\theta'_0, \ldots, \theta'_{n'-1})}$. Suppose for contradiction that $\theta' = \overline{\theta \mu^n}$. Now, suppose for contradiction that $\theta \in \text{ATERM}$. Then, we would have $\theta \in \text{CONST} \cup \text{PAR} \cup \text{VAR}$. According to Theorem 1-7-(iii) and with $\varphi'(\theta'_0, \ldots, \theta'_{n'-1}) = \theta' = \overline{\theta \mu^n}$, we would then have that $\varphi' = \theta \in \text{CONST} \cup \text{PAR} \cup \text{VAR}$. Contradiction! Therefore $\theta \notin \text{FTERM}$ and there are thus $n \in \mathbb{N}\setminus\{0\}$ and $\varphi \in \text{FUNC}, \varphi$ $n$-ary, and $\{\theta_0, \ldots, \theta_{n-1}\} \subseteq \text{TERM}$ such that $\theta = \overline{\varphi(\theta_0, \ldots, \theta_{n-1})}. \text{Then it holds with Theorem 1-7-(iii) that } \varphi' = \varphi \text{ and thus, according to Definition 1-5 and Postulate 1-1-(iv), we have } n = n'. \text{Therefore } \varphi(\theta_0, \ldots, \theta_{n-1}) = \overline{\varphi(\theta_0, \ldots, \theta_{n-1})}. \text{Note that } \text{EXPL}(\theta'), \text{EXPL}(\theta_i) < \text{EXPL}(\theta') \text{ for all } i < n.$

With $\{\mu\} \cup \text{TERM} \subseteq \text{EXP}$, it then holds that there are $\{\mu^*_{0}, \ldots, \mu^*_{\text{EXPL}(\theta)^{i}}\} \subseteq \text{BEXP}$ and $\{\mu^0_{0}, \ldots, \mu^0_{\text{EXPL}(\theta_0)^{i}}\} \cup \ldots \cup \{\mu^{n-1}_{0}, \ldots, \mu^{n-1}_{\text{EXPL}(\theta_{n-1})^{i}}\} \subseteq \text{BEXP}$ and $\{\mu^0_{0}, \ldots, \mu^0_{\text{EXPL}(\theta_0)^{i}}\} \cup \ldots \cup \{\mu^{n-1}_{0}, \ldots, \mu^{n-1}_{\text{EXPL}(\theta_{n-1})^{i}}\} \subseteq \text{BEXP}$ such that $\mu = \overline{\mu^*_{0} \cdots \mu^*_{\text{EXPL}(\theta)^{i}}}$ and for all $i < n$: $\theta_i = \overline{\mu^i_{0} \cdots \mu^i_{\text{EXPL}(\theta)^{i}}} \text{ and } \theta_i = \overline{\mu^i_{0} \cdots \mu^i_{\text{EXPL}(\theta)^{i}}} \text{ and } \theta_i = \overline{\mu^i_{0} \cdots \mu^i_{\text{EXPL}(\theta)^{i}}} \text{. With Theorem 1-5-(i), it then holds that } \overline{\varphi(\mu^0_{0} \cdots \mu^0_{\text{EXPL}(\theta_0)^{i}}, \ldots, \mu^{n-1}_{0} \cdots \mu^{n-1}_{\text{EXPL}(\theta_{n-1})^{i}})} = \overline{\varphi(\theta_0, \ldots, \theta_{n-1})} \text{.} \text{ With Theorem 1-5-(i), it then holds that } \overline{\varphi(\mu^0_{0} \cdots \mu^0_{\text{EXPL}(\theta_0)^{i}}, \ldots, \mu^{n-1}_{0} \cdots \mu^{n-1}_{\text{EXPL}(\theta_{n-1})^{i}})} = \overline{\varphi(\theta_0, \ldots, \theta_{n-1})} \text{.}
Suppose for contradiction that $\text{EXPL}(\theta') = \text{EXPL}(\theta)$ for all $i < n$. With Theorem 1-5-(iii) and Theorem 1-7-(i), we would then have that $\gamma' = \gamma_0^* \ldots \mu_1^* \text{EXPL}(\theta)$. Therefore, with Theorem 1-5-(ii) and Theorem 1-7-(i), we would then have that 

$$
\gamma_0 \ldots \mu_1^* \text{EXPL}(\theta) = \mu_0^* \ldots \mu_1^* \text{EXPL}(\theta) \gamma.
$$

Suppose for contradiction that $\text{EXPL}(\theta') = \text{EXPL}(\theta)$ for all $i < n$. With Theorem 1-5-(iii) and Theorem 1-7-(i), we would then have that $\gamma' = \gamma_0^* \ldots \mu_1^* \text{EXPL}(\theta)$. Therefore, with Theorem 1-5-(ii) and Theorem 1-7-(i), we would then have that 

$$
\gamma_0 \ldots \mu_1^* \text{EXPL}(\theta) = \mu_0^* \ldots \mu_1^* \text{EXPL}(\theta) \gamma.
$$

Therefore, with Theorem 1-7-(i):

$$
\gamma_0 \ldots \mu_1^* \text{EXPL}(\theta) = \mu_0^* \ldots \mu_1^* \text{EXPL}(\theta) \gamma.
$$

With Theorem 1-5-(iii), we then have that for all $j < \text{EXPL}(\theta')$ it holds that $\gamma_i = \gamma_{i+1}^{*} \ldots \mu_{j+1}^{*} \text{EXPL}(\theta')$. Therefore, with Theorem 1-5-(iii), we would then have that $\gamma' = \gamma_0^* \ldots \mu_1^* \text{EXPL}(\theta)$. Therefore, with Theorem 1-5-(ii) and Theorem 1-7-(i), we would then have that 

$$
\gamma_0 \ldots \mu_1^* \text{EXPL}(\theta) = \mu_0^* \ldots \mu_1^* \text{EXPL}(\theta) \gamma.
$$

Ad (ii): Now, suppose $\Delta, \Delta' \in \text{FORM}$ and $\mu \in \text{EXP}$. The proof is carried out by induction on $\text{EXPL}(\Delta')$. For this, suppose the statement holds for all $\Delta^* \in \text{FORM}$ with...
EXPL(Δ*) < EXPL(Δ'). With Δ' ∈ FORM, we have Δ' ∈ AFORM ∪ {γ−Δ*γ | Δ* ∈ FORM} ∪ {γ(Δ₀ ψ Δ₁)γ | Δ₀, Δ₁ ∈ FORM and ψ ∈ CON\{γ−γ}\} ∪ QFORM. These four cases are now considered separately.

First: Suppose Δ' ∈ AFORM. The proof is carried out analogously to the induction step for (i) by applying (i). Suppose Δ' = γΔμγ. With Δ' ∈ AFORM there are n' ∈ N\{0} and Φ' ∈ PRED and \{θ₀', ..., θₙ'-1\} ⊆ TERM such that Δ' = \(\Phi'(θ₀', ..., θₙ'-1)\). Suppose for contradiction that Δ ∈ CONFORM ∪ QFORM. Then there would be μ' ∈ {γ−γ, (γ) ∪ QUANT and μ* ∈ EXP such that Δ = γμ'μ*γ. Therefore, according to Theorem 1-6, \(\Phi'(θ₀', ..., θₙ'-1)' = Δ' = γΔγ = γμ*μ\) and thus, according to Theorem 1-7-(iii), Δ' = μ'. Thus we would have that Φ' ∈ {γ−γ, (γ) ∪ QUANT. Contradiction! Therefore Δ ∉ CONFORM ∪ QFORM and thus Δ ∈ AFORM. Thus there are n ∈ N\{0} and Φ ∈ PRED, Φ n-ary, and \{θ₀, ..., θₙ\} ⊆ TERM such that Δ = \(\Phi(θ₀, ..., θₙ)\). Therefore \(\Phi'(θ₀', ..., θₙ'-1)' = \Phi(θ₀, ..., θₙ)μ\). Then it holds with Theorem 1-7-(iii) that Φ' = Φ and thus we have according to Definition 1-5 and Postulate 1-1-(v) that n = n'. Therefore \(\Phi(θ₀, ..., θₙ)μ = \Phi(θ₀', ..., θₙ'-1)μ\). From here on, the proof for Δ' ∈ AFORM proceeds analogously to the induction step for (i), while the contradiction resulting here is not with the I.H., but with (i).

Second: Now, suppose Δ' ∈ {γ−Δ*γ | Δ* ∈ FORM}. Then there is Δ' ∈ FORM such that Δ' = γ−ΔΔ, and also EXPL(Δ') < EXPL(Δ'). Suppose Δ' = γΔΔ and thus γΔμ = γ−ΔΔ. Suppose for contradiction that Δ ∈ CONFORM ∪ {γ(Δ₀ ψ Δ₁)γ | Δ₀, Δ₁ ∈ FORM and ψ ∈ CON\{γ−γ}\} ∪ QFORM. Then there would be μ' ∈ PRED ∪ {γ} ∪ QUANT and μ* ∈ EXP such that Δ = γμ'μ*γ. Therefore according to Theorem 1-6, γ−ΔΔ = γΔμ = γμ'μ and thus according to Theorem 1-7-(iii) γ−γ = μ'. Then we would have that γ−γ ∈ PRED ∪ {γ} ∪ QUANT. Contradiction! Therefore Δ ∈ {γ−Δ*γ | Δ* ∈ FORM} and there is Δ+ ∈ FORM such that Δ = γ−ΔΔ. Therefore γ−ΔΔ = γ−ΔΔ ∧. With Theorem 1-7-(i) one then has that Δ' = γΔ'μ, which contradicts the I.H.

Third: Now, suppose Δ' ∈ {γ(Δ₀ ψ Δ₁)γ | Δ₀, Δ₁ ∈ FORM and ψ ∈ CON\{γ−γ}\}. Then there are Δ₀, Δ₁ ∈ FORM and ψ' ∈ CON\{γ−γ\} such that Δ' = γ(Δ₀ ψ' Δ₁)γ, and also EXPL(Δ₀) < EXPL(Δ) and EXPL(Δ₁) < EXPL(Δ'). Suppose Δ' = γΔμ and thus γΔμ = γ(Δ₀ ψ' Δ₁)γ. Suppose for contradiction Δ ∈ CONFORM ∪ {γ−Δ*γ | Δ* ∈ FORM} ∪ QFORM. Then there would be μ' ∈ PRED ∪ {γ−γ} ∪ QUANT and μ* ∈ EXP such that Δ = γμ'μ*γ, and therefore γ(Δ₀ ψ' Δ₁)γ = Δ' = γΔμ = γμ'μ*μ and thus according to Theorem 1-7-(iii) γ(γ) = μ'. Thus one would have that γ(γ) ∈ PRED ∪ {γ−γ} ∪ QUANT.
Contradiction! Therefore $\Delta \in \{\overline{\gamma}^\Delta | \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON} \backslash \{\overline{\gamma}^\gamma\} \} \text{ and there are } \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON} \backslash \{\overline{\gamma}^\gamma\} \text{ such that } \Delta = \overline{\gamma}^\Delta \text{, and also } \text{EXPL}(\Delta_0), \text{EXPL}(\Delta_1) < \text{EXPL}(\Delta')$. Therefore $\overline{\gamma}^\Delta \psi' \Delta_1' \gamma = \overline{\gamma}^\Delta \psi \Delta_1 \gamma$.

With Theorem \ref{th:1-7-(i)} it holds that $\overline{\gamma}^\Delta \psi' \Delta_1' \gamma = \overline{\gamma}^\Delta \psi \Delta_1 \gamma$. With $\{\mu \} \cup \text{FORM} \subseteq \text{EXP}$ it also holds that there are $\{\mu^0_*, \ldots, \mu^*_{\text{EXPL}(\mu_0)-1}\} \subseteq \text{BEXP}$ and $\{\mu^0_\Delta, \ldots, \mu^\Delta_{\text{EXPL}(\Delta_0)-1}\} \cup \{\mu^0_\Delta, \ldots, \mu^\Delta_{\text{EXPL}(\Delta_1)-1}\} \subseteq \text{BEXP}$ such that $\mu = \overline{\gamma}^\Delta \overline{\mu}^0 \ldots \mu^*_{\text{EXPL}(\mu_0)-1}$ and for all $i < 2$: $\Delta_i' = \overline{\gamma}^{\Delta_i} \overline{\mu}^0 \ldots \mu^*_{\text{EXPL}(\mu_0)-1}$ and $\Delta_i = \overline{\gamma}^{\Delta_i} \overline{\mu}^0 \ldots \mu^*_{\text{EXPL}(\mu_0)-1}$.

With Theorem \ref{th:1-5-(i)}, we then have that

$$\overline{\gamma}^{\Delta_0} \overline{\mu}^0 \ldots \mu^\Delta_{\text{EXPL}(\mu_0)-1} \psi' \overline{\mu}^0 \ldots \mu^\Delta_{\text{EXPL}(\Delta_1)-1} \gamma = \overline{\gamma}^{\Delta_0} \overline{\mu}^0 \ldots \mu^*_{\text{EXPL}(\mu_0)-1} \mu^0 \ldots \mu^*_{\text{EXPL}(\mu_0)-1} \gamma.$$ 

Now, suppose for contradiction that $\text{EXPL}(\Delta_0') < \text{EXPL}(\Delta_0)$. With Theorem \ref{th:1-5-(iii)}, it then holds for all $j < \text{EXPL}(\Delta_0')$ that $\mu^\Delta_j = \mu^\Delta_j$. With Postulate \ref{post:1-2-(i)}, we then have $\Delta'_0 = \overline{\gamma}^{\Delta'_0} \overline{\mu}^0 \ldots \mu^\Delta_{\text{EXPL}(\Delta_0')-1} \gamma = \overline{\gamma}^{\Delta_0} \overline{\mu}^0 \ldots \mu^\Delta_{\text{EXPL}(\Delta_0)-1} \gamma$. With Theorem \ref{th:1-6}, we then have that $\overline{\gamma}^{\Delta'_0} \overline{\mu}^0 \ldots \mu^\Delta_{\text{EXPL}(\Delta_0')-1} \gamma = \overline{\gamma}^{\Delta_0} \overline{\mu}^0 \ldots \mu^\Delta_{\text{EXPL}(\Delta_0)-1} \gamma = \overline{\gamma}^{\Delta_0} \overline{\mu}^0 \ldots \mu^\Delta_{\text{EXPL}(\Delta_0)-1} \gamma = \Delta_0$, which contradicts the I.H. In case of $\text{EXPL}(\Delta_0) < \text{EXPL}(\Delta_0)$, a contradiction follows analogously. Therefore one has that $\text{EXPL}(\Delta_0') = \text{EXPL}(\Delta_0)$. Thus it holds, with Theorem \ref{th:1-5-(iii)}, that $\overline{\gamma}^{\Delta_0} \overline{\mu}^0 \ldots \mu^\Delta_{\text{EXPL}(\Delta_0)-1} \gamma = \overline{\gamma}^{\Delta_0} \overline{\mu}^0 \ldots \mu^\Delta_{\text{EXPL}(\Delta_0)-1} \gamma$ and thus, with Theorem \ref{th:1-7-(i)}, also that $\overline{\gamma}^{\Delta_0} \overline{\mu}^0 \ldots \mu^\Delta_{\text{EXPL}(\Delta_0)-1} \gamma = \overline{\gamma}^{\Delta_0} \overline{\mu}^0 \ldots \mu^\Delta_{\text{EXPL}(\Delta_0)-1} \gamma = \Delta_0$, which contradicts Postulate \ref{post:1-2-(ii)}.

Fourth: Now, suppose $\Delta' \in \text{QFORM}$. Then there are $\Delta^\theta \in \text{FORM}$ and $\Pi' \in \text{QUANT}$ and $\xi' \in \text{VAR}$ such that $\Delta' = \overline{\Pi} \overline{\xi}^\Delta \Delta'^\gamma$, and also $\text{EXPL}(\Delta^\theta) < \text{EXPL}(\Delta')$. Suppose $\Delta' = \overline{\gamma}^\Delta \mu \gamma$ and thus $\Delta^\gamma = \overline{\Pi} \overline{\xi}^\Delta \Delta'^\gamma$. Suppose for contradiction $\Delta' \in \text{AFORM} \cup \text{CONFORM}$. Then there would be $\mu' \in \text{PRED} \cup \{\overline{\gamma}^\gamma, \overline{\gamma}^\gamma\}$ and $\mu^\gamma \in \text{EXP}$ such that $\Delta = \overline{\gamma}^\Delta \mu^\gamma \gamma$. Therefore according to Theorem \ref{th:1-6} $\overline{\Pi} \overline{\xi}^\Delta \Delta'^\gamma = \overline{\gamma}^\Delta \mu^\gamma \gamma \mu^\gamma \gamma$ and thus $\Pi' = \mu'$. Thus we would have that $\Pi' \in \text{PRED} \cup \{\overline{\gamma}^\gamma, \overline{\gamma}^\gamma\}$. Contradiction! Therefore $\Delta' \in \text{QFORM}$ and there are $\Delta'^\gamma \in \text{FORM}$ and $\Pi \in \text{QUANT}$ and $\xi \in \text{VAR}$ such that $\Delta = \overline{\Pi} \overline{\xi}^\Delta \Delta'^\gamma$. Therefore $\overline{\Pi} \overline{\xi}^\Delta \Delta'^\gamma = \overline{\Pi} \overline{\xi}^\Delta \overline{\mu}' \gamma$. With Theorem \ref{th:1-7-(iii)} and \ref{th:1-7-(i)}, we then have first $\overline{\xi}^\Delta \overline{\mu}' \gamma = \overline{\xi}^\Delta \overline{\mu}' \gamma$ and then $\Delta' = \overline{\Delta'}^\gamma \overline{\mu}' \gamma$, which contradicts the I.H.

Thus $\Delta' = \overline{\Delta'}^\gamma \overline{\mu}' \gamma$ leads to a contradiction in all four cases. Therefore $\Delta' \neq \overline{\Delta'}^\gamma$. ■
Theorem 1-10. Unique readability without sentences (a – unique categories)

(i) \( \text{CONST} \cap (\text{PAR} \cup \text{VAR} \cup \text{FTERM} \cup \text{QUANTOR} \cup \text{AFORM} \cup \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM}) = 0, \)

(ii) \( \text{PAR} \cap (\text{CONST} \cup \text{VAR} \cup \text{FTERM} \cup \text{QUANTOR} \cup \text{AFORM} \cup \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM}) = 0, \)

(iii) \( \text{VAR} \cap (\text{CONST} \cup \text{PAR} \cup \text{FTERM} \cup \text{QUANTOR} \cup \text{AFORM} \cup \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM}) = 0, \)

(iv) \( \text{FTERM} \cap (\text{CONST} \cup \text{PAR} \cup \text{VAR} \cup \text{QUANTOR} \cup \text{AFORM} \cup \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM}) = 0, \)

(v) \( \text{QUANTOR} \cap (\text{CONST} \cup \text{PAR} \cup \text{VAR} \cup \text{FTERM} \cup \text{AFORM} \cup \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM}) = 0, \)

(vi) \( \text{AFORM} \cap (\text{CONST} \cup \text{PAR} \cup \text{VAR} \cup \text{FTERM} \cup \text{QUANTOR} \cup \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM}) = 0, \)

(vii) \( \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cap (\text{CONST} \cup \text{PAR} \cup \text{VAR} \cup \text{FTERM} \cup \text{QUANTOR} \cup \text{AFORM} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM}) = 0, \)

(viii) \( \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cap (\text{CONST} \cup \text{PAR} \cup \text{VAR} \cup \text{FTERM} \cup \text{QUANTOR} \cup \text{AFORM} \cup \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM}) = 0, \) and

(ix) \( \text{QFORM} \cap (\text{CONST} \cup \text{PAR} \cup \text{VAR} \cup \text{FTERM} \cup \text{QUANTOR} \cup \text{AFORM} \cup \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM}) = 0. \)

Proof: Suppose \( \mu \in \text{CONST} \). According to Postulate 1-1, we then have that \( \mu \notin \text{PAR} \cup \text{VAR} \) and, according to Definition 1-7, that \( \mu \notin \text{FTERM} \). Suppose for contradiction that \( \mu \in \text{QUANTOR} \cup \text{AFORM} \cup \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM} \). Then, there would be \( \mu' \in \text{BEXP} \) and \( \mu^* \in \text{EXP} \) such that \( \mu = \varphi \mu^* \mu^* \). This contradicts Postulate 1-2-(ii). Therefore \( \mu \notin \text{QUANTOR} \cup \text{AFORM} \cup \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM} \).

For \( \mu \in \text{PAR} \) and \( \mu \in \text{VAR} \), the proof is carried out analogously.

Now, suppose \( \mu \in \text{FTERM} \). According to Definition 1-7, we then have \( \mu \notin \text{CONST} \cup \text{PAR} \cup \text{VAR} \) and we have \( \mu \in \text{TERM} \). According to Definition 1-6, there are thus \( \varphi \in \text{FUNC} \) and \( \mu^* \in \text{EXP} \) such that \( \mu = \varphi \mu^* \). Suppose for contradiction that \( \mu \in \text{QUANTOR} \cup \text{AFORM} \cup \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM} \). Then there would be \( \mu' \in \text{PRE} \cup \text{QUANT} \cup \{\lnot \gamma, \lnot \gamma\} \) and \( \mu^* \in \text{EXP} \) such that \( \mu = \mu' \mu^* \). According to Theorem 1-7-(iii), we would then have \( \mu' = \varphi \) and thus \( \mu' \in \text{FUNC} \). This contradicts Postulate 1-1. Therefore \( \mu \notin \text{QUANTOR} \cup \text{AFORM} \cup \{\lnot \Delta^n \mid \Delta \in \text{FORM}\} \cup \{\varphi(\Delta_0 \psi \Delta_1)^\gamma \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON}\{\lnot \gamma\}\} \cup \text{QFORM} \).
For $\mu \in \text{QUANTOR}$, $\mu \in \text{AFORM}$, $\mu \in \{\neg\Delta \mid \Delta \in \text{FORM}\}$, $\mu \in \{\gamma(\Delta_0 \psi \Delta_1) \mid \Delta_0, \Delta_1 \in \text{FORM} \}$ and $\psi \in \text{CON}\{\neg\gamma\}$ and $\mu \in \text{QFORM}$, the proof is carried out analogously.

**Theorem 1-11.** Unique readability without sentences (b – unique decomposability)

If $\mu \in \text{TERM} \cup \text{QUANTOR} \cup \text{FORM}$, then:

1. $\mu \in \text{ATERM}$ or
2. $\mu \in \text{FTERM}$ and there are $n \in \mathbb{N}\{0\}$, $\varphi \in \text{FUNC}$ and $\{\theta_0, \ldots, \theta_{n-1}\} \subseteq \text{TERM}$ such that $\mu = \gamma(\varphi(\theta_0, \ldots, \theta_{n-1}))$ and for all $n' \in \mathbb{N}\{0\}$, $\varphi' \in \text{FUNC}$ and $\{\theta_0, \ldots, \theta_{n'-1}\} \subseteq \text{TERM}$ with $\mu = \gamma(\varphi'(\theta_0, \ldots, \theta_{n'-1}))$ it holds that $n = n'$ and $\varphi = \varphi'$ and for all $i < n$: $\theta_i = \theta'_i$, or
3. $\mu \in \text{QUANTOR}$ and there are $\Pi \in \text{QUANT}$ and $\xi \in \text{VAR}$ such that $\mu = \gamma(\Pi \xi \eta)$ and for all $\Pi' \in \text{QUANT}$ and $\xi' \in \text{VAR}$ with $\mu = \gamma(\Pi' \xi' \eta)$ it holds that $\Pi = \Pi'$ and $\xi = \xi'$, or
4. $\mu \in \text{AFORM}$ and there are $n \in \mathbb{N}\{0\}$, $\Phi \in \text{PRED}$ and $\{\theta_0, \ldots, \theta_{n-1}\} \subseteq \text{TERM}$ such that $\mu = \gamma(\Phi(\theta_0, \ldots, \theta_{n-1}))$ and for all $n' \in \mathbb{N}\{0\}$, $\Phi' \in \text{PRED}$ and $\{\theta_0, \ldots, \theta_{n'-1}\} \subseteq \text{TERM}$ with $\mu = \gamma(\Phi'(\theta_0, \ldots, \theta_{n'-1}))$ it holds that $n = n'$ and $\Phi = \Phi'$ and for all $i < n$: $\theta_i = \theta'_i$, or
5. $\mu \in \{\neg\Delta \mid \Delta \in \text{FORM}\}$ and there is $\Delta$ in $\text{FORM}$ such that $\mu = \neg\Delta$ and for all $\Delta'$ in $\text{FORM}$ with $\mu = \neg\Delta'$ it holds that $\Delta = \Delta'$, or
6. $\mu \in \{\gamma(\Delta_0 \psi \Delta_1) \mid \Delta_0, \Delta_1 \in \text{FORM} \}$ and $\psi \in \text{CON}\{\neg\gamma\}$ and there are $\Delta_0, \Delta_1 \in \text{FORM}$ and $\psi \in \text{CON}\{\neg\gamma\}$ such that $\mu = \gamma(\Delta_0 \psi \Delta_1)$ and for all $\Delta'_0, \Delta'_1 \in \text{FORM}$ and $\psi' \in \text{CON}\{\neg\gamma\}$ with $\mu = \gamma(\Delta'_0 \psi' \Delta'_1)$ it holds that $\Delta_0 = \Delta'_0$ and $\Delta_1 = \Delta'_1$ and $\psi = \psi'$, or
7. $\mu \in \text{QFORM}$ and there are $\Pi \in \text{QUANT}$, $\xi \in \text{VAR}$ and $\Delta$ in $\text{FORM}$ such that $\mu = \gamma(\Pi \xi \Delta)$ and for all $\Pi' \in \text{QUANT}$, $\xi' \in \text{VAR}$ and $\Delta'$ in $\text{FORM}$ with $\mu = \gamma(\Pi' \xi' \Delta')$ it holds that $\Pi = \Pi'$ and $\xi = \xi'$ and $\Delta = \Delta'$.

**Proof:** Suppose $\mu \in \text{TERM} \cup \text{QUANTOR} \cup \text{FORM}$. Therefore $\mu \in \text{ATERM} \cup \text{FTERM} \cup \text{QUANTOR} \cup \text{AFORM} \cup \{\neg\Delta \mid \Delta \in \text{FORM}\} \cup \{\gamma(\Delta_0 \psi \Delta_1) \mid \Delta_0, \Delta_1 \in \text{FORM} \}$ and $\psi \in \text{CON}\{\neg\gamma\}$ and $\mu \in \text{QFORM}$. These seven cases will be treated separately. First: Suppose $\mu \in \text{ATERM}$. Then (i) is satisfied trivially.

**Second:** Suppose $\mu \in \text{FTERM}$. According to Definition 1-6 and Definition 1-7, there are then $n \in \mathbb{N}\{0\}$, $\varphi \in \text{FUNC}$ and $\{\theta_0, \ldots, \theta_{n-1}\} \subseteq \text{TERM}$ such that $\mu = \gamma(\varphi(\theta_0, \ldots, \theta_{n-1}))$. Now, let also $n' \in \mathbb{N}\{0\}$, $\varphi' \in \text{FUNC}$ and $\{\theta_0, \ldots, \theta_{n'-1}\} \subseteq \text{TERM}$ be such that $\mu = \gamma(\varphi'(\theta_0, \ldots, \theta_{n'-1}))$. $\varphi = \varphi'$ follows from Theorem 1-7-(iii). With Theorem 1-7-(i), we thus have $\gamma(\theta_0, \ldots, \theta_{n-1}) = \gamma(\theta'_0, \ldots, \theta'_{n'-1})$. By induction on $i$ we will now show that for all $i \in \mathbb{N}$: If $i < n$, then $i < n'$ and $\theta_i = \theta'_i$. For this, suppose that the statement holds for all $k < i$. Suppose $i < n$. Suppose $i = 0$. We have that $0 < n'$. We also have that there are $\{\mu_0, \ldots,
\( \mu_{\text{EXPL}(\theta_0)} \cup \{ \mu', \ldots, \mu'_{\text{EXPL}(\theta'_{0})} \} \subseteq \text{BEXP} \) such that \( \theta_0 = \mu_0 \cdots \mu_{\text{EXPL}(\theta_0)} \) and \( \theta'_0 = \mu'_0 \cdots \mu'_{\text{EXPL}(\theta'_{0})} \) and thus, with Theorem 1-6, \( \mu_0 \cdots \mu_{\text{EXPL}(\theta_0)} = \mu'_0 \cdots \mu'_{\text{EXPL}(\theta'_{0})} \). Now, suppose \( \text{EXPL}(\theta_0) < \text{EXPL}(\theta'_0) \). With Theorem 1-5-(iii), it would then hold for all \( l < \text{EXPL}(\theta_0) \) that \( \mu = \mu' \). With Postulate 1-2-(i), we would thus have that \( \theta_0 = \mu_0 \cdots \mu_{\text{EXPL}(\theta_0)} \) and thus, with Theorem 1-6, \( \theta'_0 \), which contradicts Theorem 1-9-(i). In the same way, a contradiction follows for \( \text{EXPL}(\theta'_0) < \text{EXPL}(\theta_0) \). Therefore we have that \( \text{EXPL}(\theta_0) = \text{EXPL}(\theta'_0) \) and thus, with Theorem 1-5-(iii), also \( \theta_0 = \theta'_0 \).

Now, suppose \( 0 < i \). Then it holds for all \( k < i \) that \( k < n \). With the I.H., we thus have for all \( k < i \) that \( k < n' \) and \( \theta_k = \theta'_k \). With Theorem 1-5-(iii), we then have that \( \theta_0, \ldots, \theta_{i-1} = \theta'_0, \ldots, \theta'_{i-1} \). We also have that \( i-1 < n' \) and thus that \( i \leq n' \). Suppose for contradiction that \( i = n' \). Then we would have that \( \theta_0, \ldots, \theta_{i-1} = \theta'_0, \ldots, \theta'_{i-1} \). With Theorem 1-7-(i), we would then have that \( \gamma, \theta_0, \ldots, \theta_{i-1} = \gamma' \), which contradicts Postulate 1-2-(ii). Thus we have \( i < n' \). Again with Theorem 1-7-(i), we then have that \( \theta_0, \ldots, \theta_{n-1} = \theta'_0, \ldots, \theta'_{n-1} \). From this, we can derive \( \theta_i = \theta'_i \) in the same way as \( \theta_0 = \theta'_0 \) for \( i = 0 \). Therefore it holds for all \( i < n \) that \( i < n' \) and \( \theta_i = \theta'_i \). Analogously, we can show that for all \( i < n' \) we have that \( i < n \) and \( \theta_i = \theta'_i \). Taken together, we thus have that \( n = n' \) and that for all \( i < n \): \( \theta_i = \theta'_i \).

Third: Suppose \( \mu \in \text{QUANTOR} \). According to Definition 1-8, there are then \( \Pi \in \text{QUANT} \) and \( \xi \in \text{VAR} \) such that \( \mu = \Pi\xi^\gamma \). Now, let also \( \Pi' \in \text{QUANT} \), \( \xi' \in \text{VAR} \) such that \( \mu = \Pi\xi^\gamma \). From Theorem 1-7-(iii) and -(i) follows immediately \( \Pi = \Pi' \) and \( \xi = \xi' \).

Fourth: Suppose \( \mu \in \text{AFORM} \). According to Definition 1-10-(i), there are then \( n \in \mathbb{N} \setminus \{ 0 \} \), \( \Phi \in \text{PRED} \) and \( \{ \theta_0, \ldots, \theta_{n-1} \} \subseteq \text{TERM} \) such that \( \mu = \Phi(\theta_0, \ldots, \theta_{n-1}) \). Let now also \( n' \in \mathbb{N} \setminus \{ 0 \} \), \( \Phi' \in \text{PRED} \) and \( \{ \theta'_0, \ldots, \theta'_{n'-1} \} \subseteq \text{TERM} \) such that \( \mu = \Phi(\theta'_0, \ldots, \theta'_{n'-1}) \). \( \Phi = \Phi' \) follows from Theorem 1-7-(iii). With Theorem 1-7-(i), we then get that \( \gamma, \theta_0, \ldots, \theta_{i-1} = \gamma', \theta'_0, \ldots, \theta'_{i-1} \). In the same way as in the second case, we can then show that \( n = n' \) and that for all \( i < n \): \( \theta_i = \theta'_i \).

Fifth: Suppose \( \mu \in \{ \gamma^\Delta \mid \Delta \in \text{FORM} \} \). Then there is \( \Delta \in \text{FORM} \) such that \( \mu = \gamma^\Delta \). Now, suppose \( \Delta' \in \text{FORM} \) and \( \mu = \gamma^\Delta' \). From Theorem 1-7-(i) follows immediately \( \Delta = \Delta' \).

Sixth: Suppose \( \mu \in \{ \gamma(\Delta_0 \psi \Delta_1) \mid \Delta_0, \Delta_1 \in \text{FORM} \text{ and } \psi \in \text{CON} \setminus \{ \gamma^- \} \} \). Then there are \( \Delta_0, \Delta_1 \in \text{FORM} \) and \( \psi \in \text{CON} \setminus \{ \gamma^- \} \) such that \( \mu = \gamma(\Delta_0 \psi \Delta_1) \). Let now also \( \Delta_0', \Delta_1' \)
∈ FORM and ψ' ∈ CON\{¬,\} be such that μ = (Δ₀ψΔ₁). With Theorem 1-7-(i), we then have (Δ₀ψΔ₁) = (Δ₀ψΔ₁). Also, there is {μ₀, ..., μEXPL(Δ₀)-1} ∪ {μ₀', ..., μEXPL(Δ₀)-1}' ⊆ BEXP such that Δ₀ = [μ₀...μEXPL(Δ₀)-1] and Δ₀' = [μ₀'...μEXPL(Δ₀)-1]'. Suppose for contradiction that EXPL(Δ₀) < EXPL(Δ₀'). With Theorem 1-5-(iii), we would then have μ = μ for all i < EXPL(Δ₀). But then we would have, with Postulate 1-2-(i), that Δ₀ = [μ₀...μEXPL(Δ₀)-1] = [μ₀'...μEXPL(Δ₀)-1]' = Δ₀', which contradicts Theorem 1-9-(ii). Analogously, a contradiction follows from EXPL(Δ₀') < EXPL(Δ₀). Therefore EXPL(Δ₀) = EXPL(Δ₀') and thus Δ₀ = [μ₀...μEXPL(Δ₀)-1] = [μ₀'...μEXPL(Δ₀)-1]' = Δ₀'. With Theorem 1-7, it then follows first that (Δ₀ψΔ₁) = (Δ₀ψΔ₁)' and finally that Δ₁ = Δ₁'.

Seventh: Suppose μ ∈ QFORM. According to Definition 1-10-(iii), there are then Π ∈ QUANT, ξ ∈ VAR and Δ ∈ FORM such that μ = ΠξΔ. Let now also Π' ∈ QUANT, ξ' ∈ VAR, Δ' ∈ FORM such that μ = Π'ξΔ'. From Theorem 1-7-(iii) and -(i) follows immediately Π = Π' and ξ = ξ' and Δ = Δ'. ■

With Theorem 1-10 and Theorem 1-11, one can now define functions on the sets TERM, FORM and their union by recursion on the complexity of terms and formulas. The following definitions of the degree of a term and the degree of a formula (Definition 1-11 and Definition 1-12), allow us to prove properties of terms and formulas by induction on the natural numbers more conveniently then this can be done by using EXPL.

**Definition 1-11. Degree of a term8 (TDEG)**

TDEG is a function on TERM and

(i) If θ ∈ ATERM, then TDEG(θ) = 0,

(ii) If (φ(θ₀, ..., θᵣ)) ∈ FTERM, then

TDEG((φ(θ₀, ..., θᵣ))) = max({TDEG(θ₀), ..., TDEG(θᵣ)}) + 1.

---

8 Let 'min(..)' be defined as usual for non-empty subsets of N and 'max(..)' as usual for non-empty and finite subsets of N. If X is not a non-empty subset of N, let min(X) = 0, and if X is not a non-empty finite subset of N, also let max(X) = 0.
Definition 1-12. Degree of a formula (FDEG)
FDEG is a function on FORM and

(i) If $\Delta \in \text{AFORM}$, then $\text{FDEG}(\Delta) = 0$,
(ii) If $\neg\Delta \in \text{CONFORM}$, then $\text{FDEG}(\neg\Delta) = \text{FDEG}(\Delta)+1$,
(iii) If $(\Delta_0 \psi \Delta_1) \in \text{CONFORM}$, then
    
    $\text{FDEG}(\neg(\Delta_0 \psi \Delta_1)) = \text{max}(\{\text{FDEG}(\Delta_0), \text{FDEG}(\Delta_1)\})+1$,
(iv) If $\Pi_\xi \Delta \in \text{QFORM}$, then $\text{FDEG}(\Pi_\xi \Delta) = \text{FDEG}(\Delta)+1$.

We will henceforth use the usual infix notation without parentheses for identity formulas, e.g. $\theta = \theta^*$ for $\equiv(\theta, \theta^*)$. Furthermore, we will often omit the outermost parentheses, e.g. $A \psi B$ for $(A \psi B)$. With Definition 1-13, we can now characterise the free variables of terms and formulas.

Definition 1-13. Assignment of the set of variables that occur free in a term $\theta$ or in a formula $\Gamma$ (FV)
FV is a function on TERM $\cup$ FORM and

(i) If $\alpha \in \text{CONST}$, then $\text{FV}(\alpha) = \emptyset$,
(ii) If $\beta \in \text{PAR}$, then $\text{FV}(\beta) = \emptyset$,
(iii) If $\xi \in \text{VAR}$, then $\text{FV}(\xi) = \{\xi\}$,
(iv) If $\phi(\theta_0, \ldots, \theta_{n-1}) \in \text{FTERM}$, then
    
    $\text{FV}(\phi(\theta_0, \ldots, \theta_{n-1})) = \bigcup\{\text{FV}(\theta_i) | i < n\}$,
(v) If $\Phi(\theta_0, \ldots, \theta_{n-1}) \in \text{AFORM}$, then
    
    $\text{FV}(\Phi(\theta_0, \ldots, \theta_{n-1})) = \bigcup\{\text{FV}(\theta_i) | i < n\}$,
(vi) If $\neg\Delta \in \text{CONFORM}$, then $\text{FV}(\neg\Delta) = \text{FV}(\Delta)$,
(vii) If $\neg(\Delta_0 \psi \Delta_1) \in \text{CONFORM}$, then $\text{FV}(\neg(\Delta_0 \psi \Delta_1)) = \text{FV}(\Delta_0) \cup \text{FV}(\Delta_1)$, and
(viii) If $\Pi_\xi \Delta \in \text{QFORM}$ and, then $\text{FV}(\Pi_\xi \Delta) = \text{FV}(\Delta) \setminus \{\xi\}$.

Definition 1-14. The set of closed terms (CTERM)
CTERM = $\{\theta | \theta \in \text{TERM and FV}(\theta) = \emptyset\}$.

Note that, according to Definition 1-14, parameters are closed terms.
**Definition 1-15.** The set of closed formulas (CFORM)

\[
\text{CFORM} = \{ \Delta \mid \Delta \in \text{FORM and } \text{FV}(\Delta) = \emptyset \}.
\]

Closed formulas are also called propositions. Note that closed formulas can have parameters among their subexpression (see Definition 1-20). Sentences are now defined as the result of applying a performator to a closed formula.

**Definition 1-16.** The set of sentences (SENT; metavariables: \(\Sigma, \Sigma', \Sigma^*, \ldots\))

\[
\text{SENT} = \{ \%\Xi\Gamma \mid \Xi \in \text{PERF and } \Gamma \in \text{CFORM} \}.
\]

**Definition 1-17.** Assumption- and inference-sentences (ASENT and ISENT)

(i) \(\text{ASENT} = \{ \%\text{Suppose } \Gamma \mid \Gamma \in \text{CFORM} \} \)

(ii) \(\text{ISENT} = \{ \%\text{Therefore } \Gamma \mid \Gamma \in \text{CFORM} \} \)

**Theorem 1-12.** Unique category and unique decomposability for sentences

If \(\Sigma \in \text{SENT} \), then \(\Sigma \notin \text{TERM} \cup \text{QUANTOR} \cup \text{FORM} \) and

(i) \(\Sigma \in \text{ASENT} \) and \(\Sigma \notin \text{ISENT} \) and there is \(\Gamma \in \text{CFORM} \) such that \(\Sigma = \%\text{Suppose } \Gamma \) and for all \(\Gamma' \in \text{CFORM} \) with \(\Sigma = \%\text{Suppose } \Gamma' \) holds: \(\Gamma = \Gamma' \), or

(ii) \(\Sigma \in \text{ISENT} \) and \(\Sigma \notin \text{ASENT} \) and there is \(\Gamma \in \text{CFORM} \) such that \(\Sigma = \%\text{Therefore } \Gamma \) and for all \(\Gamma' \in \text{CFORM} \) with \(\Sigma = \%\text{Therefore } \Gamma' \) holds: \(\Gamma = \Gamma' \).

**Proof:** Suppose \(\Sigma \in \text{SENT} \). Then there are \(\Xi \in \text{PERF} \) and \(\Gamma \in \text{CFORM} \) such that \(\Sigma = \%\Xi\Gamma \). If \(\Sigma \in \text{TERM} \cup \text{QUANTOR} \cup \text{FORM} \), then we would have that \(\Sigma \in \text{ATERM} \) or \(\Sigma \in \text{FTERM} \cup \text{QUANTOR} \cup \text{FORM} \). In the first case, we would have \(\Sigma \in \text{BEXP} \), which contradicts Postulate 1-2-(ii). In the second case, there would be \(\mu \in \text{FUNC} \cup \text{QUANT} \cup \text{PRED} \cup \{ \%\neg, \%\} \) and \(\mu' \in \text{EXP} \) such that \(\Sigma = \%\mu\mu' \). Thus we would have \(\Xi = \mu \) and therefore \(\Xi \in \text{FUNC} \cup \text{QUANT} \cup \text{PRED} \cup \{ \%\neg, \%\} \), which contradicts Postulate 1-1. Therefore \(\Sigma \notin \text{TERM} \cup \text{QUANTOR} \cup \text{FORM} \).

If now \(\Sigma \in \text{SENT} \), then by Postulate 1-1-(viii) \(\Sigma \in \text{ASENT} \) or \(\Sigma \in \text{ISENT} \). The two cases will be treated separately. First: Suppose \(\Sigma \in \text{ASENT} \). Then there is \(\Gamma \in \text{CFORM} \) such that \(\Sigma = \%\text{Suppose } \Gamma \). If \(\Sigma \in \text{ISENT} \), then there would be \(\Gamma* \) such that \(\Sigma = \%\text{Therefore } \Gamma* \) and thus, according to Theorem 1-7-(iii), \(\%\text{Suppose } = \%\text{Therefore} \). Then \{ \%\text{Suppose}, \%\text{Therefore} \} \) would not be a 2-element set, which contradicts Postulate 1-1-(viii). Therefore \(\Sigma \notin \text{ISENT} \). Now, suppose \(\Gamma* \in \text{CFORM} \) and \(\Sigma = \%\text{Suppose } \Gamma* \).
Then we have "Suppose \( \Gamma \) = "Suppose \( \Gamma' \). With Theorem 1-7-(i), it follows immediately that \( \Gamma = \Gamma' \).

Second: Suppose \( \Sigma \in \text{isen} \). Then there is \( \Gamma \in \text{cfom} \) such that \( \Sigma = "\text{Therefore } \Gamma" \). For \( \Sigma \in \text{ASENT} \) we would again have a contradiction to Postulate 1-1-(viii). Therefore \( \Sigma \not\in \text{ASENT} \). Now, suppose \( \Gamma' \in \text{cfom} \) and \( \Sigma = "\text{Therefore } \Gamma'" \). Then we have "Therefore \( \Gamma' = "\text{Therefore } \Gamma'" \). With Theorem 1-7-(i), it follows immediately that \( \Gamma = \Gamma' \).

With Theorem 1-12, we can now define functions on the set \( \text{term} \cup \text{form} \cup \text{sent} \) by recursion on the complexity of terms, formulas and sentences.

**Definition 1-18. Assignment of the proposition of a sentence (P)**

\[
P = \{("\Xi \Gamma", \Gamma) \mid \Xi \in \text{perf} \text{ and } \Gamma \in \text{cfom}\}.
\]

**Note:** With Definition 1-16 and Theorem 1-12, it follows immediately that \( P \) is a function on \( \text{sent} \). Because of this, we use function notation: \( P("\Xi \Gamma") = \Gamma \). We now define the set of proper expressions as the union of the set of basic expressions and the grammatical categories.

**Definition 1-19. The set of proper expressions (PEXP)**

\[
\text{PEXP} = \text{bexp} \cup \text{quantor} \cup \text{term} \cup \text{form} \cup \text{sent}.
\]

**Definition 1-20. The subexpression function (SE)**

SE is a function on PEXP and

(i) If \( \tau \in \text{bexp} \), then SE(\( \tau \)) = \{\( \tau \}\},

(ii) If "\( \varphi(\theta_0, \ldots, \theta_{n-1})" \in \text{fterm} \), then

\[
\text{SE}("\varphi(\theta_0, \ldots, \theta_{n-1})") = \{"\varphi(\theta_0, \ldots, \theta_{n-1})", \varphi\} \cup \{\text{SE}(\theta_i) \mid i < n\},
\]

(iii) If "\( \Pi \xi \) \in \text{quantor} \), then SE("\( \Pi \xi \)) = \{"\( \Pi \xi \), \Pi, \xi\},

(iv) If "\( \Phi(\theta_0, \ldots, \theta_{n-1})" \in \text{afom} \), then

\[
\text{SE}("\Phi(\theta_0, \ldots, \theta_{n-1})") = \{"\Phi(\theta_0, \ldots, \theta_{n-1})", \Phi\} \cup \{\text{SE}(\theta_i) \mid i < n\},
\]

(v) If "\( \neg\Delta" \in \text{conform} \), then SE("\( \neg\Delta"\) = \{"\( \neg\Delta"\), \( \neg\)\} \cup \text{SE}(\Delta),

(vi) If "\( \Delta \psi \Delta_i" \in \text{conform} \), then

\[
\text{SE}("\( \Delta \psi \Delta_i") = \{"\( \Delta \psi \Delta_i")", \psi\} \cup \text{SE}(\Delta_0) \cup \text{SE}(\Delta_1),
\]

(vii) If "\( \Pi \xi \Delta \) \in \text{qform} \), then

\[
\text{SE}("\( \Pi \xi \Delta") = \{"\( \Pi \xi \Delta")\} \cup \text{SE}(\Delta) \cup \text{SE}(\Delta),
\]

(viii) If "\( \Xi \Delta" \in \text{sent} \), then SE("\( \Xi \Delta") = \{"\( \Xi \Delta")\}, \Xi \} \cup \text{SE}(\Delta).
**Definition 1-21. The subterm function (ST)**

ST is a function on \( \text{TERM} \cup \text{FORM} \cup \text{SENT} \) and for all \( \tau \in \text{TERM} \cup \text{FORM} \cup \text{SENT} \):
\[
\text{ST}(\tau) = \text{SE}(\tau) \cap \text{TERM}.
\]

**Definition 1-22. The subformula function (SF)**

SF is a function on \( \text{FORM} \cup \text{SENT} \) and for all \( \tau \in \text{FORM} \cup \text{SENT} \):
\[
\text{SF}(\tau) = \text{SE}(\tau) \cap \text{FORM}.
\]

The following definitions describe the syntax of L insofar as it goes beyond the sentence level. As before, we suppress explicit references to L. Definition 1-23 characterises sentence sequences as finite sequences of inference- and assumption-sentences:

**Definition 1-23. Sentence sequence (metavariables: \( \mathcal{S}, \mathcal{S}', \mathcal{S}'', \ldots \))**

\( \mathcal{S} \) is a sentence sequence if

\[
\begin{align*}
\text{if} & \\
\mathcal{S} \text{ is a finite sequence and for all } i \in \text{Dom}(\mathcal{S}) \text{ holds: } \mathcal{S}_i \in \text{SENT}.
\end{align*}
\]

**Definition 1-24. The set of sentence sequences (SEQ)**

\[
\text{SEQ} = \{ \mathcal{S} \mid \mathcal{S} \text{ is a sentence sequence} \}.
\]

**Definition 1-25. Conclusion assignment (C)**

\[
\text{C} = \{ (\mathcal{S}, \Gamma) \mid \mathcal{S} \in \text{SEQ}\setminus\{\emptyset\} \text{ and } \Gamma = \text{P}(\mathcal{S}_{\text{Dom}(\mathcal{S})-1}) \}.
\]

*Note:* From this definition it follows directly that C is a function on \( \text{SEQ}\setminus\{\emptyset\} \).

**Definition 1-26. Assignment of the subset of a sequence \( \mathcal{S} \) whose members are the assumption-sentences of \( \mathcal{S} \) (AS)**

\[
\text{AS} = \{ (\mathcal{S}, X) \mid \mathcal{S} \in \text{SEQ} \text{ and } X = \{ (i, \mathcal{S}_i) \mid i \in \text{Dom}(\mathcal{S}) \text{ and } \mathcal{S}_i \in \text{ASENT} \} \}.
\]

**Definition 1-27. Assignment of the set of assumptions (AP)**

\[
\text{AP} = \{ (\mathcal{S}, X) \mid \mathcal{S} \in \text{SEQ} \text{ and } X = \{ \Gamma \mid \text{There is an } i \in \text{Dom}(\text{AS}(\mathcal{S})) \text{ such that } \Gamma = \text{P}(\mathcal{S}_i) \} \}.
\]

**Definition 1-28. Assignment of the subset of a sequence \( \mathcal{S} \) whose members are the inference-sentences of \( \mathcal{S} \) (IS)**

\[
\text{IS} = \{ (\mathcal{S}, X) \mid \mathcal{S} \in \text{SEQ} \text{ and } X = \{ (i, \mathcal{S}_i) \mid i \in \text{Dom}(\mathcal{S}) \text{ and } \mathcal{S}_i \in \text{ISENT} \} \}.
\]

*Note:* From these definitions it follows directly that AS, AP and IS are functions on SEQ.
**Definition 1-29. Assignment of the set of subterms of the members of a sequence \( \vec{S} \) (STSEQ)**

\[
\text{STSEQ} = \{(\vec{S}, X) \mid \vec{S} \in \text{SEQ} \text{ and } X = \bigcup \{\text{ST}(\vec{S}_i) \mid i \in \text{Dom}(\vec{S})\}\}.
\]

*Note:* From this definition it follows directly that STSEQ a function on SEQ.

**Definition 1-30. Assignment of the set of subterms of the elements of a set of formulas \( X \) (STSF)**

\[
\text{STSF} = \{(X, Y) \mid X \subseteq \text{FORM} \text{ and } Y = \bigcup \{\text{ST}(A) \mid A \in X\}\}.
\]

*Note:* From this definition, it follows directly that STSF is a function on Pot(FORM).
1.2 Substitution

Now the substitution concept is to be established. In this, we restrict the usual substitution concept: Only atomic terms are substituenda and only closed terms are substituents. This makes it superfluous to rename bound variables in order to avoid variable clashes. The tasks that are fulfilled by free variables in many calculi and usually in model-theory are fulfilled by parameters, which are closed terms (see Definition 1-14), in the Speech Act Calculus as well as in the model-theory developed here. Furthermore, also sentences and sentence sequences are substitution bases and not just terms and formulas (clauses (ix) and (x) of Definition 1-31).

**Definition 1-31.** Substitution of closed terms for atomic terms in terms, formulas, sentences and sentence sequences

Substitution is a 3-ary function on \{\langle\theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \mu\rangle \mid k \in \mathbb{N} \setminus \{0\}, \langle \theta'_0, \ldots, \theta'_{k-1} \rangle \in k\text{CTERM}, \langle \theta_0, \ldots, \theta_{l-1} \rangle \in k\text{ATERM} \text{ and } \mu \in \text{TERM} \cup \text{FORM} \cup \text{SENT} \cup \text{SEQ}\}. '[.., .., ..]' is used as substitution operator. Values are assigned as follows:

(i) If \(\theta^+ \in \text{ATERM}\) and \(\theta^+ = \theta_{k-1}\), then \(\langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \theta^+ \rangle = \langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \theta^+ \rangle\),

(ii) If \(\theta^+ \in \text{ATERM}\), \(\theta^+ \neq \theta_{k-1}\) and \(k = 1\), then \(\langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \theta^+ \rangle = \theta^+\),

(iii) If \(\theta^+ \in \text{ATERM}\), \(\theta^+ \neq \theta_{k-1}\) and \(k \neq 1\), then

\[\langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \theta^+ \rangle = \langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{k-2}, \theta_0, \ldots, \theta_{l-1}, \theta^+ \rangle\],

(iv) If \(\varphi(\theta^0_0, \ldots, \theta^0_{l-1})^\gamma \in \text{FTERM}\), then

\[\langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \varphi(\theta^0_0, \ldots, \theta^0_{l-1})^\gamma \rangle = \varphi(\langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \theta^0_0, \ldots, \theta^0_{l-1} \rangle^\gamma),\]

(v) If \(\varphi(\theta_0, \ldots, \theta_{l-1})^\gamma \in \text{AFORM}\), then

\[\langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \varphi(\theta^0_0, \ldots, \theta^0_{l-1})^\gamma \rangle = \varphi(\langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \theta^0_0, \ldots, \theta^0_{l-1} \rangle^\gamma),\]

(vi) If \(\neg \Delta^\gamma \in \text{CONFORM}\), then

\[\langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \neg \Delta^\gamma \rangle = \neg \langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \Delta \rangle^\gamma,\]

(vii) If \(\langle \Delta_0 \psi \Delta_1 \rangle^\gamma \in \text{CONFORM}\), then

\[\langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \langle \Delta_0 \psi \Delta_1 \rangle^\gamma \rangle = \langle \langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \Delta_0 \psi \theta^0_0, \ldots, \theta^0_{l-1} \rangle^\gamma, \theta_0, \ldots, \theta_{l-1}, \Delta_1 \rangle^\gamma,\]

(viii) If \(\Pi \xi \Delta^\gamma \in \text{QFORM}\), then let \(\langle i_0, \ldots, i_{s-1} \rangle\) be such that \(s = 1 - \{j \mid j < k \text{ and } \theta_j \neq \xi \}\) and for all \(1 < l \leq s; i_l \in \{j \mid j < k \text{ and } \theta_j \neq \xi \}\) and for all \(k < l < s; i_k < i_l\) and let

\[\langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \Pi \xi \Delta^\gamma \rangle = \Pi \xi \langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \Delta \rangle^\gamma, \text{ if } \{j \mid j \leq k \text{ and } \theta_j \neq \xi \} \neq 0, \langle \theta'_0, \ldots, \theta'_{k-1}, \theta_0, \ldots, \theta_{l-1}, \Pi \xi \Delta^\gamma \rangle = \Pi \xi \Delta^\gamma \text{ otherwise,}\]

---

9 Let \(Y = \{f \mid f \in \text{Pot}(X \times Y)\}\) and \(f\) is function on \(X\) and \(\text{Ran}(f) \subseteq Y\) and let \(\langle a_0, \ldots, a_{l-1} \rangle = \{(i, a) \mid i < k\}\). In the following we will designate 1-tuples by their values if we write down substitution results. So, for example, \([\theta'_0, \theta_0, \Delta]\) for \([\theta'_0, \langle \theta'_0 \rangle, \theta_0, \Delta]\).
Clause (viii) regulates the substitution in quantificational formulas. In this case, the substitution is to be carried out for and only for those members of the substituendum sequence that are not identical to the variable bound by the respective quantifier (if such members exist). Accordingly, the desired members of the substituendum sequence and the corresponding members of the substituens sequence have to be singled out. This is achieved by the (in each case uniquely determined) number sequence \(\langle i_0, \ldots, i_{s-1}\rangle\), which picks exactly those indices whose values in the substituendum sequence are different from the bound variable. The new substituendum resp. substituens sequences, which have the desired properties, are then simply the result of the composition of the original substituendum resp. substituens sequences with \(\langle i_0, \ldots, i_{s-1}\rangle\). If, however, all members of the substituendum sequence are identical to the bound variable, then the substitution result is to be identical to the substitution basis, i.e. the respective quantificational formula.

Now, some theorems are to be established which are needed for the meta-theory of the Speech Act Calculus – especially from ch. 4 onwards. We recommend that more impatient readers skip these theorems for now and return here if the need arises. The first theorem eases proofs by induction on the degree of a formula. It is proved by induction on the complexity of a formula.

**Theorem 1-13. Conservation of the degree of a formula as substitution basis**

If \(\theta \in \text{CTERM}, \theta' \in \text{ATERM} \text{ and } \Delta \in \text{FORM}\), then \(\text{FDEG}(\Delta) = \text{FDEG}([\theta, \theta', \Delta])\).

**Proof:** Suppose \(\theta \in \text{CTERM}, \theta' \in \text{ATERM} \text{ and } \Delta \in \text{FORM} \). The proof is carried out by induction on the complexity of \(\Delta\). Suppose \(\Delta = \tau\Phi(\theta_0, \ldots, \theta_{n-1})\) \(\in \text{AFORM}\). According to Definition 1-12, we then have \(\text{FDEG}(\Delta) = 0\). Then we have that \([\theta, \theta', \Delta] = [\theta, \theta', \tau\Phi(\theta_0, \ldots, \theta_{n-1})] = \tau\Phi([\theta, \theta', \theta_0], \ldots, [\theta, \theta', \theta_{n-1}]) \in \text{AFORM}\). Therefore also \(\text{FDEG}([\theta, \theta', \Delta]) = 0\). Suppose the statement holds for \(\Delta_0, \Delta_1 \in \text{FORM}\). That is: \(\text{FDEG}(\Delta_0) = \text{FDEG}([\theta, \theta', \Delta_0]) \) and \(\text{FDEG}(\Delta_1) = \text{FDEG}([\theta, \theta', \Delta_1])\).

**Ad CONFORM:** Now, suppose \(\Delta = \tau\neg\Delta_0\). Then we have that \(\text{FDEG}(\Delta) = \text{FDEG}(\tau\neg\Delta_0) = \text{FDEG}(\Delta_0) + 1 = \text{FDEG}([\theta, \theta', \Delta_0]) + 1 = \text{FDEG}(\tau\neg[\theta, \theta', \Delta_0]) = \text{FDEG}(\tau\neg\Delta_0) = \text{FDEG}(\tau\neg\Delta_0)\).
FDEG([0, θ', ¬Δ₀]) = FDEG([0, θ', Δ]). Now, suppose Δ = γ(Δ₀ θ Δ₁) for some θ ∈ CON\{¬}. Then we have that FDEG(Δ) = FDEG(γ(Δ₀ θ Δ₁)) = max({FDEG(Δ₀), FDEG(Δ₁)})+1 = max({FDEG([0, θ', Δ₀]), FDEG([0, θ', Δ₁])})+1 = FDEG(γ([0, θ', Δ₀] θ [0, θ', Δ₁])) = FDEG([0, θ', (Δ₀ θ Δ₁)⁺]) = FDEG([0, θ', Δ]).

Ad QFORM: Now, suppose Δ = γ(TIξΔ₀). First, let ξ ≠ θ. Then we have that FDEG(Δ) = FDEG(γ(TIξΔ₀⁺) = FDEG(Δ₀)⁺1 = FDEG([0, θ', Δ₀])+1 = FDEG(γ(TIξθ Δ₀⁺)) = FDEG([0, θ', γ(TIξθ Δ₀⁺)]) = FDEG([0, θ', Δ]). Now, suppose ξ = θ. Then we have that FDEG(Δ) = FDEG(γ(TIξθ Δ₀⁺) = FDEG([0, θ', γ(TIξθ Δ₀⁺)]) = FDEG([0, θ', Δ]).

Theorem 1.14. For all substituenda and substitution bases it holds that either all closed terms are subterms of the respective substitution result or that the respective substitution result is identical to the respective substitution basis for all closed terms.

If θ' ∈ ATERM, θ* ∈ TERM, Δ ∈ FORM, then:

(i) θ ∈ ST([0, θ', θ*]) for all θ ∈ CTERM or [0, θ', θ*] = θ* for all θ ∈ CTERM, and

(ii) θ ∈ ST([0, θ', Δ]) for all θ ∈ CTERM or [0, θ', Δ] = Δ for all θ ∈ CTERM.

Proof: Suppose θ' ∈ ATERM, θ* ∈ TERM, Δ ∈ FORM. Ad (i): The proof is carried out by induction on the complexity of θ*. Suppose θ* ∈ ATERM. If θ' = θ*, then we have that [0, θ', θ*] = θ and thus that θ ∈ ST([0, θ', θ*]) for all θ ∈ CTERM. If θ' ≠ θ*, then we have that [0, θ', θ*] = θ* for all θ ∈ CTERM. Suppose the statement holds for θ*₀, ..., θ*ᵣ, θ*₁ ∈ TERM and let θ* = γ(θ*₀, ..., θ*ᵣ) ∈ FTERM. Then we have that [0, θ', θ*] = [0, θ', γ(θ*₀, ..., θ*ᵣ)]⁺ = γ([0, θ', θ*₀], ..., [0, θ', θ*ᵣ])⁺ for all θ ∈ CTERM. According to the I.H., we have that for all i < r: θ ∈ ST([0, θ', θ*]) for all θ ∈ CTERM or [0, θ', θ*] = θ*, for all θ ∈ CTERM. Suppose there is an i < r such that θ ∈ ST([0, θ', θ*]) for all θ ∈ CTERM. Then we have that θ ∈ ST(γ(θ*₀, ..., θ*ᵣ⁺)) = ST(γ(θ₀, θₕ, θ*)) for all θ ∈ CTERM. Suppose there is no i < r such that θ ∈ ST([0, θ', θ*]) for all θ ∈ CTERM. According to the I.H., we then have that [0, θ', θ*] = θ*, for all θ ∈ CTERM and all i < r. Therefore [0, θ', θ*] = γ(θ*₀, ..., θ*ᵣ⁺)⁺ = γ(θ₀, θₕ, θ*)⁺ = θ* for all θ ∈ CTERM.

Ad (ii): Suppose Δ ∈ FORM. The proof is carried out by induction on the complexity of Δ. Suppose Δ = γ(φₜ₀, ..., φᵣ)⁺ ∈ AFORM. This case is proved in the same way as the FTERM-case by applying (i).

Suppose the statement holds for Δ₀, Δ₁ ∈ FORM and let Δ = γ(¬Δ₀⁺) ∈ CONFORM. Then we have that [0, θ', Δ] = [0, θ', γ(¬Δ₀⁺)] = γ(¬[0, θ', Δ₀⁺] for all θ ∈ CTERM. Accord-
ing to the I.H., we have that $\theta \in \text{ST}(\{\theta, \theta', \Delta_0\})$ for all $\theta \in \text{TERM}$ or $[\theta, \theta', \Delta_0] = \Delta_0$ for all $\theta \in \text{TERM}$. In the first case, we thus have that $\theta \in \text{ST}(\neg-[\theta, \theta', \Delta_0]) = \text{ST}(\theta)$ for all $\theta \in \text{TERM}$. In the second case, we have that $[\theta, \theta', \Delta] = \neg-[\theta, \theta', \Delta_0] = \neg-\Delta_0 = \Delta$ for all $\theta \in \text{TERM}$. Suppose $\Delta = \neg(\Delta_0 \psi \Delta_1)$. This case is proved in the same way as the negation-case.

Suppose $\Delta = \neg(\Pi \Delta_0 \neg).$ First, suppose $\xi = \theta$. Then we have that $[\theta, \theta', \Delta] = [\theta, \theta', \Pi \Delta_0 \neg] = \Delta$ for all $\theta \in \text{TERM}$. Now, suppose $\xi \neq \theta$. Then we have that $[\theta, \theta', \Delta] = [\theta, \theta', \Pi \Delta_0 \neg] = \Delta$ for all $\theta \in \text{TERM}$. According to the I.H., we then have that $\theta \in \text{ST}(\Pi \Delta_0 \neg)$ for all $\theta \in \text{TERM}$ or $[\theta, \theta', \Delta_0] = \Delta_0$ for all $\theta \in \text{TERM}$. In the first case, we thus have that $\theta \in \text{ST}(\Pi \Delta_0 \neg) = \text{ST}(\theta)$ for all $\theta \in \text{TERM}$. In the second case, we have that $[\theta, \theta', \Delta] = [\Pi \Delta_0 \neg] = \Pi \Delta_0 \neg = \Delta$ for all $\theta \in \text{TERM}$. ■

**Theorem 1-15. Bases for the substitution of closed terms in terms**

If $\theta \in \text{TERM}$, $k \in \mathbb{N}\setminus\{0\}$, $\{\theta_0, ..., \theta_{k-1}\} \subseteq \text{TERM}$ and $\{\xi_0, ..., \xi_{k-1}\} \subseteq \text{VAR}\setminus\text{ST}(\theta)$, where $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$, then there is a $\theta^* \in \text{TERM}$, where $\text{FV}(\theta^*) \subseteq \{\xi_0, ..., \xi_{k-1}\} \cup \text{FV}(\theta)$ and $\text{ST}(\theta^*) \cap \{\theta_0, ..., \theta_{k-1}\} = \emptyset$ such that $\theta = [\theta_0, ..., \theta_{k-1}, \xi_0, ..., \xi_{k-1}]$, $\theta^*$. Then we have that $[\theta, \theta', \theta_0, ..., \theta_{k-1}, \xi_0, ..., \xi_{k-1}, \theta^*] = 0$. In the second case, there is an $i < k$ such that $\theta = \theta_i$, or there is an $i < k$ such that $\theta = \theta_i$. In the first case, it follows that $\theta = [\{\theta_0, ..., \theta_{k-1}\}, \xi_0, ..., \xi_{k-1}]$ and $\theta^*$ and we have that $\text{FV}(\theta) \subseteq \{\xi_0, ..., \xi_{k-1}\} \cup \text{FV}(\theta^*)$ and $\text{ST}(\theta) \cap \{\theta_0, ..., \theta_{k-1}\} = \emptyset$. In the second case, there is an $i < k$ such that $\theta = [\{\theta_0, ..., \theta_{k-1}\}, \xi_0, ..., \xi_{k-1}, \theta^*]$. Because of $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$, we then also have that $\theta = [\{\theta_0, ..., \theta_{k-1}\}, \xi_0, ..., \xi_{k-1}] = [\theta, \theta', \theta_0, ..., \theta_{k-1}]$. With $\text{FV}(\theta) \subseteq \{\xi_0, ..., \xi_{k-1}\} \cup \text{FV}(\theta^*)$ and $\text{ST}(\theta) \cap \{\theta_0, ..., \theta_{k-1}\} = \emptyset$, now, suppose $\theta \in \text{VAR}$. Because of $\{\xi_0, ..., \xi_{k-1}\} \subseteq \text{VAR}\setminus\text{ST}(\theta)$, we then have that $\theta = [\{\theta_0, ..., \theta_{k-1}\}, \xi_0, ..., \xi_{k-1}] \subseteq \text{VAR} \cap \text{TERM} = \emptyset$ we also have that $\text{ST}(\theta) \cap \{\theta_0, ..., \theta_{k-1}\} = \emptyset$.

Suppose the statement holds for $\theta_0, ..., \theta_{r-1} \in \text{TERM}$ and let $\theta = \neg[\theta_0, ..., \theta_{r-1}] \in \text{FTERM}$. Now, suppose $k \in \mathbb{N}\setminus\{0\}$, $\{\theta_0, ..., \theta_{k-1}\} \subseteq \text{TERM}$ and $\{\xi_0, ..., \xi_{k-1}\} \subseteq \text{VAR}\setminus\text{ST}(\theta)$, where $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$. With $\cup\{\text{ST}(\theta_i') | i < r\} \subseteq \text{ST}(\theta)$, it then holds for all $i < r$ that $\{\xi_0, ..., \xi_{k-1}\} \subseteq \text{VAR}\setminus\text{ST}(\theta_i')$. According to the I.H., we then
have that for every $\theta_i$ ($i < r$) there is a $\theta^+_i \in \text{TERM}$ such that $\theta_i = \{\theta_0, \ldots, \theta_{k-1}\}, (\xi_1, \ldots, 
abla_{k-1}), \theta^+_i$ and $\text{FV}(\theta^+_i) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup \text{FV}(\theta^+_i)$ and $\text{ST}(\theta^+_i) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$. Then there is no $i < k$ such that $\varphi(\theta^+_i \ldots \theta^+_i) = 0$. In the first case, we have that $\varphi(\theta^+_i \ldots \theta^+_i) = \varphi(\{\theta_0, \ldots, \theta_{k-1}\}, (\xi_0, \ldots, \xi_{k-1}), (\xi^+_0, \ldots, \xi^+_i))$. We also have that $\text{FV}(\varphi(\theta^+_i \ldots, \theta^+_i)) = \bigcup \{\text{FV}(\theta^+_i) \mid i < r\}$ and hence, with the I.H., that $\text{FV}(\varphi(\theta^+_i \ldots, \theta^+_i)) \subseteq \bigcup \{\text{FV}(\theta^+_i) \mid i < r\} \cup \{\xi_0, \ldots, \xi_{k-1}\} = \text{FV}(\varphi(\theta^+_i \ldots, \theta^+_i)) \cup \{\xi_0, \ldots, \xi_{k-1}\}$. According to the case assumption and the I.H., we also have that $\text{ST}(\varphi(\theta^+_i \ldots, \theta^+_i)) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \{\varphi(\theta^+_i \ldots, \theta^+_i) \cup \{\text{ST}(\theta^+_i) \mid i < r\} \cap \{\theta_0, \ldots, \theta_{k-1}\} = 0 \cup \{\text{ST}(\theta^+_i) \mid i < r\} \cap \{\theta_0, \ldots, \theta_{k-1}\} = 0$. In the second case there is an $i < k$ such that $\varphi(\theta^+_i \ldots, \theta^+_i) = [\{\theta_0, \ldots, \theta_{k-1}\}, (\xi_0, \ldots, \xi_{k-1}), (\xi^+_0, \ldots, \xi^+_i)]$. Because of $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$, we then also have that $\varphi(\theta^+_i \ldots, \theta^+_i) = [\{\theta_0, \ldots, \theta_{k-1}\}, (\xi_0, \ldots, \xi_{k-1}), (\xi^+_0, \ldots, \xi^+_i)]$ and $\text{FV}(\xi_i) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup \text{FV}(\varphi(\theta^+_i \ldots, \theta^+_i))$ and because of $\xi_i \notin \text{CTERM}$ also $\text{ST}(\xi_i) \cap \{\theta_0, \ldots, \theta_{k-1}\} = 0$.  

**Theorem 1.16. Bases for the substitution of closed terms in formulas**

If $\Delta \in \text{FORM}$, $k \in \mathbb{N}\{0\}, \{\theta_0, \ldots, \theta_{k-1}\} \subseteq \text{CTERM}$ and $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq \text{VAR} \setminus \text{ST}(\Delta)$, where $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$, then there is a $\Delta' \in \text{FORM}$, where $\text{FV}(\Delta') \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup \text{FV}(\Delta)$ and $\text{ST}(\Delta') \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$ such that $\Delta = [\{\theta_0, \ldots, \theta_{k-1}\}, (\xi_0, \ldots, \xi_{k-1}), \Delta']$.

**Proof:** By induction on the complexity of $\Delta$. Suppose $\Delta = \varphi(\theta^+_0 \ldots, \theta^+_r \ldots) \in \text{AFORM}$. Now, suppose $k \in \mathbb{N}\{0\}, \{\theta_0, \ldots, \theta_{k-1}\} \subseteq \text{CTERM}$ and $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq \text{VAR} \setminus \text{ST}(\varphi(\theta^+_0 \ldots, \theta^+_r \ldots))$, where $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$. With $\{\text{ST}(\theta^+_i) \mid i < r\} = \text{ST}(\varphi(\theta^+_0 \ldots, \theta^+_r \ldots)) \cap \{\theta_0, \ldots, \theta_{k-1}\}$, it then holds for all $i < r$ that $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq \text{VAR} \setminus \text{ST}(\theta^+_i)$. According to Theorem 1.15, we then have that for every $\theta^+_i$ ($i < r$) there is a $\theta^+_j \in \text{TERM}$ such that $\theta^+_i = [\{\theta_0, \ldots, \theta_{k-1}\}, (\xi_0, \ldots, \xi_{k-1}), \theta^+_j]$ and $\text{FV}(\theta^+_j) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup \text{FV}(\theta^+_i)$ and $\text{ST}(\theta^+_j) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$. Then we also have that $\varphi(\theta^+_0 \ldots, \theta^+_r \ldots) = \varphi([\{\theta_0, \ldots, \theta_{k-1}\}, (\xi_0, \ldots, \xi_{k-1}), \theta^+_0 \ldots, \theta^+_r \ldots] \subseteq \bigcup \{\text{FV}(\theta^+_i) \mid i < r\}$ and thus $\text{FV}(\varphi(\theta^+_0 \ldots, \theta^+_r \ldots)) \subseteq \bigcup \{\text{FV}(\theta^+_i) \mid i < r\}$ and $\{\xi_0, \ldots, \xi_{k-1}\} = \text{FV}(\varphi(\theta^+_0 \ldots, \theta^+_r \ldots)) \subseteq \{\xi_0, \ldots, \xi_{k-1}\}$. We then also have that $\text{ST}(\varphi(\theta^+_0 \ldots, \theta^+_r \ldots)) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \bigcup \{\text{ST}(\theta^+_i) \mid i < r\} \cap \{\theta_0, \ldots, \theta_{k-1}\} = \bigcup \{\text{ST}(\theta^+_i) \mid i < r\} = \emptyset$.
Now, suppose that the statement holds for $\Delta_0$, $\Delta_1 \in \text{FORM}$ and let $\Delta = \gamma_\Delta \Delta_0^\gamma \in \text{CONFORM}$. Now, suppose $k \in \mathbb{N}\setminus\{0\}$, $\{\theta_0, \ldots, \theta_{k-1}\} \subseteq \text{CTERM}$ and $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq \text{VAR}\backslash \text{ST}(\gamma_\Delta \Delta_0^\gamma)$, where $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$. With $\text{ST}(\Delta_0) = \text{ST}(\gamma_\Delta \Delta_0^\gamma)$, we then have $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq \text{VAR}\backslash \text{ST}(\Delta_0)$. According to the I.H. for $\Delta_0$, there is then a $\Delta^+_0 \in \text{FORM}$ such that $\Delta_0 = [(\theta_0, \ldots, \theta_{k-1}), (\xi_0, \ldots, \xi_{k-1}), \Delta^+_0]$ and $\text{FV}(\Delta^+_0) \subseteq \text{FV}(\Delta_0) \cup \{\xi_0, \ldots, \xi_{k-1}\}$ and $\text{ST}(\Delta^+_0) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$. Then we also have that $\gamma_\Delta \Delta_0^\gamma = \gamma_\Delta [(\theta_0, \ldots, \theta_{k-1}), (\xi_0, \ldots, \xi_{k-1}), \gamma_\Delta \Delta_0^\gamma]$. Furthermore, we have that $\text{FV}(\gamma_\Delta \Delta_0^\gamma) = \text{FV}(\Delta^+_0)$ and thus, with the I.H., that $\text{FV}(\gamma_\Delta \Delta_0^\gamma) \subseteq \text{FV}(\Delta_0) \cup \{\xi_0, \ldots, \xi_{k-1}\} = \text{FV}(\gamma_\Delta \Delta_0^\gamma) \cup \{\xi_0, \ldots, \xi_{k-1}\}$. According to the I.H., we also have that $\text{ST}(\gamma_\Delta \Delta_0^\gamma) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \text{ST}(\Delta^+_0) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$.

Now, let $\Delta = \gamma_\Delta \text{QFORM}$ and suppose $k \in \mathbb{N}\setminus\{0\}$, $\{\theta_0, \ldots, \theta_{k-1}\} \subseteq \text{CTERM}$ and $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq \text{VAR}\backslash \text{ST}(\gamma_\Delta \text{QFORM})$, where $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$. Then, we have in particular $\zeta \notin \{\xi_0, \ldots, \xi_{k-1}\}$. With $\text{ST}(\Delta_0) \subseteq \text{ST}(\gamma_\Delta \text{QFORM})$, we have that $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq \text{VAR}\backslash \text{ST}(\Delta_0)$. According to the I.H. for $\Delta_0$, there is then a $\Delta^+_0 \in \text{FORM}$ such that $\Delta_0 = [(\theta_0, \ldots, \theta_{k-1}), (\xi_0, \ldots, \xi_{k-1}), \Delta^+_0]$ and $\text{FV}(\Delta^+_0) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup \text{FV}(\Delta_0)$ and $\text{ST}(\Delta^+_0) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$. Since $\zeta \notin \{\xi_0, \ldots, \xi_{k-1}\}$, we then have $\gamma_\Delta \text{QFORM} = \gamma_\Delta [(\theta_0, \ldots, \theta_{k-1}), (\xi_0, \ldots, \xi_{k-1}), \gamma_\Delta \text{QFORM}]$. We then have $\text{FV}(\gamma_\Delta \text{QFORM}) = \text{FV}(\Delta^+_0) \cup \{\zeta\} \subseteq (\text{FV}(\Delta_0) \cup \{\xi_0, \ldots, \xi_{k-1}\}) = \text{FV}(\gamma_\Delta \text{QFORM}) \cup \{\xi_0, \ldots, \xi_{k-1}\}$. With $\text{VAR} \cap \text{CTERM} = \emptyset$ we then also have $\text{ST}(\gamma_\Delta \text{QFORM}) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \text{ST}(\Delta^+_0) \cup \{\zeta\} \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$. ■
Theorem 1-17. Alternative bases for the substitution of closed terms for variables in terms
If \( \{\xi, \zeta\} \cup X \subseteq \text{VAR}, \) where \( \{\xi, \zeta\} \cap X = \emptyset, \) and \( \theta \in \text{TERM}, \) where \( \text{FV}(\theta) \subseteq \{\xi\} \cup X, \) then there is a \( \theta^* \in \text{TERM}, \) where \( \text{FV}(\theta^*) \subseteq \{\zeta\} \cup X, \) such that for all \( \theta' \in \text{TERM} \) it holds that \( [\theta', \xi, \theta] = [\theta', \zeta, \theta^*]. \)

Proof: Suppose \( \{\xi, \zeta\} \cup X \subseteq \text{VAR}, \) where \( \{\xi, \zeta\} \cap X = \emptyset, \) and \( \theta \in \text{TERM}, \) where \( \text{FV}(\theta) \subseteq \{\xi\} \cup X. \) For \( \xi = \zeta, \) the statement follows immediately with \( \theta^* = \theta. \) Now, suppose \( \xi \neq \zeta. \) The proof is now carried out by induction on the complexity of \( \theta. \) Suppose \( \theta \in \text{CONST} \cup \text{PAR}. \) Then it holds with \( \theta^* = \theta \) that \( \text{FV}(\theta^*) = \emptyset \subseteq \{\xi\} \cup X \) and that for all \( \theta' \in \text{TERM}: \]  

\[ [\theta', \xi, \theta] = [\theta', \zeta, \theta^*]. \]

Now, suppose the statement holds for \( \theta_0, \ldots, \theta_{r-1} \in \text{TERM} \) and suppose \( \theta = \varphi(\theta_0, \ldots \theta_{r-1}) \in \text{FTERM}. \) Then we have for all \( i < r: \]  

\[ \text{FV}(\theta_i) \subseteq \{\xi\} \cup X. \] According to the I.H., we then have that for all \( i < r \) there is a \( \theta^*_i \in \text{TERM}, \) with \( \text{FV}(\theta^*_i) \subseteq \{\xi\} \cup X, \) such that for all \( \theta' \in \text{TERM} \) it holds that \( [\theta', \xi, \theta_i] = [\theta', \zeta, \theta^*_i]. \) With \( \theta^* = \varphi(\theta^*_0, \ldots \theta^*_{r-1}) \in \text{FTERM} \) it then holds that \( \text{FV}(\theta^*) \subseteq \{\xi\} \cup X \) and that for all \( \theta' \in \text{TERM}: \]  

\[ [\theta', \xi, \theta] = [\theta', \zeta, \theta^*]. \] ■

Theorem 1-18. Alternative bases for the substitution of closed terms for variables in formulas
If \( \{\xi, \zeta\} \cup X \subseteq \text{VAR}, \) where \( \{\xi, \zeta\} \cap X = \emptyset, \) and \( \Delta \in \text{FORM}, \) where \( \text{FV}(\Delta) \subseteq \{\xi\} \cup X \) and \( \zeta \notin \text{ST}(\Delta), \) then there is a \( \Delta^* \in \text{FORM}, \) where \( \text{FV}(\Delta^*) \subseteq \{\zeta\} \cup X, \) such that for all \( \theta' \in \text{TERM} \) it holds that \( [\theta', \xi, \Delta] = [\theta', \zeta, \Delta^*]. \)

Proof: The proof is carried out by induction on the complexity of \( \Delta. \) Suppose \( \Delta = \varphi(\theta_0, \ldots \theta_{r-1}) \in \text{AFORM}. \) Let \( \{\xi, \zeta\} \cup X \subseteq \text{VAR}, \) where \( \{\xi, \zeta\} \cap X = \emptyset, \) and \( \text{FV}(\Delta) \subseteq \{\xi\} \cup X \) and \( \zeta \notin \text{ST}(\Delta). \) Then we have for all \( i < r: \]  

\[ \text{FV}(\theta_i) \subseteq \{\xi\} \cup X. \] According to Theorem 1-17, there is then for all \( i < r \) a \( \theta^*_i \in \text{TERM}, \) where \( \text{FV}(\theta^*_i) \subseteq \{\xi\} \cup X, \) such that for all \( \theta' \in \text{TERM} \) holds: \( [\theta', \xi, \theta_i] = [\theta', \zeta, \theta^*_i]. \) Then it holds with \( \Delta^* = \varphi(\theta^*_0, \ldots \theta^*_{r-1}) \) that \( \text{FV}(\Delta^*) \subseteq \{\zeta\} \cup X \) and that for all \( \theta' \in \text{TERM} \) holds: \( [\theta', \xi, \varphi(\theta_0, \ldots \theta_{r-1})] = \varphi([\theta', \xi, \theta_0], \ldots [\theta', \xi, \theta_{r-1}]). \)
ɕ, 0], ... [θ', ɕ, θ_{r-1}])^\gamma = \gamma \Phi([0', ɕ, θ_{*0}], ... [0', ɕ, θ_{*r-1}])^\gamma = [θ', ɕ, \theta(θ_{*0}, ... θ_{*r-1})]^\gamma = [θ', ɕ, Δ^*].

Now, suppose the statement holds for Δ₀, Δ₁ ∈ FORM and let Δ ∈ CONFORM. Let {ξ, ζ} ⊂ X ⊂ VAR, where {ξ, ζ} ∩ X = ∅, and FV(Δ) ⊂ {ξ} ∪ X and ζ ∉ ST(Δ). First, suppose Δ = \gamma θ Δ_0^\gamma. Then we have FV(Δ₀) = FV(Δ) ⊂ {ξ} ∪ X and ζ ∉ ST(Δ₀). According to the I.H., we have a Δ^*₀ ∈ FORM, where FV(Δ^*₀) ⊂ {ξ} ∪ X, such that for all θ ∈ CTERM holds: [θ', ξ, Δ₀] = [θ', ζ, Δ^*₀]. With Δ^* = \gamma θ Δ^*₀, it then holds that FV(Δ^*) ⊂ {ξ} ∪ X and that for all θ' ∈ CTERM: [θ', ξ, \gamma θ Δ₀^\gamma] = \gamma [θ', ξ, Δ₀] = \gamma [θ', ζ, Δ^*₀]^\gamma = [θ', ξ, \gamma θ Δ^*₀]^\gamma = [θ', ξ, Δ^*].

Now, suppose Δ = \gamma θ Δ₀^\gamma ∈ CONFORM. Then we have FV(Δ₀) ⊂ FV(Δ) ⊂ {ξ} ∪ X and ζ ∉ ST(Δ₀) and FV(Δ₁) ⊂ FV(Δ) ⊂ {ξ} ∪ X and ζ ∉ ST(Δ₁). According to the I.H., there are then Δ^*₀, Δ^*₁ ∈ FORM, where FV(Δ^*₀) ⊂ {ξ} ∪ X and FV(Δ^*₁) ⊂ {ξ} ∪ X, such that for all θ' ∈ CTERM holds: [θ', ξ, Δ₀] = [θ', ζ, Δ^*₀] and [θ', ξ, Δ₁] = [θ', ζ, Δ^*₁]. With Δ^* = \gamma θ Δ^*₀ ∪ Δ^*₁, it then holds that FV(Δ^*) ⊂ {ξ} ∪ X and that for all θ' ∈ CTERM: [θ', ξ, \gamma θ Δ₀^\gamma] = \gamma ([θ', ζ, Δ₀] ∪ [θ', ξ, Δ₁])^\gamma = \gamma ([θ', ζ, Δ^*₀] ∪ [θ', ξ, Δ^*₁])^\gamma = [θ', ζ, \gamma θ Δ^*₀ ∪ Δ^*₁]^\gamma = [θ', ζ, Δ^*].

Now, suppose Δ = \gamma Πξ Δ₀^\gamma ∈ QFORM. Let {ξ, ζ} ⊂ X ⊂ VAR, where {ξ, ζ} ∩ X = ∅, and FV(Δ) ⊂ {ξ} ∪ X and ζ ∉ ST(Δ). Then we have in particular ξ ≠ ζ'. First, suppose ζ = ζ'. Then we have [θ', ξ, \gamma Πξ Δ₀^\gamma] = \gamma Πξ Δ₀^\gamma for all θ' ∈ CTERM and FV(Δ) ⊂ X. Let Δ^* = Δ = \gamma Πξ Δ₀^\gamma. Since ζ ∉ ST(Δ), we also have [θ', ξ, \gamma Πξ Δ₀^\gamma] = \gamma Πξ Δ₀^\gamma for all θ' ∈ CTERM and FV(Δ^*) = FV(Δ) ⊂ X ⊂ {ξ} ∪ X. Now, suppose ζ ≠ ζ'. Then we have FV(Δ₀) ⊂ FV(Δ) ∪ {ξ} ⊂ {ξ} ∪ X ∪ {ξ'} and ζ ∉ ST(Δ₀). According to the I.H., there is then Δ^*₀ ∈ FORM, where FV(Δ^*₀) ⊂ {ξ} ∪ X ∪ {ξ'}, such that for all θ' ∈ CTERM it holds that [θ', ξ, Δ₀] = [θ', ζ, Δ^*₀]. With Δ^* = \gamma Πξ Δ^*₀, it then holds that FV(Δ^*) ⊂ {ξ} ∪ X and that for all θ' ∈ CTERM it holds that [θ', ξ, \gamma Πξ Δ₀^\gamma] = \gamma Πξ[θ', θ₀, θ₁] = \gamma Πξ[θ', θ₀, θ₁] = \gamma Πξ[θ', θ₀, θ₁] = [θ', ξ, Δ^*₀].
**Theorem 1-19.** Unique substitution bases (a) for terms

If \( \theta, \theta^+ \in \text{TERM} \), \( \theta^* \in \text{CTERM}(\text{ST}(\theta) \cup \text{ST}(\theta^+)) \) and \( \theta^\delta \in \text{ATERM} \) and if \([\theta^*, \theta^\delta, \theta] = [\theta^*, \theta^\delta, \theta^+]\), then \( \theta = \theta^+ \).

**Proof:** By induction on the complexity of \( \theta \). Suppose \( \theta \in \text{ATERM} \). Now, suppose \( \theta^+ \in \text{TERM} \), \( \theta^* \in \text{CTERM}(\text{ST}(\theta) \cup \text{ST}(\theta^+)) \) and \( \theta^\delta \in \text{ATERM} \) and suppose \([\theta^*, \theta^\delta, \theta] = [\theta^*, \theta^\delta, \theta^+]\). Now, suppose \( \theta^\delta = \theta \). Then we have \([\theta^*, \theta^\delta, \theta] = \theta^*\). Then we also have \( \theta^* = [\theta^*, \theta^\delta, \theta^+] \). Since, according to the hypothesis, \( \theta^* \not\in \text{ST}(\theta^+) \) and thus \( \theta^+ \neq \theta^* \), we then have \( \theta = \theta^+ \). Now, suppose \( \theta^\delta \neq \theta \). Then we have \([\theta^*, \theta^\delta, \theta] = \theta^*\). Then we have \( \theta = [\theta^*, \theta^\delta, \theta^+] \). Because of \( \theta^* \not\in \text{ST}(\theta) \) and Theorem 1-14-(i), we then also have \( \theta = \theta^+ \).

Now, suppose the statement holds for \( \{\theta_0, ..., \theta_{r-1}\} \subseteq \text{TERM} \) and let \( \tilde{\varphi}(\theta_0, ..., \theta_{r-1})^\gamma \in \text{FTERM} \). Now, suppose \( \theta^+ \in \text{TERM} \), \( \theta^* \in \text{CTERM}(\text{ST}(\tilde{\varphi}(\theta_0, ..., \theta_{r-1})^\gamma) \cup \text{ST}(\theta^+)) \) and \( \theta^\delta \in \text{ATERM} \) and suppose \([\theta^*, \theta^\delta, \tilde{\varphi}(\theta_0, ..., \theta_{r-1})^\gamma] = [\theta^*, \theta^\delta, \theta^+]\). Therefore \([\theta^*, \theta^\delta, \theta^+] = \tilde{\varphi}(\theta_0, \theta^\delta, \theta_{r-1})^\gamma \in \text{FTERM} \). Suppose for contradiction that \( \theta^+ \in \text{ATERM} \). We have \( \theta^\delta \neq \theta^+ \) or \( \theta^\delta = \theta^+ \). Suppose \( \theta^\delta \neq \theta^+ \). Then we have \( \theta^+ = [\theta^*, \theta^\delta, \theta^+] = \tilde{\varphi}(\theta_0, \theta^\delta, \theta_{r-1})^\gamma \in \text{FTERM} \). Contradiction! Suppose \( \theta^\delta = \theta^+ \). Then we have \( \theta^+ = [\theta^*, \theta^\delta, \theta^+] = \tilde{\varphi}(\theta_0, \theta^\delta, \theta_{r-1})^\gamma \in \text{FTERM} \). Contradiction! Suppose \( \theta^\delta = \theta^+ \). Then we have \( \theta^+ = [\theta^*, \theta^\delta, \theta^+] = \tilde{\varphi}(\theta_0, \theta^\delta, \theta_{r-1})^\gamma \in \text{FTERM} \). Now, suppose \( \theta^* \in \text{ATERM} \) and \( \theta^\delta \in \text{ATERM} \) and suppose \([\theta^*, \theta^\delta, \theta] = [\theta^*, \theta^\delta, \theta^+]\). With Theorem 1-14-(i), it then follows that for all \( i < r \): \([\theta^*, \theta^\delta, \theta] = [\theta^*, \theta^\delta, \theta^+] \). If \( [\theta^*, \theta^\delta, \theta] = [\theta^*, \theta^\delta, \theta^+] \), then \( \theta^* \) would be a proper subterm of \( \tilde{\varphi}(\theta_0, \theta^\delta, \theta_{r-1})^\gamma \) and therefore a proper subterm of itself, which contradicts Theorem 1-8. Therefore \( \theta^* \not\in \text{ATERM} \), but \( \theta^+ \in \text{FTERM} \). Therefore there are \( \{\theta'_0, ..., \theta'_{k-1}\} \subseteq \text{TERM} \) and \( \varphi' \in \text{FUNC} \) such that \( \theta^+ = \tilde{\varphi}'(\theta'_0, ..., \theta'_{k-1})^\gamma \). Thus we have \( \tilde{\varphi}'(\theta'_0, \theta^\delta, \theta_{r-1})^\gamma = [\theta^*, \theta^\delta, \tilde{\varphi}'(\theta'_0, ..., \theta'_{k-1})^\gamma] = \tilde{\varphi}'(\theta'_0, \theta^\delta, \theta_{r-1})^\gamma \). With Theorem 1-11-(ii), it then follows that \( k = r \) and \( \varphi' = \varphi \) and \( [\theta^*, \theta^\delta, \theta] = [\theta^*, \theta^\delta, \theta'] \) for all \( i < r \). With the I.H., it follows that \( \theta_i = \theta_i \) for all \( i < r \). Thus we then have \( \tilde{\varphi}'(\theta_0, ..., \theta_{r-1})^\gamma = \tilde{\varphi}'(\theta'_0, ..., \theta'_{k-1})^\gamma = \theta^+ \). □
Theorem 1-20. Unique substitution bases (a) for formulas

If $\Delta, \Delta^+ \in \text{FORM}$, $\theta^* \in \text{CTERM}(\text{ST}(\Delta) \cup \text{ST}(\Delta^+))$ and $\theta^\delta \in \text{ATERM}$ and if $[\theta^*, \theta^\delta, \Delta] = [\theta^*, \theta^\delta, \Delta']$, then $\Delta = \Delta^+$.

Proof: Suppose $\Delta, \Delta^+ \in \text{FORM}$, $\theta^* \in \text{CTERM}(\text{ST}(\Delta) \cup \text{ST}(\Delta^+))$ and $\theta^\delta \in \text{ATERM}$ and $[\theta^*, \theta^\delta, \Delta] = [\theta^*, \theta^\delta, \Delta']$. In the same way as we did in the inductive step of the preceding proof for functional terms, one can show for all formulas that substitution bases ($\Delta$ and $\Delta^+$) belong to the same category and have the same main operator (predicate, connective or quantifier) as the respective substitution results ($[\theta^*, \theta^\delta, \Delta]$ and $[\theta^*, \theta^\delta, \Delta^+]$). The proof is carried out by induction on the complexity of $\Delta$. Suppose $\Delta = \Gamma(\theta_0, \ldots, \theta_{r-1})^\gamma \in \text{AFORM}$.

Then we have $[\theta^*, \theta^\delta, \Delta] = \Gamma([\theta^*, \theta^\delta, \theta_0], \ldots, [\theta^*, \theta^\delta, \theta_{r-1}])^\gamma \in \text{AFORM}$ and there are $[\theta^0, \ldots, \theta^r] \subseteq \text{TERM}$ with $\Gamma(\theta_0, \ldots, \theta_{r-1})^\gamma = \Delta^+$. Therefore also $\Gamma([\theta^*, \theta^\delta, \theta_0], \ldots, [\theta^*, \theta^\delta, \theta_{r-1}])^\gamma = [\theta^*, \theta^\delta, \Delta] = [\theta^*, \theta^\delta, \Delta' = [\theta^*, \theta^\delta, \Gamma(\theta_0, \ldots, \theta_{r-1})^\gamma] = \Gamma([\theta^*, \theta^\delta, \theta_0], \ldots, \theta^0, \ldots, \theta^r])^\gamma \in \text{AFORM}$.

With Theorem 1-11-(iv), it then follows that $[\theta^*, \theta^\delta, \theta_i] = [\theta^*, \theta^\delta, \theta_i]$ for all $i < r$. With Theorem 1-19, it then follows that $\theta_i = \theta_i$ for all $i < r$. Thus we have $\Gamma(\theta_0, \ldots, \theta_{r-1})^\gamma = \Gamma(\theta_0, \ldots, \theta_{r-1})^\gamma = \Delta^+$.

Now, suppose the statement holds for $\Delta_0$, $\Delta_1 \in \text{FORM}$ and let $\Delta = \Gamma(\Delta_0^\gamma, \Delta_1^\gamma) \in \text{CONFORM}$. Then we also have $[\theta^*, \theta^\delta, \Delta] = \Gamma([\theta^*, \theta^\delta, \Delta_0^\gamma, \Delta_1^\gamma] \in \text{AFORM}$ and there is $\Delta_0^\gamma \in \text{FORM}$ with $\Gamma(\Delta_0^\gamma, \Delta_1^\gamma) = \Delta^+$. Therefore also $\Gamma(\Delta_0^\gamma, \Delta_0^\gamma) = [\theta^*, \theta^\delta, \Delta] = [\theta^*, \theta^\delta, \Delta'] = [\theta^*, \theta^\delta, \Gamma(\theta_0, \ldots, \theta_{r-1})^\gamma]$ = $\Gamma([\theta^*, \theta^\delta, \theta_0], \ldots, \theta^0, \ldots, \theta^r])^\gamma \in \text{AFORM}$.

With Theorem 1-11-(v), it then follows that $\Delta_0 = \Delta_0^\gamma$ and $\Delta = \Gamma(\Delta_0^\gamma, \Delta_1^\gamma) \in \text{CONFORM}$. Then we also have $[\theta^*, \theta^\delta, \Delta] = \Gamma([\theta^*, \theta^\delta, \Delta, \Delta_0^\gamma, \Delta_1^\gamma]) \in \text{AFORM}$ and there are $\Delta^\gamma, \Delta_1^\gamma \in \text{FORM}$ with $\Gamma(\Delta_0^\gamma, \Delta_1^\gamma) = \Delta^+$. Therefore also $\Gamma([\theta^*, \theta^\delta, \Delta, \Delta_0^\gamma, \Delta_1^\gamma]) = [\theta^*, \theta^\delta, \Delta] = [\theta^*, \theta^\delta, \Delta'] = [\theta^*, \theta^\delta, \Gamma(\theta_0, \ldots, \theta_{r-1})^\gamma]$ = $\Gamma([\theta^*, \theta^\delta, \theta_0], \ldots, \theta^0, \ldots, \theta^r])^\gamma \in \text{AFORM}$.

With Theorem 1-11-(vi), it then follows that $\Delta_0 = \Delta^\gamma_0$ and $\Delta_0 = \Delta_1^\gamma$ and thus $\Delta = \Gamma(\Delta_0^\gamma, \Delta_1^\gamma) = \Gamma(\Delta_0^\gamma, \Delta_1^\gamma) = \Delta^+$. Suppose $\Delta = \Gamma(\Delta_0^\gamma, \Delta_1^\gamma) \in \text{QFORM}$. Then we also have $[\theta^*, \theta^\delta, \Delta] \in \text{QFORM}$ and there is $\Delta_0^\gamma \in \text{FORM}$ with $\Gamma(\Delta_0^\gamma, \Delta_1^\gamma) = \Delta^+$. Suppose $\xi = \theta^\delta$. Then we have $\Delta = \Gamma(\Delta_0^\gamma, \Delta_1^\gamma) = [\theta^*, \theta^\delta, \Delta] = [\theta^*, \theta^\delta, \Delta'] = [\theta^*, \theta^\delta, \Gamma(\theta_0, \ldots, \theta_{r-1})^\gamma] = \Gamma([\theta^*, \theta^\delta, \theta_0], \ldots, \theta^0, \ldots, \theta^r])^\gamma \in \text{AFORM}$. With Theorem 1-11-(vii), it then follows that $\Delta_0 = \Delta_0^\gamma$ and thus $\Delta = \Gamma(\Delta_0^\gamma, \Delta_1^\gamma) = \Gamma(\Delta_0^\gamma, \Delta_1^\gamma) = \Delta^+$. 


If $\Sigma, \Sigma' \in \text{SENT}$, $\theta^* \in \text{CTERM}(\Sigma) \cup \text{ST}(\Sigma')$ and $\theta^0 \in \text{ATERM}$ and if $[\theta^*, \theta^0, \Sigma] = [\theta^*, \theta^0, \Sigma']$, then $\Sigma = \Sigma'$.

**Proof:** The theorem is proved analogously to the negation-case in the proof of Theorem 1-20 by applying Theorem 1-20 and Theorem 1-12. $\blacksquare$

**Theorem 1-22. Unique substitution bases (b) for terms**

If $\theta, \theta^0 \in \text{TERM}, \theta^* \in \text{CTERM}(\text{ST}(\theta) \cup \text{ST}(\theta^0)), \xi \in \text{VAR}, \beta \in \text{PAR}$ and $[\theta^*, \xi, \theta] = [\theta^*, \beta, \theta^+]$, then $\theta^+ = [\beta, \xi, \theta]$.

**Proof:** By induction on the complexity of $\theta$. Suppose $\theta \in \text{ATERM}$. Now, suppose $\theta^* \in \text{TERM}, \theta^* \in \text{CTERM}(\text{ST}(\theta) \cup \text{ST}(\theta^0)), \xi \in \text{VAR}, \beta \in \text{PAR}$ and $[\theta^*, \xi, \theta] = [\theta^*, \beta, \theta^+]$. Then we have $\theta \in \text{CONST} \cup \text{PAR} \cup \text{VAR}$. Now, suppose $\theta \in \text{CONST}$. Then we have $[\theta^*, \xi, \theta] = \theta$. Then we have $\theta = [\beta, \xi, \theta^+]$. Because of $\theta^* \not\in \text{ST}(\theta)$ and Theorem 1-14-(i), we then have that $\theta = \theta^+$ and because of $\xi \neq \beta$ we have $\theta^+ = \theta = [\beta, \xi, \theta]$. Now, suppose $\theta \in \text{PAR}$. Then we have $[\theta^*, \xi, \theta] = \theta$. Then we have $\theta = [\theta^*, \beta, \theta^+]$. Because of $\theta^* \not\in \text{ST}(\theta)$ and Theorem 1-14-(i), we then have again $\theta = \theta^+$ and because of $\xi \neq \beta: \theta^+ = \theta = [\beta, \xi, \theta]$. Now, suppose $\theta \in \text{VAR}$. Suppose $\theta = \xi$. Then we have $[\theta^*, \xi, \theta] = \theta^*$. Then we have $\theta = [\theta^*, \beta, \theta^+]$. Because of $\theta^* \not\in \theta^*$, we then have $\beta \in \text{ST}(\theta^0)$. Thus we have $\theta^* \in \text{ST}(\theta^0)$. If $\theta^+ \neq \beta$, we would have, with $\theta^* = [\theta^*, \beta, \theta^+]$, that $\theta^*$ is a proper subterm of itself, which contradicts Theorem 1-8. Therefore we have $\theta^+ = \beta = [\beta, \xi, \theta]$. Now, suppose $\theta \neq \xi$. Then we have $\theta = [\theta^*, \xi, \theta]$. Then we have $\theta = [\theta^*, \beta, \theta^+]$. Because of $\theta^* \not\in \text{ST}(\theta)$ and Theorem 1-14-(i), we then have $\theta = \theta^+$ and, because of $\theta \neq \xi$, we thus have $\theta^+ = \theta = [\beta, \xi, \theta]$. Now, suppose the statement holds for $\{\theta_0, \ldots, \theta_{r-1}\} \subseteq \text{TERM}$ and suppose $\varphi(\theta_0, \ldots, \theta_{r-1}) \in \text{FTERM}$. Now, suppose $\theta^+ \in \text{TERM}, \theta^* \in \text{CTERM}(\text{ST}(\varphi(\theta_0, \ldots, \theta_{r-1})^+) \cup \text{ST}(\theta^0)), \xi \in \text{VAR}, \beta \in \text{PAR}$ and $[\theta^*, \xi, \varphi(\theta_0, \ldots, \theta_{r-1})^+] = [\theta^*, \beta, \theta^+]$. Therefore $[\theta^*, \beta, \theta^+] = \varphi(\theta^*, \xi, \theta_0, \ldots, \theta_{r-1})^+) \in \text{FTERM}$. Suppose for contradiction that $\theta^* \in \text{ATERM}$. We have $\beta \neq \theta^+ or \beta = \theta^+$. Suppose $\beta \neq \theta^+$. Then we have $\theta^+ = [\theta^*, \beta, \theta^+] = \varphi(\theta^*, \xi, \theta_0, \ldots, \theta_{r-1})^+) \in \text{FTERM}$. Contradiction! Suppose $\beta = \theta^+$. Then we have $\theta^* = [\theta^*, \beta, \theta^+] = \varphi(\theta^*, \xi, \theta_0, \ldots, \theta_{r-1})^+) \in \text{FTERM}$. With Theorem 1-14-(i), it then follows that for all $i < r: [\theta^*, \xi, \theta_i] = \theta_i$ or there is an $i < r$ such that $\theta^* \in \text{ST}([\theta^*, \xi, \theta_i])$. If $[\theta^*, \xi, \theta_i] = \theta_i$ for all $i < r$, then we would have $\theta^* = \varphi(\theta^*, \xi, \theta_0, \ldots, \theta_{r-1})^+) = \varphi(\theta_0, \ldots,
\[ \theta_{r-1} \] and thus \( \theta^* \in \text{ST}(\varphi(\theta_0, \ldots, \theta_{r-1}) \gamma) \), which contradicts the hypothesis. If, on the other hand, there was an \( i < r \) such that \( \theta^* \in \text{ST}([\theta^*, \xi, \theta_i]) \), then \( \theta^* \) would be a proper subterm of \( \varphi([\theta^*, \xi, \theta_0], \ldots, [\theta^*, \xi, \theta_{r-1}] \gamma) \) and therefore a proper subterm of itself, which contradicts Theorem 1-8. Therefore \( \theta^* \notin \text{ATERM} \), but \( \theta^* \in \text{FTERM} \). Therefore there are \( \{ \theta'_0, \ldots, \theta'_{k-1} \} \subset \text{TERM} \) and \( \varphi' \in \text{FUNC} \) such that \( \theta^+ = \varphi'(\theta'_0, \ldots, \theta'_{k-1}) \gamma \). Thus we have \( \varphi'(\theta^*, \beta, \theta'_0), \ldots, [\theta^*, \beta, \theta'_{k-1}] \gamma = [\theta^*, \beta, \varphi'(\theta'_0, \ldots, \theta'_{k-1}) \gamma] = [\theta^*, \beta, \theta^+] = \varphi([\theta^*, \xi, \theta_0], \ldots, [\theta^*, \xi, \theta_{r-1}] \gamma) \). With Theorem 1-11-(ii), it then follows that \( k = r \) and \( \varphi' = \varphi \) and \( [\theta^*, \beta, \theta'_i] = [\theta^*, \xi, \theta_i] \) for all \( i < r \). With the I.H., it follows that \( \theta'_i = [\beta, \xi, \theta_i] \) for all \( i < r \). Thus we have \( \theta^+ = \varphi'(\theta'_0, \ldots, \theta'_{k-1}) \gamma = \varphi([\beta, \xi, \theta_0], \ldots, [\beta, \xi, \theta_{r-1}] \gamma) = [\beta, \xi, \varphi(\theta_0, \ldots, \theta_{r-1}) \gamma] \). □

**Theorem 1-23. Unique substitution bases (b) for formulas**

If \( \Delta, \Delta^+ \in \text{FORM} \), \( \theta^* \in \text{TERM}(\text{ST}(\Delta) \cup \text{ST}(\Delta^+)) \), \( \xi \in \text{VAR} \), \( \beta \in \text{PAR} \) and \( [\theta^*, \xi, \Delta] = [\theta^*, \beta, \Delta^+] \), then \( \Delta^+ = [\beta, \xi, \Delta] \).

**Proof:** Let \( \Delta, \Delta^+ \in \text{FORM} \), \( \theta^* \in \text{CTERM}(\text{ST}(\Delta) \cup \text{ST}(\Delta^+)) \) and \( \xi \in \text{VAR} \), \( \beta \in \text{PAR} \) and \( [\theta^*, \xi, \Delta] = [\theta^*, \beta, \Delta^+] \). In the same way as we did in the inductive step of the preceding proof for functional terms, one can show for all formulas that substitution bases (\( \Delta \) and \( \Delta^+ \)) belong to the same category and have the same main operator (predicate, connective or quantifier) as the respective substitution results ([\( \theta^*, \xi, \Delta \) and \( [\theta^*, \beta, \Delta^+] \)]). The proof is carried out by induction on the complexity of \( \Delta \). Suppose \( \Delta = \varphi(\theta_0, \ldots, \theta_{r-1}) \gamma \in \text{AFORM} \). Then we also have \( [\theta^*, \xi, \Delta] = \varphi([\theta^*, \xi, \theta_0], \ldots, [\theta^*, \xi, \theta_{r-1}] \gamma) \in \text{AFORM} \) and there are \( \{ \theta'_0, \ldots, \theta'_{r-1} \} \subset \text{TERM} \) with \( \varphi(\theta'_0, \ldots, \theta'_{r-1}) \gamma = \Delta^+ \). Therefore we also have \( \varphi([\theta^*, \xi, \theta_0], \ldots, [\theta^*, \xi, \theta_{r-1}] \gamma) = [\theta^*, \xi, \Delta] = [\theta^*, \beta, \Delta^+] = [\theta^*, \beta, \varphi(\theta'_0, \ldots, \theta'_{r-1}) \gamma] = \varphi([\theta^*, \beta, \theta'_0], \ldots, [\theta^*, \beta, \theta'_{r-1}] \gamma) \in \text{AFORM} \). With Theorem 1-11-(iv), it then follows that \( [\theta^*, \xi, \theta_i] = [\theta^*, \beta, \theta_i] \) for all \( i < r \). With Theorem 1-22, it follows that \( \theta'_i = [\beta, \xi, \theta_i] \) for all \( i < r \). Thus we then have \( \Delta^+ = \varphi(\theta'_0, \ldots, \theta'_{r-1}) \gamma = \varphi([\beta, \xi, \theta_0], \ldots, [\beta, \xi, \theta_{r-1}] \gamma) = [\beta, \xi, \varphi(\theta_0, \ldots, \theta_{r-1}) \gamma] = [\beta, \xi, \Delta] \).

Now, suppose the statement holds for \( \Delta_0, \Delta_1 \in \text{FORM} \) and let \( \Delta = \varphi(\Delta_0, \Delta_1) \gamma \in \text{CONFORM} \). Then we also have \( [\theta^*, \xi, \Delta] = \varphi([\theta^*, \xi, \Delta_0], \Delta_1) \gamma \in \text{CONFORM} \) and there is \( \Delta_0 \in \text{FORM} \) with \( \varphi(\Delta_0) \gamma = \Delta^+ \). Therefore we also have \( \varphi([\theta^*, \xi, \Delta_0], \Delta_1) \gamma = [\theta^*, \beta, \Delta^+] \). With Theorem 1-11-(v), it then follows that \( [\theta^*, \xi, \Delta_0] = [\theta^*, \beta, \Delta_0] \). With the I.H., it follows that \( \Delta_0 = [\beta, \xi, \Delta_0] \) and thus that \( \Delta^+ = \varphi(\Delta_0) \gamma = [\beta, \xi, \varphi(\Delta_0, \Delta_1) \gamma] = [\beta, \xi, \Delta] \). Suppose \( \Delta = \varphi(\Delta_0, \Delta_1) \gamma \in \text{CONFORM} \). Then we also have \( [\theta^*, \xi, \Delta] = \varphi([\theta^*, \xi, \Delta_0], \Delta_1) \gamma \in \text{CONFORM} \).
and there are $\Delta_0, \Delta'_1 \in \text{FORM}$ with $\gamma((\Delta_0 \psi \Delta'_1)\gamma = \Delta'$. Therefore we also have $\gamma([\theta^*, \xi, \Delta_0] \psi [\theta^*, \xi, \Delta_1])\gamma = [\theta^*, \beta, \Delta'] = \gamma([\theta^*, \beta, \Delta_0]) \psi [\theta^*, \beta, \Delta'_1])\gamma \in \text{CONFORM}$. With Theorem 1-11-(vi), it then follows that $[\theta^*, \xi, \Delta_0] = [\theta^*, \beta, \Delta'_0]$ and $[\theta^*, \xi, \Delta_1] = [\theta^*, \beta, \Delta'_1]$. With the I.H., it follows that $\Delta_0 = [\beta, \xi, \Delta_0]$ and $\Delta'_1 = [\beta, \xi, \Delta_1]$ and thus we have $\Delta' = \gamma((\Delta_0 \psi \Delta'_1)\gamma = \gamma([\beta, \xi, \Delta_0] \psi [\beta, \xi, \Delta_1])\gamma = [\beta, \xi, \gamma((\Delta_0 \psi \Delta'_1)\gamma] = \gamma([\beta, \xi, \Delta]).$

Suppose $\Delta = \gamma(\Pi \xi \Delta_0) \in \text{QFORM}$. Suppose $\xi' = \xi$. Then we have $\Delta = \gamma(\Pi \xi \Delta_0) = [\theta^*, \xi, \gamma(\Pi \xi \Delta_0)] = [\theta^*, \xi, \Delta]$. With Theorem 1-14-(ii), we then have $\theta^* \in \text{ST}([\theta^*, \beta, \Delta])$ or $[\theta^*, \beta, \Delta'] = \Delta'$. This first case is excluded by the hypothesis. In the second case, we have $\Delta' = \gamma(\Pi \xi \Delta_0) = [\beta, \xi, \gamma(\Pi \xi \Delta_0)] = [\beta, \xi, \Delta]$. Suppose $\xi' \neq \xi$. Then we have $[\theta^*, \xi, \Delta] = \gamma(\Pi \xi [\theta^*, \xi, \Delta_0]) \in \text{QFORM}$ and there is $\Delta_0 \in \text{FORM}$ with $\gamma(\Pi \xi [\theta^*, \xi, \Delta_0]) = \Delta'$. Therefore we also have $\gamma(\Pi \xi [\theta^*, \xi, \Delta_0]) = \gamma([\theta^*, \beta, \Delta]) = [\theta^*, \beta, \gamma(\Pi \xi \Delta_0)] = \gamma(\Pi \xi [\theta^*, \beta, \Delta_0]) \in \text{QFORM}$. With Theorem 1-11-(vii), it then follows that $[\theta^*, \xi, \Delta_0] = \gamma(\Pi \xi \Delta'_0) \in \text{QFORM}$ and there is $\Delta'_0 \in \text{FORM}$ with $\Delta'_0 \in \text{FORM}$. With Theorem 1-11-(vii), it then follows that $[\theta^*, \xi, \Delta_0] = [\theta^*, \Delta'_0]$. With the I.H., it follows that $\Delta'_0 = [\beta, \xi, \Delta_0]$ and thus $\Delta' = \gamma(\Pi \xi \Delta'_0) = \gamma(\Pi \xi [\beta, \xi, \Delta_0]) = [\beta, \xi, \Delta]$. $\blacksquare$

**Theorem 1-24. Cancellation of parameters in substitution results**

If $\theta \in \text{TERM}, \Delta \in \text{FORM}, \Sigma \in \text{SENT}, \theta^* \in \text{CTERM}, \beta \in \text{PAR}(\text{ST}(\theta) \cup \text{ST}(\Delta) \cup \text{ST}(\Sigma))$ and $\theta^* \in \text{ATERM}$, then:

(i) $[\theta^*, \theta^*, \theta] = [\theta^*, \beta, \gamma(\theta^*, \theta^*, \theta)]$,

(ii) $[\theta^*, \theta^*, \Delta] = [\theta^*, \beta, \gamma(\theta^*, \theta^*, \Delta)]$, and

(iii) $[\theta^*, \theta^*, \Sigma] = [\theta^*, \beta, \gamma(\theta^*, \theta^*, \Sigma)]$.

**Proof:** Let $\theta \in \text{TERM}, \Delta \in \text{FORM}, \Sigma \in \text{SENT}, \theta^* \in \text{CTERM}, \beta \in \text{PAR}(\text{ST}(\theta) \cup \text{ST}(\Delta) \cup \text{ST}(\Sigma))$ and $\theta^* \in \text{ATERM}$. Ad (i): The proof is carried out by induction on the complexity of $\theta$. Suppose $\theta \in \text{ATERM}$. Then we have $\theta = \theta^*$ or $\theta ^* \neq \theta^*$. First, suppose $\theta = \theta^*$. Then we have $[\theta^*, \theta^*, \theta] = [\theta^*, \beta, \gamma(\theta^*, \theta^*, \theta)]$ and thus $\theta = [\theta^*, \beta, \theta^*, \gamma(\theta^*, \theta^*, \theta)]$. Now, suppose $\theta \neq \theta^*$. Then we have $[\theta^*, \theta^*, \theta] = [\theta^*, \beta, \gamma(\theta^*, \theta^*, \theta)]$ and thus $\beta = \gamma(\theta^*, \beta, \gamma(\theta^*, \theta^*, \theta)]$. Therefore we have $[\theta^*, \theta^*, \theta] = [\theta^*, \beta, \gamma(\theta^*, \theta^*, \theta)]$. $\blacksquare$

Now, suppose the statement holds for $\{\theta_0, \ldots, \theta_{r-1}\} \subseteq \text{TERM}$ and suppose $\theta = \gamma(\phi(\theta_0, \ldots, \theta_{r-1})) \in \text{FTERM}$. Because of $\beta \notin \text{ST}(\theta)$, we also have that $\beta \notin \text{ST}(\theta_i)$ for all $i < r$. With the I.H., it then holds that $[\theta^*, \theta^*, \theta] = [\theta^*, \beta, \gamma(\theta^*, \theta^*, \theta)]$ for all $i < r$. Then we have $[\theta^*, \theta^*, \theta] = [\theta^*, \beta, \gamma(\theta^*, \theta^*, \theta)]$. $\blacksquare$
Thus we have proved analogously to the negation-case.

Proof

Now, suppose the statement holds for $\Delta_0, \Delta_1 \in$ FORM. First, let $\Delta = \overline{r\Delta_0 \gamma} \in$ CONFORM. Then we have $\beta \not\in ST(\Delta_0)$ and $[\theta*, \theta^+, \Delta] = [\theta*, \theta^+, \overline{r\Delta_0 \gamma}] = [\theta*, \theta^+, \Delta_0]$. With (i), it holds that $[\theta*, \theta^+, \Delta] = [\theta*, \beta, [\beta, \theta^+, \Delta_i]]$ for all $i < r$. Therefore we have $[\theta*, \theta^+, \Delta] = [\theta*, \beta, [\beta, \theta^+, \Delta_0]] = [\theta*, \beta, [\beta, \theta^+, \Delta]]$. Suppose $\Delta = [\Delta_0 \psi \Delta_1] \in$ CONFORM. This case is proved analogously to the negation-case.

Suppose $\Delta = \overline{\Pi\xi\Delta_0 \gamma} \in$ QFORM. Suppose $\xi = \theta^+$. Then we have $[\theta*, \theta^+, \Delta] = [\theta*, \theta^+, \overline{\Pi\xi\Delta_0 \gamma}] = [\theta*, \theta^+, \Delta_0 \gamma]$ for all $i < r$. Therefore $[\theta*, \theta^+, \Delta] = [\theta*, \beta, [\beta, \theta^+, \Delta]]$. Suppose $\xi \neq \theta^+$. This case is proved analogously to the negation-case.

Ad (iii): This case is proved analogously to the negation-case.

Theorem 1.25. A sufficient condition for the commutativity of a substitution in terms and formulas

If $\theta_*^*, \theta_*^1 \in$ CTERM, $\theta_0, \theta_1 \in$ ATERM, $\theta_0 \neq \theta_1, \theta_1 \not\in ST(\theta_*^0)$ and $\theta_0 \not\in ST(\theta_*^1)$, then:

(i) If $\theta^+ \in$ TERM, then $[\theta_*^1, \theta_1, [\theta_*^0, \theta_0, \theta^+]] = [\theta_*^1, \theta_1, \theta_*^0, \theta_0, \theta^+]$, and

(ii) If $\Delta \in$ FORM, then $[\theta_*^1, \theta_1, [\theta_*^0, \theta_0, \Delta]] = [\theta_*^1, \theta_0, [\theta_*^0, \theta_1, \Delta]]$.

Proof: Let $\theta_*^0, \theta_*^1 \in$ CTERM, $\theta_0, \theta_1 \in$ ATERM, $\theta_0 \neq \theta_1, \theta_1 \not\in ST(\theta_*^0)$ and $\theta_0 \not\in ST(\theta_*^1)$. Ad (i): Suppose $\theta^+ \in$ TERM. The proof is carried out by induction on the complexity of $\theta^+$. Suppose $\theta^+ \in$ ATERM. Suppose $\theta^+ = \theta_0$. Then we have $[\theta_*^0, \theta_0, \theta_*^0]$ for all $i < r$. Therefore we have $[\theta_*^0, \theta_0, \theta_*^0] = [\theta_*^0, \theta_0]$ for all $i < r$. Therefore $[\theta_*^0, \theta_0, \theta_*^0] = [\theta_*^0, \theta_0]$ for all $i < r$. Therefore $[\theta_*^0, \theta_0, \theta_*^0] = [\theta_*^0, \theta_0]$ for all $i < r$. Therefore $[\theta_*^0, \theta_0, \theta_*^0] = [\theta_*^0, \theta_0]$ for all $i < r$. Therefore $[\theta_*^0, \theta_0, \theta_*^0] = [\theta_*^0, \theta_0]$. Suppose $\theta^+ \neq \theta_0$. Suppose $\theta^+ = \theta_1$. Then we have $[\theta_*^0, \theta_1, [\theta_*^0, \theta_0, \theta^+]] = [\theta_*^0, \theta_1, [\theta_*^0, \theta_0, \theta^+]]$. Because of $\theta_0 \not\in ST(\theta_*^0)$, we have $[\theta_*^0, \theta_1, [\theta_*^0, \theta_0, \theta^+]] = [\theta_*^0, \theta_0]$. On the other hand, we have $[\theta_*^0, \theta_0, [\theta_*^0, \theta_1, \theta^+]] = [\theta_*^0, \theta_0, \theta^+]$. Therefore $[\theta_*^0, \theta_1, [\theta_*^0, \theta_0, \theta^+]] = [\theta_*^0, \theta_0, \theta^+]$. Now, suppose $\theta^+ \neq \theta_0$. Suppose $\theta^+ = \theta_1$. Then we have $[\theta_*^0, \theta_1, [\theta_*^0, \theta_0, \theta^+]] = [\theta_*^0, \theta_1, [\theta_*^0, \theta_0, \theta^+]]$. Because of $\theta_0 \not\in ST(\theta_*^0)$, we have $[\theta_*^0, \theta_0, [\theta_*^0, \theta_1, \theta^+]] = [\theta_*^0, \theta_0]$. Therefore $[\theta_*^0, \theta_1, [\theta_*^0, \theta_0, \theta^+]] = [\theta_*^0, \theta_0]$. Suppose $\theta^+ \neq \theta_0$. Then we have $[\theta_*^0, \theta_1, [\theta_*^0, \theta_0, \theta^+]] = [\theta_*^0, \theta_0]$. Therefore $[\theta_*^0, \theta_1, [\theta_*^0, \theta_0, \theta^+]] = [\theta_*^0, \theta_0]$. Suppos
\( \theta' = 0^+ \) and \([\theta^*, 0, 0, [\theta^*, 1, \theta_1, 0^+]]) = [0^*, 0, 0, 0^+] = \theta'. \) Therefore we have again that \([\theta^*, 1, \theta_1, [\theta^*, 0, 0, 0^+]]) = [0^*, 0, 0, [\theta^*, 1, \theta_1, 0^+]]) = [0^*, 0, 0, 0^+].

Now, suppose the statement holds for \(\{\theta', \ldots, \theta'_{r-1}\} \subseteq \text{TERM and suppose } \theta' = \overline{\varphi(\theta'_1, \ldots, \theta'_{r-1})}\in \text{FTERM. Then we have } [\theta^*, 1, [\theta^*, 0, 0, \theta''_1]]) = [0^*, 1, [\theta^*, 0, 0, \overline{\varphi(\theta'_1, \ldots, \theta'_{r-1})}]] = \overline{\varphi([\theta^*, 1, \theta_1, [\theta^*, 0, 0, \theta''_1]])}. \) Therefore we have \([\theta^*, 1, [\theta^*, 0, 0, \theta'']]) = [0^*, 0, 0, [\theta^*, 1, \theta_1, \theta''_1]]) for all \(i < r. \) Therefore we have \([\theta^*, 1, [\theta^*, 0, 0, \theta'']]) = [0^*, 0, 0, [\theta^*, 1, \theta_1, \theta''_1]]) = [0^*, 0, 0, [\theta^*, 1, \theta_1, \overline{\varphi(\theta'_1, \ldots, \theta'_{r-1})}]] = [0^*, 0, 0, [\theta^*, 1, \theta'']]).

**Ad (ii):** Suppose \(\Delta \in \text{FORM}. \) The proof is carried out by induction on the complexity of \(\Delta. \) Suppose \(\Delta = \overline{\Phi(\theta'_1, \ldots, \theta'_{r-1})} \in \text{AFORM. Then we have } [\theta^*, 1, [\theta^*, 0, 0, \Delta]]) = [0^*, 1, \theta_1, [\theta^*, 0, 0, \overline{\Phi(\theta'_1, \ldots, \theta'_{r-1})}]] = \overline{\Phi([\theta^*, 1, \theta_1, [\theta^*, 0, 0, \theta'0]])}. \) With (i), we have that \([\theta^*, 1, [\theta^*, 0, 0, \theta'0]]) = [0^*, 0, 0, [\theta^*, 1, \theta_1, \theta''_1]]) for all \(i < r. \) Therefore we have \([\theta^*, 1, [\theta^*, 0, 0, \Delta]]) = \overline{\Phi([\theta^*, 0, 0, [\theta^*, 1, \theta_1, \theta''_1]])}. \) Therefore \([\theta^*, 1, [\theta^*, 0, 0, \Delta]] = [0^*, 0, 0, [\theta^*, 1, \theta_1, \Delta]]) = [0^*, 0, 0, [\theta^*, 1, \overline{\Phi(\theta'_1, \ldots, \theta'_{r-1})}]] = [0^*, 0, 0, [\theta^*, 1, \theta']].

Now, suppose the statement holds for \(\Delta_0, \Delta_1 \in \text{FORM and suppose } \Delta = \overline{\Delta_0} \in \text{CONFORM. Then we have } [\theta^*, 1, [\theta^*, 0, 0, \Delta_0]]) = [0^*, 1, \theta_1, [\theta^*, 0, 0, \overline{\Delta_0}]] = [0^*, 1, \theta_1, [\theta^*, 0, 0, \Delta]]) = [0^*, 0, 0, [\theta^*, 1, \theta_1, \Delta_0]]. \) Therefore we have \([\theta^*, 1, [\theta^*, 0, 0, \Delta_0]] = [0^*, 0, 0, [\theta^*, 1, \theta_1, \Delta_0]] = [0^*, 0, 0, [\theta^*, 1, \overline{\Delta_0}]] = [0^*, 0, 0, [\theta^*, 1, \theta_1, \Delta]]) = [0^*, 0, 0, [\theta^*, 1, \theta_1, \Delta]]. \) Suppose \(\Delta = \overline{\Phi(\theta'_1, \ldots, \theta'_{r-1})}) \in \text{CONFORM. This case is proved analogously to the negation-case.}

Suppose \(\Delta = \overline{\Phi(\theta'_1, \ldots, \theta'_{r-1})} \in \text{QFORM. Suppose } \xi = \theta_0. \) Then we have \(\xi = \theta_1 \) and \([\theta^*, 1, \theta_1, [\theta^*, 0, 0, \Delta]]) = [0^*, 1, \theta_1, [\theta^*, 0, 0, \overline{\Phi(\theta'_1, \ldots, \theta'_{r-1})}]] = [0^*, 1, \overline{\Phi(\theta'_1, \ldots, \theta'_{r-1})}] = [0^*, 0, 0, [\theta^*, 1, \theta_1, \Delta]]) = [0^*, 0, 0, [\theta^*, 1, \theta_1, \Delta]]). \) Suppose \(\xi = \theta_1. \) Then we have \(\xi = \theta_0 \) and \([\theta^*, 1, [\theta^*, 0, 0, \Delta]]) = [0^*, 1, [\theta^*, 0, 0, \overline{\Phi(\theta'_1, \ldots, \theta'_{r-1})}]] = [0^*, 1, [\theta^*, 0, 0, \overline{\Phi(\theta'_1, \ldots, \theta'_{r-1})}] = [0^*, 0, 0, [\theta^*, 1, \theta_1, \Delta]]). \) Suppose \(\Delta = \overline{\Phi(\theta'_1, \ldots, \theta'_{r-1})} \in \text{QFORM. This case is proved analogously to the negation-case.} \)
**Theorem 1-26. Substitution in substitution results**

If $\zeta \in \text{VAR}$, $\theta', \theta^* \in \text{CTERM}$ and $\theta^+ \in \text{CONST} \cup \text{PAR}$, then:

(i) If $\theta \in \text{TERM}$, then $[[\theta', \theta^*, [\theta^*, \zeta, \theta]] = [[\theta', \theta^*, \theta^*], \zeta, [\theta^+, \theta^+], \theta]]$, and

(ii) If $\Delta \in \text{FORM}$, then $[[\theta', \theta^+, [\theta^*, \zeta, \Delta]] = [[\theta', \theta^+, \theta^*], \zeta, [\theta^+, \theta^+, \Delta]]$.

**Proof:** Suppose $\zeta \in \text{VAR}$, $\theta', \theta^* \in \text{CTERM}$ and $\theta^+ \in \text{CONST} \cup \text{PAR}$. Ad (i): Suppose $\theta \in \text{TERM}$. The proof is carried out by induction on the complexity of $\theta$. Suppose $\theta \in \text{ATERM}$. First, suppose $\theta \in \text{CONST} \cup \text{PAR}$. Suppose $\theta = \theta^+$. Then we have $[\theta', \theta^+, [\theta^*, \zeta, \theta]] = [\theta', \theta^+, \theta] = \theta$. We have $\zeta \not\in \text{ST}(\theta') \in \text{CTERM}$ and thus $[\theta', \theta^+, [\theta^*, \zeta, \theta]] = \theta' = [[\theta', \theta^+, \theta^*], \zeta, \theta'] = [[\theta', \theta^+, \theta^*], \zeta, [\theta^+, \theta^+, \theta]]$. Suppose $\theta \neq \theta^+$. Then we have $[\theta', \theta^+, [\theta^*, \zeta, \theta]] = [\theta', \theta^+, \theta^*] = [[\theta', \theta^+, \theta^*], \zeta, \theta] = [[\theta', \theta^+, \theta^*], \zeta, [\theta', \theta^+, \theta]]$. Now, suppose $\theta \in \text{VAR}$. Suppose $\theta = \zeta$. Then we have $[\theta', \theta^+, [\theta^*, \zeta, \theta]] = [\theta', \theta^+, \theta^*] = [[\theta', \theta^+, \theta^*], \zeta, \theta] = [[\theta', \theta^+, \theta^*], \zeta, [\theta', \theta^+, \theta]]$. Suppose $\theta \neq \zeta$. Then we have $[\theta', \theta^+, [\theta^*, \zeta, \theta]] = [\theta', \theta^+, \theta] = \theta = [[\theta', \theta^+, \theta^*], \zeta, \theta] = [[\theta', \theta^+, \theta^*], \zeta, [\theta', \theta^+, \theta]]$.

Now, suppose the statement holds for $\{\theta_0, \ldots, \theta_{r-1}\} \subseteq \text{TERM}$ and suppose $\theta = \text{r}(\theta_0, \ldots, \theta_{r-1})^\gamma \in \text{FTERM}$. Then we have $[\theta', \theta^+, [\theta^*, \zeta, \theta]] = [\theta', \theta^+, [\theta^*, \zeta, \theta]] = [\text{r}(\theta_0, \ldots, \theta_{r-1})^\gamma] = [\text{r}(\theta_0, \theta^+, [\theta^*, \zeta, \theta_0]), \ldots, [\theta', \theta^+, [\theta^*, \zeta, \theta_{r-1}]]]^\gamma]$. With the I.H., it holds that $[\theta', \theta^+, [\theta^*, \zeta, \theta_i]] = [\theta', \theta^*, [\theta^*, \zeta, \theta_i]]$ for all $i < r$. Therefore we have $[\theta', \theta^+, [\theta^*, \zeta, \theta]] = [\text{r}(\theta', \theta^+, [\theta^*, \zeta, \theta]), [\theta', \theta^+, [\theta^*, \zeta, \theta_0]], \ldots, [\theta', \theta^+, [\theta^*, \zeta, \theta_{r-1}]]]^\gamma] = [[\theta', \theta^+, \theta^*], \zeta, [\theta', \theta^+, \theta]]$.

Ad (ii): Suppose $\Delta \in \text{FORM}$. The proof is carried out by induction on the complexity of $\Delta$. Suppose $\Delta = \text{r}(\theta_0, \ldots, \theta_{r-1})^\gamma \in \text{AFORM}$. This case is proved analogously to the FTERMCASE by applying (i).

Now, suppose the statement holds for $\Delta_0$, $\Delta_1 \in \text{FORM}$ and suppose $\Delta = \text{r}(\Delta_0, \Delta_1)^\gamma$. With the I.H., it holds that $[\theta', \theta^+, [\theta^*, \zeta, \Delta]] = [\theta', \theta^+, [\theta^*, \zeta, \Delta]] = [\theta', \theta^+, [\theta^*, \zeta, \Delta]] = [\theta', \theta^+, [\theta^*, \zeta, \Delta]]$. Therefore $[\theta', \theta^+, [\theta^*, \zeta, \Delta]] = [\text{r}([\theta', \theta^+, \theta^*], \zeta, [\theta', \theta^+, \theta^*], \zeta, [\theta', \theta^+, \theta^*], \zeta, [\theta', \theta^+, \theta^*, \Delta_0]]^\gamma] = [[\theta', \theta^+, \theta^*], \zeta, [\theta', \theta^+, \theta^*, \Delta_0]]$. Suppose $\Delta = [\text{r}(\Delta_0, \Delta_1)^\gamma \in \text{CONFORM}$, then: $\Delta = [\text{r}(\Delta_0, \Delta_1)^\gamma].$
Theorem 1.27. Multiple substitution of new and pairwise different parameters for pairwise different parameters in terms, formulas, sentences and sequences

If \( \theta \in \text{TERM}, \Delta \in \text{FORM}, \Sigma \in \text{SENT}, \phi \in \text{SEQ}, k \in \mathbb{N} \setminus \{0\} \) and \( \{\beta^*_0, \ldots, \beta^*_k\} \subseteq \text{PAR}(\text{ST}(\theta) \cup \text{ST}(\Delta) \cup \Sigma) \cup \text{STSEQ}(\phi)) \) and \( \{\beta_0, \ldots, \beta_k\} \subseteq \text{PAR}\{\beta^*_0, \ldots, \beta^*_k\} \), where \( \beta^*_i \neq \beta^*_j \) and \( \beta_i \neq \beta_j \) for all \( i, j < k+1 \) with \( i \neq j \), then:

(i) \( \{\beta^*_i, \beta_k, \{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \theta\} = \{\{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \theta\} \),

(ii) \( \{\beta^*_i, \beta_i, \{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \Delta\} = \{\{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \Delta\} \),

(iii) \( \{\beta^*_i, \beta_k, \{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \Sigma\} = \{\{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \Sigma\} \), and

(iv) \( \{\beta^*_i, \beta_i, \{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \phi\} = \{\{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \phi\} \).

Proof: Suppose \( \theta \in \text{TERM}, \Delta \in \text{FORM}, \Sigma \in \text{SENT}, \phi \in \text{SEQ}, k \in \mathbb{N} \setminus \{0\} \) and \( \{\beta^*_0, \ldots, \beta^*_k\} \subseteq \text{PAR}(\text{ST}(\theta) \cup \text{ST}(\Delta) \cup \Sigma) \cup \text{STSEQ}(\phi)) \) and \( \{\beta_0, \ldots, \beta_k\} \subseteq \text{PAR}\{\beta^*_0, \ldots, \beta^*_k\} \), where \( \beta^*_i \neq \beta^*_j \) and \( \beta_i \neq \beta_j \) for all \( i, j < k+1 \) with \( i \neq j \). Ad (i): The proof is carried out by induction on the complexity of \( \theta \). Suppose \( \theta \in \text{CONST} \cup \text{PAR} \cup \text{VAR} \). Now, suppose \( \theta \in \text{CONST} \cup \text{VAR} \cup \text{PAR}\{\beta_0, \ldots, \beta_k\} \). Then we have \( \theta = \{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, 0 \) and we have \( \theta = \{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \theta \} \) and thus \( \{\beta^*_i, \beta_k, \{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \theta\} = \{\beta^*_i, \beta_k, \{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \theta\} \).

Now, suppose \( \theta \in \{\beta_0, \ldots, \beta_k\} \). Then we have \( \theta = \beta_i \) for an \( i < k+1 \). According to the hypothesis, we then have that for all \( j < k+1 \) with \( j \neq i \) it holds that \( \theta \neq \beta_j \). Thus we have \( \{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \theta\} = \beta^*_i \). Now, suppose \( i < k \). Then we have \( \{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \theta\} = \{\beta^*_0, \ldots, \beta^*_k\}, \{\beta_0, \ldots, \beta_k\}, \theta\} \).

Ad (ii): The proof is carried out by induction on the complexity of \( \Delta \). Suppose \( \Delta = \{\Phi(\theta_0, \ldots, \theta_r)\} \in \text{AFORM} \). This case is proved analogously to the FTERM-case by applying (i).
Now, suppose the statement holds for \( \Delta_0, \Delta_1 \in \text{FORM} \) and suppose \( \Delta = \lnot \Delta_0 \).\(^7\) \in \text{CONFORM}. Then we have \([\beta*_{k}, \beta_{i}, [\beta*_{0}, \ldots, \beta*_{k-1}], \langle \beta_{0}, \ldots, \beta_{k-1}, \Delta \rangle] = [\beta*_{k}, \beta_{i}, [\beta*_{0}, \ldots, \beta*_{k-1}], \langle \beta_{0}, \ldots, \beta_{k-1}, \Delta_0 \rangle] \). With the I.H., it holds that \([\beta*_{k}, \beta_{i}, [\beta*_{0}, \ldots, \beta*_{k-1}], \langle \beta_{0}, \ldots, \beta_{k-1}, \Delta_0 \rangle] = [\beta*_{0}, \ldots, \beta*_{k}], \langle \beta_{0}, \ldots, \beta_{k}, \Delta \rangle \). Therefore we have \([\beta*_{k}, \beta_{i}, [\beta*_{0}, \ldots, \beta*_{k-1}], \langle \beta_{0}, \ldots, \beta_{k-1}, \Delta \rangle] = [\beta*_{0}, \ldots, \beta*_{k}], \langle \beta_{0}, \ldots, \beta_{k}, \Delta \rangle \). Suppose \( \Delta = \lnot (\Delta_0 \psi \Delta_1) \). This case is proved analogously to the negation-case. Suppose \( \Delta = \lnot \forall \xi \Delta_0 \). This case is also proved analogously to the negation-case.

Ad (iii) and (iv): (iii) follows analogously to the negation-case by applying (ii), and (iv) follows analogously to the FTERM-case by applying (iii). ■

**Note:** For sets of formulas, a theorem that is analogous to Theorem 1-27 can be proved.

**Theorem 1-28.** Multiple substitution of closed terms for pairwise different variables in terms and formulas (a)

If \( k \in \mathbb{N}\setminus\{0\}, \{\theta*_{0}, \ldots, \theta*_{k}\} \subseteq \text{CTERM} \) and \( \{\xi_{0}, \ldots, \xi_{k}\} \subseteq \text{VAR} \), where \( \xi_{i} \neq \xi_{j} \) for all \( i, j < k + 1 \) with \( i \neq j \), then:

(i) If \( \theta \in \text{TERM} \), then

\[
[\theta*_{0}, \xi_{0}, [\theta*_{0}, \ldots, \theta*_{k-1}], \langle \xi_{0}, \ldots, \xi_{k-1}, \theta \rangle] = [\theta*_{0}, \xi_{0}, [\theta*_{0}, \ldots, \theta*_{k-1}], \langle \xi_{0}, \ldots, \xi_{0}, \theta \rangle, \theta],
\]

and

(ii) If \( \Delta \in \text{FORM} \), then

\[
[\theta*_{0}, \xi_{0}, [\theta*_{0}, \ldots, \theta*_{k-1}], \langle \xi_{0}, \ldots, \xi_{k-1}, \Delta \rangle] = [\theta*_{0}, \xi_{0}, [\theta*_{0}, \ldots, \theta*_{k-1}], \langle \xi_{0}, \ldots, \xi_{0}, \Delta \rangle, \Delta].
\]

**Proof:** Let \( k \in \mathbb{N}\setminus\{0\}, \{\theta*_{0}, \ldots, \theta*_{k}\} \subseteq \text{CTERM} \) and \( \{\xi_{0}, \ldots, \xi_{k}\} \subseteq \text{VAR} \), where \( \xi_{i} \neq \xi_{j} \) for all \( i, j < k + 1 \) with \( i \neq j \). Ad (i): Suppose \( \theta \in \text{TERM} \). The proof is carried out by induction on the complexity of \( \theta \). Suppose \( \theta \in \text{ATERM} \). Suppose \( \xi_{i} \neq \theta \) for all \( i < k + 1 \). Then we have \([\theta*_{0}, \xi_{0}, [\theta*_{0}, \ldots, \theta*_{k-1}], \langle \xi_{0}, \ldots, \xi_{k-1}, \theta \rangle] = [\theta*_{0}, \xi_{0}, [\theta*_{0}, \ldots, \theta*_{k-1}], \langle \xi_{0}, \ldots, \xi_{0}, \theta \rangle, \theta] \). Suppose \( \xi_{i} = \theta \) for an \( i < k \). Then we have \( \xi_{j} \neq \theta \) for all \( i < j < k + 1 \). Then we have \([\theta*_{0}, \ldots, \theta*_{k}], \langle \xi_{0}, \ldots, \xi_{k} \rangle, \theta] = [\theta*_{0}, \ldots, \theta*_{k}], \langle \xi_{0}, \ldots, \xi_{k}, \theta \rangle, \theta] = [\theta*_{0}, \ldots, \theta*_{k}], \langle \xi_{0}, \ldots, \xi_{k} \rangle, \theta] = [\theta*_{0}, \ldots, \theta*_{k}], \langle \xi_{0}, \ldots, \xi_{k} \rangle, \theta] = [\theta*_{0}, \ldots, \theta*_{k}], \langle \xi_{0}, \ldots, \xi_{k}, \theta \rangle, \theta]. \) Suppose \( \xi_{k} = \theta \). Then we have \( \xi_{i} \neq \theta \) for all \( i < k \) and \([\theta*_{0}, \ldots, \theta*_{k}], \langle \xi_{0}, \ldots, \xi_{k-1}, \theta \rangle, \theta] = [\theta*_{0}, \ldots, \theta*_{k}], \langle \xi_{0}, \ldots, \xi_{k}, \theta \rangle, \theta] = [\theta*_{0}, \ldots, \theta*_{k}], \langle \xi_{0}, \ldots, \xi_{k}, \theta \rangle, \theta]. \) Therefore \([\theta*_{0}, \ldots, \theta*_{k}], \langle \xi_{0}, \ldots, \xi_{k-1}, \theta \rangle, \theta] = [\theta*_{0}, \ldots, \theta*_{k}], \langle \xi_{0}, \ldots, \xi_{k}, \theta \rangle, \theta]. \)
Now, suppose the statement holds for \( \{\theta_0, \ldots, \theta_{r-1}\} \subseteq \text{TERM} \) and suppose \( \theta = \varphi(\theta_0, \ldots, \theta_{r-1}) \in \text{FERM} \). Then we have \( [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \theta)] = [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \theta)] \). With the I.H., it holds that \( [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \theta)] \) for all \( i < r \). Therefore we have \( [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \theta)] = \varphi(\theta_0, \ldots, \theta_{k-1}, \langle \xi_0, \ldots, \xi_{k-1} \rangle, \theta) \). Therefore \( \Delta \) is proved analogously to the FERM-case by applying (i).

\( Ad \ (ii): \) Suppose \( \Delta \in \text{FORM} \). The proof is carried out by induction on the complexity of \( \Delta \). Suppose \( \Delta = \varphi(\theta_0, \ldots, \theta_{r-1}) \in \text{AFORM} \). This case is proved analogously to the FERM-case by applying (i).

Now, suppose the theorem holds for \( \Delta_0, \Delta_1 \in \text{FORM} \). Suppose \( \Delta = \lnot \Delta_0 \in \text{CONFORM} \). Then we have \( [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \Delta)] = [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \Delta_0)] \). With the I.H., it holds that \( [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \Delta)] = [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \Delta_0)]. \) Therefore \( \Delta = \lnot (\Delta_0 \psi \Delta_1) \in \text{CONFORM} \). This case is proved analogously to the negation-case.

Suppose \( \Delta = \varphi \xi_k \Delta_0 \in \text{QFORM} \). Suppose \( \xi_j = \xi \) for one \( i < k \). Then we have \( \xi_j \neq \xi \) for all \( j < k+1 \) with \( i \neq j \). Then we have \( [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \Delta)] = [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \Delta_0)]. \) Therefore we have \( [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \Delta)] = [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \Delta_0)]. \) Suppose \( \xi_j = \xi \) for all \( i < k \) and \( [\theta_0, \xi_k, \xi] \subseteq \text{QFORM} \). Therefore we have \( [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \Delta)] = [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \Delta_0)]. \) With the I.H., it holds that \( [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \Delta)] = [\theta_0, \xi_k, ([\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1} \rangle, \Delta_0)]. \)
Theorem 1-29. Multiple substitution of closed terms for pairwise different variables in terms and formulas (b)

If \( k \in \mathbb{N} \setminus \{0\} \), \( \{\theta^*_0, \ldots, \theta^*_k\} \subseteq \text{TERM} \) and \( \{\xi_0, \ldots, \xi_k\} \subseteq \text{VAR} \), where \( \xi_i \neq \xi_j \) for all \( i, j < k+1 \) with \( i \neq j \), then:

(i) If \( \theta \in \text{TERM} \), then

\[
[\theta^*_0, \ldots, \theta^*_k, \langle \xi_0, \ldots, \xi_k \rangle, \theta^*_0, \langle \xi_0, \ldots, \xi_k \rangle, \theta] = [\theta^*_0, \ldots, \theta^*_k, \langle \xi_0, \ldots, \xi_k \rangle, \theta],
\]

and

(ii) If \( \Delta \in \text{FORM} \), then

\[
[\theta^*_0, \ldots, \theta^*_k, \langle \xi_0, \ldots, \xi_k \rangle, \theta^*_0, \langle \xi_0, \ldots, \xi_k \rangle, \Delta] = [\theta^*_0, \ldots, \theta^*_k, \langle \xi_0, \ldots, \xi_k \rangle, \Delta].
\]

Proof: Suppose \( k \in \mathbb{N} \setminus \{0\} \), \( \{\theta^*_0, \ldots, \theta^*_k\} \subseteq \text{TERM} \) and \( \{\xi_0, \ldots, \xi_k\} \subseteq \text{VAR} \), where \( \xi_i \neq \xi_j \) for all \( i, j < k+1 \) with \( i \neq j \). Ad (i): Suppose \( \theta \in \text{TERM} \). The proof is carried out by induction on \( k \). Suppose \( k = 1 \). With Theorem 1-25-(i) and Theorem 1-28-(i), we then have

\[
[\theta^*_0, \xi_0, \langle \theta^*_1, \xi_1, \theta \rangle] = [\theta^*_0, \xi_0, \langle \theta^*_1, \xi_1 \rangle, \theta] = [\theta^*_0, \theta^*_1, \langle \xi_0, \xi_1 \rangle, \theta].
\]

Now, suppose \( 1 < k \). Applying the I.H., Theorem 1-25-(i), the I.H., Theorem 1-28-(i), the I.H. and Theorem 1-28-(i) (in this order) yields

\[
[\theta^*_0, \ldots, \theta^*_k, \langle \xi_0, \ldots, \xi_k \rangle, \theta^*_0, \langle \xi_0, \ldots, \xi_k \rangle, \theta] = [\theta^*_0, \ldots, \theta^*_k, \langle \xi_0, \ldots, \xi_k \rangle, \theta].
\]

(ii) follows analogously from Theorem 1-25-(ii) and Theorem 1-28-(ii).
2 The Availability of Propositions

In this chapter, the availability concepts that are needed for the calculus are established. Our course of action can be sketched as follows: First, preliminary concepts concerning segments and segment sequences are to be established, where a segment in a sentence sequence $S$ will be a non-empty, uninterrupted subset of $S$ (2.1). Second, closed segments will be characterised as certain CdI-, NI- and RA-like segments, i.e. certain segments of the kinds that are connected to inferences by conditional introduction (CdI), negation introduction (NI) and particular-quantifier elimination (PE) (2.2). The availability concepts themselves will then be established with recourse to closed segments. This will be done in such a way that exactly those propositions are available in a sentence sequence at a position that do not lie within a proper initial segment of a closed segment in this sentence sequence at this position (2.3). With the theorems that are established in this chapter, we can later show that CdI, NI and PE and only CdI, NI and PE can discharge assumptions.

2.1 Segments and Segment Sequences

First, segments in a non-empty sequence $S$ will be characterised as non-empty and uninterrupted subsets of $S$. Second, some theorems on segments will be proved. Then, some concepts and theorems concerning segment sequences for sentence sequences will be established, where a segment sequence for a sentence sequence $S$ is a finite sequence that enumerates disjunct segments in $S$. Then, AS-comprising segment sequences for segments in sentence sequences will be defined with recourse to segment sequences. An AS-comprising segment sequence for a segment $A$ in $S$ will be a segment sequence for $S$ for which it holds that all values of the sequence are disjunct subsegments of $A$ and that all assumption-sentences in $A$ lie in one of the values of the sequence. These AS-comprising segment sequences will later play a crucial role in the inductive generation of closed segments. The end of the chapter contains the proofs of theorems about AS-comprising segment sequences that are needed for the establishment of closed segments and of theorems on these. We start with the segment definition:
Definition 2-1. Segment in a sequence (metavariabes: $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{A'}, \mathfrak{B'}, \mathfrak{C'}, \mathfrak{A^*}, \mathfrak{B^*}, \mathfrak{C^*}, ...$)

$\mathfrak{A}$ is a segment in $\mathfrak{H}$

iff

$\mathfrak{H} \in \text{SEQ}, \mathfrak{A} \neq \emptyset, \mathfrak{A} \subseteq \mathfrak{H}$ and $\mathfrak{A} = \{(i, \mathfrak{H}_i) \mid \min(\text{Dom}(\mathfrak{A})) \leq i \leq \max(\text{Dom}(\mathfrak{A}))\}$.

Definition 2-2. Assignment of the set of segments of $\mathfrak{H}$ (SG)

$SG = \{ (\mathfrak{H}, X) \mid \mathfrak{H} \in \text{SEQ} \text{ and } X = \{ \mathfrak{A} \mid \mathfrak{A} \text{ is a segment in } \mathfrak{H} \} \}$.

Definition 2-3. Segment

$\mathfrak{A}$ is a segment iff there is an $\mathfrak{H}$ such that $\mathfrak{A}$ is a segment in $\mathfrak{H}$.

Definition 2-4. Subsegment

$\mathfrak{A}$ is a subsegment of $\mathfrak{A'}$ iff $\mathfrak{A}, \mathfrak{A'}$ are segments and $\mathfrak{A} \subseteq \mathfrak{A'}$.

Definition 2-5. Proper subsegment

$\mathfrak{A}$ is a proper subsegment of $\mathfrak{A'}$ iff $\mathfrak{A}$ is a subsegment of $\mathfrak{A'}$ and $\mathfrak{A} \neq \mathfrak{A'}$.

Theorem 2-1. A sentence sequence $\mathfrak{H}$ is non-empty if and only if $SG(\mathfrak{H})$ is non-empty

If $\mathfrak{H} \in \text{SEQ}$, then: $\mathfrak{H} \neq \emptyset$ iff $SG(\mathfrak{H}) \neq \emptyset$.

Proof: Suppose $\mathfrak{H} \in \text{SEQ}$. Suppose $\mathfrak{H} \neq \emptyset$. Then $\mathfrak{H}$ is a segment in $\mathfrak{H}$ and thus $\mathfrak{H} \in SG(\mathfrak{H})$. Now, suppose $SG(\mathfrak{H}) \neq \emptyset$. Then there is an $\mathfrak{A}$ such that $\mathfrak{A}$ is a segment in $\mathfrak{H}$. Then we have $\mathfrak{A} \neq \emptyset$ and $\mathfrak{A} \subseteq \mathfrak{H}$ and thus $\mathfrak{H} \neq \emptyset$. ■

Theorem 2-2. The segment predicate is monotone relative to inclusion between sequences

If $\mathfrak{H}, \mathfrak{H'} \in \text{SEQ}$, $\mathfrak{H} \subseteq \mathfrak{H'}$ and $\mathfrak{A}$ is a segment in $\mathfrak{H}$, then $\mathfrak{A}$ is a segment in $\mathfrak{H'}$.

Proof: Suppose $\mathfrak{H}, \mathfrak{H'} \in \text{SEQ}, \mathfrak{H} \subseteq \mathfrak{H'}$ and $\mathfrak{A}$ is a segment in $\mathfrak{H}$. Then we have $\mathfrak{A} \neq \emptyset$ and $\mathfrak{A} \subseteq \mathfrak{H} \subseteq \mathfrak{H'}$. Moreover, we have $\mathfrak{H} = \mathfrak{H'} \upharpoonright \text{Dom}(\mathfrak{H})$. Thus we have

$\mathfrak{A} = \{(i, \mathfrak{H}_i) \mid \min(\text{Dom}(\mathfrak{A})) \leq i \leq \max(\text{Dom}(\mathfrak{A}))\}$

$= \{(i, \mathfrak{H'}_i) \mid \min(\text{Dom}(\mathfrak{A})) \leq i \leq \max(\text{Dom}(\mathfrak{A}))\}$

and hence we have that $\mathfrak{A}$ is a segment in $\mathfrak{H'}$. ■
Remark 2-1. All of the segment predicates defined in the following are monotone relative to inclusion between sequences. The respective instances of this result are used in the further account without being proven individually.

If $F$ is one of the segment predicates defined in the following, then: If $\mathcal{S}, \mathcal{S}' \in \text{SEQ}, \mathcal{S} \subseteq \mathcal{S}'$ and $\mathfrak{A}$ is an $F$-segment in $\mathcal{S}$, then $\mathfrak{A}$ is an $F$-segment in $\mathcal{S}'$.

Comment: All following definitions of segment predicates have one of the following two forms:

- $\mathfrak{A}$ is an $F$-segment in $\mathcal{S}$ iff $\mathfrak{A} \in \text{SEQ}, \mathfrak{A} \in \text{SG}(\mathcal{S})$ and $H(\mathfrak{A}, \mathcal{S})$.
- $\mathfrak{A}$ is an $F$-segment in $\mathcal{S}$ iff $\mathfrak{A}$ is a segment in $\mathcal{S}$ and $H(\mathfrak{A}, \mathcal{S})$.

In each case, $H$ is the variable part of the definiens, which distinguishes the different definitions. For $H$ it holds in each case that if $\mathcal{S}, \mathcal{S}' \in \text{SEQ}, \mathcal{S} \subseteq \mathcal{S}'$ and $\mathfrak{A} \in \text{SG}(\mathcal{S})$ (or, equivalently: $\mathfrak{A}$ is a segment in $\mathcal{S}$) and $H(\mathfrak{A}, \mathcal{S})$, then $H(\mathfrak{A}, \mathcal{S}')$. With Theorem 2-2 and the respective definition it then follows in each case that if $\mathcal{S}, \mathcal{S}'$ are sequences, $\mathcal{S} \subseteq \mathcal{S}'$ and $\mathfrak{A}$ is an $F$-segment in $\mathcal{S}$, then $\mathfrak{A}$ is an $F$-segment in $\mathcal{S}'$.

From this, it also follows that if $\mathcal{S}, \mathcal{S}'$ are sequences and $\mathfrak{A}$ is an $F$-segment in $\mathcal{S}$, then $\mathfrak{A}$ is also an $F$-segment in $\mathcal{S}'$.\(^{10}\) Note, however, that for many of the sequence predicates defined in the following, it does not hold that if $\mathcal{S}, \mathcal{S}'$ are sequences, and $\mathfrak{A}$ is an $F$-segment in $\mathcal{S}$, then $\mathfrak{A}$ is also an $F$-segment in $\mathcal{S}'$.

Theorem 2-3. Segments in restrictions\(^{11}\)

If $\mathcal{S} \in \text{SEQ}$, then: $\mathfrak{A}$ is a segment in $\mathcal{S}$ iff $\mathfrak{A}$ is a segment in $\mathcal{S} \upharpoonright \text{max}(\text{Dom}(\mathfrak{A}))+1$.

Proof: Suppose $\mathcal{S} \in \text{SEQ}$. (L-R): Suppose $\mathfrak{A}$ is a segment in $\mathcal{S}$. Then we have $\mathfrak{A} \neq \emptyset, \mathfrak{A} \subseteq \mathcal{S}$ and thus: $\mathcal{S} \upharpoonright \text{max}(\text{Dom}(\mathfrak{A}))+1 \in \text{SEQ}$. We also have that $\mathfrak{A} \subseteq \mathcal{S} \upharpoonright \text{max}(\text{Dom}(\mathfrak{A}))+1 \subseteq \mathcal{S}$ and hence that $\mathcal{S} \upharpoonright \text{max}(\text{Dom}(\mathfrak{A}))+1 \in \text{SEQ}\setminus \emptyset$ and also that

\(^{10}\) Let $f^\uparrow g = f \cup \{(\text{Dom}(f)+i, g_i) \mid i \in \text{Dom}(g)\}$ if $f$ is a finite sequence and $g$ is a sequence, else $f^\uparrow g = \emptyset$.
We omit parentheses and assume that they are nested from left to right, i.e., $'(a_0 \sim a_1 \sim a_2 \sim \ldots \sim a_n)' = '('(\ldots((a_0 \sim a_1) \sim a_2) \sim \ldots) \sim a_n)'$.

\(^{11}\) Let $R\upharpoonright X = \{(a, b) \mid (a, b) \in R \text{ and } a \in X\}$.
\[\mathfrak{A} = \{(i, \mathfrak{f}_i) \mid \min(\text{Dom}(\mathfrak{A})) \leq i \leq \max(\text{Dom}(\mathfrak{A}))\}\]
\[= \{(i, (\mathfrak{f}_i + \max(\text{Dom}(\mathfrak{A})) + 1)) \mid \min(\text{Dom}(\mathfrak{A})) \leq i \leq \max(\text{Dom}(\mathfrak{A}))\}.\]

Thus, \(\mathfrak{A}\) is a segment in \(\mathfrak{f}_i + \max(\text{Dom}(\mathfrak{A})) + 1\). \(\Box\) 

**(Remark 2-2. F-segments in restrictions)**

If \(F\) is one of the segment predicates defined in the following, then: If \(\mathfrak{f} \in \text{SEQ}\), then \(\mathfrak{A}\) is an \(F\)-segment in \(\mathfrak{f}\) iff \(\mathfrak{A}\) is an \(F\)-segment in \(\mathfrak{f}_i + \max(\text{Dom}(\mathfrak{A})) + 1\).

**Comment:** All of the following definitions of segment predicates have one of the two forms noted in Remark 2-1, where for \(H\) it holds that if \(\mathfrak{f} \in \text{SEQ}\), \(\mathfrak{A} \in \text{SG}(\mathfrak{f})\) (or, equivalently: \(\mathfrak{A}\) is a segment in \(\mathfrak{f}\)) and \(H(\mathfrak{A}, \mathfrak{f})\), then \(H(\mathfrak{A}, \mathfrak{f}_i + \max(\text{Dom}(\mathfrak{A})) + 1)\). The reason for this is in each case that the respective definientia only refer to conditions in \(\mathfrak{f}_i + \max(\text{Dom}(\mathfrak{A})) + 1\). With Theorem 2-3 and the respective definitions it thus follows in each case that if \(\mathfrak{f}\) is a sentence sequence and \(\mathfrak{A}\) is an \(F\)-segment in \(\mathfrak{f}\) ist, then \(\mathfrak{A}\) is an \(F\)-segment in \(\mathfrak{f}_i + \max(\text{Dom}(\mathfrak{A})) + 1\). For the right-left-direction see Remark 2-1. \(\Box\)

**(Theorem 2-4. Segments with identical beginning and end are identical)**

If \(\mathfrak{f} \in \text{SEQ}\), \(\mathfrak{A}, \mathfrak{A}' \in \text{SG}(\mathfrak{f})\), \(\min(\text{Dom}(\mathfrak{A})) = \min(\text{Dom}(\mathfrak{A}'))\) and \(\max(\text{Dom}(\mathfrak{A})) = \max(\text{Dom}(\mathfrak{A}'))\), then \(\mathfrak{A} = \mathfrak{A}'\).

**Proof:** Suppose \(\mathfrak{f} \in \text{SEQ}\), \(\mathfrak{A}, \mathfrak{A}' \in \text{SG}(\mathfrak{f})\), \(\min(\text{Dom}(\mathfrak{A})) = \min(\text{Dom}(\mathfrak{A}'))\) and \(\max(\text{Dom}(\mathfrak{A})) = \max(\text{Dom}(\mathfrak{A}'))\). Then we have for all \((i, \mathfrak{f}_i)\): \((i, \mathfrak{f}_i) \in \mathfrak{A}\) iff \(\min(\text{Dom}(\mathfrak{A})) \leq i \leq \max(\text{Dom}(\mathfrak{A}))\) iff \(\min(\text{Dom}(\mathfrak{A}')) \leq i \leq \max(\text{Dom}(\mathfrak{A}'))\) iff \((i, \mathfrak{f}_i) \in \mathfrak{A}'\). \(\Box\)

**(Theorem 2-5. Inclusion between segments)**

If \(\mathfrak{f} \in \text{SEQ}\) and \(\mathfrak{A}, \mathfrak{A}' \in \text{SG}(\mathfrak{f})\), then:

(i) \(\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(\mathfrak{A}'))\) and \(\max(\text{Dom}(\mathfrak{A})) \leq \max(\text{Dom}(\mathfrak{A}'))\) iff \(\mathfrak{A}' \subseteq \mathfrak{A}\), and

(ii) If \(\min(\text{Dom}(\mathfrak{A})) = \min(\text{Dom}(\mathfrak{A}'))\), then \(\mathfrak{A} \subseteq \mathfrak{A}'\) or \(\mathfrak{A}' \subseteq \mathfrak{A}\).

**Proof:** Suppose \(\mathfrak{f} \in \text{SEQ}\) and \(\mathfrak{A}, \mathfrak{A}' \in \text{SG}(\mathfrak{f})\). Then we have

\[\mathfrak{A} = \{(l, \mathfrak{f}_l) \mid \min(\text{Dom}(\mathfrak{A})) \leq l \leq \max(\text{Dom}(\mathfrak{A}))\}\]
and
\[ \mathcal{A}' = \{(l, \mathcal{A}) | \min(\text{Dom}(\mathcal{A})) \leq l \leq \max(\text{Dom}(\mathcal{A}'))\}. \]

**Ad (i):** Suppose \( \min(\text{Dom}(\mathcal{A})) \leq \min(\text{Dom}(\mathcal{A}')) \) and \( \max(\text{Dom}(\mathcal{A})) \leq \max(\text{Dom}(\mathcal{A}')) \). Suppose \( (l, \mathcal{A}) \in \mathcal{A}' \). Then we have \( \min(\text{Dom}(\mathcal{A}')) \leq l \leq \max(\text{Dom}(\mathcal{A}')) \) and thus according to the hypothesis \( \min(\text{Dom}(\mathcal{A})) \leq \min(\text{Dom}(\mathcal{A}')) \leq l \leq \max(\text{Dom}(\mathcal{A}')) \leq \max(\text{Dom}(\mathcal{A})). \) Therefore we have \( (l, \mathcal{A}) \in \mathcal{A} \).

Now, suppose \( \mathcal{A}' \subseteq \mathcal{A} \). Then we have that \( \min(\text{Dom}(\mathcal{A}')), \max(\text{Dom}(\mathcal{A}')) \in \text{Dom}(\mathcal{A}) \) and hence \( \min(\text{Dom}(\mathcal{A})) \leq \min(\text{Dom}(\mathcal{A}')) \) and \( \max(\text{Dom}(\mathcal{A})) \leq \max(\text{Dom}(\mathcal{A}')). \)**

**Ad (ii):** Suppose \( \min(\text{Dom}(\mathcal{A})) = \min(\text{Dom}(\mathcal{A}')) \). Then we have \( \max(\text{Dom}(\mathcal{A})) \leq \max(\text{Dom}(\mathcal{A}')) \) or \( \max(\text{Dom}(\mathcal{A}')) \leq \max(\text{Dom}(\mathcal{A})). \) In the first case, it follows with (i) that \( \mathcal{A} \subseteq \mathcal{A}' \). In the second case, it follows with (i) that \( \mathcal{A}' \subseteq \mathcal{A} \). ■

**Theorem 2-6.** Non-empty restrictions of segments are segments

If \( \mathcal{A} \in \text{SEQ} \) and \( \mathcal{A} \in \text{SG}(\mathcal{A}) \), then for all \( k \in \text{Dom}(\mathcal{A}) \): \( \mathcal{A}|k+1 \in \text{SG}(\mathcal{A}) \).

**Proof:** Suppose \( \mathcal{A} \in \text{SEQ} \) and \( \mathcal{A} \in \text{SG}(\mathcal{A}) \) and suppose \( k \in \text{Dom}(\mathcal{A}) \). Then we have that \( \min(\text{Dom}(\mathcal{A})) < k+1 \leq \max(\text{Dom}(\mathcal{A}))+1 \). Thus we have that \( \mathcal{A}|k+1 = \{(i, \mathcal{A}) | \min(\text{Dom}(\mathcal{A})) \leq i \leq \max(\text{Dom}(\mathcal{A}))\}|k+1 = \{(i, \mathcal{A}) | \min(\text{Dom}(\mathcal{A})) \leq i \leq k\} = \{(i, \mathcal{A}) | \min(\text{Dom}(\mathcal{A}|k+1)) \leq i \leq \max(\text{Dom}(\mathcal{A}|k+1))\} \) and also that \( \mathcal{A}|k+1 \subseteq \mathcal{A} \subseteq \mathcal{A} \). We also have \( k \in \text{Dom}(\mathcal{A}|k+1) \) and thus that \( \mathcal{A}|k+1 \neq \emptyset \). Hence we have \( \mathcal{A}|k+1 \in \text{SG}(\mathcal{A}). \) ■

**Theorem 2-7.** Restrictions of segments that are segments themselves have the same beginning as the restricted segment

If \( \mathcal{A} \) is a segment in \( \mathcal{A} \), then for all \( k \in \text{Dom}(\mathcal{A}) \): If \( \mathcal{A}|k \) is a segment in \( \mathcal{A} \), then \( \min(\text{Dom}(\mathcal{A}|k)) = \min(\text{Dom}(\mathcal{A})). \)

**Proof:** Suppose \( \mathcal{A} \) is a segment in \( \mathcal{A} \). Now, suppose \( k \in \text{Dom}(\mathcal{A}) \) and suppose \( \mathcal{A}|k \) is a segment in \( \mathcal{A} \) and hence \( \mathcal{A}|k \neq \emptyset \). Then we have \( \mathcal{A}|k = \{(i, \mathcal{A}) | \min(\text{Dom}(\mathcal{A})) \leq i \leq \max(\text{Dom}(\mathcal{A}))\}|k = \{(i, \mathcal{A}) | \min(\text{Dom}(\mathcal{A})) \leq i \leq k-1\} \) and hence with \( \mathcal{A}|k \neq \emptyset \) that \( \min(\text{Dom}(\mathcal{A}|k)) = \min(\text{Dom}(\mathcal{A})). \) ■
**Theorem 2-8.** Two segments are disjunct if and only if one of them lies before the other

If $s \in \text{SEQ}$ and $\mathfrak{A}, \mathfrak{A}' \in \text{SG}(s)$, then:

$\mathfrak{A} \cap \mathfrak{A}' = \emptyset$

iff

(i) $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}'))$ and $\max(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}'))$, or

(ii) $\min(\text{Dom}(\mathfrak{A}')) < \min(\text{Dom}(\mathfrak{A}))$ and $\max(\text{Dom}(\mathfrak{A}')) < \min(\text{Dom}(\mathfrak{A}))$.

*Proof:* Suppose $s \in \text{SEQ}$ and $\mathfrak{A}, \mathfrak{A}' \in \text{SG}(s)$. (*L-R*): Suppose $\mathfrak{A} \cap \mathfrak{A}' = \emptyset$. Then we have

- $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}'))$
  or
- $\min(\text{Dom}(\mathfrak{A})) = \min(\text{Dom}(\mathfrak{A}'))$
  or
- $\min(\text{Dom}(\mathfrak{A}')) < \min(\text{Dom}(\mathfrak{A}))$.

The second case, i.e. $\min(\text{Dom}(\mathfrak{A})) = \min(\text{Dom}(\mathfrak{A}'))$, is impossible because otherwise we would have that $(\min(\text{Dom}(\mathfrak{A})), \mathfrak{A}_{\min(\text{Dom}(\mathfrak{A}))}) \in \mathfrak{A}$ and $(\min(\text{Dom}(\mathfrak{A})), \mathfrak{A}_{\min(\text{Dom}(\mathfrak{A}))}) \in \mathfrak{A}'$ and thus that $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$.

Suppose $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}'))$. If $\min(\text{Dom}(\mathfrak{A}')) \leq \max(\text{Dom}(\mathfrak{A}))$, then we would have $(\min(\text{Dom}(\mathfrak{A}')), \mathfrak{A}_{\min(\text{Dom}(\mathfrak{A}'))}) \in \mathfrak{A}$ and $(\min(\text{Dom}(\mathfrak{A}')), \mathfrak{A}_{\min(\text{Dom}(\mathfrak{A}'))}) \in \mathfrak{A}'$. Thus we would have $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$, which contradicts the hypothesis. In the first case, we thus have $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}'))$ and $\max(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}'))$.

Suppose $\min(\text{Dom}(\mathfrak{A}')) < \min(\text{Dom}(\mathfrak{A}))$. If $\min(\text{Dom}(\mathfrak{A})) \leq \max(\text{Dom}(\mathfrak{A}'))$, then we would have $(\min(\text{Dom}(\mathfrak{A})), \mathfrak{A}_{\min(\text{Dom}(\mathfrak{A}))}) \in \mathfrak{A}$ and $(\min(\text{Dom}(\mathfrak{A})), \mathfrak{A}_{\min(\text{Dom}(\mathfrak{A}))}) \in \mathfrak{A}'$. Thus we would again have $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$. In the third case, we thus have $\min(\text{Dom}(\mathfrak{A}')) < \min(\text{Dom}(\mathfrak{A}))$ and $\max(\text{Dom}(\mathfrak{A}')) < \min(\text{Dom}(\mathfrak{A}))$.

(*R-L*): Now, suppose $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}'))$ and $\max(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}'))$ or $\min(\text{Dom}(\mathfrak{A}')) < \min(\text{Dom}(\mathfrak{A}))$ and $\max(\text{Dom}(\mathfrak{A}')) < \min(\text{Dom}(\mathfrak{A}))$.

Now, suppose for contradiction that $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$. Then there would be an $i$ such that $(i, \mathfrak{A}_i) \in \mathfrak{A} \cap \mathfrak{A}'$. Then we would have $\min(\text{Dom}(\mathfrak{A})) \leq i \leq \max(\text{Dom}(\mathfrak{A}))$ and $\min(\text{Dom}(\mathfrak{A}')) \leq i \leq \max(\text{Dom}(\mathfrak{A}'))$. Thus we would have $\min(\text{Dom}(\mathfrak{A}')) < \min(\text{Dom}(\mathfrak{A}'))$ or $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}'))$. Contradiction! Therefore we have $\mathfrak{A} \cap \mathfrak{A}' = \emptyset$. ■
Theorem 2-9. Two segments have a common element if and only if the beginning of one of them lies within the other
If \( \mathcal{S} \in \text{SEQ} \) and \( \mathfrak{A}, \mathfrak{A}' \in \text{SG}(\mathcal{S}) \), then:
\( \mathfrak{A} \cap \mathfrak{A}' \neq \emptyset \)
iff
(i) \( \min(\text{Dom}(\mathfrak{A})) \in \text{Dom}(\mathfrak{A}') \) or
or
(ii) \( \min(\text{Dom}(\mathfrak{A}')) \in \text{Dom}(\mathfrak{A}) \).

Proof: Suppose \( \mathcal{S} \in \text{SEQ} \) and \( \mathfrak{A}, \mathfrak{A}' \in \text{SG}(\mathcal{S}) \). (L-R): Suppose \( \mathfrak{A} \cap \mathfrak{A}' \neq \emptyset \). Then there is an \( i \in \text{Dom}(\mathcal{S}) \) such that \( (i, \mathfrak{S}_i) \in \mathfrak{A} \cap \mathfrak{A}' \). Then we have
\[
\min(\text{Dom}(\mathfrak{A})) \leq i \leq \max(\text{Dom}(\mathfrak{A})) \quad \text{and} \quad \min(\text{Dom}(\mathfrak{A}')) \leq \min(\text{Dom}(\mathfrak{A})) \leq \max(\text{Dom}(\mathfrak{A}')).
\]
Thus we then have
\[
\min(\text{Dom}(\mathfrak{A}')) \leq \min(\text{Dom}(\mathfrak{A})) \leq i \leq \max(\text{Dom}(\mathfrak{A}'))
\]
or
\[
\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(\mathfrak{A}')) \leq i \leq \max(\text{Dom}(\mathfrak{A})).
\]
Thus we have eventually that
\[
\min(\text{Dom}(\mathfrak{A})) \in \text{Dom}(\mathfrak{A}') \quad \text{or} \quad \min(\text{Dom}(\mathfrak{A}')) \in \text{Dom}(\mathfrak{A}).
\]
(R-L): If \( \min(\text{Dom}(\mathfrak{A})) \in \text{Dom}(\mathfrak{A}') \) or \( \min(\text{Dom}(\mathfrak{A}')) \in \text{Dom}(\mathfrak{A}) \), then we have
\( (\min(\text{Dom}(\mathfrak{A})), \mathcal{S}_{\min(\text{Dom}(\mathfrak{A}))}) \in \mathfrak{A} \cap \mathfrak{A}' \) or \( (\min(\text{Dom}(\mathfrak{A}')), \mathcal{S}_{\min(\text{Dom}(\mathfrak{A}'))}) \in \mathfrak{A} \cap \mathfrak{A}' \) and thus in both cases \( \mathfrak{A} \cap \mathfrak{A}' \neq \emptyset \). ■

Definition 2-6. Suitable sequences of natural numbers for subsets of sentence sequences
\( g \) is a suitable sequence of natural numbers for \( \mathfrak{A} \)
iff
There is an \( \mathcal{S} \in \text{SEQ} \) such that \( \mathfrak{A} \subseteq \mathcal{S} \) and \( g \) is a strictly monotone increasing sequence of natural numbers with \( \text{Ran}(g) = \text{Dom}(\mathfrak{A}) \).

The immediate purpose of the definition is to enable us to enumerate the elements (of the domain) of a subset of a sequence in a way that preserves their natural order. Moreover, suitable sequences can be used to turn segments of sequences into sequences by compos-
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The respective segments with a suitable sequence of natural numbers. Such a procedure could be considered as an inverse operation to the concatenation of sequences.

**Theorem 2-10. Existence of suitable sequences of natural numbers**

If \( \mathcal{A} \in \text{SEQ} \) and \( \mathcal{A} \subseteq \mathfrak{s} \), then there is a \( g \) such that \( g \) is a suitable sequence of natural numbers for \( \mathcal{A} \).

**Proof:** Suppose \( \mathcal{A} \in \text{SEQ} \) and \( \mathcal{A} \subseteq \mathfrak{s} \). The proof is carried out by induction on \( |\mathcal{A}| \). Suppose \( |\mathcal{A}| = 0 \). Let \( g = \emptyset \). Then \( g \) is trivially a strictly monotone increasing sequence of natural numbers with \( \text{Ran}(g) = \text{Dom}(\mathcal{A}) \). Now, suppose \( |\mathcal{A}| = k+1 \). Then we have \( k = 0 \) or \( k > 0 \). In the first case, \( \{(0, \max(\text{Dom}(\mathfrak{s})))\} \) is a suitable sequence of natural numbers for \( \mathcal{A} \). Now, suppose \( k > 0 \). Since \( \mathcal{A} \) is a finite function, we have that \( |\mathcal{A} \setminus \{(\max(\text{Dom}(\mathcal{A})), \mathcal{A}_{\max(\text{Dom}(\mathcal{A}))})\}| = k \). Furthermore, we have \( \mathcal{A} \setminus \{(\max(\text{Dom}(\mathcal{A})), \mathcal{A}_{\max(\text{Dom}(\mathcal{A}))})\} \subseteq \mathfrak{s} \). According to the I.H., we thus have a \( g \) such that \( g \) is a suitable sequence of natural numbers for \( \mathcal{A} \setminus \{(\max(\text{Dom}(\mathcal{A})), \mathcal{A}_{\max(\text{Dom}(\mathcal{A}))})\} \). Now, let \( g' = g \cup \{(\text{ Dom}(g), \max(\text{Dom}(\mathcal{A}))\}) \). Obviously it holds that \( \text{Ran}(g') = \text{Dom}(\mathcal{A}) \). Because of

\[
g(\max(\text{Dom}(g))) = \max(\text{Ran}(g)) = \max(\text{Dom} \setminus \{(\max(\text{Dom}(\mathfrak{s})), \mathcal{A}_{\max(\text{Dom}(\mathcal{A}))})\})
\]

\[
< \max(\text{Dom}(\mathcal{A})) = \max(\text{Ran}(g')) = g'(\text{Dom}(g)) = g'(\max(\text{Dom}(g'))),
\]

the strict monotony of \( g \) carries over to \( g' \). Therefore we have that \( g' \) is a suitable sequence of natural numbers for \( \mathcal{A} \). ■

**Theorem 2-11. Bijectivity of suitable sequences of natural numbers**

If \( \mathcal{A} \in \text{SEQ} \), \( \mathcal{A} \subseteq \mathfrak{s} \), and \( g \) is a suitable sequence of natural numbers for \( \mathcal{A} \), then \( g \) is a bijection between \( \text{Dom}(g) \) and \( \text{Dom}(\mathcal{A}) \).

**Proof:** Suppose \( \mathcal{A} \in \text{SEQ} \), \( \mathcal{A} \subseteq \mathfrak{s} \) and suppose \( g \) is a suitable sequence of natural numbers for \( \mathcal{A} \). Then we have \( \text{Ran}(g) = \text{Dom}(\mathcal{A}) \) and hence that \( g \) is a surjection of \( \text{Dom}(g) \) onto \( \text{Dom}(\mathcal{A}) \). Furthermore, because \( g \) is a strictly monotone sequence of natural numbers, we have that \( g \) is an injection of \( \text{Dom}(g) \) into \( \text{Dom}(\mathcal{A}) \). Hence \( g \) is a bijection between \( \text{Dom}(g) \) and \( \text{Dom}(\mathcal{A}) \). ■
Theorem 2-12. Uniqueness of suitable sequences of natural numbers
If $\mathcal{S} \in \text{SEQ}, \mathcal{A} \subseteq \mathcal{S}$, and $g, g'$ are suitable sequences of natural numbers for $\mathcal{A}$, then: $g = g'$.

Proof: Suppose $\mathcal{S} \in \text{SEQ}, \mathcal{A} \subseteq \mathcal{S}$ and suppose $g, g'$ are suitable sequences of natural numbers for $\mathcal{A}$. Then we have $\text{Ran}(g) = \text{Dom}(\mathcal{A}) = \text{Ran}(g')$. With Theorem 2-11, we also have that $\text{Dom}(g) = |\text{Ran}(g)| = |\text{Ran}(g')| = \text{Dom}(g')$. Now, it holds that strictly monotone increasing sequences of natural numbers with identical domains and identical ranges are identical. Therefore we have $g = g'$. \hfill $\blacksquare$

Theorem 2-13. Non-recursive characterisation of the suitable sequence for a segment
If $\mathcal{A}$ is a segment in $\mathcal{S}$, then $\{(l, \text{min}(|\text{Dom}(\mathcal{S})|)+l) \mid l < |\text{Dom}(\mathcal{S})|\}$ is a suitable sequence of natural numbers for $\mathcal{A}$.

Proof: Suppose $\mathcal{S} \in \text{SEQ}$ and $\mathcal{A}$ is a segment in $\mathcal{S}$. Then we have $\mathcal{A} \neq \emptyset$. The proof is carried out by induction on $|\text{Dom}(\mathcal{A})|$. Suppose $|\text{Dom}(\mathcal{A})| = 1$. Then we have $\text{Dom}(\mathcal{A}) = \{\text{min}(\text{Dom}(\mathcal{A}))\}$ and $\{(0, \text{min}(\text{Dom}(\mathcal{A})))\}$ is a suitable sequence of natural numbers for $\mathcal{A}$ and $\{(0, \text{min}(\text{Dom}(\mathcal{A})))\} = \{(l, \text{min}(\text{Dom}(\mathcal{A}))+l) \mid l < 1\} = \{(l, \text{min}(\text{Dom}(\mathcal{A}))+l) \mid l < |\text{Dom}(\mathcal{A})|\}$.

Now, suppose the statement holds for $k \geq 1$ and suppose $|\text{Dom}(\mathcal{A})| = k+1$. Since $\mathcal{A}$ is a finite function, we have that $|\mathcal{A}\setminus\{(\text{max}(\text{Dom}(\mathcal{A})), \text{max}(\text{Dom}(\mathcal{A})))\}| = k$. Furthermore, we have that $\mathcal{A}^* = \mathcal{A}\setminus\{(\text{max}(\text{Dom}(\mathcal{A})), \text{max}(\text{Dom}(\mathcal{A})))\}$ is a segment in $\mathcal{S}$. According to the I.H., we therefore have that $g = \{(l, \text{min}(\text{Dom}(\mathcal{A}^*))+l) \mid l < |\text{Dom}(\mathcal{A}^*)|\} = \{(l, \text{min}(\text{Dom}(\mathcal{A}))+l) \mid l < |\text{Dom}(\mathcal{A})|-1\}$ is a suitable sequences of natural numbers for $\mathcal{A}^*$. Let $g' = g \cup \{(|\text{Dom}(\mathcal{A})|-1, \text{max}(\text{Dom}(\mathcal{A})))\}$. Then we have $\text{Ran}(g') = \text{Dom}(\mathcal{A}^*) \cup \{\text{max}(\text{Dom}(\mathcal{A}))\} = \text{Dom}(\mathcal{A})$ and we have $\text{Dom}(g') = \text{Dom}(g) \cup \{\text{Dom}(g)\} = \text{Dom}(g)+1 = |\text{Dom}(\mathcal{A}^*)|+1 = |\text{Dom}(\mathcal{A})|$. Since $\mathcal{A}$ is a segment in $\mathcal{S}$, it also holds that $\text{max}(\text{Dom}(\mathcal{A}^*))+1 = \text{max}(\text{Dom}(\mathcal{A}))$. Thus we have $g'(\text{Dom}(\mathcal{A})|-1) = \text{max}(\text{Dom}(\mathcal{A}^*))+1 = g(\text{Dom}(\mathcal{A})|-2)+1 = (\text{min}(\text{Dom}(\mathcal{A}^*))+|\text{Dom}(\mathcal{A})|-2)+1 = (\text{min}(\text{Dom}(\mathcal{A}))+|\text{Dom}(\mathcal{A})|-2)+1 = \text{min}(\text{Dom}(\mathcal{A}))+|\text{Dom}(\mathcal{A})|-1$. Hence we then have $g' = \{(l, \text{min}(\text{Dom}(\mathcal{A}))+l) \mid l < |\text{Dom}(\mathcal{A})|-1\} \cup \{(|\text{Dom}(\mathcal{A})|-1, \text{min}(\text{Dom}(\mathcal{A}))+|\text{Dom}(\mathcal{A})|-1)\} = \{(l, \text{min}(\text{Dom}(\mathcal{A}))+l) \mid l < |\text{Dom}(\mathcal{A})|\}$. Thus we have that $g'$ is also a strictly monotone increasing sequence of natural numbers and hence we have that $g'$ is a suitable sequence of natural numbers for $\mathcal{A}$. \hfill $\blacksquare$
Definition 2-7. Segment sequences for sentence sequences

\( G \) is a segment sequence for \( \mathcal{S} \)
if
\( \mathcal{S} \in \text{SEQ} \) and \( G \) is a sequence with \( \text{Ran}(G) \subseteq \text{SG}(\mathcal{S}) \) and for all \( i, j \in \text{Dom}(G) \): If \( i < j \), then \( \min(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \) and \( \max(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \).

Definition 2-8. Assignment of the set of segment sequences for \( \mathcal{S} \) (SGS)

\( \text{SGS} = \{ (\mathcal{S}, X) \mid \mathcal{S} \in \text{SEQ} \text{ and } X = \{ G \mid G \text{ is a segment sequence for } \mathcal{S} \} \} \)

Theorem 2-14. A sentence sequence \( \mathcal{S} \) is non-empty if and only if there is a non-empty segment sequence for \( \mathcal{S} \)

If \( \mathcal{S} \in \text{SEQ} \), then: \( \mathcal{S} \neq \emptyset \) iff there is a \( G \in \text{SGS}(\mathcal{S}) \) with \( G \neq \emptyset \).

Proof: Suppose \( \mathcal{S} \in \text{SEQ} \). (L-R): Suppose \( \mathcal{S} \neq \emptyset \). Then we have \( \emptyset \neq \{(i, \{(i, \mathcal{S}_i)\}) \mid i \in \text{Dom}(\mathcal{S})\} \in \text{SGS}(\mathcal{S}) \). (R-L): Now, suppose there is a \( G \in \text{SGS}(\mathcal{S}) \) such that \( G \neq \emptyset \). Then there is an \( i \in \text{Dom}(G) \). Also, we have \( \text{Ran}(G) \subseteq \text{SG}(\mathcal{S}) \) and thus \( G(i) \in \text{SG}(\mathcal{S}) \). With Theorem 2-1, we then have \( \mathcal{S} \neq \emptyset \). ■

Theorem 2-15. \( \emptyset \) is a segment sequence for all sequences

If \( \mathcal{S} \in \text{SEQ} \), then \( \emptyset \in \text{SGS}(\mathcal{S}) \).

Proof: Suppose \( \mathcal{S} \in \text{SEQ} \). Then we have that \( \emptyset \) is a sequence with \( \text{Ran}(\emptyset) = \emptyset \subseteq \text{SG}(\mathcal{S}) \) and for all \( i, j \in \text{Dom}(\emptyset) = \emptyset \) we trivially have: If \( i < j \), then \( \min(\text{Dom}(\emptyset(i))) < \min(\text{Dom}(\emptyset(j))) \) and \( \max(\text{Dom}(\emptyset(i))) < \min(\text{Dom}(\emptyset(j))) \). ■

Theorem 2-16. Properties of segment sequences

If \( \mathcal{S} \in \text{SEQ} \) and \( G \in \text{SGS}(\mathcal{S}) \), then:

(i) \( G \) is an injection of \( \text{Dom}(G) \) into \( \text{Ran}(G) \),
(ii) \( G \) is a bijection between \( \text{Dom}(G) \) and \( \text{Ran}(G) \),
(iii) \( \text{Dom}(G) = |\text{Ran}(G)| \), and
(iv) \( G \) is a finite sequence.

Proof: Suppose \( \mathcal{S} \in \text{SEQ} \) and \( G \in \text{SGS}(\mathcal{S}) \). Then we have that \( G \) is a sequence with \( \text{Ran}(G) \subseteq \text{SG}(\mathcal{S}) \) and for all \( i, j \in \text{Dom}(G) \): If \( i < j \), then \( \min(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \) and \( \max(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \).
Ad (i): Now, suppose \( i, j \in \text{Dom}(G) \) and suppose \( G(i) = G(j) \). Then we have \( \min(\text{Dom}(G(i))) = \min(\text{Dom}(G(j))) \). Suppose for contradiction that \( i \neq j \). Then we would have \( i < j \) or \( j < i \) and thus we would have \( \min(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \) or \( \min(\text{Dom}(G(j))) < \min(\text{Dom}(G(i))) \), which both contradict \( \min(\text{Dom}(G(i))) = \min(\text{Dom}(G(j))) \). Therefore we have for \( i, j \in \text{Dom}(G) \) with \( G(i) = G(j) \) that \( i = j \). Hence \( G \) is an injection of \( \text{Dom}(G) \) in \( \text{Ran}(G) \).

Ad (ii): \( G \) is a surjection of \( \text{Dom}(G) \) onto \( \text{Ran}(G) \) and with (i) \( G \) is then a bijection between \( \text{Dom}(G) \) and \( \text{Ran}(G) \).

Ad (iii): Since \( G \) is a sequence, it holds with (ii): \( \text{Dom}(G) = |\text{Ran}(G)| \)

Ad (iv): \( G \) is a sequence and with (iii) \( G \) is then a finite sequence, because we have \( \text{Ran}(G) \subseteq \text{SG}(\mathcal{F}) \subseteq \text{POT}(\mathcal{F}) \) and hence (because with \( \mathcal{F} \in \text{SEQ} \) it holds that \( |\mathcal{F}| \in \mathbb{N} \)):

\[
\text{Dom}(G) = |\text{Ran}(G)| \leq |\text{SG}(\mathcal{F})| \leq |\text{POT}(\mathcal{F})| = 2^{|\mathcal{F}|} \in \mathbb{N}.\]

\[\blacksquare\]

**Theorem 2-17.** Existence of segment sequences that enumerate all elements of a set of disjunct segments

If \( \mathcal{F} \in \text{SEQ} \) and \( X \subseteq \text{SG}(\mathcal{F}) \) and for all \( \mathfrak{A}, \mathfrak{A}' \in X \) it holds that if \( \mathfrak{A} \neq \mathfrak{A}' \), then \( \mathfrak{A} \cap \mathfrak{A}' = \emptyset \), then: There is a \( G \in \text{SGS}(\mathcal{F}) \) such that \( \text{Ran}(G) = X \).

Proof: Suppose \( \mathcal{F} \in \text{SEQ} \) and \( X \subseteq \text{SG}(\mathcal{F}) \) and suppose for all \( \mathfrak{A}, \mathfrak{A}' \in X \) If \( \mathfrak{A} \neq \mathfrak{A}' \), then \( \mathfrak{A} \cap \mathfrak{A}' = \emptyset \). We have \( \mathcal{B} = \{(l, \mathcal{F}) | \text{ There is an } \mathfrak{A} \in X \text{ and } l = \min(\text{Dom}(\mathfrak{A})) \} \subseteq \mathcal{F} \). According to Theorem 2-10, there is thus a suitable sequence of natural numbers \( g \) for \( \mathcal{B} \). With Theorem 2-11, we then have that \( g \) is a bijection between \( \text{Dom}(g) \) and \( \text{Dom}(\mathcal{B}) \). According to the definition of \( \mathcal{B} \), we then have for all \( \mathfrak{A} \in X \): \( \min(\text{Dom}(\mathfrak{A})) = g(i) \) for an \( i \in \text{Dom}(g) \). Because \( g \) is strictly monotone increasing we also have: If \( i, j \in \text{Dom}(g) \) and \( i < j \), then \( g(i) < g(j) \).

We then have for all \( i \in \text{Dom}(g) \): There is exactly one \( \mathfrak{A} \in X \) such that \( g(i) = \min(\text{Dom}(\mathfrak{A})) \). To see this, suppose that \( i \in \text{Dom}(g) \). Then we have \( g(i) = \min(\text{Dom}(\mathfrak{A})) \) for an \( \mathfrak{A} \in X \). Now, suppose \( \mathfrak{A}' \in X \) and \( g(i) = \min(\text{Dom}(\mathfrak{A}')) \). According to the hypothesis, we have \( X \subseteq \text{SG}(\mathcal{F}) \) and hence, with Theorem 2-9, we have \( \mathfrak{A} \cap \mathfrak{A}' \neq \emptyset \). By hypothesis, we have that \( \mathfrak{A} = \mathfrak{A}' \).
Now, let \( G = \{(i, \mathfrak{A}) \mid i \in \text{Dom}(g) \text{ and } \mathfrak{A} \in X \text{ and } g(i) = \min(\text{Dom}(\mathfrak{A}))\}. \) First, we have that \( G \) is a sequence with \( \text{Ran}(G) \subseteq X \subseteq \text{SG}(\mathfrak{A}) \). Also, we have for all \( i, j \in \text{Dom}(G) \): If \( i < j \), then \( \min(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \) and \( \max(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \).

To see this, suppose \( i, j \in \text{Dom}(G) \) and suppose \( i < j \). Then we have \( \min(\text{Dom}(G(i))) = g(i) < g(j) = \min(\text{Dom}(G(j))) \). Then we have \( G(i) \neq G(j) \) and hence, by hypothesis, \( G(i) \cap G(j) = \emptyset \). Furthermore, we have \( G(i), G(j) \in \text{SG}(\mathfrak{A}) \). Because of \( \min(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \), it then follows with Theorem 2-8 that \( \max(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \).

Last, we have \( \text{Ran}(G) = X \). We already have \( \text{Ran}(G) \subseteq X \). Now, suppose \( \mathfrak{A} \in X \). Then we have \( \min(\text{Dom}(\mathfrak{A})) = g(i) \text{ for an } i \in \text{Dom}(g) \). Then we have \( (i, \mathfrak{A}) \in G \) and hence \( \mathfrak{A} \in \text{Ran}(G) \).

**Theorem 2-18.** Sufficient conditions for the identity of arguments of a segment sequence

If \( \mathfrak{A} \in \text{SEQ} \) and \( G \in \text{SGS}(\mathfrak{A}) \), then for all \( i, j \in \text{Dom}(G) \):

(i) If \( \min(\text{Dom}(G(i))) = \min(\text{Dom}(G(j))) \), then \( i = j \), and

(ii) If \( \max(\text{Dom}(G(i))) = \max(\text{Dom}(G(j))) \), then \( i = j \).

**Proof:** Suppose \( \mathfrak{A} \in \text{SEQ} \) and \( G \in \text{SGS}(\mathfrak{A}) \) and suppose \( i, j \in \text{Dom}(G) \). Now, suppose \( \min(\text{Dom}(G(i))) = \min(\text{Dom}(G(j))) \). With Definition 2-7, it follows that if \( i < j \), then \( \min(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \), and if \( j < i \), then \( \min(\text{Dom}(G(j))) < \min(\text{Dom}(G(i))) \). Both cases contradict the assumption. Therefore we have \( i = j \).

Now, suppose \( \max(\text{Dom}(G(i))) = \max(\text{Dom}(G(j))) \). If \( i < j \) or \( j < i \), then we would have \( \max(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \) or \( \max(\text{Dom}(G(j))) < \min(\text{Dom}(G(i))) \). Therefore we would have \( \max(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \leq \max(\text{Dom}(G(j))) \) or \( \max(\text{Dom}(G(j))) < \min(\text{Dom}(G(i))) \leq \max(\text{Dom}(G(i))) \). Both cases contradict the assumption. Therefore we have \( i = j \). ■
Theorem 2-19. Different members of a segment sequence are disjunct
If $\mathfrak{S} \in \text{SEQ}$ and $G \in \text{SGS}(\mathfrak{S})$, then for all $i, j \in \text{Dom}(G)$: If $G(i) \neq G(j)$, then $G(i) \cap G(j) = \emptyset$.

Proof: Suppose $\mathfrak{S} \in \text{SEQ}$ and $G \in \text{SGS}(\mathfrak{S})$. Then $G$ is a sequence with $\text{Ran}(G) \subseteq \text{SG}(\mathfrak{S})$ and for all $i, j \in \text{Dom}(G)$: If $i < j$, then $\min(\text{Dom}(G(i))) < \min(\text{Dom}(G(j)))$ and $\max(\text{Dom}(G(i))) < \min(\text{Dom}(G(j)))$. Let $i, j \in \text{Dom}(G)$. Then it holds that $G(i), G(j) \in \text{SG}(\mathfrak{S})$. Now, suppose $G(i) \neq G(j)$. With Theorem 2-16-(i) it then holds that $i \neq j$. Then we have $i < j$ or $j < i$. Thus we have

$$
\min(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \quad \text{and} \quad \max(\text{Dom}(G(i))) < \min(\text{Dom}(G(j)))
$$

or

$$
\min(\text{Dom}(G(j))) < \min(\text{Dom}(G(i))) \quad \text{and} \quad \max(\text{Dom}(G(j))) < \min(\text{Dom}(G(i))).
$$

With Theorem 2-8, we thus have $G(i) \cap G(j) = \emptyset$. ■

Definition 2-9. AS-comprising segment sequence for a segment in $\mathfrak{S}$
$G$ is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{S}$ iff

(i) $\mathfrak{S} \in \text{SEQ},$
(ii) $\mathfrak{A} \in \text{SG}(\mathfrak{S}),$
(iii) $G \in \text{SGS}(\mathfrak{S}) \setminus \{\emptyset\}$, and

a) $\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(G(0))),$

b) $\max(\text{Dom}(G(\max(\text{Dom}(G)))))) \leq \max(\text{Dom}(\mathfrak{A})), \text{ and}$

c) for all $l \in \text{Dom}(\text{AS}(\mathfrak{S})) \cap \text{Dom}(\mathfrak{A})$ it holds that there is an $i \in \text{Dom}(G)$ such that $l \in \text{Dom}(G(i))$.

Definition 2-10. Assignment of the set of AS-comprising segment sequences in $\mathfrak{S}$ (ASCS)
$\text{ASCS} = \{(\mathfrak{S}, X) \mid \mathfrak{S} \in \text{SEQ} \text{ and } X = \{ G \mid \text{There is an } \mathfrak{A} \in \text{SG}(\mathfrak{S}) \text{ and } G \text{ is an AS-comprising segment sequence for } \mathfrak{A} \text{ in } \mathfrak{S} \}}$

Theorem 2-20. Existence of AS-comprising segment sequences for all segments
If $\mathfrak{S} \in \text{SEQ}$ and $\mathfrak{A} \in \text{SG}(\mathfrak{S})$, then there is an AS-comprising segment sequence $G$ for $\mathfrak{A}$ in $\mathfrak{S}$.

Proof: Suppose $\mathfrak{S} \in \text{SEQ}$ and $\mathfrak{A} \in \text{SG}(\mathfrak{S})$. Then we have that $\{(0, \mathfrak{A})\}$ is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{S}$. ■
Theorem 2-21. A sentence sequence $\mathcal{S}$ is non-empty if and only if $\text{ASCS}(\mathcal{S})$ is non-empty

If $\mathcal{S} \in \text{SEQ}$, then: $\mathcal{S} \neq \emptyset$ iff $\text{ASCS}(\mathcal{S}) \neq \emptyset$.

**Proof:** Suppose $\mathcal{S} \in \text{SEQ}$. Suppose $\mathcal{S} \neq \emptyset$. Then there is with Theorem 2-1 an $\mathfrak{A}$ such that $\mathfrak{A} \in \text{SG}(\mathcal{S})$. With Theorem 2-20, we then have $\text{ASCS}(\mathcal{S}) \neq \emptyset$. Now, suppose $\text{ASCS}(\mathcal{S}) \neq \emptyset$. According to Definition 2-10 there is then an $\mathfrak{A} \in \text{SG}(\mathcal{S})$. From this it follows with Theorem 2-1 that $\mathcal{S} \neq \emptyset$.

Theorem 2-22. Properties of AS-comprising segment sequences

If $\mathcal{S} \in \text{SEQ}$ and $G \in \text{ASCS}(\mathcal{S})$, then:

(i) $G$ is an injection of $\text{Dom}(G)$ into $\text{Ran}(G)$,
(ii) $G$ is a bijection between $\text{Dom}(G)$ and $\text{Ran}(G)$,
(iii) $\text{Dom}(G) = |\text{Ran}(G)|$, and
(iv) $G$ is a finite sequence.

**Proof:** Suppose $\mathcal{S} \in \text{SEQ}$ and $G \in \text{ASCS}(\mathcal{S})$. With Definition 2-9, we have that $G \in \text{SG}(\mathcal{S}) \setminus \{\emptyset\}$. From this, the statement follows with Theorem 2-16.

Theorem 2-23. All members of an AS-comprising segment sequence lie within the respective segment

If $G$ is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathcal{S}$, then for all $i \in \text{Dom}(G)$:

$\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(G(i)))$ and $\max(\text{Dom}(G(i))) \leq \max(\text{Dom}(G))$.

**Proof:** Suppose $G$ is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathcal{S}$ and suppose $i \in \text{Dom}(G)$. Then we have $0 \leq i \leq \max(\text{Dom}(G))$. According to Definition 2-9, we have that $G \in \text{SG}(\mathcal{S}) \setminus \{\emptyset\}$. With Definition 2-7 we then have that for all $k, j \in \text{Dom}(G)$: If $k < j$, then $\min(\text{Dom}(G(k))) < \min(\text{Dom}(G(j)))$ and $\max(\text{Dom}(G(k))) < \min(\text{Dom}(G(j)))$. Therefore we have that $\min(\text{Dom}(G(0))) \leq \min(\text{Dom}(G(i)))$ and $\max(\text{Dom}(G(i))) \leq \max(\text{Dom}(G(\max(\text{Dom}(G))))$. It also follows from the assumption and Definition 2-9 that $\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(G(0)))$ and $\max(\text{Dom}(G(\max(\text{Dom}(G)))) \leq \max(\text{Dom}(\mathfrak{A}))$. Thus it then follows that: $\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(G(i)))$ and $\max(\text{Dom}(G(i))) \leq \max(\text{Dom}(\mathfrak{A}))$. ■
Theorem 2-24. All members of an AS-comprising segment sequence are subsets of the respective segment

If $G$ is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{S}$, then for all $i \in \text{Dom}(G)$: $G(i) \subseteq \mathfrak{A}$.

Proof: Suppose $G$ is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{S}$ and suppose $i \in \text{Dom}(G)$. With Definition 2-9 and Definition 2-7 we then have $\text{Ran}(G) \subseteq \text{SG}(\mathfrak{S})$ and thus that $G(i)$ is a segment in $\mathfrak{S}$. With Theorem 2-23 we also have that $\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(G(i)))$ and $\max(\text{Dom}(G(i))) \leq \max(\text{Dom}(\mathfrak{A}))$. It then follows with Theorem 2-5 that $G(i) \subseteq \mathfrak{A}$. ■

Theorem 2-25. Non-empty restrictions of AS-comprising segment sequences are AS-comprising segment sequences

If $G$ is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{S}$, then for all $j \in \text{Dom}(G)$: $G^!(j+1)$ is an AS-comprising segment sequence for $\mathfrak{A}^!(\max(\text{Dom}(G(j)))+1)$.

Proof: Suppose $G$ is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{S}$ and suppose $j \in \text{Dom}(G)$. According to Definition 2-9 we then have that $\mathfrak{S} \in \text{SEQ}$ and $\mathfrak{A} \in \text{SG}(\mathfrak{S})$ and $G \in \text{SGS}(\mathfrak{S})\setminus\{\emptyset\}$ and $\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(G(0)))$ and $\max(\text{Dom}(G(\max(\text{Dom}(G)))))) \leq \max(\text{Dom}(\mathfrak{A}))$ and that it holds for all $l \in \text{Dom}(\text{AS}(\mathfrak{S})) \cap \text{Dom}(\mathfrak{A})$ that there is an $i \in \text{Dom}(G)$ such that $l \in \text{Dom}(G(i))$. With Definition 2-7, we can easily show that $G^!(j+1) \in \text{SGS}(\mathfrak{S})\setminus\{\emptyset\}$. With Theorem 2-23, we have that $\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(G(j))) \leq \max(\text{Dom}(G(j))) \leq \max(\text{Dom}(\mathfrak{A}))$ and thus that $\max(\text{Dom}(G(j))) \in \text{Dom}(\mathfrak{A})$. With Theorem 2-6, we thus have that $\mathfrak{A}^!(\max(\text{Dom}(G(j)))+1) \in \text{SG}(\mathfrak{S})$.

Now, the three sub-clauses of clause (iii) of Definition 2-9 have to be shown. Ad a): First, we have $0 < j+1$. Thus we have $0 \in \text{Dom}(G^!(j+1))$ and hence $(G^!(j+1))(0) = G(0)$ and thus $\min(\text{Dom}(!\mathfrak{A}^!(\max(\text{Dom}(G(j)))+1))) = \min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(G(0))) \leq \min(\text{Dom}(((G^!(j+1))(0))))$. Ad b): $\max(\text{Dom}((G^!(j+1))(\max(\text{Dom}(G^!(j+1))))) = \max(\text{Dom}(G(j))) = \max(\text{Dom}(\mathfrak{A}^!(\max(\text{Dom}(G(j)))+1)))$. Ad c): Now, suppose $l \in \text{Dom}(\text{AS}(\mathfrak{S})) \cap \text{Dom}(\mathfrak{A}^!(\max(\text{Dom}(G(j)))+1))$. Then there is an $i \in \text{Dom}(G)$ such that $l \in \text{Dom}(G(i))$. Suppose for contradiction that $j+1 < i$. With $G \in \text{SGS}(\mathfrak{S})$ and Definition 2-7, we would then have that $\max(\text{Dom}(G(j))) < \min(\text{Dom}(G(i))) \leq l \leq \max(\text{Dom}(G(i)))$
and, at the same time, we would have that \( l \leq \max(\text{Dom}(G(j))) \). Contradiction! Therefore we have \( i < j+1 \) and thus \( G(i) = (G\upharpoonright(j+1))(i) \). Therefore we have that for all \( l \in \text{Dom}(\text{AS}(\mathcal{A})) \cap \text{Dom}(\mathfrak{a}(\max(\text{Dom}(G(j))) + 1)) \) it holds that there is an \( i \in \text{Dom}(G\upharpoonright(j+1)) \) such that \( l \in \text{Dom}((G\upharpoonright(j+1))(i)) \). According to Definition 2-9, we thus have that \( G\upharpoonright(j+1) \) is an AS-comprising segment sequence for \( \mathfrak{a}(\max(\text{Dom}(G(j))) + 1) \). ■

**Theorem 2-26. Sufficient conditions for the identity of arguments of an AS-comprising segment sequence**

If \( \mathcal{A} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathcal{A}) \), then for all \( i, j \in \text{Dom}(G) \):

(i) If \( \min(\text{Dom}(G(i))) = \min(\text{Dom}(G(j))) \), then \( i = j \), and

(ii) If \( \max(\text{Dom}(G(i))) = \max(\text{Dom}(G(j))) \), then \( i = j \).

**Proof:** Suppose \( \mathcal{A} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathcal{A}) \). It then follows with Definition 2-9 and Definition 2-10 that \( G \in \text{SGS}(\mathcal{A}) \setminus \{\emptyset\} \) and thus the theorem follows with Theorem 2-18. ■

**Theorem 2-27. Different members of an AS-comprising segment sequence are disjunct**

If \( \mathcal{A} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathcal{A}) \), then for all \( i, j \in \text{Dom}(G) \): If \( G(i) \neq G(j) \), then \( G(i) \cap G(j) = \emptyset \).

**Proof:** Suppose \( \mathcal{A} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathcal{A}) \). It then follows with Definition 2-9 and Definition 2-10 that \( G \in \text{SGS}(\mathcal{A}) \setminus \{\emptyset\} \) and thus the theorem follows with Theorem 2-19. ■
2.2 Closed Segments

In the following section, we introduce CdI-, NI- and RA-like segments. These kinds of segments show forms that are connected to inferences by conditional introduction (CdI-like), negation introduction (NI-like) and particular-quantifier elimination (RA-like), respectively. Among these segments, we will then distinguish so called minimal CdI-, NI, and PE-closed segments, which will form the minimal closed segments. Then, we will define the generation relation GEN, with which we can generate further non-redundant CdI-, NI- and RA-like segments from minimal closed segments. Then, we will define the set of GEN-inductive relations. The intersection of the set of GEN-inductive relations will then be singled out as that relation which assigns a sentence sequence all and only those segments that are closed in this sentence sequence. Thus, closed segments in a sentence sequence will be exactly those CdI-, NI- and RA-like segments in this sequence that are either minimal closed segments or that can be generated by the generation relation from minimal closed segments.

Then, we will prove some general theorems on closed segments. Subsequently, we will define CdI-, NI- and PE-closed segments. This will be done in such a way that CdI-, NI- and PE-closed segments will be closed segments that are CdI-, NI- and RA-like, respectively, and that all closed segments will be CdI- or NI- or PE-closed. At the end of the chapter, we will prove theorems (Theorem 2-66, Theorem 2-67, Theorem 2-68, Theorem 2-69), with which we can later show that CdI-, NI-, PE-closed segments (and thus any closed segments) can be generated by (and only by) CdI, NI and PE, respectively. In the next chapter (2.3), the availability conception will be established with direct recourse to this chapter: A proposition $\Gamma$ will be available in a sequence $\mathcal{S}$ at a position $i$ if and only if $\Gamma$ is the proposition of $\mathcal{S}_i$ and $(i, \mathcal{S}_i)$ lies in all closed segments in $\mathcal{S}$ at most at the end. We will then have that assumptions can be discharged by and only by CdI, NI and PE.

The first three definitions introduce CdI-, NI- and RA-like segments. Then, following some theorems, we will define minimal (CdI- resp. NI- resp. PE-)closed segments.
**Definition 2-11. Cdl-like segment**

\( \mathfrak{A} \) is a Cdl-like segment in \( \mathfrak{S} \)
iff
\( \mathfrak{S} \in \text{SEQ}, \mathfrak{A} \in \text{SG}(\mathfrak{S}) \) and there are \( \Delta, \Gamma \in \text{CFORM} \) such that

(i) \( \mathfrak{S}_{\text{min}}(\text{Dom}(\mathfrak{A})) = "\text{Suppose } \Delta", \)
(ii) \( \mathfrak{P}(\mathfrak{S}_{\text{max}}(\text{Dom}(\mathfrak{A})) - 1) = \Gamma, \) and
(iii) \( \mathfrak{S}_{\text{max}}(\text{Dom}(\mathfrak{A})) = "\text{Therefore } \Delta \rightarrow \Gamma". \)

**Definition 2-12. NI-like segment**

\( \mathfrak{A} \) is an NI-like segment in \( \mathfrak{S} \)
iff
\( \mathfrak{S} \in \text{SEQ}, \mathfrak{A} \in \text{SG}(\mathfrak{S}) \) and there are \( \Delta, \Gamma \in \text{CFORM} \) and \( i \in \text{Dom}(\mathfrak{S}) \) such that

(i) \( \min(\text{Dom}(\mathfrak{A})) \leq i < \max(\text{Dom}(\mathfrak{A})), \)
(ii) \( \mathfrak{S}_{\text{min}}(\text{Dom}(\mathfrak{A})) = "\text{Suppose } \Delta", \)
(iii) \( \mathfrak{P}(\mathfrak{S}_i) = \Gamma \) and \( \mathfrak{P}(\mathfrak{S}_{\text{max}}(\text{Dom}(\mathfrak{A})) - 1) = "\text{Therefore } \neg \Delta". \)

In clause (iii) of Definition 2-12, two contradictory propositions, such as one needs for negation introduction, are localised in the respective sentence sequence. Either the negative ("\( \neg \Gamma \)") or the positive (\( \Gamma \)) part of the contradiction is the proposition of the penultimate member of the respective segment \( \mathfrak{A} \). The position of the other part of the contradiction is left open. It is only required that this other part occurs at some position (i) between the first and the penultimate member of the segment. This is unproblematic in the case of minimal NI-closed segments (Definition 2-15). However, if we want to generate not-minimal closed segments from closed segments, we have to take care that the part of the contradiction whose exact position is not specified does not lie in a proper subsegment of \( \mathfrak{A} \) that is already closed. This we have to keep in mind when we construct the generation relation (cf. especially Definition 2-18).
Definition 2-13. RA-like segment

A is an RA-like segment in $\mathcal{H}$
iff
$\mathcal{H} \in \text{SEQ}, A \in \text{SG}(\mathcal{H})$ and there is $\xi \in \text{VAR}, \Delta \in \text{FORM}, \text{where } \text{FV}(\Delta) \subseteq \{\xi\}, \beta \in \text{PAR}, \Gamma \in \text{CFORM}$ and $B \in \text{SG}(\mathcal{H})$ such that

(i) $P(\mathcal{H}_{\min(\text{Dom}(B))}) = \forall \xi \Delta$,
(ii) $\mathcal{H}_{\min(\text{Dom}(B))} \supseteq \text{"Suppose } [\beta, \xi, \Delta]$, 
(iii) $P(\mathcal{H}_{\max(\text{Dom}(B))}) = \Gamma$,
(iv) $\mathcal{H}_{\max(\text{Dom}(B))} \supseteq \text{"Therefore } \Gamma$,
(v) $\beta \notin \text{STSF}(\{\Delta, \Gamma\})$,
(vi) There is no $j$ such that $j \leq \min(\text{Dom}(B))$ and $\beta \in \text{ST}(\mathcal{H}_j)$, and
(vii) $A = B \setminus \{\min(\text{Dom}(B)), \mathcal{H}_{\min(\text{Dom}(B))}\}$.

Note: 'RA' stands for representative instance assumption, that is, for the representative instance assumption one has to make before one can carry out a particular-quantifier elimination.

Theorem 2-28. No segment is at the same time a CdI- and an NI- or a CdI- and an RA-like segment

(i) There are no $A, \mathcal{H}$ such that $A$ is a CdI- and an NI-like segment in $\mathcal{H}$,
(ii) There are no $A, \mathcal{H}$ such that $A$ is a CdI- and an RA-like segment in $\mathcal{H}$.

Proof: Follows from the definitions and the theorems on unique readability (Theorem 1-10 to Theorem 1-12).

Note that it is possible that an $A$ is an NI- and RA-like segment in $\mathcal{H}$. This is for example the case if the assumption for an indirect proof does not contain parameters and provides one part of the contradiction, while the (empty) particular-quantification of the indirect assumption has been gained immediately before this assumption.

Theorem 2-29. The last member of a CdI- or NI- or RA-like segment is not an assumption-sentence

If $A$ is a CdI- or NI- or RA-like segment in $\mathcal{H}$, then $\max(\text{Dom}(A)) \notin \text{Dom}(\text{AS}(\mathcal{H}))$.

Proof: Follows from Definition 2-11-(iii), Definition 2-12-(iv), Definition 2-13-(iv) and the theorem on the unique readability of sentences (Theorem 1-12).
**Theorem 2-30.** All assumption-sentences in a CdI- or NI- or RA-like segment lie in a proper subsegment that does not include the last member of the respective segment

If $\AA$ is a CdI- or NI- or RA-like segment in $\phi$, and $i \in \text{Dom}(\AA) \cap \text{Dom}(\text{AS}(\phi))$, then $\min(\text{Dom}(\AA)) \leq i < \max(\text{Dom}(\AA))$.

*Proof:* Follows from Theorem 2-29. ■

**Theorem 2-31.** Cardinality of CdI-, NI-, and RA-like segments

(i) If $\AA$ is a CdI- or RA-like segment in $\phi$, then $2 \leq |\AA|$, and

(ii) If $\AA$ is an NI-like segment in $\phi$, then $3 \leq |\AA|$.

*Proof:* The theorem follows with the theorems on unique readability (Theorem 1-10 to Theorem 1-12) directly from Definition 2-11, Definition 2-12 and Definition 2-13. ■

**Definition 2-14.** Minimal CdI-closed segment

$\AA$ is a minimal CdI-closed segment in $\phi$ iff $\AA$ is a CdI-like segment in $\phi$ and

(i) $\text{AS}(\phi) \cap \AA = \{(\min(\text{Dom}(\AA)), \min(\text{Dom}(\AA)))\}$, and

(ii) For all $i \in \text{Dom}(\AA)$ it holds that $\AA \upharpoonright i$ is not a CdI- or NI- or RA-like segment in $\phi$.

**Definition 2-15.** Minimal NI-closed segment

$\AA$ is a minimal NI-closed segment in $\phi$ iff $\AA$ is an NI-like segment in $\phi$ and

(i) $\text{AS}(\phi) \cap \AA = \{(\min(\text{Dom}(\AA)), \min(\text{Dom}(\AA)))\}$, and

(ii) For all $i \in \text{Dom}(\AA)$ it holds that $\AA \upharpoonright i$ is not a CdI- or NI- or RA-like segment in $\phi$.

**Definition 2-16.** Minimal PE-closed segment

$\AA$ is a minimal PE-closed segment in $\phi$ iff $\AA$ is a RA-like segment in $\phi$ and

(i) $\text{AS}(\phi) \cap \AA = \{(\min(\text{Dom}(\AA)), \min(\text{Dom}(\AA)))\}$, and

(ii) For all $i \in \text{Dom}(\AA)$ holds that $\AA \upharpoonright i$ is not a CdI- or NI- or RA-like segment in $\phi$. 
**Definition 2-17. Minimal closed segment**

\(\mathcal{A}\) is a minimal closed segment in \(\mathcal{E}\) if

iff

\(\mathcal{A}\) is a minimal CdI- or a minimal NI- or a minimal PE-closed segment in \(\mathcal{E}\).

**Theorem 2-32. CdI-, NI- and RA-like segments with just one assumption-sentence have a minimal closed segment as an initial segment**

If \(\mathcal{A}\) is a CdI- or NI- or RA-like segment in \(\mathcal{E}\) and \(|AS(\mathcal{E}) \cap \mathcal{A}| = 1\), then \(\mathcal{A}\) is a minimal closed segment in \(\mathcal{E}\) or there is an \(i \in \text{Dom}(\mathcal{A})\) such that \(\mathcal{A}|i\) is a minimal closed segment in \(\mathcal{E}\).

**Proof:** Suppose \(\mathcal{A}\) is a CdI- or NI- or RA-like segment in \(\mathcal{E}\) and \(|AS(\mathcal{E}) \cap \mathcal{A}| = 1\). With Definition 2-11, Definition 2-12 and Definition 2-13, we then have \(AS(\mathcal{E}) \cap \mathcal{A} = \{(\min(\text{Dom}(\mathcal{A})), \mathcal{E}_{\min(\text{Dom}(\mathcal{A}))})\}\). Suppose \(\mathcal{A}\) is not a minimal closed segment in \(\mathcal{E}\). By hypothesis, we then have, with Definition 2-17 and Definition 2-14, Definition 2-15 and Definition 2-16, that there is a \(j \in \text{Dom}(\mathcal{A})\) such that \(\mathcal{A}|j\) is a CdI- or NI- or RA-like segment in \(\mathcal{E}\). Now, let \(i = \min(\{j \mid j \in \text{Dom}(\mathcal{A})\text{ and }\mathcal{A}|j\text{ is a CdI-, NI- or RA-like segment in }\mathcal{E}\})\). Then we have \(AS(\mathcal{E}) \cap \mathcal{A}|i \subseteq AS(\mathcal{E}) \cap \mathcal{A}\) and, with Theorem 2-7, we have \(\min(\text{Dom}(\mathcal{A}|i)) = \min(\text{Dom}(\mathcal{A}))\) and thus \(AS(\mathcal{E}) \cap \mathcal{A}|i = \{(\min(\text{Dom}(\mathcal{A}|i)), \mathcal{E}_{\min(\text{Dom}(\mathcal{A}|i))})\}\). Because of the minimality of \(i\), we also have that for all \(l \in \text{Dom}(\mathcal{A}|i)\) it holds that \((\mathcal{A}|i)|l = \mathcal{A}|l\) is not a CdI-, NI- or RA-like segment in \(\mathcal{E}\). Thus we have that \(\mathcal{A}|i\) is a minimal CdI- or NI- or PE-closed segment and thus a minimal closed segment in \(\mathcal{E}\).

**Theorem 2-33. Ratio of inference- and assumption-sentences in minimal closed segments**

If \(\mathcal{A}\) is a minimal closed segment in \(\mathcal{E}\), then \(|AS(\mathcal{E}) \cap \mathcal{A}| \leq |IS(\mathcal{E}) \cap \mathcal{A}|\).

**Proof:** Suppose \(\mathcal{A}\) is a minimal closed segment and thus a minimal CdI- or NI- or PE-closed segment in \(\mathcal{E}\). Then it holds with the definitions and Theorem 2-29 that \(|AS(\mathcal{E}) \cap \mathcal{A}| = 1 \leq |IS(\mathcal{E}) \cap \mathcal{A}|\).

Now, we will define a generation relation for segments with which we can generate further non-redundant CdI-, NI-, and RA-like segments from minimal closed segments, where all assumption-sentences of the generated segments are first members of a non-redundant CdI-, NI- or RA-like subsegment. To do this, we first define the following proto-generation relation:
Definition 2-18. Proto-generation relation for non-redundant CdI-, NI- and RA-like segments in sequences (PGEN)

\[
PGEN = \{ ((\mathcal{A}, G), X) \mid \mathcal{A} \in \text{SEQ} \text{ and } G \in \text{ASCS} (\mathcal{A}) \text{ and } X = \{ \mathfrak{A} \mid \mathfrak{A} \in \text{SG}(\mathcal{A}) \text{ and there is a } \mathfrak{B} \in \text{SG}(\mathcal{A}) \text{ such that} \}
\]

(i) \( G \) is an AS-comprising segment sequence for \( \mathfrak{B} \) in \( \mathcal{A} \),

(ii) \( \text{AS}(\mathcal{A}) \cap \mathfrak{B} \neq \emptyset \),

(iii) \( \min(\text{Dom}(\mathfrak{A})) + 1 = \min(\text{Dom}(\mathfrak{B})) \text{ and } \max(\text{Dom}(\mathfrak{A})) = \max(\text{Dom}(\mathfrak{B})) + 1 \),

(iv) \( \mathfrak{A} \) is a CdI- or NI- or RA-like segment in \( \mathcal{A} \) and if \( \mathfrak{A} \) is an NI-like segment in \( \mathcal{A} \), then there are \( \Delta, \Gamma \in \text{CFORM} \text{ and } i \in \text{Dom}(\mathfrak{A}) \) such that

(a) \( \min(\text{Dom}(\mathfrak{A})) \leq i < \max(\text{Dom}(\mathfrak{A})) \),

(b) \( \mathcal{A}_{\min(\text{Dom}(\mathfrak{A}))} = \text{"Suppose } \Delta" \),

(c) \( P(\mathfrak{A}_i) = \Gamma \) and \( P(\mathfrak{A}_{\max(\text{Dom}(\mathfrak{A}))-1}) = \text{"} \neg \Gamma \text{"} \)

or

\( P(\mathfrak{A}_i) = \text{"} \neg \Gamma \text{"} \) and \( P(\mathfrak{A}_{\max(\text{Dom}(\mathfrak{A}))-1}) = \Gamma \),

(d) For all \( r \in \text{Dom}(G) \): \( i < \min(\text{Dom}(G(r))) \) or \( \max(\text{Dom}(G(r))) \leq i \),

(e) \( \mathcal{A}_{\max(\text{Dom}(\mathfrak{A}))} = \text{"Therefore } \neg \Delta" \), and

(v) For all \( i \in \text{Dom}(\mathfrak{A}) \): \( \mathfrak{A} \vert i \text{ is not a minimal closed segment in } \mathcal{A} \} \}.

In clause (iv) of Definition 2-18, a special requirement is made for NI-like segments. The reason is that the values of the AS-comprising segment sequence \( G \) are to be the \textit{material} when we construct further closed segments from closed segments. In the NI-case, we have to make sure that only such segments \( \mathfrak{A} \) are generated as NI-closed in which both parts of the required contradiction actually lie in \( \mathfrak{A} \vert \max(\text{Dom}(\mathfrak{A})) \) and are both not included in any closed subsegment of \( \mathfrak{A} \vert \max(\text{Dom}(\mathfrak{A})) \). For the first part of the contradiction, this is ensured by (iv-d) (cf. the proof of Theorem 2-68).

Theorem 2-34. Some properties of PGEN

If \( \mathcal{A} \in \text{SEQ} \text{ and } G \in \text{ASCS}(\mathcal{A}) \text{ and } \mathfrak{A} \in \text{PGEN}(\mathcal{A}, G) \), then:

(i) There is \( \mathfrak{B} \in \text{SG}(\mathcal{A}) \text{ such that } G \text{ is an AS-comprising segment sequence for } \mathfrak{B} \text{ in } \mathcal{A} \text{ and } \text{AS}(\mathcal{A}) \cap \mathfrak{B} \neq \emptyset \text{, } \min(\text{Dom}(\mathfrak{A})) + 1 = \min(\text{Dom}(\mathfrak{B})) \text{ and } \max(\text{Dom}(\mathfrak{A})) = \max(\text{Dom}(\mathfrak{B})) + 1 \),

(ii) \( \mathfrak{A} \in \text{SG}(\mathcal{A}) \) is a CdI- or NI- or RA-like segment in \( \mathcal{A} \),

(iii) For all \( i \in \text{Dom}(\mathfrak{A}) \): \( \mathfrak{A} \vert i \text{ is not a minimal closed segment in } \mathcal{A} \),

(iv) There is an \( i \in \text{Dom}(\mathfrak{A}) \text{ such that } \min(\text{Dom}(\mathfrak{A})) < i \text{ and } i \in \text{Dom}(\text{AS}(\mathcal{A})) \),

(v) \( \mathfrak{A} \) is not a minimal closed segment in \( \mathcal{A} \),
2.2 Closed Segments

(vi) \( G \neq \emptyset \), and

(vii) For all \( \mathcal{C} \in \text{PGEN}(\langle \mathcal{H}, G \rangle) \) it holds that \( \min(\text{Dom}(\mathcal{C})) = \min(\text{Dom} (\mathcal{A})) \).

Proof: Suppose \( \mathcal{H} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathcal{H}) \) and \( \mathcal{A} \in \text{PGEN}(\langle \mathcal{H}, G \rangle) \). Then clauses (i)-(iii) follow directly from Definition 2-18. Now, suppose \( \mathcal{B} \) satisfies clause (i). Then we have \( \text{AS}(\mathcal{H}) \cap \mathcal{B} \neq \emptyset \) and hence there is an \( i \in \text{Dom}(\text{AS}(\mathcal{H})) \cap \text{Dom}(\mathcal{B}) \subseteq \text{Dom}(\text{AS}(\mathcal{H})) \cap \text{Dom}(\mathcal{A}) \) where, because of \( \min(\text{Dom}(\mathcal{H}))+1 = \min(\text{Dom}(\mathcal{B})) \), we have that \( \min(\text{Dom}(\mathcal{A})) < i \). It then follows that clause (iv) holds. From this follows with Definition 2-14, Definition 2-15, Definition 2-16 and Definition 2-17 that clause (v) also holds. With \( \text{AS}(\mathcal{H}) \cap \mathcal{B} \neq \emptyset \) and Definition 2-9, we also have that there is an \( i \in \text{Dom}(G) \), and hence that \( G \neq \emptyset \). Therefore we have (vi).

According to Definition 2-9, we have that \( \min(\text{Dom}(\mathcal{B})) \leq \min(\text{Dom}(G(0))) \leq \max(\text{Dom}(\mathcal{B})) \) and thus that \( \min(\text{Dom}(\mathcal{A})) < \min(\text{Dom}(G(0))) \). Now, suppose \( \mathcal{C} \in \text{PGEN}(\langle \mathcal{H}, G \rangle) \). Then there is a \( \mathcal{B}' \in \text{SG}(\mathcal{H}) \) such that \( G \) is an AS-comprising segment sequence for \( \mathcal{B}' \) in \( \mathcal{H} \) and \( \min(\text{Dom}(\mathcal{C}))+1 = \min(\text{Dom}(\mathcal{B}')) \) and \( \max(\text{Dom}(\mathcal{C})) = \max(\text{Dom}(\mathcal{B}'))+1 \) and \( \mathcal{C} \) is a CdI- or NI- or RA-like segment in \( \mathcal{H} \). Then we have \( \min(\text{Dom}(\mathcal{A})), \min(\text{Dom}(\mathcal{C})) \in \text{Dom}(\text{AS}(\mathcal{H})) \). According to Definition 2-9, we have that \( \min(\text{Dom}(\mathcal{B}')) \leq \min(\text{Dom}(G(0))) \leq \max(\text{Dom}(\mathcal{B}')) \) and thus \( \min(\text{Dom}(\mathcal{C})) < \min(\text{Dom}(G(0))) \). It thus follows that \( \min(\text{Dom}(\mathcal{A})), \min(\text{Dom}(\mathcal{C})) < \min(\text{Dom}(G(0))) \leq \max(\text{Dom}(\mathcal{B})), \max(\text{Dom}(\mathcal{B}')). \)

Now, suppose for contradiction that \( \min(\text{Dom}(\mathcal{C})) < \min(\text{Dom}(\mathcal{A})) \). Then we would have that \( \min(\text{Dom}(\mathcal{B}')) \leq \min(\text{Dom}(\mathcal{A})) \leq \max(\text{Dom}(\mathcal{B}')) \). Then we would also have that \( \min(\text{Dom}(\mathcal{A})) \in \text{Dom}(\text{AS}(\mathcal{H})) \cap \text{Dom}(\mathcal{B}'). \) Now, \( G \) is an AS-comprising segment sequence for \( \mathcal{B}' \) in \( \mathcal{H} \). With Definition 2-9, we would thus have that \( \min(\text{Dom}(\mathcal{A})) \in \text{Dom}(G(l)) \) for an \( l \in \text{Dom}(G) \). Since \( G \) is an AS-comprising segment sequence for \( \mathcal{B} \) in \( \mathcal{H} \), we would have, with Theorem 2-24, that \( \min(\text{Dom}(\mathcal{A}))+1 = \min(\text{Dom}(\mathcal{B})) \leq \min(\text{Dom}(\mathcal{A})) \). Contradiction! Now, suppose for contradiction that \( \min(\text{Dom}(\mathcal{A})) < \min(\text{Dom}(\mathcal{C})) \). Then we would have that \( \min(\text{Dom}(\mathcal{B}')) \leq \min(\text{Dom}(\mathcal{C})) \leq \max(\text{Dom}(\mathcal{B}')). \)

Thus we would now have \( \min(\text{Dom}(\mathcal{C})) \in \text{Dom}(\text{AS}(\mathcal{H})) \cap \text{Dom}(\mathcal{B}) \) and thus \( \min(\text{Dom}(\mathcal{C})) \in \text{Dom}(G(l')) \) for an \( l' \in \text{Dom}(G) \) and thus \( \min(\text{Dom}(\mathcal{C}))+1 = \min(\text{Dom}(\mathcal{B}')) \leq \min(\text{Dom}(\mathcal{C})). \) Contradiction! Therefore we have \( \min(\text{Dom}(\mathcal{C})) = \min(\text{Dom}(\mathcal{A})) \) and hence that clause (vii) holds. \( \blacksquare \)
For given $\mathfrak{f}$, $G$, the desired generation relation singles out the non-redundant segments from $\text{PGEN}(\langle \mathfrak{f}, G \rangle)$:

**Definition 2-19.** Generation relation for non-redundant CdI-, NI- and RA-like segments in sequences (GEN)  
$\text{GEN} = \{ (\langle \mathfrak{f}, G \rangle, X) \mid \mathfrak{f} \in \text{SEQ}, G \in \text{ASCS}(\mathfrak{f}) \text{ and } X = \{ \mathfrak{A} \mid \mathfrak{A} \in \text{PGEN}(\langle \mathfrak{f}, G \rangle) \text{ and there is no } i \in \text{Dom}(\mathfrak{A}) \text{ and } j \in \text{Dom}(G) \text{ such that } \mathfrak{A}[i] \in \text{PGEN}(\langle \mathfrak{f}, G[j+1] \rangle) \} \}.$

$\text{GEN}$ is a 2-ary function that assigns each sentence sequence $\mathfrak{f}$ and AS-comprising segment sequence $G$ for a segment $\mathfrak{B}$ in $\mathfrak{f}$ a subset $X$ of the set of CdI-, NI- or RA-like segments in $\mathfrak{f}$ that have the members of $G$ as proper subsegments. This subset is then either empty or it is the singleton of the shortest segment that can be generated with $\text{PGEN}$ for $\mathfrak{f}$ and restrictions of $G$ on $j+1$ with $j \in \text{Dom}(G)$. This ensures later that not only minimal, but also $\text{GEN}$-generated and thus all closed segments are uniquely determined by their beginning (cf. Theorem 2-50). The following theorem sums up some properties of $\text{GEN}$ for $\text{GEN}(\langle \mathfrak{f}, G \rangle) \neq \emptyset$.

**Theorem 2-35.** Some consequences of Definition 2-19  
If $\mathfrak{f} \in \text{SEQ}$ and $G \in \text{ASCS}(\mathfrak{f})$ and $\mathfrak{A} \in \text{GEN}(\langle \mathfrak{f}, G \rangle)$, then:

(i) There is $\mathfrak{B} \in \text{SG}(\mathfrak{f})$ such that $G$ is an AS-comprising segment sequence for $\mathfrak{B}$ in $\mathfrak{f}$ and $\text{AS}(\mathfrak{f}) \cap \mathfrak{B} \neq \emptyset$, $\min(\text{Dom}(\mathfrak{A}))+1 = \min(\text{Dom}(\mathfrak{B}))$ and $\max(\text{Dom}(\mathfrak{A})) = \max(\text{Dom}(\mathfrak{B}))+1$,  
(ii) $\mathfrak{A} \in \text{SG}(\mathfrak{f})$ is a CdI- or NI- or RA-like segment in $\mathfrak{f}$,  
(iii) For all $i \in \text{Dom}(\mathfrak{A})$: $\mathfrak{A}[i]$ is not a minimal closed segment in $\mathfrak{f}$,  
(iv) There is an $i \in \text{Dom}(\mathfrak{A})$ such that $\min(\text{Dom}(\mathfrak{A})) < i$ and $i \in \text{Dom}(\text{AS}(\mathfrak{f}))$,  
(v) $\mathfrak{A}$ is not a minimal closed segment in $\mathfrak{f}$,  
(vi) There is no $i \in \text{Dom}(\mathfrak{A})$ and $j \in \text{Dom}(G)$ such that $\mathfrak{A}[i] \in \text{PGEN}(\langle \mathfrak{f}, G[j+1] \rangle)$, and  
(vii) $\text{GEN}(\langle \mathfrak{f}, G \rangle) = \{ \mathfrak{A} \}$.  

**Proof:** Suppose $\mathfrak{f} \in \text{SEQ}$ and $G \in \text{ASCS}(\mathfrak{f})$ and $\mathfrak{A} \in \text{GEN}(\langle \mathfrak{f}, G \rangle)$. Then clauses (i)-(v) follow directly from Definition 2-19 and Theorem 2-34. Clause (vi) follows directly from Definition 2-19. Now, suppose $\mathfrak{C} \in \text{GEN}(\langle \mathfrak{f}, G \rangle)$. With Definition 2-19, we then have with $\mathfrak{A}$, $\mathfrak{C} \in \text{GEN}(\langle \mathfrak{f}, G \rangle)$, that also $\mathfrak{A}$, $\mathfrak{C} \in \text{PGEN}(\langle \mathfrak{f}, G \rangle)$ and thus with Theorem 2-34-(vii) that $\min(\text{Dom}(\mathfrak{A})) = \min(\text{Dom}(\mathfrak{C}))$. Now, suppose for contradiction that $\max(\text{Dom}(\mathfrak{A})) < \max(\text{Dom}(\mathfrak{C}))$. Then we would have that $\min(\text{Dom}(\mathfrak{C})) \leq$
max(Dom(𝒜)) + 1 ≤ max(Dom(𝒞)) and thus max(Dom(𝒜)) + 1 ∈ Dom(𝒞). At the same time we would have that ( maxx(Dom(𝒜)) + 1 = 𝒫 ∈ PGEN(⟨建设工程, 𝑆⟩) = PGEN(⟨建设工程, 𝑆, 𝑆⟩ max(Dom(𝑆)) + 1)). With Definition 2-19, we would thus have 𝒫 ∉ GEN(⟨建设工程, 𝑆⟩). Contradiction! For max(Dom(𝒞)) < max(Dom(𝒜)), a contradiction follows analogously.

Therefore we have that also max(Dom(𝒜)) = max(Dom(𝒞)) and thus, with Theorem 2-4, that 𝒫 = 𝒫 ∈ {建设工程}. Therefore we have GEN(⟨建设工程, 𝑆⟩) ⊆ {建设工程} and thus (vii). ■

**Theorem 2-36.** GEN-generated segments are greater than the members of the respective AS-comprising segment sequence

If建设工程 ∈ SEQ and 𝑆 ∈ ASCS(建设工程), then for all 𝒫 ∈ Ran(𝑆) and 𝒫 ∈ GEN(⟨建设工程, 𝑆⟩): |𝒫| < |𝒜|.

**Proof:** Suppose建设工程 ∈ SEQ and 𝑆 ∈ ASCS(建设工程). Now, suppose 𝒫 ∈ Ran(𝑆) and 𝒫 ∈ GEN(⟨建设工程, 𝑆⟩). Then there is a 𝑆 ∈ SG(建设工程) such that 𝑆 is an AS-comprising segment sequence for 𝑆 in建设工程 and min(Dom(𝒜)) + 1 = min(Dom(𝑆)) and max(Dom(𝒜)) = max(Dom(𝑆)) + 1 and 𝒫 is a Cdl- or NI- or RA-like segment in建设工程. Then we have |薜| < |𝒜|. Because of 𝒫 ∈ Ran(𝑆), we also have, with Theorem 2-24, that |𝒫| ≤ |薜| and hence that |𝒫| < |薜|. ■

**Theorem 2-37.** Preparatory theorem for Theorem 2-39 (a)

{建设工程, 𝒫} | 𝒫 is a minimal closed segment in建设工程} ⊆ SEQ × {建设工程 |建设工程 is a segment}.

**Proof:** Suppose建设工程, 𝒫 ∈ {建设工程, 𝒫 | 𝒫 is a minimal closed segment in建设工程}. It then follows from Definition 2-14, Definition 2-15 and Definition 2-16 that 𝒫 is a segment in建设工程 and thus that建设工程 ∈ SEQ. Thus:建设工程, 𝒫 ∈ SEQ × {建设工程 |建设工程 is a segment}. ■

**Theorem 2-38.** Preparatory for Theorem 2-39 (b)

For all建设工程 ∈ SEQ and 𝑆 ∈ ASCS(建设工程) it holds that建设工程 × GEN(⟨建设工程, 𝑆⟩) ⊆ SEQ × {建设工程 |建设工程 is a segment}.

**Proof:** Suppose建设工程 ∈ SEQ and 𝑆 ∈ ASCS(建设工程). Now, suppose建设工程, 𝒫 ∈建设工程 × GEN(⟨建设工程, 𝑆⟩). It then follows by hypothesis and Theorem 2-35-(ii) that 𝒫 ∈ SG(建设工程) and thus follows the whole statement. ■
Now, we can define the set of GEN-inductive relations:

**Definition 2-20.** The set of GEN-inductive relations (CSR)

\[
\text{CSR} = \{R \mid R \subseteq \text{SEQ} \times \{\mathcal{A} \mid \mathcal{A} \text{ is a segment}\} \text{ and}
\]

(i) \(\{(\mathcal{F}, \mathcal{A}) \mid \mathcal{A} \text{ is a minimal closed segment in } \mathcal{F}\} \subseteq R\), and

(ii) For all \(\mathcal{F} \in \text{SEQ}\) and \(G \in \text{ASCS}(\mathcal{F})\) with \(\{\mathcal{F}\} \times \text{Ran}(G) \subseteq R\) it holds that \(\{\mathcal{F}\} \times \text{GEN}(\langle \mathcal{F}, G \rangle) \subseteq R\).

Definition 2-20 is essentially a supporting definition for Definition 2-21, in which we define the relation that relates a given sentence sequence \(\mathcal{F}\) to all and only the segments that are closed in this sequence. Informally, we can say that CSR consists of all relations \(R\) that relate a given sentence sequence \(\mathcal{F}\) to all minimal closed segments in \(\mathcal{F}\) (if such segments exist) and further to all segments \(\mathcal{A}\) in \(\mathcal{F}\) that can be generated by GEN from segments \(\mathcal{B}_0, \ldots, \mathcal{B}_n\) with \(\{(\mathcal{F}, \mathcal{B}_0), \ldots, (\mathcal{F}, \mathcal{B}_n)\}\) \(\subseteq R\).

**Theorem 2-39.** Preparatory theorem for Theorem 2-40

\(\text{SEQ} \times \{\mathcal{A} \mid \mathcal{A} \text{ is a segment}\} \in \text{CSR}\).

**Proof:** First, we have \(\text{SEQ} \times \{\mathcal{A} \mid \mathcal{A} \text{ is a segment}\} \subseteq \text{SEQ} \times \{\mathcal{A} \mid \mathcal{A} \text{ is a segment}\}\). With Theorem 2-37, we also have that \(\{(\mathcal{F}, \mathcal{A}) \mid \mathcal{A} \text{ is a minimal closed segment in } \mathcal{F}\} \subseteq \text{SEQ} \times \{\mathcal{A} \mid \mathcal{A} \text{ is a segment}\}\). With Theorem 2-38, we also have that for all \(\mathcal{F} \in \text{SEQ}\) and \(G \in \text{ASCS}(\mathcal{F})\) with \(\{\mathcal{F}\} \times \text{Ran}(G) \subseteq \text{SEQ} \times \{\mathcal{A} \mid \mathcal{A} \text{ is a segment}\}\) it holds that \(\{\mathcal{F}\} \times \text{GEN}(\langle \mathcal{F}, G \rangle) \subseteq \text{SEQ} \times \{\mathcal{A} \mid \mathcal{A} \text{ is a segment}\}\). \(\blacksquare\)

Now, we define the relation that relates a given sentence sequence \(\mathcal{F}\) to all and only the segments that are minimal closed segments in \(\mathcal{F}\) or that can be generated from minimal closed segments in \(\mathcal{F}\) by successive applications of GEN:

**Definition 2-21.** The smallest GEN-inductive relation (CS)

\(\text{CS} = \cap \text{CSR}\).

The following theorem assures us that CS is, first, indeed a relation, that relates a given sentence sequence \(\mathcal{F}\) to all and only the segments that are minimal closed segments in \(\mathcal{F}\) or that can be generated from minimal closed segments in \(\mathcal{F}\) by successive applications of GEN, and, second, that CS is a subset of all such relations and hence the smallest such
relation. Thus, we have that CS relates a given sentence sequence only to segments of the kind indicated.

**Theorem 2-40.** CS is the smallest GEN-inductive relation

(i) \( CS \subseteq CSR \) and
(ii) If \( R \in CSR \), then \( CS \subseteq R \).

**Proof:** (ii) follows from Definition 2-21. **Ad (i):** We have to show that a) \( CS \subseteq \text{SEQ} \times \{ \mathfrak{A} \mid \mathfrak{A} \text{ is a segment} \} \), b) \( \{(\mathfrak{g}, \mathfrak{r}) \mid \mathfrak{r} \text{ is a minimal closed segment in } \mathfrak{g} \} \subseteq CS \) and c) for all \( \mathfrak{g} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathfrak{g}) \) with \( \{\mathfrak{g}\} \times \text{Ran}(G) \subseteq CS \) it holds that \( \{\mathfrak{g}\} \times \text{GEN}(\langle \mathfrak{g}, G \rangle) \subseteq CS \).

a), i.e. \( CS \subseteq \text{SEQ} \times \{ \mathfrak{A} \mid \mathfrak{A} \text{ is a segment} \} \), follows with Theorem 2-39 and (ii). Since for all \( R \in CSR \) we have that \( \{(\mathfrak{g}, \mathfrak{r}) \mid \mathfrak{r} \text{ is a minimal closed segment in } \mathfrak{g} \} \subseteq R \), we have, with Definition 2-21, also b), i.e. \( \{(\mathfrak{g}, \mathfrak{r}) \mid \mathfrak{r} \text{ is a minimal closed segment in } \mathfrak{g} \} \subseteq CS \).

We still have to show that c), i.e. that for all \( \mathfrak{g} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathfrak{g}) \) with \( \{\mathfrak{g}\} \times \text{Ran}(G) \subseteq CS \) it holds holds that \( \{\mathfrak{g}\} \times \text{GEN}(\langle \mathfrak{g}, G \rangle) \subseteq CS \). For this, suppose first that \( \mathfrak{g} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathfrak{g}) \) and \( \{\mathfrak{g}\} \times \text{Ran}(G) \subseteq CS \). According to Definition 2-21, what we have to show in order to prove that \( \{\mathfrak{g}\} \times \text{GEN}(\langle \mathfrak{g}, G \rangle) \subseteq CS \) is that for all \( R \in CSR \) it holds that \( \{\mathfrak{g}\} \times \text{GEN}(\langle \mathfrak{g}, G \rangle) \subseteq R \). Now, suppose \( R \in CSR \). It then follows from \( \{\mathfrak{g}\} \times \text{Ran}(G) \subseteq CS \) (from our first hypothesis) and (ii) that \( \{\mathfrak{g}\} \times \text{Ran}(G) \subseteq R \). By hypothesis, we have \( R \in CSR \). With Definition 2-20, we thus have \( \{\mathfrak{g}\} \times \text{GEN}(\langle \mathfrak{g}, G \rangle) \subseteq R \) and thus we have that \( \{\mathfrak{g}\} \times \text{GEN}(\langle \mathfrak{g}, G \rangle) \subseteq CS \). Therefore we have for all \( \mathfrak{g} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathfrak{g}) \) with \( \{\mathfrak{g}\} \times \text{Ran}(G) \subseteq CS: \{\mathfrak{g}\} \times \text{GEN}(\langle \mathfrak{g}, G \rangle) \subseteq CS \). \( \blacksquare \)

With the preceding theorem, we can informally say that the following definition characterises exactly those segments in a sentence sequence as segments that are closed in this sequence that are minimal closed segments in this sequence or that can be generated from these minimal segments by successive application of GEN.
Definition 2-22. Closed segments
\( \mathfrak{A} \) is a closed segment in \( \mathfrak{f} \) iff \( (\mathfrak{f}, \mathfrak{A}) \in \text{CS} \).

Theorem 2-41. Closed segments are minimal or GEN-generated
\((\mathfrak{f}, \mathfrak{A}) \in \text{CS} \)
iff
(i) \( \mathfrak{A} \) is a minimal closed segment in \( \mathfrak{f} \)
or
(ii) \( \mathfrak{f} \in \text{SEQ} \) and there is a \( G \in \text{ASCS}(\mathfrak{f}) \) with \( \{\mathfrak{f}\} \times \text{Ran}(G) \subseteq \text{CS} \) and \( \mathfrak{A} \in \text{GEN}(\langle \mathfrak{f}, G \rangle) \).

Proof: The right-left-direction follows with Theorem 2-40-(i) and Definition 2-20. Now, for the left-right-direction, suppose \( X = \{(\mathfrak{f}, \mathfrak{A}) \mid \mathfrak{A} \) is a minimal closed segment in \( \mathfrak{f} \) or \( \mathfrak{f} \in \text{SEQ} \) and there is a \( G \in \text{ASCS}(\mathfrak{f}) \) with \( \{\mathfrak{f}\} \times \text{Ran}(G) \subseteq \text{CS} \) and \( \mathfrak{A} \in \text{GEN}(\langle \mathfrak{f}, G \rangle) \} \cap \text{CS} \). To prove the theorem, it suffices to show that \( X \in \text{CSR} \), then the statement follows with Theorem 2-40-(ii).

With Theorem 2-40-(i), we have \( \text{CS} \in \text{CSR} \). According to Definition 2-20 and the definition of \( X \), we then have \( X \subseteq \text{CS} \subseteq \text{SEQ} \times \{\mathfrak{A} \mid \mathfrak{A} \) is a segment\} \) and \( \{(\mathfrak{f}, \mathfrak{A}) \mid \mathfrak{A} \) is a minimal closed segment in \( \mathfrak{f} \} \subseteq X \).

We still have to show that for all \( \mathfrak{f} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathfrak{f}) \) with \( \{\mathfrak{f}\} \times \text{Ran}(G) \subseteq X \) it holds that \( \{\mathfrak{f}\} \times \text{GEN}(\langle \mathfrak{f}, G \rangle) \subseteq X \). First, suppose \( \mathfrak{f} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathfrak{f}) \) and \( \{\mathfrak{f}\} \times \text{Ran}(G) \subseteq X \). Then we have that \( \{\mathfrak{f}\} \times \text{Ran}(G) \subseteq \text{CS} \) and thus, with Theorem 2-40-(i) and Definition 2-20, that also \( \{\mathfrak{f}\} \times \text{GEN}(\langle \mathfrak{f}, G \rangle) \subseteq \text{CS} \). Now, suppose \( (\mathfrak{f}, \mathfrak{A}) \in \{\mathfrak{f}\} \times \text{GEN}(\langle \mathfrak{f}, G \rangle) \). Then we have \( \mathfrak{A} \in \text{GEN}(\langle \mathfrak{f}, G \rangle) \). Thus there is a \( G \in \text{ASCS}(\mathfrak{f}) \) with \( \{\mathfrak{f}\} \times \text{Ran}(G) \subseteq \text{CS} \) and \( \mathfrak{A} \in \text{GEN}(\langle \mathfrak{f}, G \rangle) \) and we also have \( (\mathfrak{f}, \mathfrak{A}) \in \text{CS} \). Therefore we have \( (\mathfrak{f}, \mathfrak{A}) \in X \). Hence we have \( X \in \text{CSR} \).

Theorem 2-42. Closed segments are CdI- or NI- or RA-like segments
If \( (\mathfrak{f}, \mathfrak{A}) \in \text{CS} \), then \( \mathfrak{A} \) is a CdI-, NI- or RA-like segment in \( \mathfrak{f} \).

Proof: Suppose \( (\mathfrak{f}, \mathfrak{A}) \in \text{CS} \). Then it holds with Theorem 2-41 and Theorem 2-37 that \( \mathfrak{f} \in \text{SEQ} \) and that \( \mathfrak{A} \) is a minimal closed segment in \( \mathfrak{f} \) or that there is a \( G \in \text{ASCS}(\mathfrak{f}) \) with
\( \{ \bar{s} \} \times \text{Ran}(G) \subseteq \text{CS} \) and \( \bar{\alpha} \in \text{GEN}(\langle \bar{s}, G \rangle) \). The statement then follows immediately with Definition 2-14, Definition 2-15, Definition 2-16, Definition 2-17 and Theorem 2-35-(ii).

**Theorem 2-43.** \( \emptyset \) is neither in \( \text{Dom}(\text{CS}) \) nor in \( \text{Ran}(\text{CS}) \)

If \( (\bar{s}, \bar{\alpha}) \in \text{CS} \), then \( \bar{s} \neq \emptyset \) and \( \bar{\alpha} \neq \emptyset \).

**Proof:** Suppose \( (\bar{s}, \bar{\alpha}) \in \text{CS} \). It then holds with Theorem 2-42 that \( \bar{\alpha} \) is a CdI- or an NI- or an RA-like segment in \( \bar{s} \). It then holds with Definition 2-11, Definition 2-12 and Definition 2-13 that \( \bar{s} \in \text{SEQ} \) and \( \bar{\alpha} \in \text{SG}(\bar{s}) \). With Theorem 2-1 and Definition 2-1, we then have \( \bar{s} \neq \emptyset \) and \( \bar{\alpha} \neq \emptyset \).

Theorem 2-42 shows that CS only contains pairs of sentence sequences and CdI- or NI- or RA-like segments in these sequences. So, the first and last members of the segments give them the form that is known from the corresponding patterns of inference (for NE with the contradictory statements included in a proper initial segment of the respective segment and for PE with the particular-quantification before the respective RA-like segment). However, not every pair of a sentence sequence and a segment in this sentence sequence that shows such a form is in CS. This can be shown using Theorem 2-41 and Theorem 2-42. Here an example for a sentence sequence and a CdI-like segment in this sequence for which the ordered pair of both is not an element of CS:

**Example [2.1]** Let \( \bar{s}^{[2.1]} \) be the following sequence:

\[
\begin{align*}
0 & \quad \text{Suppose} \quad P_{1,1}(c_1) \\
1 & \quad \text{Suppose} \quad P_{1,1}(c_1) \\
2 & \quad \text{Therefore} \quad P_{1,1}(c_1) \rightarrow P_{1,1}(c_1)
\end{align*}
\]

**Comment:** Suppose \( (\bar{s}^{[2.1]}, \bar{s}^{[2.1]}) \in \text{CS} \). According to Theorem 2-41, we would then have that \( \bar{s}^{[2.1]} \) is a minimal closed segment in \( \bar{s}^{[2.1]} \) or that there would be a \( G \in \text{ASCS}(\bar{s}^{[2.1]}) \) with \( \{ \bar{s}^{[2.1]} \} \times \text{Ran}(G) \subseteq \text{CS} \) and \( \bar{s}^{[2.1]} \in \text{GEN}(\langle \bar{s}^{[2.1]}, G \rangle) \). Since \( |\text{AS}(\bar{s}^{[2.1]})| = 2 \), \( \bar{s}^{[2.1]} \) is not a minimal closed segment in \( \bar{s}^{[2.1]} \). Therefore there has to be a \( G \in \text{ASCS}(\bar{s}^{[2.1]}) \) with \( \{ \bar{s}^{[2.1]} \} \times \text{Ran}(G) \subseteq \text{CS} \) and \( \bar{s}^{[2.1]} \in \text{GEN}(\langle \bar{s}^{[2.1]}, G \rangle) \).

Then we have \( \bar{s}^{[2.1]} \in \text{GEN}(\langle \bar{s}^{[2.1]}, G \rangle) \). Then there is a \( \mathcal{B} \in \text{SG}(\bar{s}^{[2.1]}) \) such that \( G \) is an AS-comprising segment sequence for \( \mathcal{B} \) in \( \bar{s}^{[2.1]} \) and \( \min(\text{Dom}(\bar{s}^{[2.1]}))+1 = \)
min(Dom(\(\mathfrak{B}\))) and max(Dom(\(\mathfrak{A}^{[2.1]}\))) = max(Dom(\(\mathfrak{B}\))) + 1. Then we have \(\mathfrak{B} = \{(1, \text{Suppose } P_{1.1}(c_1)\})\). Since \(G\) is an AS-comprising segment sequence for \(\mathfrak{B}\) in \(\mathfrak{A}^{[2.1]}\), we then have \(\text{Ran}(G) = \{(1, \text{Suppose } P_{1.1}(c_1)\})\).

Yet, \(\{(1, \text{Suppose } P_{1.1}(c_1)\})\) is not a CdI- or NI- or RA-like segment in \(\mathfrak{A}^{[2.1]}\). By hypothesis, however, we have \(\mathfrak{A}^{[2.1]} \times \text{Ran}(G) \subseteq \text{CS}\) and thus \(\{(1, \text{Suppose } P_{1.1}(c_1)\})\) \(\in\) \(\text{CS}\). According to Theorem 2-42, we would then have that \(\{(1, \text{Suppose } P_{1.1}(c_1)\})\) is a CdI- or NI- or RA-like segment in \(\mathfrak{A}^{[2.1]}\). Thus, the assumption that \((\mathfrak{A}^{[2.1]}, \mathfrak{A}^{[2.1]})) \in \text{CS}\) leads to a contradiction. Therefore \((\mathfrak{A}^{[2.1]}, \mathfrak{A}^{[2.1]})) \notin \text{CS}\).

**Theorem 2-44.** Closed segments have at least two elements

If \((\mathfrak{A}, \mathfrak{A}) \in \text{CS}\), then \(2 \leq |\mathfrak{A}|\).

**Proof:** With Theorem 2-31 it holds for all CdI- or NI- or RA-like segments \(\mathfrak{A}\) in \(\mathfrak{A}\) that \(2 \leq |\mathfrak{A}|\). From this the theorem follows with Theorem 2-42.

**Theorem 2-45.** Every closed segment has a minimal closed segment as subsegment

If \((\mathfrak{A}, \mathfrak{A}) \in \text{CS}\), then there is a minimal closed segment \(\mathfrak{B}\) in \(\mathfrak{A}\) such that \(\mathfrak{B} \subseteq \mathfrak{A}\).

**Proof:** Let \(X = \{(\mathfrak{A}, \mathfrak{A}) | \text{There is a minimal closed segment } \mathfrak{B} \text{ in } \mathfrak{A} \text{ such that } \mathfrak{B} \subseteq \mathfrak{A}\} \cap \text{CS}\). To prove the theorem, it suffices to show that \(X \in \text{CSR}\), then the statement follows with Theorem 2-40-(ii).

First, we have \(X \subseteq \text{CS} \subseteq \text{SEQ} \times \{\mathfrak{A} | \mathfrak{A} \text{ is a segment}\} \times \{(\mathfrak{A}, \mathfrak{A}) | \mathfrak{A} \text{ is a minimal closed segment in } \mathfrak{A}\} \subseteq X\).

We still have to show that it holds for all \(\mathfrak{A} \in \text{SEQ}\) and \(G \in \text{ASCS}(\mathfrak{A})\) with \(\{(\mathfrak{A}) \times \text{Ran}(G) \subseteq X\) that \(\{(\mathfrak{A}) \times \text{GEN}((\mathfrak{A}, G)) \subseteq X\). First, suppose \(\mathfrak{A} \in \text{SEQ}\) and \(G \in \text{ASCS}(\mathfrak{A})\) and \(\{(\mathfrak{A}) \times \text{Ran}(G) \subseteq X\). Then we have \(\{(\mathfrak{A}) \times \text{Ran}(G) \subseteq X\). Now, suppose \((\mathfrak{A}, \mathfrak{A}) \in \text{CS}\). Because of \(\mathfrak{A} \in \text{GEN}((\mathfrak{A}, G))\) there is then, with Theorem 2-35, a \(\mathfrak{B} \in \text{SG}(\mathfrak{A})\) such that \(G\) is an AS-comprising segment sequence for \(\mathfrak{B}\) in \(\mathfrak{A}\), \(\text{AS}(\mathfrak{A}) \cap \mathfrak{B} \neq \emptyset\) and \(\min(\text{Dom}(\mathfrak{A})) + 1 = \min(\text{Dom}(\mathfrak{B}))\) and \(\max(\text{Dom}(\mathfrak{A})) = \max(\text{Dom}(\mathfrak{B})) + 1\) and \(\mathfrak{A}\) is a CdI- or NI- or RA-like segment in \(\mathfrak{A}\).

Then there is an \(i \in \text{Dom}(\text{AS}(\mathfrak{A})) \cap \text{Dom}(\mathfrak{B})\). We have that \(G\) is an AS-comprising segment sequence for \(\mathfrak{B}\). With Definition 2-9, it thus holds for all \(r \in \text{Dom}(\text{AS}(\mathfrak{A})) \cap \text{Dom}(\mathfrak{B})\).
2.2 Closed Segments

Dom(Ø) that there is an \( s \in \text{Dom}(G) \) such that \( r \in \text{Dom}(G(s)) \). Therefore there is such an \( s \) for \( i \). By hypothesis, we have \( \{\delta\} \times \text{Ran}(G) \subseteq X \) and hence \( (\delta, G(s)) \in X \) and thus there is a minimal closed segment \( \mathcal{C} \) in \( \delta \) such that \( \mathcal{C} \subseteq G(s) \). With Theorem 2-24, we have \( G(s) \subseteq \mathcal{B} \) and hence \( \mathcal{C} \subseteq \mathcal{B} \) and thus, because of \( \mathcal{B} \subseteq \mathcal{A} \), we have \( \mathcal{C} \subseteq \mathcal{A} \). Hence we have \( (\delta, \mathcal{A}) \in X \). □

**Theorem 2-46. Ratio of inference- and assumption-sentences in closed segments**

If \( (\delta, \mathcal{A}) \in \text{CS} \), then \( |\text{AS}(\delta) \cap \mathcal{A}| \leq |\text{IS}(\delta) \cap \mathcal{A}| \).

**Proof:** Let \( X = \{(\delta, \mathcal{A}) \mid \text{If } \mathcal{A} \text{ is a CdI- or NI- or RA-like segment in } \delta, \text{ then } |\text{AS}(\delta) \cap \mathcal{A}| \leq |\text{IS}(\delta) \cap \mathcal{A}| \} \cap \text{CS} \). To prove the theorem, it suffices to show that \( X \in \text{CSR} \), then the statement follows with Theorem 2-40-(ii) and Theorem 2-42.

First, we have \( X \subseteq \text{CS} \subseteq \text{SEQ} \times \{A \mid \mathcal{A} \text{ is a segment}\} \). With Theorem 2-33, we also have \( \{(\delta, \mathcal{A}) \mid \mathcal{A} \text{ is a minimal closed segment in } \delta\} \subseteq X \).

We have to show that for all \( \delta \in \text{SEQ} \) and \( G \in \text{ASCS}(\delta) \) with \( \{\delta\} \times \text{Ran}(G) \subseteq X \) it holds that \( \{\delta\} \times \text{GEN}((\delta, G)) \subseteq X \). First, suppose \( \delta \in \text{SEQ} \) and \( G \in \text{ASCS}(\delta) \) and \( \{\delta\} \times \text{Ran}(G) \subseteq X \). Then we have \( \{\delta\} \times \text{Ran}(G) \subseteq \text{CS} \). Now, suppose \( (\delta, \mathcal{A}) \in \{\delta\} \times \text{GEN}((\delta, G)) \). Then we have \( (\delta, \mathcal{A}) \in \text{CS} \). Because of \( \mathcal{A} \in \text{GEN}((\delta, G)) \), there is then, with Theorem 2-35, a \( \mathcal{B} \in \text{SG}(\delta) \) such that \( G \) is an AS-comprising segment sequence for \( \mathcal{B} \) in \( \delta \) and \( \min(\text{Dom}(\mathcal{A}))+1 = \min(\text{Dom}(\mathcal{B})) \) and \( \max(\text{Dom}(\mathcal{A})) = \max(\text{Dom}(\mathcal{B}))+1 \) and \( \mathcal{A} \) is a CdI- or NI- or RA-like segment in \( \delta \). With Theorem 2-29, we then have \( |\text{AS}(\delta) \cap \mathcal{A}| \leq 1+|\text{AS}(\mathcal{A}) \cap \mathcal{B}| \) and \( 1+|\text{IS}(\delta) \cap \mathcal{B}| \leq |\text{IS}(\delta) \cap \mathcal{A}| \). With Definition 2-9-(iii-c), we have for all \( l \in \text{Dom}(\text{AS}(\delta)) \) \& \( \text{Dom}(\mathcal{B}) \): There is an \( i \in \text{Dom}(G) \) such that \( l \in \text{Dom}(G(i)) \) and with Theorem 2-24 it holds for all \( i \in \text{Dom}(G) \) that \( G(i) \subseteq \mathcal{B} \). Thus we have \( \bigcup\{\text{AS}(\delta) \cap G(i) \mid i \in \text{Dom}(G)\} = \text{AS}(\delta) \cap \mathcal{B} \). Also, we have \( \bigcup\{\text{IS}(\delta) \cap G(i) \mid i \in \text{Dom}(G)\} = \text{IS}(\delta) \cap \mathcal{B} \).

Because of \( \{\delta\} \times \text{Ran}(G) \subseteq X \), we have that for all \( i \in \text{Dom}(G) \) it holds that \( (\delta, G(i)) \in X \) and thus that \( |\text{AS}(\delta) \cap G(i)| \leq |\text{IS}(\delta) \cap G(i)| \). With Theorem 2-22-(i) and Theorem 2-27, it holds for all \( i, j \in \text{Dom}(G) \) that if \( i \neq j \), then \( G(i) \cap G(j) = \emptyset \). Thus we have for
all \( i, j \in \text{Dom}(G) \): If \( i \neq j \), then \((\text{AS}(\mathcal{F}) \cap G(i)) \cap (\text{AS}(\mathcal{F}) \cap G(j)) = \emptyset\) and \((\text{IS}(\mathcal{F}) \cap G(i)) \cap (\text{IS}(\mathcal{F}) \cap G(j)) = \emptyset\).

Hence we have

\[
|\bigcup \{\text{AS}(\mathcal{F}) \cap G(j) \mid j \in \text{Dom}(G)\}| = \sum_{j=0}^{\text{Dom}(G)-1} |\text{AS}(\mathcal{F}) \cap G(j)|
\]

and

\[
|\bigcup \{\text{IS}(\mathcal{F}) \cap G(j) \mid j \in \text{Dom}(G)\}| = \sum_{j=0}^{\text{Dom}(G)-1} |\text{IS}(\mathcal{F}) \cap G(j)|.
\]

Because of \(|\text{AS}(\mathcal{F}) \cap G(j)| \leq |\text{IS}(\mathcal{F}) \cap G(j)|\) for all \( j \in \text{Dom}(G) \), we also have:

\[
\sum_{j=0}^{\text{Dom}(G)-1} |\text{AS}(\mathcal{F}) \cap G(j)| \leq \sum_{j=0}^{\text{Dom}(G)-1} |\text{IS}(\mathcal{F}) \cap G(j)|.
\]

Thus we have

\[
|\text{AS}(\mathcal{F}) \cap \mathcal{A}| \leq 1+|\text{AS}(\mathcal{F}) \cap \mathcal{B}| = 1+\sum_{j=0}^{\text{Dom}(G)-1} |\text{AS}(\mathcal{F}) \cap G(j)| \leq 1+\sum_{j=0}^{\text{Dom}(G)-1} |\text{IS}(\mathcal{F}) \cap G(j)| \leq |\text{IS}(\mathcal{F}) \cap \mathcal{A}|.
\]

Therefore we have \((\mathcal{F}, \mathcal{A}) \in X\). \(\blacksquare\)

**Theorem 2-47.** Every assumption-sentence in a closed segment \(\mathcal{A}\) lies at the beginning of \(\mathcal{A}\) or at the beginning of a proper closed subsegment of \(\mathcal{A}\).

If \((\mathcal{F}, \mathcal{A}) \in \text{CS}\), then for all \( i \in \text{Dom}(\text{AS}(\mathcal{F})) \cap \text{Dom}(\mathcal{A})\):

(i) \( i = \min(\text{Dom}(\mathcal{A})) \)

or

(ii) There is a \( \mathcal{B} \) with \((\mathcal{F}, \mathcal{B}) \in \text{CS}\) such that

a) \( i = \min(\text{Dom}(\mathcal{B})) \) and

b) \( \min(\text{Dom}(\mathcal{A})) < \min(\text{Dom}(\mathcal{B})) < \max(\text{Dom}(\mathcal{B})) < \max(\text{Dom}(\mathcal{A})). \)

**Proof:** Let \( X = \{(\mathcal{F}, \mathcal{A}) \mid \text{For all } i \in \text{Dom}(\text{AS}(\mathcal{F})) \cap \text{Dom}(\mathcal{A}): i = \min(\text{Dom}(\mathcal{A})) \text{ or there is a } \mathcal{B} \text{ with } (\mathcal{F}, \mathcal{B}) \in \text{CS} \text{ such that } i = \min(\text{Dom}(\mathcal{B})) \text{ and } \min(\text{Dom}(\mathcal{A})) < \min(\text{Dom}(\mathcal{B})) < \max(\text{Dom}(\mathcal{B})) < \max(\text{Dom}(\mathcal{A})). \}\}

We have \( X \subseteq \text{CS} \subseteq \text{SEQ} \times \{\mathcal{A} \mid \mathcal{A} \text{ is segment}\} \) and with Definition 2-17, Definition 2-14-(i), Definition 2-15-(i), Definition 2-16-(i) and Theorem 2-41 it holds that \( \{(\mathcal{F}, \mathcal{A}) \mid \mathcal{A} \text{ is a minimal closed segment in } \mathcal{F}\} \subseteq X. \)
We still have to show that for all \( \mathfrak{A} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathfrak{A}) \) with \( \{ \mathfrak{A} \} \times \text{Ran}(G) \subseteq X \) it holds that \( \{ \mathfrak{A} \} \times \text{GEN}((\mathfrak{A}, G)) \subseteq X \). First, suppose \( \mathfrak{F} \in \text{SEQ} \) and \( G \in \text{ASCS}(\mathfrak{F}) \) and \( \{ \mathfrak{F} \} \times \text{Ran}(G) \subseteq X \). Then we have \( \{ \mathfrak{F} \} \times \text{Ran}(G) \subseteq \text{CS} \). Now, suppose \( (\mathfrak{F}, \mathfrak{A}) \in \{ \mathfrak{F} \} \times \text{GEN}((\mathfrak{F}, G)) \). Then we have \((\mathfrak{F}, \mathfrak{A}) \in \text{CS}\). With \( \mathfrak{A} \in \text{GEN}((\mathfrak{F}, G)) \), there is then a \( \mathfrak{B} \in \text{SG}(\mathfrak{F}) \) such that \( G \) is an AS-comprising segment sequence for \( \mathfrak{A} \) in \( \mathfrak{F} \), \( \text{AS}(\mathfrak{F}) \cap \mathfrak{A} \neq \emptyset \) and \( \min(\text{Dom}(\mathfrak{A}))+1 = \min(\text{Dom}(\mathfrak{B})) \) and \( \max(\text{Dom}(\mathfrak{A})) = \max(\text{Dom}(\mathfrak{B}))+1 \) and \( \mathfrak{A} \) is a Cdl- or NI- or RA-like segment in \( \mathfrak{F} \).

Now, suppose \( i \in \text{Dom}(\text{AS}(\mathfrak{F})) \cap \text{Dom}(\mathfrak{A}) \) and \( i \neq \min(\text{Dom}(\mathfrak{A})) \). With Theorem 2-30, we then have \( \min(\text{Dom}(\mathfrak{A})) < i < \max(\text{Dom}(\mathfrak{A})) \). Then we have \( \min(\text{Dom}(\mathfrak{B})) \leq i \leq \max(\text{Dom}(\mathfrak{B})) \). Then we have \( i \in \text{Dom}(\text{AS}(\mathfrak{F})) \cap \text{Dom}(\mathfrak{B}) \). We have that \( G \) is an AS-comprising segment sequence for \( \mathfrak{A} \) in \( \mathfrak{F} \) and \( \text{AS}(\mathfrak{F}) \cap \mathfrak{A} \neq \emptyset \) and \( \min(\text{Dom}(\mathfrak{A}))+1 = \min(\text{Dom}(\mathfrak{B})) \) and \( \max(\text{Dom}(\mathfrak{A})) = \max(\text{Dom}(\mathfrak{B}))+1 \) and \( \mathfrak{A} \) is a Cdl- or NI- or RA-like segment in \( \mathfrak{F} \).

Moreover, it then follows by hypothesis that \( \min(\text{Dom}(\mathfrak{A})) < i = \min(\text{Dom}(\mathfrak{A})) \) and \( \max(\text{Dom}(\mathfrak{A})) \leq \max(\text{Dom}(\mathfrak{B})) \). Then we have \( \min(\text{Dom}(\mathfrak{A})) < i = \min(\text{Dom}(\mathfrak{A})) \). With Theorem 2-44, we also have \( \min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{B})) \). Suppose for the second case that \( \mathfrak{C} \) is as required. Then we have \( \min(\text{Dom}(\mathfrak{A})) < i = \min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{C})) < \max(\text{Dom}(\mathfrak{C})) \) and hence \( \mathfrak{C} \) is the desired segment.

Therefore we have for all \( i \in \text{Dom}(\text{AS}(\mathfrak{F})) \cap \text{Dom}(\mathfrak{A}) \): \( i = \min(\text{Dom}(\mathfrak{A})) \) or there is a \( \mathfrak{B} \) with \( (\mathfrak{F}, \mathfrak{B}) \in \text{CS} \) such that \( i = \min(\text{Dom}(\mathfrak{B})) \) and \( \min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{B})) < \max(\text{Dom}(\mathfrak{B})) < \max(\text{Dom}(\mathfrak{A})) \). Hence we have \( (\mathfrak{F}, \mathfrak{A}) \in X \).
Theorem 2-48. Every closed segment is a minimal closed segment or a CdI- or NI- or RA-like segment whose assumption-sentences lie at the beginning or in a proper closed subsegment.

If \((\mathfrak{H}, \mathfrak{A}) \in \text{CS}\), then:
(i) \(\mathfrak{A}\) is a minimal closed segment in \(\mathfrak{H}\)

or
(ii) \(\mathfrak{A}\) is a CdI- or NI- or RA-like segment \(\mathfrak{H}\), where for all \(i \in \text{Dom}(\text{AS}(\mathfrak{H})) \cap \text{Dom}(\mathfrak{A})\) with \(\min(\text{Dom}(\mathfrak{A})) < i\) it holds that there is a \(\mathfrak{B}\) such that

\[
\begin{align*}
\text{a)} & \quad (i, \mathfrak{H}) \in \mathfrak{B}, \\
\text{b)} & \quad (\mathfrak{H}, \mathfrak{B}) \in \text{CS}, \\
\text{c)} & \quad i = \min(\text{Dom}(\mathfrak{B})) \quad \text{and} \\
\text{d)} & \quad \min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{B})) < \max(\text{Dom}(\mathfrak{B})) < \max(\text{Dom}(\mathfrak{A})).
\end{align*}
\]

Proof: Suppose \((\mathfrak{H}, \mathfrak{A}) \in \text{CS}\). Now, suppose \(\mathfrak{A}\) is not a minimal closed segment in \(\mathfrak{H}\). Then it holds with Theorem 2-42 that \(\mathfrak{A}\) is a CdI- or NI- or RA-like segment in \(\mathfrak{H}\) and, with Theorem 2-47, that for all \(i \in \text{Dom}(\text{AS}(\mathfrak{H})) \cap \text{Dom}(\mathfrak{A})\) with \(\min(\text{Dom}(\mathfrak{A})) < i\) there is a suitable \(\mathfrak{B}\). ■

Theorem 2-49. Closed segments are non-redundant, i.e. proper initial segments of closed segments are not closed segments.

If \((\mathfrak{H}, \mathfrak{A}) \in \text{CS}\), then for all \(i \in \text{Dom}(\mathfrak{A})\): \((\mathfrak{H}, \mathfrak{A}|i) \notin \text{CS}\).

Proof: Suppose \(X = \{(\mathfrak{H}, \mathfrak{A}) \mid (\mathfrak{H}, \mathfrak{A}) \in \text{CS} \text{ and for all } i \in \text{Dom}(\mathfrak{A}) \mid (\mathfrak{H}, \mathfrak{A}|i) \notin \text{CS} \}/\text{CSR}\). To prove the theorem, it suffices to show that \(X \in \text{CSR}\), then the statement follows with Theorem 2-40-(ii).

First, we have \(X \subseteq \text{CS} \subseteq \text{SEQ} \times \{\mathfrak{A} \mid \mathfrak{A}\ is \ a \ segment\}\) and with Definition 2-17, Definition 2-14-(ii), Definition 2-15-(ii), Definition 2-16-(ii), Theorem 2-41 and Theorem 2-42 it holds that \(\{(\mathfrak{H}, \mathfrak{A}) \mid \mathfrak{A}\ is \ a \ minimal \ closed \ segment \ in \ \mathfrak{H}\}\ \subseteq \ X\).

We have to show that for all \(\mathfrak{H} \in \text{SEQ} \) and \(G \in \text{ASCS}(\mathfrak{H})\) with \(\{\mathfrak{H}\} \times \text{Ran}(G) \subseteq X\) it holds that \(\{\mathfrak{H}\} \times \text{GEN}((\mathfrak{H}, G)) \subseteq X\). First, suppose \(\mathfrak{H} \in \text{SEQ} \) and \(G \in \text{ASCS}(\mathfrak{H})\) and \(\{\mathfrak{H}\} \times \text{Ran}(G) \subseteq X\). Then we have \(\{\mathfrak{H}\} \times \text{Ran}(G) \subseteq X\). Now, suppose \((\mathfrak{H}, \mathfrak{A}) \in \{\mathfrak{H}\} \times \text{GEN}((\mathfrak{H}, G))\). Then we have \(\mathfrak{A} \in \text{GEN}((\mathfrak{H}, G))\) and thus \((\mathfrak{H}, \mathfrak{A}) \in \text{CS}\). Also, there is then a \(\mathfrak{B} \in \text{SG}(\mathfrak{H})\) such that \(G\) is an AS-comprising segment sequence for \(\mathfrak{B}\) in \(\mathfrak{H}\) and \(\text{AS}(\mathfrak{H}) \cap \mathfrak{B} \neq \emptyset\) and \(\min(\text{Dom}(\mathfrak{A}))+1 = \min(\text{Dom}(\mathfrak{B}))\) and \(\max(\text{Dom}(\mathfrak{A})) = \max(\text{Dom}(\mathfrak{B}))+1\) and \(\mathfrak{A}\) is a CdI- or NI- or RA-like segment in \(\mathfrak{H}\). Now, suppose for contradiction that \((\mathfrak{H}, \mathfrak{A}) \notin \text{CS}\).
$\mathfrak{A}\upharpoonright i \in \text{CS}$ for an $i \in \text{Dom}(\mathfrak{A})$. Then we have that $\mathfrak{A}\upharpoonright i$ is a segment in $\mathfrak{y}$. With Theorem 2-7, we then have $\min(\text{Dom}(\mathfrak{A}\upharpoonright i)) = \min(\text{Dom}(\mathfrak{A}))$ and thus with Theorem 2-23 that for all $j \in \text{Dom}(G)$ it holds that $\min(\text{Dom}(\mathfrak{A}\upharpoonright i)) < \min(\text{Dom}(\mathfrak{B})) \leq \min(\text{Dom}(G(j)))$.

With Theorem 2-35-(iii), we then have that $\mathfrak{A}\upharpoonright i$ is not a minimal closed segment in $\mathfrak{y}$. Then it holds with Theorem 2-41 that there is a $G^* \in \text{ASCS}(\mathfrak{y})$ with $\{\mathfrak{y}\} \times \text{Ran}(G^*) \subseteq \text{CS}$ and $\mathfrak{A}\upharpoonright i \in \text{GEN}(\langle \mathfrak{y}, G^* \rangle)$. With Theorem 2-35, we then have that there is a $\mathfrak{B}' \in \text{SG}(\mathfrak{y})$ such that $\min(\text{Dom}(\mathfrak{A})) + 1 = \min(\text{Dom}(\mathfrak{A}\upharpoonright i)) + 1 = \min(\text{Dom}(\mathfrak{B}'))$ and $\max(\text{Dom}(\mathfrak{A}\upharpoonright i)) = i-1 = \max(\text{Dom}(\mathfrak{B}')) + 1$. We will now show that there is an $s \in \text{Dom}(G)$ such that $\mathfrak{A}\upharpoonright s \in \text{PGEN}(\langle \mathfrak{y}, G(\langle s+1 \rangle) \rangle)$, which, according to Theorem 2-35-(vi), contradicts $\mathfrak{A} \in \text{GEN}(\langle \mathfrak{y}, G \rangle)$.

It holds with Theorem 2-35-(iv) that there is an $l \in \text{Dom}(\text{AS}(\mathfrak{y})) \cap \text{Dom}(\mathfrak{A}\upharpoonright i)$ such that $\min(\text{Dom}(\mathfrak{A}\upharpoonright i)) = \min(\text{Dom}(\mathfrak{A})) < l$. Now, suppose $l_0 = \max\{l \mid l \in \text{Dom}(\text{AS}(\mathfrak{y})) \cap \text{Dom}(\mathfrak{A}\upharpoonright i) \text{ and } \min(\text{Dom}(\mathfrak{A}\upharpoonright i)) < l\}$. It then follows with $i \leq \max(\text{Dom}(\mathfrak{A}))$ and $\text{Dom}(\mathfrak{A}\upharpoonright i)$ that $\min(\text{Dom}(\mathfrak{A})) = \min(\text{Dom}(\mathfrak{A}\upharpoonright i)) < l_0 < \max(\text{Dom}(\mathfrak{A}))$. Then we have $\min(\text{Dom}(\mathfrak{B})) \leq l_0 \leq \max(\text{Dom}(\mathfrak{B}))$. Then we have $l_0 \in \text{Dom}(\text{AS}(\mathfrak{y})) \cap \text{Dom}(\mathfrak{B})$. We have that $G$ is an AS-comprising segment sequence for $\mathfrak{B}$. With Definition 2-9, it therefore holds that there is an $s \in \text{Dom}(G)$ such that $l_0 \in \text{Dom}(G(s))$. Then we have that $l_0 \in \text{Dom}(\text{AS}(\mathfrak{y})) \cap \text{Dom}(G(s))$ and hence, because of $\{\mathfrak{y}\} \times \text{Ran}(G) \subseteq X \subseteq \text{CS}$ and with Theorem 2-47, that $\min(\text{Dom}(G(s))) \leq l_0 < \max(\text{Dom}(G(s)))$. We also have that $(\mathfrak{y}, \mathfrak{A}\upharpoonright i) \in \text{CS}$ and thus, with Theorem 2-47, that $l_0 < i-1$. Hence, we have that $\min(\text{Dom}(\mathfrak{A}\upharpoonright i)) < \min(\text{Dom}(\mathfrak{y}(s))) < l_0 < i-1$.

Now, suppose $k \leq s$. Since $G$ is an AS-comprising segment sequence for $\mathfrak{B}$ in $\mathfrak{y}$, it then follows with Definition 2-9 and Definition 2-7 that $\min(\text{Dom}(\mathfrak{A}\upharpoonright i)) < \min(\text{Dom}(G(k))) \leq \min(\text{Dom}(G(s))) < l_0 < \max(\text{Dom}(G(k)))$. Since $\{\mathfrak{y}\} \times \text{Ran}(G) \subseteq X \subseteq \text{CS}$, it then holds with Theorem 2-42 that $\min(\text{Dom}(G(k))) \in \text{Dom}(\text{AS}(\mathfrak{y})) \cap \text{Dom}(\mathfrak{B'})$. Since $G^*$ is an AS-comprising segment sequence for $\mathfrak{B}'$ in $\mathfrak{y}$, there is then an $r \in \text{Dom}(G^*)$ such that $\min(\text{Dom}(G(k))) \leq \min(\text{Dom}(G^*(r))) = \min(\text{Dom}(G(k)))$. Then it holds with $\{\mathfrak{y}\} \times \text{Ran}(G) \subseteq X$ and $\{\mathfrak{y}\} \times \text{Ran}(G^*) \subseteq \text{CS}$ that $\max(\text{Dom}(G(k))) \leq \max(\text{Dom}(G^*(r)))$. Suppose $\min(\text{Dom}(G^*(r))) = \min(\text{Dom}(G(k)))$. Then it holds with
\{\delta\} \times \text{Ran}(G^*) \subseteq \text{CS} \text{ and Theorem 2-47 that there is a } \mathcal{C} \text{ such that } (\delta, \mathcal{C}) \in \text{CS} \text{ and } \min(\text{Dom}(G(k))) = \min(\text{Dom}(\mathcal{C})) \text{ and } \min(\text{Dom}(G^*(r))) < \min(\text{Dom}(\mathcal{C})) < \max(\text{Dom}(\mathcal{C})) < \max(\text{Dom}(G^*(r))). \text{ Then it holds with } \{\delta\} \times \text{Ran}(G) \subseteq X \text{ that } \max(\text{Dom}(G(k))) \leq \max(\text{Dom}(\mathcal{C})). \text{ Thus holds with Theorem 2-5-(i) in both cases } G(k) \subseteq G^*(r). \text{ Therefore we have for all } k \leq s \text{ that there is an } r \in \text{Dom}(G^*) \text{ such that } G(k) \subseteq G^*(r).

Since } G^* \text{ is an AS-comprising segment sequence for } \mathcal{B}' \text{ and } \max(\text{Dom}(\mathcal{B}')) = i-2 \text{ we thus have in particular that } \max(\text{Dom}(G(s))) \leq i-2. \text{ We also have that if } \mathfrak{A} \vDash i \text{ is an NI-like segment in } \delta, \text{ then there is } j \in \text{Dom}(\mathfrak{A} \vDash i) \text{ such that } P(\delta_j) = \Gamma \text{ and } P(\delta_{j-2}) = \Gamma' \text{ or } P(\delta_j) = \Gamma' \text{ and } P(\delta_{j-2}) = \Gamma \text{ and for all } r \in \text{Dom}(G^*) \text{ it holds that } j < \min(\text{Dom}(G^*(r))) \text{ or } \max(\text{Dom}(G^*(r))) \leq j. \text{ If there was a } k \leq s \text{ such that } \min(\text{Dom}(G(k))) \leq j < \max(\text{Dom}(G(k))), \text{ then there would be, as we have just shown, an } r \in \text{Dom}(G^*) \text{ such that } G(k) \subseteq G^*(r) \text{ and thus } \min(\text{Dom}(G^*(r))) \leq j < \max(\text{Dom}(G^*(r))). \text{ Therefore, if } \mathfrak{A} \vDash i \text{ is an NI-like segment in } \delta, \text{ then there is } j \in \text{Dom}(\mathfrak{A} \vDash i) \text{ such that } P(\delta_j) = \Gamma \text{ and } P(\delta_{j-2}) = \Gamma' \text{ or } P(\delta_j) = \Gamma' \text{ and } P(\delta_{j-2}) = \Gamma \text{ and for all } k \leq s \text{ it holds that } j < \min(\text{Dom}(G(k))) \text{ or } \max(\text{Dom}(G(k))) \leq j. \text{ Also, we have for all } l \in \text{Dom}(\text{AS}(\delta)) \cap \text{Dom}(\mathcal{B}') \text{ that there is a } k \leq s \text{ such that } l \in \text{Dom}(G(k)). \text{ First, we have } \mathcal{B}' \subseteq \mathcal{B} \text{ and thus there is for every such } l \text{ a } k \in \text{Dom}(G) \text{ such that } l \in \text{Dom}(G(k)). \text{ Also, if } s < k, \text{ we would have, with Definition 2-9 and Definition 2-7, that } l_0 < \max(\text{Dom}(G(s))) < \min(\text{Dom}(G(k))) \leq l, \text{ while, on the other hand, we have } l \leq l_0.

With Definition 2-9 and Definition 2-7, we can easily show that } G|i(s+1) \in \text{SGS}(\delta). \text{ Hence, we have that } G|i(s+1) \text{ is an AS-comprising segment sequence for } \mathcal{B}' \text{ and thus also that } G|i(s+1) \in \text{ASCS}(\delta) \text{ and hence that } \mathfrak{A} \vDash i \in \text{PGEN}(\delta, G|i(s+1))). \text{ This, however contradicts Theorem 2-35-(vi). Therefore there is no } i \in \text{Dom}(\mathfrak{A}) \text{ such that } (\delta, \mathfrak{A} \vDash i) \in \text{CS} \text{ and, because } (\delta, \mathfrak{A}) \in \text{CS}, \text{ we have } (\delta, \mathfrak{A}) \in X. \blacksquare

**Theorem 2-50.** Closed segments are uniquely determined by their beginnings

If } \mathfrak{A}, \mathfrak{A}' \text{ are closed segments in } \delta \text{ and } \min(\text{Dom}(\mathfrak{A})) = \min(\text{Dom}(\mathfrak{A}')), \text{ then } \mathfrak{A} = \mathfrak{A}'.

**Proof:** Let } \mathfrak{A}, \mathfrak{A}' \text{ be closed segments in } \delta \text{ and } \min(\text{Dom}(\mathfrak{A})) = \min(\text{Dom}(\mathfrak{A}')). \text{ Suppose for contradiction that } \max(\text{Dom}(\mathfrak{A})) < \max(\text{Dom}(\mathfrak{A}')). \text{ Then we would have have
min(Dom(\(\mathcal{A}'\))) = min(Dom(\(\mathcal{A}\))) < max(Dom(\(\mathcal{A}\))) + 1 \leq max(Dom(\(\mathcal{A}'\))). Since \(\mathcal{A}'\) is a segment, we would thus have max(Dom(\(\mathcal{A}\))) + 1 \in Dom(\(\mathcal{A}'\)) and thus that \(\mathcal{A}'\) is a closed segment in \(\mathcal{H}\). Together with Theorem 2-49 this contradicts our assumption that \(\mathcal{A}'\) is a closed segment in \(\mathcal{H}\). In the same way, it follows for max(Dom(\(\mathcal{A}'\))) < max(Dom(\(\mathcal{A}\))) that \(\mathcal{A}\) would not be a closed segment in \(\mathcal{H}\). Therefore we have max(Dom(\(\mathcal{A}\))) = max(Dom(\(\mathcal{A}'\))) and thus \(\mathcal{A} = \mathcal{A}'\). 

**Theorem 2-51.** AS-comprising segment sequences for one and the same segment for which all values are closed segments are identical.

If \(\mathcal{A}\) is a segment in \(\mathcal{H}\) and \(G, G^*\) are AS-comprising segment sequences for \(\mathcal{A}\) in \(\mathcal{H}\) and \(\{\mathcal{H}\} \times \text{Ran}(G) \subseteq \text{CS} \) and \(\{\mathcal{H}\} \times \text{Ran}(G^*) \subseteq \text{CS} \), then \(G = G^*\).

**Proof:** Suppose \(\mathcal{A}\) is a segment in \(\mathcal{H}\) and suppose \(G, G^*\) are AS-comprising segment sequences for \(\mathcal{A}\) in \(\mathcal{H}\) and \(\{\mathcal{H}\} \times \text{Ran}(G) \subseteq \text{CS} \) and \(\{\mathcal{H}\} \times \text{Ran}(G^*) \subseteq \text{CS} \). With Definition 2-9, we then have \(G, G^* \in SGS(\mathcal{H}) \setminus \{\emptyset\}\) and with Theorem 2-24 it holds for all \(i \in \text{Dom}(G)\) that \(G(i) \subseteq \mathcal{A}\), and for all \(j \in \text{Dom}(G^*)\) that \(G^*(j) \subseteq \mathcal{A}\). Also, we have \(\text{Ran}(G) \subseteq \text{Ran}(G^*)\). To see this, suppose \(i \in \text{Dom}(G)\). Then we have \((\mathcal{H}, G(i)) \in \text{CS}\) and thus we have that \(\min(\text{Dom}(G(i))) \in \text{Dom}(\text{AS}(\mathcal{H})) \cap \text{Dom}(\mathcal{A})\). Thus there is a \(j \in \text{Dom}(G^*)\) such that \(\min(\text{Dom}(G(i))) \in \text{Dom}(G^*(j))\). With \((\mathcal{H}, G^*(j)) \in \text{CS}\) and Theorem 2-47 and Theorem 2-49, we then have \(G(i) \subseteq G^*(j)\). Analogously, it follows that there is an \(i^* \in \text{Dom}(G)\) such that \(G^*(j) \subseteq G(i^*)\). Then we have \(G(i) \subseteq G(i^*)\). Since we have, with Theorem 2-43, that \(G(i) \neq \emptyset\) and thus \(G(i) \cap G(i^*) \neq \emptyset\), it then follows with Theorem 2-27 that \(G(i) = G(i^*)\) and thus that \(G^*(j) \subseteq G(i)\). Hence we have \(G^*(j) = G(i)\). Therefore we have \(G(i) \in \text{Ran}(G^*)\). Hence, we have \(\text{Ran}(G) \subseteq \text{Ran}(G^*)\). Analogously, it follows that \(\text{Ran}(G^*) \subseteq \text{Ran}(G)\). Hence, we have \(\text{Ran}(G) = \text{Ran}(G^*)\). With Theorem 2-22-(iii), it then follows that \(\text{Dom}(G) = \text{Dom}(G^*)\).

Now, we show by induction on \(i\) that it holds for all \(i \in \text{Dom}(G) = \text{Dom}(G^*)\) that \(G(i) = G^*(i)\) and thus that \(G = G^*\). For this, suppose that for all \(l < i\) it holds that if \(l \in \text{Dom}(G)\), then \(G(l) = G^*(l)\). Now, suppose \(i \in \text{Dom}(G)\). Suppose for contradiction that \(G(i) \neq G^*(i)\). With \((\mathcal{H}, G(i)) \in \text{CS}\) and \((\mathcal{H}, G^*(i)) \in \text{CS}\) and with Theorem 2-50, we then have \(\min(\text{Dom}(G(i))) \neq \min(\text{Dom}(G^*(i)))\). Suppose \(\min(\text{Dom}(G(i))) <
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\[ \min(\text{Dom}(G(i))) \]. It holds with \((f, G(i)) \in \text{CS}\) that \(\min(\text{Dom}(G(i))) \in \text{Dom}(\text{AS}(f)) \cap \text{Dom}(\mathcal{A})\). Thus there is a \(j \in \text{Dom}(G\ast)\) such that \(\min(\text{Dom}(G(i))) \in \text{Dom}(G\ast(j))\). In the same way as above, it then follows that \(G\ast(j) = G(i)\). Since, by hypothesis, \(G(i) \neq G\ast(i)\), we then have \(G\ast(j) \neq G\ast(i)\) and thus \(j \neq i\). Since \(G, G\ast \in \text{SGS}(f)\), it then follows with Definition 2-7 and \(\min(\text{Dom}(G\ast(j))) = \min(\text{Dom}(G(i))) < \min(\text{Dom}(G\ast(i)))\) that \(j < i\).

According to the I.H., it then follows that \(G(j) = G\ast(j) = G(i)\), whereas it holds with Theorem 2-22-(i) and \(j < i\) that \(G(j) \neq G(i)\). Contradiction! Using the I.H., we can show a contradiction for \(\min(\text{Dom}(G\ast(i))) < \min(\text{Dom}(G(i)))\) in the same way. Hence we have \(\min(\text{Dom}(G(i))) = \min(\text{Dom}(G\ast(i)))\) and thus we have \(G(i) = G\ast(i)\). ■

Theorem 2-52. If the beginning of a closed segments \(\mathcal{A}'\) lies in a closed segment \(\mathcal{A}\), then \(\mathcal{A}'\) is a subsegment of \(\mathcal{A}\).

If \(\mathcal{A}, \mathcal{A}'\) are closed segments in \(f\) and \(\min(\text{Dom}(\mathcal{A}')) \in \text{Dom}(\mathcal{A})\), then \(\mathcal{A}' \subseteq \mathcal{A}\).

_Proof:_ Let \(\mathcal{A}, \mathcal{A}'\) be closed segments in \(f\) and suppose \(\min(\text{Dom}(\mathcal{A}')) \in \text{Dom}(\mathcal{A})\). Then we have \(\min(\text{Dom}(\mathcal{A}')) \in \text{Dom}(\text{AS}(f)) \cap \text{Dom}(\mathcal{A})\). With Theorem 2-47, there is then a \(\mathcal{B} \subseteq \mathcal{A}\) such that \(\mathcal{B}\) is a closed segment in \(f\) and \(\min(\text{Dom}(\mathcal{A}')) = \min(\text{Dom}(\mathcal{B}))\). It then follows with Theorem 2-50 that \(\mathcal{A}' = \mathcal{B}\) and therefore that \(\mathcal{A}' \subseteq \mathcal{A}\). ■

Theorem 2-53. Closed segments are uniquely determined by their end.

If \(\mathcal{A}, \mathcal{A}'\) are closed segments in \(f\) and \(\max(\text{Dom}(\mathcal{A})) = \max(\text{Dom}(\mathcal{A}'))\), then \(\mathcal{A} = \mathcal{A}'\).

_Proof:_ Let \(\mathcal{A}, \mathcal{A}'\) be closed segments in \(f\) and \(\max(\text{Dom}(\mathcal{A})) = \max(\text{Dom}(\mathcal{A}'))\). Suppose \(\min(\text{Dom}(\mathcal{A})) < \min(\text{Dom}(\mathcal{A}'))\). Then we have \(\min(\text{Dom}(\mathcal{A})) < \min(\text{Dom}(\mathcal{A}')) < \max(\text{Dom}(\mathcal{A}')) = \max(\text{Dom}(\mathcal{A}))\). Then we have \(\min(\text{Dom}(\mathcal{A}')) \in \text{Dom}(\text{AS}(f)) \cap \text{Dom}(\mathcal{A})\) and \(\min(\text{Dom}(\mathcal{A})) < \min(\text{Dom}(\mathcal{A}'))\). With Theorem 2-48 there is thus a closed segment \(\mathcal{B}\) in \(f\) such that \(\min(\text{Dom}(\mathcal{A}')) = \min(\text{Dom}(\mathcal{B}))\) and \(\min(\text{Dom}(\mathcal{A})) < \min(\text{Dom}(\mathcal{B})) < \max(\text{Dom}(\mathcal{B})) < \max(\text{Dom}(\mathcal{A}))\). It then holds with Theorem 2-50 that \(\mathcal{A}' = \mathcal{B}\). But then we have \(\max(\text{Dom}(\mathcal{A}')) = \max(\text{Dom}(\mathcal{B})) < \max(\text{Dom}(\mathcal{A}))\), which contradicts the hypothesis. Therefore we have \(\min(\text{Dom}(\mathcal{A}')) \leq \min(\text{Dom}(\mathcal{A}))\). In the same way, we can show that for \(\min(\text{Dom}(\mathcal{A}')) < \min(\text{Dom}(\mathcal{A}))\) we would have \(\max(\text{Dom}(\mathcal{A})) < \max(\text{Dom}(\mathcal{A}'))\), which also contradicts the assumption. Hence we have \(\min(\text{Dom}(\mathcal{A}')) \leq \min(\text{Dom}(\mathcal{A}))\).
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min(Dom(\(\mathfrak{A}\))) and min(Dom(\(\mathfrak{A}'\))) \leq min(Dom(\(\mathfrak{A}\'))) and thus min(Dom(\(\mathfrak{A}\))) = min(Dom(\(\mathfrak{A}'\))). From this, it follows with Theorem 2-50 that \(\mathfrak{A} = \mathfrak{A}'\).

\[\square\]

**Theorem 2-54.** Proper subsegment relation between closed segments

If \(\mathfrak{A}, \mathfrak{A}'\) are closed segments in \(\mathfrak{H}\), then:

min(Dom(\(\mathfrak{A}'\))) \in Dom(\(\mathfrak{A}\)) \setminus \{\text{min}(\text{Dom}(\mathfrak{A}))\}

iff

\(\mathfrak{A}' \subset \mathfrak{A}\).

**Proof:** Let \(\mathfrak{A}, \mathfrak{A}'\) be closed segments in \(\mathfrak{H}\). \((L-R):\) Suppose min(Dom(\(\mathfrak{A}'\))) \in Dom(\(\mathfrak{A}\)) \setminus \{\text{min}(\text{Dom}(\mathfrak{A}))\}. Hence min(Dom(\(\mathfrak{A}'\))) \neq \text{min}(\text{Dom}(\mathfrak{A})) and therefore \(\mathfrak{A}' \neq \mathfrak{A}\). Furthermore min(Dom(\(\mathfrak{A}'\))) \in Dom(\(\mathfrak{A}\)) and hence by Theorem 2-52 \(\mathfrak{A}' \subset \mathfrak{A}\). Thus \(\mathfrak{A}' \subset \mathfrak{A}\).

\((R-L):\) Now, suppose \(\mathfrak{A}' \subset \mathfrak{A}\). Then we have min(Dom(\(\mathfrak{A}'\))) \in Dom(\(\mathfrak{A}\)). We also have min(Dom(\(\mathfrak{A}'\))) \neq \text{min}(\text{Dom}(\mathfrak{A}))), because otherwise it would hold with Theorem 2-50 that \(\mathfrak{A}' = \mathfrak{A}\). Hence we have min(Dom(\(\mathfrak{A}'\))) \in Dom(\(\mathfrak{A}\)) \setminus \{\text{min}(\text{Dom}(\mathfrak{A}))\}. \[\square\]

**Theorem 2-55.** Proper and improper subsegment relations between closed segments

If \(\mathfrak{A}, \mathfrak{A}'\) are closed segments in \(\mathfrak{H}\) and min(Dom(\(\mathfrak{A}'\))) \in Dom(\(\mathfrak{A}\)), then \(\mathfrak{A}' \subset \mathfrak{A}\) or \(\mathfrak{A}' = \mathfrak{A}\).

**Proof:** Let \(\mathfrak{A}, \mathfrak{A}'\) be closed segments in \(\mathfrak{H}\) and suppose min(Dom(\(\mathfrak{A}'\))) \in Dom(\(\mathfrak{A}\)). Suppose min(Dom(\(\mathfrak{A}'\))) \in Dom(\(\mathfrak{A}\)) \setminus \{\text{min}(\text{Dom}(\mathfrak{A}))\}. With Theorem 2-54, we then have \(\mathfrak{A}' \subset \mathfrak{A}\). Suppose min(Dom(\(\mathfrak{A}'\))) = min(Dom(\(\mathfrak{A}\))). With Theorem 2-50, we then have \(\mathfrak{A}' = \mathfrak{A}\). \[\square\]

**Theorem 2-56.** Inclusion relations between non-disjunct closed segments

If \(\mathfrak{A}, \mathfrak{A}'\) are closed segments in \(\mathfrak{H}\) and \(\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset\), then:

(i) \(\text{min}(\text{Dom}(\mathfrak{A})) < \text{min}(\text{Dom}(\mathfrak{A}'))\) iff \(\mathfrak{A}' \subset \mathfrak{A}\),
(ii) \(\text{min}(\text{Dom}(\mathfrak{A})) = \text{min}(\text{Dom}(\mathfrak{A}'))\) iff \(\mathfrak{A}' = \mathfrak{A}\),
(iii) \(\text{min}(\text{Dom}(\mathfrak{A})) < \text{min}(\text{Dom}(\mathfrak{A}'))\) iff \(\max(\text{Dom}(\mathfrak{A}')) < \max(\text{Dom}(\mathfrak{A}))\),
(iv) \(\text{min}(\text{Dom}(\mathfrak{A})) = \text{min}(\text{Dom}(\mathfrak{A}'))\) iff \(\max(\text{Dom}(\mathfrak{A})) = \max(\text{Dom}(\mathfrak{A}'))\).

**Proof:** Let \(\mathfrak{A}\) and \(\mathfrak{A}'\) be closed segments in \(\mathfrak{H}\) and let \(\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset\).

Ad (i): \((L-R):\) Suppose \(\text{min}(\text{Dom}(\mathfrak{A})) < \text{min}(\text{Dom}(\mathfrak{A}'))\). Since \(\mathfrak{A}\) and \(\mathfrak{A}'\) are segments and \(\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset\), it holds with Theorem 2-9 that \(\text{min}(\text{Dom}(\mathfrak{A})) \in \text{Dom}(\mathfrak{A}')\) or \(\text{min}(\text{Dom}(\mathfrak{A}')) \in \text{Dom}(\mathfrak{A})\). With the hypothesis, it then holds that \(\text{min}(\text{Dom}(\mathfrak{A}')) \in \text{Dom}(\mathfrak{A})\).

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Dom(\(\mathfrak{A}\)) \{ \min(\text{Dom}(\mathfrak{A})) \}. With Theorem 2-54, we thus have \(\mathfrak{A}' \subset \mathfrak{A}\). (R-L): Suppose \(\mathfrak{A}' \subset \mathfrak{A}\). Again with Theorem 2-54, we then have \(\min(\text{Dom}(\mathfrak{A}')) \in \text{Dom}(\mathfrak{A})) \{ \min(\text{Dom}(\mathfrak{A})) \} \) and therefore: \(\min(\text{Dom}(\mathfrak{A})) \prec \min(\text{Dom}(\mathfrak{A}'))\).

Ad (ii): Follows with Theorem 2-50

Ad (iii): (L-R): Suppose \(\min(\text{Dom}(\mathfrak{A}')) \prec \min(\text{Dom}(\mathfrak{A}))\). Then we have with (i) that \(\mathfrak{A}' \subset \mathfrak{A}\). With Theorem 2-5-(i) we then have \(\max(\text{Dom}(\mathfrak{A}'')) \leq \max(\text{Dom}(\mathfrak{A}))\). With \(\mathfrak{A}' \subset \mathfrak{A}\) and Theorem 2-53, we then have \(\max(\text{Dom}(\mathfrak{A}'')) \neq \max(\text{Dom}(\mathfrak{A}))\). Hence we have \(\max(\text{Dom}(\mathfrak{A}'')) < \max(\text{Dom}(\mathfrak{A}))\). (R-L): Suppose \(\max(\text{Dom}(\mathfrak{A}'')) < \max(\text{Dom}(\mathfrak{A}))\). It then holds with Theorem 2-5-(i) that \(\mathfrak{A} \not\subset \mathfrak{A}'\). With (i) and (ii) we then have that neither \(\min(\text{Dom}(\mathfrak{A}')) \prec \min(\text{Dom}(\mathfrak{A}))\) nor \(\min(\text{Dom}(\mathfrak{A}')) = \min(\text{Dom}(\mathfrak{A}))\). Therefore we have \(\min(\text{Dom}(\mathfrak{A})) \prec \min(\text{Dom}(\mathfrak{A}'))\).

Ad (iv): Follows with (ii) and Theorem 2-53. ■

**Theorem 2-57.** Closed segments are either disjunct or one is a subsegment of the other.

If \(\mathfrak{A}\) and \(\mathfrak{A}'\) are closed segments in \(\mathfrak{F}\), then: \(\mathfrak{A} \cap \mathfrak{A}' = \emptyset\) or \(\mathfrak{A} \subset \mathfrak{A}'\) or \(\mathfrak{A}' \subset \mathfrak{A}\).

**Proof:** Let \(\mathfrak{A}\) and \(\mathfrak{A}'\) be closed segments in \(\mathfrak{F}\). Suppose \(\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset\). Then we have \(\min(\text{Dom}(\mathfrak{A}')) \leq \min(\text{Dom}(\mathfrak{A}))\) or \(\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(\mathfrak{A}'))\). With Theorem 2-56-(i) and -(ii), it then follows that \(\mathfrak{A} \subset \mathfrak{A}'\) or \(\mathfrak{A}' \subset \mathfrak{A}\). ■

**Theorem 2-58.** A minimal closed segment \(\mathfrak{A}'\) is either disjunct from a closed segment \(\mathfrak{A}\) or it is a subsegment of \(\mathfrak{A}\).

If \(\mathfrak{A}\) is a closed segment in \(\mathfrak{F}\) and \(\mathfrak{A}'\) is a minimal closed segment in \(\mathfrak{F}\), then: \(\mathfrak{A} \cap \mathfrak{A}' = \emptyset\) or \(\mathfrak{A}' \subset \mathfrak{A}\)

**Proof:** Let \(\mathfrak{A}\) be a closed segment in \(\mathfrak{F}\) and suppose \(\mathfrak{A}'\) is a minimal closed segment in \(\mathfrak{F}\). Then \(\mathfrak{A}'\) is also a closed segment in \(\mathfrak{F}\). Suppose \(\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset\). Then we have \(\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(\mathfrak{A}'))\). For if \(\min(\text{Dom}(\mathfrak{A}')) < \min(\text{Dom}(\mathfrak{A}))\), we would have with Theorem 2-56-(i) that \(\mathfrak{A} \subset \mathfrak{A}'\). Then we would have with Theorem 2-54 \(\min(\text{Dom}(\mathfrak{A})) \in \text{Dom}(\mathfrak{A}')) \{ \min(\text{Dom}(\mathfrak{A}')) \}. Thus we would have \(\min(\text{Dom}(\mathfrak{A})) \neq \min(\text{Dom}(\mathfrak{A}'))\). Since \(\mathfrak{A}\) is a closed segment, we would also have that \(\min(\text{Dom}(\mathfrak{A})) \in \text{Dom}(\mathfrak{A}) \cap \text{Dom}(\text{AS}(\mathfrak{F}))\) and thus, according to Definition 2-17, Definition 2-14, Definition 2-15 and Definition 2-16, that \(\min(\text{Dom}(\mathfrak{A})) = \min(\text{Dom}(\mathfrak{A}'))\). Contradiction! Therefore \(\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(\mathfrak{A}'))\). With \(\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset\) and Theorem 2-56-(i) and -(ii), it then follows that \(\mathfrak{A}' \subset \mathfrak{A}\). ■
The next theorem tells us that for every segment $A$ that contains at least one assumption-sentence and in which for every assumption-sentence there is a closed subsegment of $A$ that contains this assumption-sentence there is an AS-comprising segment sequence $G$ for $A$ that enumerates the greatest closed disjunct subsegments of $A$ in such a way that all closed subsegments of $A$ are covered.

Theorem 2-59 will play an important role in the proofs of Theorem 2-67, Theorem 2-68, Theorem 2-69, which are crucial for arriving at a proof of the correctness and completeness of the Speech Act Calculus: With these theorems we can later show that assumptions can be discharged by CdI, NI and PE and only by CdI, NI and PE. Theorem 2-59 itself is essential for showing that CdI, NI and PE can discharge assumptions and thus for the proof of completeness.

**Theorem 2-59. GEN-material-provision theorem**

If $A$ is a segment in $\mathcal{F}$, $\text{AS}(\mathcal{F}) \cap A \neq \emptyset$, and for every $i \in \text{Dom}(A) \cap \text{Dom}(\text{AS}(\mathcal{F}))$ there is a closed segment $B$ in $\mathcal{F}$ such that $(i, \mathcal{F}_i) \in B$ and $B \subseteq A$, then:

There is a $G \in \text{ASCS}(\mathcal{F})$ such that

(i) $G$ is an AS-comprising segment sequence for $A$ in $\mathcal{F}$,

(ii) $\text{Ran}(G) = \{B \mid B \subseteq A \text{ is a closed segment in } \mathcal{F}\}$, and

(iii) $\{\mathcal{F}_i\} \times \text{Ran}(G) \subseteq \{\mathcal{F}_i\} \times \{B \mid B \subseteq A \text{ is a closed segment in } \mathcal{F}\} \subseteq \text{CS}$.

**Proof:** Suppose $A$ is a segment in $\mathcal{F}$, $\text{AS}(\mathcal{F}) \cap A \neq \emptyset$, and for every $i \in \text{Dom}(A) \cap \text{Dom}(\text{AS}(\mathcal{F}))$ there is a closed segment $B$ in $\mathcal{F}$ such that $(i, \mathcal{F}_i) \in B$ and $B \subseteq A$. It follows with Definition 2-1 that $\mathcal{F}_i \in \text{SEQ}$.

Suppose $X = \{B \mid B \subseteq A \text{ and } (\mathcal{F}_i, B) \in \text{CS}\}$ and for all $C \subseteq A$: If $(\mathcal{F}_i, C) \in \text{CS}$ and $B \subseteq C$, then $B = C$. Then it holds that $X \subseteq \text{SG}(\mathcal{F})$. To apply Theorem 2-17 we show that for all $A^*, A' \in X$ with $A^* \neq A'$ it holds, that $A^* \cap A' = \emptyset$. To that end suppose $A^*, A' \in X$ with $A^* \neq A'$. From $A^*, A' \in X$ it follows that $(\mathcal{F}_i, A^*)$, $(\mathcal{F}_i, A') \in \text{CS}$. Theorem 2-57 yields $A^* \cap A' = \emptyset$ or $A^* \subseteq A'$ or $A' \subseteq A^*$. The second and the third alternative lead to a contradiction: Assume $A^* \subseteq A'$. Since $A^* \in X$ we have that for all $C \subseteq A$: If $(\mathcal{F}_i, C) \in \text{CS}$ and $A^* \subseteq C$, then $A^* = C$. Since $A' \in X$ we have $A' \subseteq A$ and $(\mathcal{F}_i, A') \in \text{CS}$. From the last assumption we can derive $A^* = A'$, which contradicts an earlier assumption. From the assumption of $A' \subseteq A^*$ we can analogously derive a contradiction. Hence $A^* \cap A' = \emptyset$.
must be the case. So we have for all $\mathfrak{A}^*, \mathfrak{A}' \in X$ with $\mathfrak{A}^* \neq \mathfrak{A}'$, that $\mathfrak{A}^* \cap \mathfrak{A}' = \emptyset$. With Theorem 2-17 it holds that there is a $G \in \text{SGS}(\mathfrak{A})$ such that $\text{Ran}(G) = X$.

Now we can show that $G$ satisfies conditions (i) to (iii). From (i) it follows that $G \in \text{ASCS}(\mathfrak{A})$. \textit{Ad (i):} We have to show that

a) $G \neq \emptyset$,

b) $\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(G(0)))$, 

c) $\max(\text{Dom}(G(\max(\text{Dom}(G)))))) \leq \max(\text{Dom}(\mathfrak{A}))$, 

d) for all $l \in \text{Dom}(\text{AS}(\mathfrak{A})) \cap \text{Dom}(\mathfrak{A})$ it holds that there is an $i \in \text{Dom}(G)$ such that $l \in \text{Dom}(G(i))$.

By Definition 2-9 it then follows that $G$ is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{A}$. Since $\text{AS}(\mathfrak{A}) \cap \mathfrak{A} \neq \emptyset$ and thus $\text{Dom}(\text{AS}(\mathfrak{A})) \cap \text{Dom}(\mathfrak{A}) \neq \emptyset$, we get a) from d). Furthermore since for every $i \in \text{Dom}(\mathfrak{A}) \cap \text{Dom}(\text{AS}(\mathfrak{A}))$ there is a closed segment $\mathfrak{B}$ in $\mathfrak{A}$ such that $(i, \mathfrak{B}) \in \mathfrak{A}$ and $\mathfrak{B} \subseteq \mathfrak{A}$, both d) and a) follow from

e) for all $\mathfrak{B} \subseteq \mathfrak{A}$ with $(\mathfrak{A}, \mathfrak{B}) \in \text{CS}$: There is an $i \in \text{Dom}(G)$, such that $\mathfrak{B} \subseteq G(i)$.

\textit{Ad e):} Suppose $\mathfrak{B} \subseteq \mathfrak{A}$ with $(\mathfrak{A}, \mathfrak{B}) \in \text{CS}$, such that there is no $i \in \text{Dom}(G)$ with $\mathfrak{B} \subseteq G(i)$. Suppose $k = \min(\{j \mid \text{There is a } \mathfrak{C} \subseteq \mathfrak{A} \text{ with } (\mathfrak{A}, \mathfrak{C}) \in \text{CS}, \text{ such that there is no } i \in \text{Dom}(G) \text{ with } \mathfrak{C} \subseteq G(i), \text{ and } j = \min(\text{Dom}(\mathfrak{C}))\})$. Then there is a $\mathfrak{C} \subseteq \mathfrak{A}$ with $(\mathfrak{A}, \mathfrak{C}) \in \text{CS}$, such that there is no $i \in \text{Dom}(G)$ with $\mathfrak{C} \subseteq G(i)$, and $k = \min(\text{Dom}(\mathfrak{C}))$. Now suppose $\mathfrak{C}' \subseteq \mathfrak{A}$ and $(\mathfrak{A}, \mathfrak{C}') \in \text{CS}$ and $\mathfrak{C} \subseteq \mathfrak{C}'$. Then we have $\min(\text{Dom}(\mathfrak{C}')) \leq k$. From that it follows that there is no $i \in \text{Dom}(G)$, such that $\mathfrak{C}' \subseteq G(i)$, else it would also hold that $\mathfrak{C} \subseteq G(i)$ for the same $i$. Since $k$ is minimal, we get $\min(\text{Dom}(\mathfrak{C}')) = k$. With Theorem 2-50 we can derive that $\mathfrak{C} = \mathfrak{C}'$. Hence for all $\mathfrak{C}' \subseteq \mathfrak{A}$ with $(\mathfrak{A}, \mathfrak{C}') \in \text{CS}$ and $\mathfrak{C} \subseteq \mathfrak{C}'$ we get $\mathfrak{C} = \mathfrak{C}'$. Therefore $\mathfrak{C} \in X$ and by that there is an $i \in \text{Dom}(G)$, such that $\mathfrak{C} = G(i)$. Contradiction! Thus for all $\mathfrak{B} \subseteq \mathfrak{A}$ with $(\mathfrak{A}, \mathfrak{B}) \in \text{CS}$ there is an $i \in \text{Dom}(G)$, such that $\mathfrak{B} \subseteq G(i)$.

\textit{Ad b):} For all $\mathfrak{B} \in \text{Ran}(G) = X$ it holds that $\mathfrak{B} \subseteq \mathfrak{A}$. Because of $G \neq \emptyset$ we get $G(0) \in \text{Ran}(G) = X$ and thereby $G(0) \subseteq \mathfrak{A}$. Hence $\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(G(0)))$. \textit{Ad c):} With $G \neq \emptyset$ we get $\max(\text{Dom}(G)) \in \text{Dom}(G)$ and thereby $G(\max(\text{Dom}(G))) \in \text{Ran}(G) = X$. Hence $\max(\text{Dom}(G(\max(\text{Dom}(G)))))) \leq \max(\text{Dom}(\mathfrak{A}))$. 

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Ad (ii): Suppose \((i, \bar{\alpha}_i) \in \cup \text{Ran}(G)\). Therefore \((i, \bar{\alpha}_i) \in \cup X\). Hence we have a \(\mathcal{B} \in X\) with \((i, \bar{\alpha}_i) \in \mathcal{B}\). From that we can infer \(\mathcal{B} \subseteq \mathcal{A}\) and \((\bar{\alpha}, \mathcal{B}) \in \mathcal{CS}\). Thus \(\mathcal{B} \in \{\mathcal{B} \mid \mathcal{B} \subseteq \mathcal{A}\text{ is a closed segment in } \bar{\alpha}\}\) and \((i, \bar{\alpha}_i) \in \cup \{\mathcal{B} \mid \mathcal{B} \subseteq \mathcal{A}\text{ is a closed segment in } \bar{\alpha}\}\). From e) we get vice versa \(\cup \{\mathcal{B} \mid \mathcal{B} \subseteq \mathcal{A}\text{ is a closed segment in } \bar{\alpha}\} \subseteq \cup \text{Ran}(G)\).

Ad (iii): (iii) follows from the definition of \(X\) and \(\text{Ran}(G) = X\). ■

Theorem 2-60. If all members of an AS-comprising segment sequence for \(\mathcal{A}\) are closed segments, then every closed subsegment of \(\mathcal{A}\) is a subsegment of a sequence member

If \(\bar{\alpha} \subseteq \text{SEQ}, \mathcal{A} \in \text{SG}(\bar{\alpha})\) and \(G \in \text{ASCS}(\bar{\alpha})\) is an AS-comprising segment sequence for \(\mathcal{A}\) in \(\bar{\alpha}\) and \(\{\bar{\alpha}\} \times \text{Ran}(G) \subseteq \mathcal{CS}\), then for all \(\mathcal{C}\): If \(\mathcal{C} \subseteq \mathcal{A}\) is a closed segment in \(\bar{\alpha}\), then there is an \(i \in \text{Dom}(G)\) such that \(\mathcal{C} \subseteq G(i)\).

Proof: Suppose \(\bar{\alpha} \subseteq \text{SEQ}, \mathcal{A} \in \text{SG}(\bar{\alpha})\) and \(G \in \text{ASCS}(\bar{\alpha})\) is an AS-comprising segment sequence for \(\mathcal{A}\) in \(\bar{\alpha}\) and \(\{\bar{\alpha}\} \times \text{Ran}(G) \subseteq \mathcal{CS}\). Now, suppose \(\mathcal{C} \subseteq \mathcal{A}\) is a closed segment in \(\bar{\alpha}\). With Definition 2-11 to Definition 2-13 and Theorem 2-42, we then have \(\min(\text{Dom}(\mathcal{C})) \in \text{Dom}(\text{AS}(\bar{\alpha}) \cap \mathcal{A})\). According to Definition 2-9-(iii-c), there is thus an \(i \in \text{Dom}(G)\) such that \(\min(\text{Dom}(\mathcal{C})) \in \text{Dom}(G(i))\). By hypothesis, we have \((\bar{\alpha}, G(i)) \in \mathcal{CS}\). It then follows with Theorem 2-52 that \(\mathcal{C} \subseteq G(i)\). ■

Up to now, we have primarily proved theorems that hold for all closed segments. Later, we will also be interested in those properties of closed segments that depend on whether they are the result of the application of conditional introduction (CdI-closed) or negation introduction (NI-closed) or particular-quantifier elimination (PE-closed). Accordingly, we will now define different predicates for these kinds of closed segments. We will then have that every closed segment belongs to one of these kinds (Theorem 2-61).

Definition 2-23. CdI-closed segment

\(\mathcal{A}\) is a CdI-closed segment in \(\bar{\alpha}\)

iff

\(\mathcal{A}\) is a closed segment and a CdI-like segment in \(\bar{\alpha}\).
Definition 2-24. *NI-closed segment*

\( \mathfrak{A} \) is an \( NI \)-closed segment in \( \mathfrak{H} \)

iff

\( \mathfrak{A} \) is a closed segment and an \( NI \)-like segment in \( \mathfrak{H} \).

Definition 2-25. *PE-closed segment*

\( \mathfrak{A} \) is a PE-closed segment in \( \mathfrak{H} \)

iff

\( \mathfrak{A} \) is a closed segment and an RA-like segment in \( \mathfrak{H} \).

Theorem 2-61. *CdI-, NI- and PE-closed segments and only these are closed segments*

\( \mathfrak{A} \) is a closed segment in \( \mathfrak{H} \)

iff

\( \mathfrak{A} \) is a CdI- or NI- or PE-closed segment in \( \mathfrak{H} \).

*Proof:* Follows from Definition 2-22, Definition 2-23, Definition 2-24, Definition 2-25 and Theorem 2-42. \( \blacksquare \)

Theorem 2-62. *Monotony of \((F\text{-})closed segment\)'-predicates*

If \( \mathfrak{H}, \mathfrak{H}' \in \text{SEQ} \) and \( \mathfrak{H} \subseteq \mathfrak{H}' \), then:

(i) If \( \mathfrak{A} \) is a CdI-closed segment in \( \mathfrak{H} \), then \( \mathfrak{A} \) is a CdI-closed segment in \( \mathfrak{H}' \),

(ii) If \( \mathfrak{A} \) is an NI-closed segment in \( \mathfrak{H} \), then \( \mathfrak{A} \) is an NI-closed segment in \( \mathfrak{H}' \),

(iii) If \( \mathfrak{A} \) is a PE-closed segment in \( \mathfrak{H} \), then \( \mathfrak{A} \) is a PE-closed segment in \( \mathfrak{H}' \),

(iv) If \( \mathfrak{A} \) is a minimal CdI-closed segment in \( \mathfrak{H} \), then \( \mathfrak{A} \) is a minimal CdI-closed segment in \( \mathfrak{H}' \),

(v) If \( \mathfrak{A} \) is a minimal NI-closed segment in \( \mathfrak{H} \), then \( \mathfrak{A} \) a minimal NI-closed segment in \( \mathfrak{H}' \),

(vi) If \( \mathfrak{A} \) is a minimal PE-closed segment in \( \mathfrak{H} \), then \( \mathfrak{A} \) is a minimal PE-closed segment in \( \mathfrak{H}' \),

(vii) If \( \mathfrak{A} \) is a minimal closed segment in \( \mathfrak{H} \), then \( \mathfrak{A} \) is a minimal closed segment in \( \mathfrak{H}' \), and

(viii) If \( \mathfrak{A} \) is a closed segment in \( \mathfrak{H} \), then \( \mathfrak{A} \) is a closed segment in \( \mathfrak{H}' \).

*Proof:* See Remark 2-1. \( \blacksquare \)
Theorem 2-63. **Closed segments in the first sequence of a concatenation remain closed**

If $\mathfrak{y}_1', \mathfrak{y}_1 \in \text{SEQ}$, then:

(i) If $\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{y}_1$, then $\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{y}_1 \cap \mathfrak{y}_1'$,

(ii) If $\mathfrak{A}$ is an NI-closed segment in $\mathfrak{y}_1$, then $\mathfrak{A}$ is an NI-closed segment in $\mathfrak{y}_1 \cap \mathfrak{y}_1'$,

(iii) If $\mathfrak{A}$ is a PE-closed segment in $\mathfrak{y}_1$, then $\mathfrak{A}$ is a PE-closed segment in $\mathfrak{y}_1 \cap \mathfrak{y}_1'$, and

(iv) If $\mathfrak{A}$ is a closed segment in $\mathfrak{y}_1$, then $\mathfrak{A}$ is a closed segment in $\mathfrak{y}_1 \cap \mathfrak{y}_1'$.

**Proof:** Follows with $\mathfrak{y}_1 \subseteq \mathfrak{y}_1 \cap \mathfrak{y}_1'$ and Theorem 2-62-(i), -(ii), -(iii) and -(viii). ■

Theorem 2-64. **($F$-)closed segments in restrictions**

If $\mathfrak{y}_1$ is a sequence, then:

(i) $\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{y}_1$ iff $\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{y}_1|\text{max}(\text{Dom}(\mathfrak{A}))+1$,

(ii) $\mathfrak{A}$ is an NI-closed segment in $\mathfrak{y}_1$ iff $\mathfrak{A}$ is an NI-closed segment in $\mathfrak{y}_1|\text{max}(\text{Dom}(\mathfrak{A}))+1$,

(iii) $\mathfrak{A}$ is a PE-closed segment in $\mathfrak{y}_1$ iff $\mathfrak{A}$ is a PE-closed segment in $\mathfrak{y}_1|\text{max}(\text{Dom}(\mathfrak{A}))+1$,

(iv) $\mathfrak{A}$ is a minimal CdI-closed segment in $\mathfrak{y}_1$ iff $\mathfrak{A}$ is a minimal CdI-closed segment in $\mathfrak{y}_1|\text{max}(\text{Dom}(\mathfrak{A}))+1$,

(v) $\mathfrak{A}$ is a minimal NI-closed segment in $\mathfrak{y}_1$ iff $\mathfrak{A}$ is a minimal NI-closed segment in $\mathfrak{y}_1|\text{max}(\text{Dom}(\mathfrak{A}))+1$,

(vi) $\mathfrak{A}$ is a minimal PE-closed segment in $\mathfrak{y}_1$ iff $\mathfrak{A}$ is a minimal PE-closed segment in $\mathfrak{y}_1|\text{max}(\text{Dom}(\mathfrak{A}))+1$,

(vii) $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{y}_1$ iff $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{y}_1|\text{max}(\text{Dom}(\mathfrak{A}))+1$, and

(viii) $\mathfrak{A}$ is a closed segment in $\mathfrak{y}_1$ iff $\mathfrak{A}$ is a closed segment in $\mathfrak{y}_1|\text{max}(\text{Dom}(\mathfrak{A}))+1$.

**Proof:** See Remark 2-2. ■

Theorem 2-65. **Preparatory theorem for Theorem 2-67, Theorem 2-68 and Theorem 2-69**

If $\mathfrak{y}_1$ is a segment in $\mathfrak{y}_1$ and if it holds for all closed segments $\mathfrak{B}$ in $\mathfrak{y}_1|\text{max}(\text{Dom}(\mathfrak{A}))$ that $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{B}))$ or $\max(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(\mathfrak{A}))$, then for all $i \in \text{Dom}(\mathfrak{A})$:

(i) $\mathfrak{A}|i$ is not a closed segment in $\mathfrak{y}_1$, and

(ii) There is no $G \in \text{ASCS}(\mathfrak{y}_1)$ such that $\{\mathfrak{y}_1\} \times \text{Ran}(G) \subseteq \text{CS}$ and $\mathfrak{A}|i \in \text{PGEN}(\langle \mathfrak{y}_1, G \rangle)$.

**Proof:** Suppose $\mathfrak{A}$ is a segment in $\mathfrak{y}_1$ and suppose it holds for all closed segments $\mathfrak{B}$ in $\mathfrak{y}_1|\text{max}(\text{Dom}(\mathfrak{A}))$ that $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{B}))$ or $\max(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(\mathfrak{A}))$.

Next, suppose $i \in \text{Dom}(\mathfrak{A})$. First, we have $\mathfrak{y}_1 \in \text{SEQ}$. Ad (i): Suppose for contradiction that $\mathfrak{A}|i$ is a closed segment in $\mathfrak{y}_1$. With Theorem 2-64-(viii), we would then have that $\mathfrak{A}|i$ is a closed segment in $\mathfrak{y}_1|i$. Furthermore, we have $i \leq \text{max}(\text{Dom}(\mathfrak{A}))$ and hence $\mathfrak{y}_1|i \subseteq \mathfrak{y}_1|\text{max}(\text{Dom}(\mathfrak{A}))$ and thus it holds with Theorem 2-62-(viii) that $\mathfrak{A}|i$ is a closed segment.
The Availability of Propositions in $\Delta_{\max(\text{Dom}(\mathfrak{A}))}$. With Theorem 2-7, we have that $\min(\text{Dom}(\mathfrak{A}|i)) = \min(\text{Dom}(\mathfrak{A}))$ and hence, with Theorem 2-31, that neither $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}|i))$ nor $\max(\text{Dom}(\mathfrak{A}|i)) \leq \min(\text{Dom}(\mathfrak{A}))$, which contradicts the hypothesis.

Ad (ii): Suppose for contradiction that there is a $G \in \text{ASCS}(\mathfrak{A})$ such that $\{\mathfrak{A}\} \times \text{Ran}(G) \subseteq CS$ and $\mathfrak{A}|i \in \text{PGEN}(\langle \mathfrak{A}, G \rangle)$. Now, suppose $j = \min(\{i \mid i \in \text{Dom}(\mathfrak{A}) \text{ and there is } G \in \text{ASCS}(\mathfrak{A}) \text{ such that } \{\mathfrak{A}\} \times \text{Ran}(G) \subseteq CS \text{ and } \mathfrak{A}|i \in \text{PGEN}(\langle \mathfrak{A}, G \rangle)\})$. Then there is a $G^* \in \text{ASCS}(\mathfrak{A})$ such that $\{\mathfrak{A}\} \times \text{Ran}(G^*) \subseteq CS$ and $\mathfrak{A}|j \in \text{PGEN}(\langle \mathfrak{A}, G^* \rangle)$. Now, suppose for contradiction that there are a $k \in \text{Dom}(\mathfrak{A}|j)$ and an $l \in \text{Dom}(G^*)$ such that $\mathfrak{A}|k \in \text{PGEN}(\langle \mathfrak{A}, G^*|\langle l+1 \rangle \rangle)$. According to Theorem 2-25, $\mathfrak{A}|(l+1)$ is then an AS-comprising segment sequence for $\mathfrak{A}|\max(\text{Dom}(G^*(l)))+1$. According to Definition 2-10, we then have that $G^*|\langle l+1 \rangle \in \text{ASCS}(\mathfrak{A})$ and, by hypothesis, that $\mathfrak{A}|k \in \text{PGEN}(\langle \mathfrak{A}, G^*|\langle l+1 \rangle \rangle)$. On the other hand, we also have $k < j$. Thus, we have a contradiction to the minimality of $j$. Therefore there are no $k \in \text{Dom}(\mathfrak{A}|j)$ and $l \in \text{Dom}(G^*)$ such that $\mathfrak{A}|k \in \text{PGEN}(\langle \mathfrak{A}, G^*|\langle l+1 \rangle \rangle)$. According to Definition 2-19, we then have that $\mathfrak{A}|j \in \text{GEN}(\langle \mathfrak{A}, G^* \rangle)$ and thus, with $\{\mathfrak{A}\} \times \text{Ran}(G^*) \subseteq CS$ and Theorem 2-41, that $(\mathfrak{A}, \mathfrak{A}|j) \in CS$ and therefore that $\mathfrak{A}|j$ is a closed segment in $\mathfrak{A}$, which contradicts (i).

We close ch. 2.2 with four theorems that provide the basis for the proof of the correctness and the completeness of the Speech Act Calculus. With these theorems we can later show that CdI, NI and PE and only CdI, NI and PE can generate CdI-, NI- and PE-closed segments and thus any closed segments.
Theorem 2-66. Every closed segment is a minimal closed segment or a CdI- or NI- or PE-closed segment whose assumption-sentences lie at the beginning or in a proper closed sub-segment.

If $\mathfrak{A}$ is a closed segment in $\mathfrak{S}$, then:

(i) $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{S}$

or

(ii) $\mathfrak{A}$ is a CdI- or NI- or PE-closed segment in $\mathfrak{S}$, where for all $i \in \text{Dom}(\text{AS}(\mathfrak{S})) \cap \text{Dom}(\mathfrak{A})$ with $\min(\text{Dom}(\mathfrak{A})) < i$ it holds that there is a $\mathfrak{B}$ such that

a) $(i, \mathfrak{A}_i) \in \mathfrak{B}$,
b) $\mathfrak{B}$ is a closed segment in $\mathfrak{S}$,
c) $i = \min(\text{Dom}(\mathfrak{B}))$, and
d) $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{B})) < \max(\text{Dom}(\mathfrak{B})) < \max(\text{Dom}(\mathfrak{A}))$.

Proof: Follows from Definition 2-22, Definition 2-23, Definition 2-24, Definition 2-25 and Theorem 2-48.

Theorem 2-67. Lemma for Theorem 2-91

$\mathfrak{A}$ is a segment in $\mathfrak{S}$ and there are $\Delta, \Gamma \in \text{CFORM}$ such that

(i) $\mathfrak{S}_{\min(\text{Dom}(\mathfrak{S}))} = \{\text{"Suppose } \Delta\}$,

(ii) For all closed segments $\mathfrak{B}$ in $\mathfrak{S}_{\max(\text{Dom}(\mathfrak{A}))}: \min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{B}))$ or $\max(\text{Dom}(\mathfrak{B})) \leq \min(\text{Dom}(\mathfrak{A}))$,

(iii) $\mathfrak{P}(\mathfrak{S}_{\max(\text{Dom}(\mathfrak{S}))-1}) = \Gamma$,

(iv) For every $r \in \text{Dom}(\text{AS}(\mathfrak{S})) \cap \text{Dom}(\mathfrak{A})$ with $\min(\text{Dom}(\mathfrak{A})) < r \leq \max(\text{Dom}(\mathfrak{A}))$-1 there is a closed segment $\mathfrak{B}$ in $\mathfrak{S}_{\max(\text{Dom}(\mathfrak{A}))}$ such that $(r, \mathfrak{A}_r) \in \mathfrak{B}$, and

(v) $\mathfrak{S}_{\max(\text{Dom}(\mathfrak{S}))} = \{\text{"Therefore } \Delta \rightarrow \Gamma\}$,

iff $\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{S}$.

Proof: (L-R): Let $\mathfrak{S}$ and $\mathfrak{A}$ satisfy the requirements and let $\Delta$ and $\Gamma$ be as demanded. First, we have $\mathfrak{S} \in \text{SEQ}$. With Definition 2-11, we have that $\mathfrak{A}$ is a CdI-like segment in $\mathfrak{S}$.

Also, from clause (ii) of our hypothesis and Theorem 2-65-(i), it follows for all $k \in \text{Dom}(\mathfrak{A})$ that $\mathfrak{A} \upharpoonright k$ is not a closed segment in $\mathfrak{S}$.

We have that $\text{AS}(\mathfrak{S}) \cap \mathfrak{A} = \{(\min(\text{Dom}(\mathfrak{A})), \mathfrak{S}_{\min(\text{Dom}(\mathfrak{S}))})\}$ or that there is an $i \in \text{Dom}(\text{AS}(\mathfrak{S})) \cap \text{Dom}(\mathfrak{A})$ with $\min(\text{Dom}(\mathfrak{A})) < i \leq \max(\text{Dom}(\mathfrak{A}))$-1.

Now, suppose $\text{AS}(\mathfrak{S}) \cap \mathfrak{A} = \{(\min(\text{Dom}(\mathfrak{A})), \mathfrak{S}_{\min(\text{Dom}(\mathfrak{S}))})\}$. Because we have for all $k \in \text{Dom}(\mathfrak{A})$ that $\mathfrak{A} \upharpoonright k$ is not a closed segment in $\mathfrak{S}$, we have, with Theorem 2-32, that $\mathfrak{A}$ is a
minimal closed and thus a closed segment in $\mathcal{Y}$. Since $\mathcal{A}$ is a Cdl-like segment in $\mathcal{Y}$, $\mathcal{A}$ is thus a Cdl-closed segment in $\mathcal{Y}$.

Now, suppose there is an $i \in \text{Dom}(\text{AS}(\mathcal{Y})) \cap \text{Dom}(\mathcal{A})$ with $\min(\text{Dom}(\mathcal{A})) < i \leq \max(\text{Dom}(\mathcal{A}))-1$. Now, let $\mathcal{E} = \{(l, \mathcal{Y}) \mid \min(\text{Dom}(\mathcal{A}))+1 \leq l \leq \max(\text{Dom}(\mathcal{A}))-1\}$. Then $\mathcal{E}$ is a segment in $\mathcal{Y}$ and $i \in \text{Dom}(\text{AS}(\mathcal{Y})) \cap \text{Dom}(\mathcal{A})$. Also, for every $r \in \text{Dom}(\text{AS}(\mathcal{Y})) \cap \text{Dom}(\mathcal{E})$ there is a closed segment $\mathcal{B}$ in $\mathcal{Y}$ such that $(r, \mathcal{Y}) \in \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{E}$. To see this, suppose $r \in \text{Dom}(\text{AS}(\mathcal{Y})) \cap \text{Dom}(\mathcal{E})$. Then we have $\min(\text{Dom}(\mathcal{A})) < r \leq \max(\text{Dom}(\mathcal{A}))-1$. According to clause (iv) of our hypothesis, there is thus a closed segment $\mathcal{B}$ in $\mathcal{Y}|\max(\text{Dom}(\mathcal{A}))$ such that $(r, \mathcal{Y}) \in \mathcal{B}$. Then we have $\min(\text{Dom}(\mathcal{E})) \leq \min(\text{Dom}(\mathcal{B}))$, because otherwise we would have $\min(\text{Dom}(\mathcal{B})) \leq \min(\text{Dom}(\mathcal{A})) < r \leq \max(\text{Dom}(\mathcal{B}))$, which contradicts clause (ii). From $\mathcal{B}$ being a segment in $\mathcal{Y}|\max(\text{Dom}(\mathcal{A}))$, we then have $\max(\text{Dom}(\mathcal{B})) \leq \max(\text{Dom}(\mathcal{A}))-1 = \max(\text{Dom}(\mathcal{E}))$. With Theorem 2-5, we hence have $\mathcal{B} \subseteq \mathcal{E}$.

Thus $\mathcal{E}$ satisfies the requirements of Theorem 2-59. Therefore there is a $G \in \text{ASCS}(\mathcal{Y})$ such that $G$ is an AS-comprising segment sequence for $\mathcal{E}$ in $\mathcal{Y}$ and $\{\mathcal{Y}\} \times \text{Ran}(G) \subseteq \text{CS}$. According to the definition of $\mathcal{E}$, we have $\mathcal{E} \in \text{SG}(\mathcal{Y})$ and $\min(\text{Dom}(\mathcal{A}))+1 = \min(\text{Dom}(\mathcal{E}))$ and $\max(\text{Dom}(\mathcal{A})) = \max(\text{Dom}(\mathcal{E}))+1$ and $\text{AS}(\mathcal{Y}) \cap \mathcal{E} \neq \emptyset$. We also have that $\mathcal{A}$ is a Cdl-like segment in $\mathcal{Y}$. It thus holds with Theorem 2-28 that $\mathcal{A}$ is not an NI-like segment in $\mathcal{Y}$. Furthermore, we have that it holds for all $i \in \text{Dom}(\mathcal{A})$ that $\mathcal{A}|i$ is not a closed segment in $\mathcal{Y}$. Thus we also have for all $i \in \text{Dom}(\mathcal{A})$ that $\mathcal{A}|i$ is not a minimal closed segment in $\mathcal{Y}$.

According to Definition 2-18, we thus have $\mathcal{A} \in \text{PGEN}(\langle \mathcal{Y}, G \rangle)$. Now, suppose for contradiction that there are $k \in \text{Dom}(\mathcal{A})$ and $l \in \text{Dom}(G)$ such that $\mathcal{A}|k \in \text{PGEN}(\langle \mathcal{Y}, G|l+1 \rangle)$. According to Theorem 2-25, $G|l+1$ is an AS-comprising segment sequence for $\mathcal{A}|\max(\text{Dom}(G(l)))+1$, and thus, with Definition 2-10, we have $G|l+1 \in \text{ASCS}(\mathcal{Y})$. By hypothesis, we have $\mathcal{A}|k \in \text{PGEN}(\langle \mathcal{Y}, G|l+1 \rangle)$ and we have $\mathcal{Y} \in \text{SEQ}$ and $\{\mathcal{Y}\} \times \text{Ran}(G|l+1) \in \mathcal{Y} \times \text{Ran}(G) \subseteq \text{CS}$. Altogether, we would thus have a contradiction to Theorem 2-65-(ii). Therefore there are no $k \in \text{Dom}(\mathcal{A})$ and $l \in \text{Dom}(G)$ such that $\mathcal{A}|k \in \text{PGEN}(\langle \mathcal{Y}, G|l+1 \rangle)$. According to Definition 2-19, we thus have $\mathcal{A} \in \text{GEN}(\langle \mathcal{Y}, G \rangle)$.

Since $\{\mathcal{Y}\} \times \text{Ran}(G) \subseteq \text{CS}$, it thus follows with Theorem 2-41 that $\langle \mathcal{Y}, \mathcal{A} \rangle \in \text{CS}$. Hence
Á is a closed segment in Ñ and a CdI-like segment in Ñ and thus a CdI-closed segment in Ñ.

(R-L): Now, suppose Á is a CdI-closed segment in Ñ. Then Á is a closed segment and a CdI-like segment in Ñ. From Á being a CdI-like segment in Ñ it then follows that there are Δ, Γ ∈ CFORM such that (i), (iii) and (v) are satisfied. With Theorem 2-48, we also have that (iv) holds. (If Á is a minimal closed segment, (iv) holds trivially.)

Now, suppose Ñ is a CdI-closed segment in Ñ. Then is a closed segment and a CdI-like segment in Ñ. From being a CdI-like segment in Ñ it then follows that there are Δ, Γ ∈ CFORM such that (i), (iii) and (v) are satisfied. With Theorem 2-48, we also have that (iv) holds. (If Á is a minimal closed segment, (iv) holds trivially.)

Now, suppose is a closed segment in Ñ max(Dom(Á)). Suppose min(Dom(Ñ)) ≤ min(Dom(Á)) and min(Dom(Á)) < max(Dom(Ñ)). Then we would have min(Dom(Á)) ∈ Dom(Ñ) and hence | ∩ | ≠ Δ and min(Dom(Ñ)) ≤ min(Dom(Á)). With Theorem 2-56-(i) and -(ii), we would thus have Á ⊆ Ñ. But then we would have Á ⊆ Ñ ⊆ Ñ max(Dom(Á)) and hence max(Dom(Á)) ∉ Dom(Á) ≠ Δ. Contradiction! Therefore we have min(Dom(Á)) < min(Dom(Ñ)) or max(Dom(Ñ)) ≤ min(Dom(Á)). Therefore we also have (iii).

\[ \Box \]

Theorem 2-68. Lemma for Theorem 2-92
Á is a segment in Ñ and there are Δ, Γ ∈ CFORM and i ∈ Dom(Ñ) such that
(i) \( \min(Dom(Á)) \leq i < \max(Dom(Á)) \),
(ii) \( \bar{\mathcal{N}}_{\min(Dom(Á))} = "\text{Suppose } \Delta" \),
(iii) For all closed segments Ñ in Ñ max(Dom(Á)): min(Dom(Á)) < min(Dom(Ñ)) or max(Dom(Ñ)) ≤ min(Dom(Á)),
(iv) \( P(\bar{\mathcal{N}}_{\min(Dom(Á))}) = \Gamma \) and \( P(\bar{\mathcal{N}}_{\max(Dom(Á))}) = \neg\Gamma \)
or
\( P(\bar{\mathcal{N}}_{\min(Dom(Á))}) = \neg\Gamma \) and \( P(\bar{\mathcal{N}}_{\max(Dom(Á))}) = \Gamma \),
(v) For all closed segments Ñ in Ñ max(Dom(Á)): \( i < \min(Dom(Ñ)) \) or \( \max(Dom(Ñ)) \leq i \),
(vi) For every \( r \in Dom(AS(Ñ)) \cap Dom(Á) \) with \( \min(Dom(Á)) < r \leq \max(Dom(Á)) \) there is a closed segment Ñ in Ñ max(Dom(Á)) such that \( (r, \bar{\mathcal{N}}_r) \in \bar{\mathcal{N}} \), and
(vii) \( \bar{\mathcal{N}}_{\max(Dom(Á))} = "\text{Therefore } \neg\Delta" \)
\[ \text{iff} \]
Á is an NI-closed segment in Ñ.

Proof: (L-R): Let Ñ and Á satisfy the requirements and let Δ, Γ and i be as demanded. First, we have Ñ ∈ SEQ. With Definition 2-12, we have that Á is an NI-like segment in Ñ. Also, from clause (iii) of our hypothesis and Theorem 2-65-(i), it follows for all \( k \in \text{Dom}(Á) \) that Á|k is not a closed segment in Ñ.
We have that $\text{AS}(\mathcal{A}) \cap \mathcal{A} = \{(\min(\text{Dom}(\mathcal{A})), \mathcal{A}_{\min(\text{Dom}(\mathcal{A}))})\}$ or that there is an $i \in \text{Dom}(\text{AS}(\mathcal{A})) \cap \text{Dom}(\mathcal{A})$ with $\min(\text{Dom}(\mathcal{A})) < i \leq \max(\text{Dom}(\mathcal{A})) - 1$.

Now, suppose $\text{AS}(\mathcal{A}) \cap \mathcal{A} = \{(\min(\text{Dom}(\mathcal{A})), \mathcal{A}_{\min(\text{Dom}(\mathcal{A}))})\}$. Because we have for all $k \in \text{Dom}(\mathcal{A})$ that $\mathcal{A}|k$ is not a closed segment in $\mathcal{A}$, we have, with Theorem 2-32, that $\mathcal{A}$ is a minimal closed and thus a closed segment in $\mathcal{A}$. Since $\mathcal{A}$ is an NI-like segment in $\mathcal{A}$, $\mathcal{A}$ is thus an NI-closed segment in $\mathcal{A}$.

Now, suppose there is an $s \in \text{Dom}(\text{AS}(\mathcal{A})) \cap \text{Dom}(\mathcal{A})$ with $\min(\text{Dom}(\mathcal{A})) < s \leq \max(\text{Dom}(\mathcal{A})) - 1$. Now, let $\mathcal{C} = \{(l, \mathcal{A}_l) \mid \min(\text{Dom}(\mathcal{A}))+1 \leq l \leq \max(\text{Dom}(\mathcal{A}))-1\}$. Then we have that $\mathcal{C}$ is a segment in $\mathcal{A}$ and $s \in \text{Dom}(\text{AS}(\mathcal{A})) \cap \text{Dom}(\mathcal{C})$. Also, there is for every $r \in \text{Dom}(\text{AS}(\mathcal{A})) \cap \text{Dom}(\mathcal{C})$ a closed segment $\mathcal{B}$ in $\mathcal{A}$ such that $(r, \mathcal{A}_r) \in \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{C}$. To see this, suppose $r \in \text{Dom}(\text{AS}(\mathcal{A})) \cap \text{Dom}(\mathcal{C})$. Then we have $\min(\text{Dom}(\mathcal{A})) < r \leq \max(\text{Dom}(\mathcal{A})) - 1$ and hence there is, according to clause (vi), a closed segment $\mathcal{B}$ in $\mathcal{A}|\max(\text{Dom}(\mathcal{A}))$ such that $(r, \mathcal{A}_r) \in \mathcal{B}$. Then we have $\min(\text{Dom}(\mathcal{C})) = \min(\text{Dom}(\mathcal{A})) < r \leq \max(\text{Dom}(\mathcal{A}))$, which contradicts clause (iii). It also follows from $\mathcal{B}$ being a segment in $\mathcal{A}|\max(\text{Dom}(\mathcal{A}))$ that $\max(\text{Dom}(\mathcal{B})) \leq \max(\text{Dom}(\mathcal{A})) - 1 = \max(\text{Dom}(\mathcal{C}))$. With Theorem 2-5, we therefore have $\mathcal{B} \subseteq \mathcal{C}$.

Thus $\mathcal{C}$ satisfies the conditions of Theorem 2-59. Therefore there is a $G \in \text{ASCS}(\mathcal{A})$ such that $G$ is an AS-comprising segment sequence for $\mathcal{C}$ in $\mathcal{A}$ and $\{\mathcal{A}\} \times \text{Ran}(G) \subseteq \{\mathcal{A}\} \times \{\mathcal{C}^* \mid \mathcal{C}^* \subseteq \mathcal{C} \text{ is a closed segment in } \mathcal{A}\} \subseteq \{\mathcal{A}\} \times \{\mathcal{C}^* \mid \mathcal{C}^* \subseteq \mathcal{A} \text{ is a closed segment in } \mathcal{A}\} \subseteq \text{CS}$. According to the definition of $\mathcal{C}$, we have that $\mathcal{C} \in \text{SG}(\mathcal{A})$ and that $\min(\text{Dom}(\mathcal{A}))+1 = \min(\text{Dom}(\mathcal{C}))$ and $\max(\text{Dom}(\mathcal{A})) = \max(\text{Dom}(\mathcal{C}))+1$ and we have that $\mathcal{A}$ is an NI-like segment in $\mathcal{A}$. Also, we have for all $r \in \text{Dom}(G)$: $i < \min(\text{Dom}(G(r)))$ or $\max(\text{Dom}(G(r))) \leq i$. To see this, suppose $r \in \text{Dom}(G)$. Then we have $G(r) \subseteq \mathcal{C}$ is a closed segment in $\mathcal{A}|\max(\text{Dom}(\mathcal{A}))$. By clause (v), then have $i < \min(\text{Dom}(G(r)))$ or $\max(\text{Dom}(G(r))) \leq i$. Furthermore, because for all $i \in \text{Dom}(\mathcal{A})$ it holds that $\mathcal{A}|i$ is not a closed segment in $\mathcal{A}$, we also have that for all $i \in \text{Dom}(\mathcal{A})$ it holds that $\mathcal{A}|i$ is not a minimal closed segment in $\mathcal{A}$.

Thus, according to Definition 2-18, we have $\mathcal{A} \in \text{PGEN}((\mathcal{A}, G))$. Now, suppose for contradiction that there are a $k \in \text{Dom}(\mathcal{A})$ and an $l \in \text{Dom}(G)$ such that $\mathcal{A}|k \in \text{PGEN}((\mathcal{A}, G|l+1)))$. According to Theorem 2-25, $G|l+1)$ is an AS-comprising segment sequence
for $\forall l \mid \max(Dom(G(l))) + 1$ and thus we have, according to Definition 2-10, that $G \mid (l+1) \in \text{ASCS}(\mathcal{H})$. By hypothesis, we have $\forall l \mid k \in \text{PGEN}(\langle \mathcal{H}, G \mid (l+1) \rangle)$. On the other hand, we have $k \in \text{SEQ}$ and $\{k\} \times \text{Ran}(G) \subseteq \{k\} \times \text{Ran}(G) \subseteq \text{CS}$. Altogether, we would thus have a contradiction to Theorem 2-65-(ii). Therefore there are no $k \in \text{Dom}(\mathcal{H})$ and $l \in \text{Dom}(G)$ such that $\forall l \mid k \in \text{PGEN}(\langle \mathcal{H}, G \mid (l+1) \rangle)$. According to Definition 2-19, we thus have $\mathcal{H} \in \text{GEN}(\langle \mathcal{H}, G \rangle)$ and thus with $\{k\} \times \text{Ran}(G) \subseteq \text{CS}$ and Theorem 2-41 ($\mathcal{H}, \mathcal{A} \in \text{CS}$). Hence we have that $\mathcal{A}$ is a closed segment in $\mathcal{H}$ and an NI-like segment in $\mathcal{H}$ and thus an NI-closed segment in $\mathcal{H}$.

$(R-L)$: Now, suppose $\mathcal{A}$ is an NI-closed segment in $\mathcal{H}$. Then $\mathcal{A}$ is a closed segment and an NI-like segment in $\mathcal{H}$. We have $\text{AS}(\mathcal{H}) \cap \mathcal{A} = \{(\min(Dom(\mathcal{A})), \mathcal{H}_{\min(Dom(\mathcal{A}))})\}$ or there is a $l \in \text{Dom}(\text{AS}(\mathcal{H})) \cap \text{Dom}(\mathcal{A})$ with $\min(Dom(\mathcal{A})) < l \leq \max(Dom(\mathcal{A}))-1$.

**First case:** Suppose $\text{AS}(\mathcal{H}) \cap \mathcal{A} = \{(\min(Dom(\mathcal{A})), \mathcal{H}_{\min(Dom(\mathcal{A}))})\}$. Then it holds, with Theorem 2-35-(iv) and Theorem 2-41, that $\mathcal{A}$ is a minimal closed segment in $\mathcal{H}$. Since $\mathcal{A}$ is an NI-like segment in $\mathcal{H}$, we then have that $\mathcal{A}$ is a minimal NI-closed segment in $\mathcal{H}$. From this it follows that there are $\Delta, \Gamma \in \text{CFORM}$ and $i \in \text{Dom}(\mathcal{H})$ such that (i), (ii), (iv) and (v) hold. Also, we have trivially that (vi) holds. Let now $\Delta, \Gamma$ and $i$ be as demanded in clauses (i), (ii), (iv) and (vii).

Then we also have (iii) and (v). To see this, suppose $\mathcal{B}$ is a closed segment in $\mathcal{H}_{\max(Dom(\mathcal{A}))}$. Then we have for $l = \min(Dom(\mathcal{A}))$ or $l = i$ that $l < \min(Dom(\mathcal{B}))$ or $\max(Dom(\mathcal{B})) \leq l$. Since $\mathcal{A}$ is a minimal NI-closed segment and thus a minimal closed segment in $\mathcal{H}$, it holds with Theorem 2-58 that $\mathcal{A} \cap \mathcal{B} = \emptyset$ or $\mathcal{A} \subseteq \mathcal{B}$. Since, by hypothesis, we have $\mathcal{B} \subseteq \mathcal{H}_{\max(Dom(\mathcal{A}))}$, it follows that $\{(\max(Dom(\mathcal{A})), \mathcal{H}_{\max(Dom(\mathcal{A}))})\} \in \mathcal{A} \cap \mathcal{B}$ and hence that $\mathcal{A} \subseteq \mathcal{B}$ and thus that $\mathcal{B} \cap \mathcal{A} = \emptyset$. On the other hand, for $l = \min(Dom(\mathcal{A}))$ or $l = i$ and $\min(Dom(\mathcal{B})) \leq l < \max(Dom(\mathcal{B}))$ we would have $\mathcal{B} \cap \mathcal{A} \neq \emptyset$ and thus a contradiction.

**Second case:** Now, suppose there is a $l \in \text{Dom}(\text{AS}(\mathcal{H})) \cap \text{Dom}(\mathcal{A})$ with $\min(Dom(\mathcal{A})) < l \leq \max(Dom(\mathcal{A}))-1$. Then $\mathcal{A}$ is not a minimal closed segment in $\mathcal{H}$. With Theorem 2-41, there is then a $G \in \text{ASCS}(\mathcal{H})$ with $\{k\} \times \text{Ran}(G) \subseteq \text{CS}$ and $\mathcal{A} \in \text{GEN}(\langle \mathcal{H}, G \rangle)$. Then $G$ is an AS-comprising segment sequence for $\mathcal{E} = \{(l, \mathcal{H}) \mid \min(Dom(\mathcal{A}))+1 \leq l \leq \max(Dom(\mathcal{A}))-1\}$ in $\mathcal{H}$. We have that $\mathcal{A}$ is an NI-like segment in $\mathcal{H}$ and thus, according to Definition 2-18 and Definition 2-19:
There is $\Delta, \Gamma \in \text{CFORM}$ and $i \in \text{Dom}(\mathfrak{A})$ such that

a) $\min(\text{Dom}(\mathfrak{A})) \leq i < \max(\text{Dom}(\mathfrak{A}))$,

b) $\mathfrak{A}_{\min(\text{Dom}(\mathfrak{A}))} = ^{\sim} \Delta$,

c) $P(\mathfrak{A}_i) = \Gamma$ and $P(\mathfrak{A}_{\max(\text{Dom}(\mathfrak{A}))-1}) = ^{\sim} \Gamma$, or

$d) \text{For all } r \in \text{Dom}(G): i < \min(\text{Dom}(G(r))) \text{ or } \max(\text{Dom}(G(r))) \leq i$,

e) $\mathfrak{A}_{\max(\text{Dom}(\mathfrak{A}))} = ^{\sim} \Gamma$.

Then clauses (i), (ii), (iv) and (vii) are satisfied. With Theorem 2-48, we also have (vi).

Also, we have (iii) and (v). To see this, suppose $\mathfrak{B}$ is a closed segment in $\mathfrak{A}_{\max(\text{Dom}(\mathfrak{A}))}$. Then it holds that $\mathfrak{B} \subseteq \mathfrak{A}_{\max(\text{Dom}(\mathfrak{A}))}$ and hence that $\{(\max(\text{Dom}(\mathfrak{A})), \mathfrak{A}_{\max(\text{Dom}(\mathfrak{A}))})\} \in \mathfrak{A} \setminus \mathfrak{B}$ and hence that $\mathfrak{A} \not\subseteq \mathfrak{B}$. It also follows that $\max(\text{Dom}(\mathfrak{B})) < \max(\text{Dom}(\mathfrak{A}))$. Thus we have that $\mathfrak{A} \cap \mathfrak{A} = \emptyset$ or $\mathfrak{B} \subseteq \mathfrak{C}$. To see this, suppose $\mathfrak{A} \cap \mathfrak{A} \neq \emptyset$. Because of $\mathfrak{A} \not\subseteq \mathfrak{B}$, we then have, with Theorem 2-57, that $\mathfrak{A} \subseteq \mathfrak{A}$ and hence, with Theorem 2-56, that $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{B}))$. Altogether, we thus have $\min(\text{Dom}(\mathfrak{C})) = \min(\text{Dom}(\mathfrak{A})) + 1 \leq \min(\text{Dom}(\mathfrak{B})) < \max(\text{Dom}(\mathfrak{B})) \leq \max(\text{Dom}(\mathfrak{A}))-1 = \max(\text{Dom}(\mathfrak{C}))$ and hence, with Theorem 2-5, $\mathfrak{B} \subseteq \mathfrak{C}$.

With Theorem 2-52 it then follows immediately that (iii) holds, i.e. that $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{B}))$ or $\max(\text{Dom}(\mathfrak{B})) \leq \min(\text{Dom}(\mathfrak{A}))$. Furthermore, we also have (v), i.e. that $i < \min(\text{Dom}(\mathfrak{B}))$ or $\max(\text{Dom}(\mathfrak{B})) \leq i$. To see this, suppose for contradiction that $\min(\text{Dom}(\mathfrak{B})) \leq i < \max(\text{Dom}(\mathfrak{B}))$. Then we would have $(i, \mathfrak{A}_i) \in \mathfrak{B}$. We have that $\mathfrak{B} \subseteq \mathfrak{A}$ is a closed segment in $\mathfrak{A}$ and thus, with Theorem 2-60, that there is an $r \in \text{Dom}(G)$ such that $\mathfrak{B} \subseteq G(r)$. Then we would have $\min(\text{Dom}(G(r))) \leq \min(\text{Dom}(\mathfrak{B})) \leq i < \max(\text{Dom}(\mathfrak{B})) \leq \max(\text{Dom}(G(r)))$. But, because of (d) we would also have that $i < \min(\text{Dom}(G(r)))$ or $\max(\text{Dom}(G(r))) \leq i$. Contradiction! Therefore we have $i < \min(\text{Dom}(\mathfrak{B}))$ or $\max(\text{Dom}(\mathfrak{B})) \leq i$. ■
2.2 Closed Segments

**Theorem 2-69. Lemma for Theorem 2-93**

\( \mathfrak{A} \) is a segment in \( \mathfrak{H} \) and there are \( \xi \in \text{VAR}, \beta \in \text{PAR}, \Delta \in \text{FORM}, \) where \( \text{FV}(\Delta) \subseteq \{\xi\}, \Gamma \in \text{CFORM} \) and \( \mathfrak{B} \in \text{SG}(\mathfrak{H}) \) such that

(i) \( P(\mathfrak{S}_{\min(\text{Dom}(\mathfrak{B}))}) = \{\forall \xi \Delta\}, \)

(ii) For all closed segments \( \mathfrak{C} \) in \( \mathfrak{H}[\max(\text{Dom}(\mathfrak{A}))]: \min(\text{Dom}(\mathfrak{B})) < \min(\text{Dom}(\mathfrak{C})) \) or \( \max(\text{Dom}(\mathfrak{C})) \leq \min(\text{Dom}(\mathfrak{B}))), \)

(iii) \( \mathfrak{S}_{\min(\text{Dom}(\mathfrak{B}))+1} = \{\sup \beta, \xi, \Delta\}, \)

(iv) For all closed segments \( \mathfrak{C} \) in \( \mathfrak{H}[\max(\text{Dom}(\mathfrak{A}))]: \min(\text{Dom}(\mathfrak{B}))+1 < \min(\text{Dom}(\mathfrak{C})) \) or \( \max(\text{Dom}(\mathfrak{C})) \leq \min(\text{Dom}(\mathfrak{B}))+1, \)

(v) \( P(\mathfrak{S}_{\max(\text{Dom}(\mathfrak{B}))-1}) = \Gamma, \)

(vi) \( \mathfrak{S}_{\max(\text{Dom}(\mathfrak{B}))} = \{\forall \xi \Delta\}, \)

(vii) \( \beta \not\in \text{STSF}(\{\Delta, \Gamma\}), \)

(viii) There is no \( j \leq \min(\text{Dom}(\mathfrak{B})) \) such that \( \beta \in \text{ST}(\mathfrak{H}_j), \)

(ix) \( \mathfrak{A} = \mathfrak{B} \setminus \{\min(\text{Dom}(\mathfrak{B})), \mathfrak{S}_{\min(\text{Dom}(\mathfrak{B}))}\}, \) and

(x) For every \( r \in \text{Dom}(\text{AS}(\mathfrak{H})) \cap \text{Dom}(\mathfrak{A}) \) with \( \min(\text{Dom}(\mathfrak{A})) < r \leq \max(\text{Dom}(\mathfrak{A}))-1 \) there is a closed segment \( \mathfrak{C} \) in \( \mathfrak{H}[\max(\text{Dom}(\mathfrak{A}))] \) such that \((r, \mathfrak{A}_r) \in \mathfrak{C} \)

iff \( \mathfrak{A} \) is a PE-closed segment in \( \mathfrak{H}. \)

**Proof:** (L-R): Let \( \mathfrak{A} \) be a segment in \( \mathfrak{H} \) and let \( \xi, \beta, \Delta, \Gamma \) and \( \mathfrak{B} \) be as demanded. Then we have \( \mathfrak{H} \in \text{SEQ}. \) With Definition 2-13, we have that \( \mathfrak{A} \) is an RA-like segment in \( \mathfrak{H} \) and we have \( \min(\text{Dom}(\mathfrak{A})) = \min(\text{Dom}(\mathfrak{B}))+1. \) With clause (iv) of our hypothesis and Theorem 2-65-(i), we have that for all \( k \in \text{Dom}(\mathfrak{A}) \) it holds that \( \mathfrak{A}[k] \) is not a closed segment in \( \mathfrak{H}. \)

We have that \( \text{AS}(\mathfrak{H}) \cap \mathfrak{A} = \{\min(\text{Dom}(\mathfrak{A})), \mathfrak{S}_{\min(\text{Dom}(\mathfrak{A}))}\} \) or that there is an \( i \in \text{Dom}(\text{AS}(\mathfrak{H})) \cap \text{Dom}(\mathfrak{A}) \) with \( \min(\text{Dom}(\mathfrak{A})) < i \leq \max(\text{Dom}(\mathfrak{A}))-1. \)

Suppose \( \text{AS}(\mathfrak{H}) \cap \mathfrak{A} = \{\min(\text{Dom}(\mathfrak{A})), \mathfrak{S}_{\min(\text{Dom}(\mathfrak{A}))}\}. \) Since it holds for all \( k \in \text{Dom}(\mathfrak{A}) \) that \( \mathfrak{A}[k] \) is not a closed segment in \( \mathfrak{H}, \) we have, with Theorem 2-32, that \( \mathfrak{A} \) is a minimal closed and thus a closed segment in \( \mathfrak{H}. \) Since \( \mathfrak{A} \) is an RA-like segment in \( \mathfrak{H}, \mathfrak{A} \) is thus a PE-closed segment in \( \mathfrak{H}. \)

Now, suppose there is an \( i \in \text{Dom}(\text{AS}(\mathfrak{H})) \cap \text{Dom}(\mathfrak{A}) \) with \( \min(\text{Dom}(\mathfrak{A})) < i \leq \max(\text{Dom}(\mathfrak{A}))-1. \) Now, let \( \mathfrak{C}^* = \{(l, \mathfrak{H}_l) | \min(\text{Dom}(\mathfrak{A}))+1 \leq l \leq \max(\text{Dom}(\mathfrak{A}))-1\}. \) Then we have that \( \mathfrak{C}^* \) is a segment in \( \mathfrak{H} \) and \( i \in \text{Dom}(\text{AS}(\mathfrak{H})) \cap \text{Dom}(\mathfrak{C}^*). \) We also have that for every \( r \in \text{Dom}(\text{AS}(\mathfrak{H})) \cap \text{Dom}(\mathfrak{C}^*) \) there is a closed segment \( \mathfrak{C} \) in \( \mathfrak{H} \) such that \((r, \mathfrak{C}_r) \in \mathfrak{C} \) and \( \mathfrak{C} \subseteq \mathfrak{C}^*. \) To see this, suppose \( r \in \text{Dom}(\text{AS}(\mathfrak{H})) \cap \text{Dom}(\mathfrak{C}^*). \) Then we have \( \min(\text{Dom}(\mathfrak{A})) < r \leq \max(\text{Dom}(\mathfrak{A}))-1 \) and hence there, is according to clause (x), a closed segment \( \mathfrak{C} \) in \( \mathfrak{H}[\max(\text{Dom}(\mathfrak{A}))] \) such that \((r, \mathfrak{C}_r) \in \mathfrak{C}. \) Then we have \( \min(\text{Dom}(\mathfrak{C}^*)) \leq \)
min(Dom(\(\mathcal{C}\))) because otherwise we would have min(Dom(\(\mathcal{C}\))) \leq min(Dom(\(\mathcal{A}\))) < r \leq max(Dom(\(\mathcal{C}\))) which contradicts clause (iv). On the other hand, it follows from \(\mathcal{C}\) being a segment in \(\mathcal{A}\) that max(Dom(\(\mathcal{C}\))) \leq max(Dom(\(\mathcal{A}\))) - 1 = max(Dom(\(\mathcal{C}^*\))). With Theorem 2-5, we therefore have \(\mathcal{C} \subseteq \mathcal{C}^*\).

Thus \(\mathcal{C}^*\) satisfies the requirements of Theorem 2-59. Therefore there is a \(G \in \text{ASCS}(\mathcal{A})\) such that \(G\) is an AS-comprising segment sequence for \(\mathcal{C}^*\) in \(\mathcal{A}\) and \(\\{\mathcal{A}\} \times \text{Ran}(G) \subseteq \text{CS}\). According to the definition of \(\mathcal{C}^*\), we have that \(\mathcal{C}^* \in \text{SG}(\mathcal{A})\) and min(Dom(\(\mathcal{A}\)))+1 = min(Dom(\(\mathcal{C}^*\))) and max(Dom(\(\mathcal{A}\))) = max(Dom(\(\mathcal{C}^*\)))+1 and that \(\mathcal{A}\) is an RA-like segment in \(\mathcal{A}\). Suppose, \(\mathcal{A}\) is an NI-like segment in \(\mathcal{A}\). Then we have \(\Gamma = \lnot[\beta, \xi, \Delta]\) and \(P(\mathcal{A}_{\text{min}(\text{Dom}(\mathcal{A}))}) = [\beta, \xi, \Delta]\) and \(P(\mathcal{A}_{\text{max}(\text{Dom}(\mathcal{A}))}) = \lnot[\beta, \xi, \Delta]\). Also, we have that for all \(r \in \text{Dom}(G)\) it holds that min(Dom(\(\mathcal{A}\))) < min(Dom(\(G(r)\))). Furthermore, since it holds for all \(i \in \text{Dom}(\mathcal{A})\) that \(\mathcal{A}|i\) is not a closed segment in \(\mathcal{A}\), we also have that for all \(i \in \text{Dom}(\mathcal{A})\) it holds that \(\mathcal{A}|i\) is not a minimal closed segment in \(\mathcal{A}\).

According to Definition 2-18, we thus have \(\mathcal{A} \in \text{PGEN}(\langle \mathcal{A}, G \rangle)\). Now, suppose for contradiction that there are a \(k \in \text{Dom}(\mathcal{A})\) and an \(l \in \text{Dom}(G)\) such that \(\mathcal{A}|k \in \text{PGEN}(\langle \mathcal{A}, G|l+1\rangle)\). According to Theorem 2-25, \(G|l+1\) is an AS-comprising segment sequence for \(\mathcal{A}|\text{max}(\text{Dom}(G(l)))\) and thus, according to Definition 2-10, we have \(G|l+1 \in \text{ASCS}(\mathcal{A})\). By hypothesis, we have \(\mathcal{A}|k \in \text{PGEN}(\langle \mathcal{A}, G|l+1\rangle)\). On the other hand, we have \(\mathcal{A} \in \text{SEQ}\) and \(\\{\mathcal{A}\} \times \text{Ran}(G|l+1) \subseteq \{\mathcal{A}\} \times \text{Ran}(G) \subseteq \text{CS}\). Altogether, we thus have a contradiction to Theorem 2-65-(ii). Therefore there are no \(k \in \text{Dom}(\mathcal{A})\) and \(l \in \text{Dom}(G)\) such that \(\mathcal{A}|k \in \text{PGEN}(\langle \mathcal{A}, G|l+1\rangle)\). According to Definition 2-19, we hence have that \(\mathcal{A} \in \text{GEN}(\langle \mathcal{A}, G \rangle)\) and thus, with \(\{\mathcal{A}\} \times \text{Ran}(G) \subseteq \text{CS}\) and Theorem 2-41, that \(\langle \mathcal{A}, \mathcal{A}\rangle \in \text{CS}\). Hence \(\mathcal{A}\) is a closed segment in \(\mathcal{A}\) and an RA-like segment in \(\mathcal{A}\) and thus a PE-closed segment in \(\mathcal{A}\).

\((R-L)\): Now, suppose \(\mathcal{A}\) is a PE-closed segment in \(\mathcal{A}\). Then we have that \(\mathcal{A}\) is a closed segment and an RA-like segment in \(\mathcal{A}\). From \(\mathcal{A}\) being an RA-like segment in \(\mathcal{A}\) it follows that there are \(\xi \in \text{VAR}\), \(\beta \in \text{PAR}\), \(\Delta \in \text{FORM}\), where \(\text{FV}(\Delta) \subseteq \{\xi\}\), \(\Gamma \in \text{CFORM}\) and a \(\mathcal{B} \in \text{SG}(\mathcal{A})\) for which clauses (i), (iii), and (v)-(ix) are satisfied. We also have with Theorem 2-48 that (x) holds (if \(\mathcal{A}\) is a minimal closed segment, (x) holds trivially). Also, we have that min(Dom(\(\mathcal{A}\))) = min(Dom(\(\mathcal{B}\))) + 1.
Now, we still have to show that clauses (ii) and (iv) hold. For this, we first show (iv). Suppose $\mathcal{C}$ is a closed segment in $\mathfrak{H}\backslash\max(\text{Dom}(\mathfrak{A}))$. Suppose for contradiction that $\min(\text{Dom}(\mathcal{C})) \leq \min(\text{Dom}(\mathfrak{A})) < \max(\text{Dom}(\mathcal{C}))$. Then we would have $\min(\text{Dom}(\mathfrak{A})) \in \text{Dom}(\mathcal{C})$ and hence $\mathfrak{A} \cap \mathcal{C} \neq \emptyset$. With Theorem 2-56, we would then have $\mathfrak{A} \subseteq \mathcal{C}$. Thus we would have $\mathfrak{A} \subseteq \mathcal{C} \subseteq \mathfrak{H}\backslash\max(\text{Dom}(\mathfrak{A}))$ and hence $\max(\text{Dom}(\mathfrak{A})) \notin \text{Dom}(\mathfrak{A}) \neq \emptyset$. Contradiction! Therefore we have $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathcal{C}))$ or $\max(\text{Dom}(\mathcal{C})) \leq \min(\text{Dom}(\mathfrak{A}))$.

We still have to show (ii). Suppose again that $\mathcal{C}$ is a closed segment in $\mathfrak{H}\backslash\max(\text{Dom}(\mathfrak{A}))$. Suppose $\min(\text{Dom}(\mathcal{C})) \leq \min(\text{Dom}(\mathfrak{B})) < \max(\text{Dom}(\mathcal{C}))$. Then we would have $\min(\text{Dom}(\mathcal{C})) < \min(\text{Dom}(\mathfrak{A})) \leq \max(\text{Dom}(\mathfrak{A}))$. As we have just shown, it holds with (iv) that $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathcal{C}))$ or $\max(\text{Dom}(\mathcal{C})) \leq \min(\text{Dom}(\mathfrak{A}))$. Since the first case is excluded, it follows that $\max(\text{Dom}(\mathcal{C})) \leq \min(\text{Dom}(\mathfrak{A}))$ and thus that $\max(\text{Dom}(\mathcal{C})) = \min(\text{Dom}(\mathfrak{A}))$. Then we would have $\max(\text{Dom}(\mathcal{C})) \in \text{Dom}(\text{AS}(\mathfrak{H}))$. But with Theorem 2-42, $\mathcal{C}$ is a Cdl- or NI- or RA-like segment in $\mathfrak{H}$ and thus we have, with Theorem 2-29, that $\max(\text{Dom}(\mathcal{C})) \notin \text{Dom}(\text{AS}(\mathfrak{H}))$. Contradiction! Thus we have $\min(\text{Dom}(\mathfrak{B})) < \min(\text{Dom}(\mathcal{C}))$ or $\max(\text{Dom}(\mathcal{C})) \leq \min(\text{Dom}(\mathfrak{B}))$. Therefore we also have (ii). ■
2.3 AVS, AVAS, AVP and AVAP

Now, the availability conception is established with recourse to ch. 2.2. This is done in such a way that a proposition is available in a sentence sequence $\mathcal{S}$ at an $i \in \text{Dom}(\mathcal{S})$ if and only if $(i, \mathcal{S}_i)$ does not lie within a proper initial segment of any closed segment in $\mathcal{S}$ (Definition 2-26). Of all the propositions of the members of a closed segment $\mathfrak{A}$ in $\mathcal{S}$ it is thus at most the proposition of the last member of $\mathfrak{A}$ that is available in $\mathcal{S}$ at any $i \in \text{Dom}(\mathfrak{A})$, namely at $\max(\text{Dom}(\mathfrak{A}))$. The function AVS then assigns exactly that subset of $\mathcal{S}$ to a sentence sequence $\mathcal{S}$ for whose elements $(i, \mathcal{S}_i)$ it holds that the proposition of $\mathcal{S}_i$ is available in $\mathcal{S}$ at $i$ (Definition 2-28). The propositions of the sentences from AVS($\mathcal{S}$) are then collected by the function AVP to form AVP($\mathcal{S}$), the set of the propositions that are available in $\mathcal{S}$ at some position (Definition 2-30). The function AVAS assigns a sentence sequence $\mathcal{S}$ that subset of $\mathcal{S}$ for whose elements $(i, \mathcal{S}_i)$ it holds that $\mathcal{S}_i$ is an assumption-sentence and that the proposition of $\mathcal{S}_i$ is available in $\mathcal{S}$ at $i$ (Definition 2-29). The propositions of the assumption-sentences from AVAS($\mathcal{S}$) are then collected by the function AVAP to form AVAP($\mathcal{S}$), the set of propositions that have been assumed in $\mathcal{S}$ at some position and are still available at that position, i.e. the set of available assumptions of $\mathcal{S}$ (Definition 2-31).

Then, we will prove some theorems which will, on the one hand, establish connections between AVS, AVAS, AVP and AVAP and, on the other hand, show connections between the extension of a sentence sequence and changes of availability. The most important theorems for the understanding of the calculus and for the further development are Theorem 2-82, Theorem 2-83, Theorem 2-91, Theorem 2-92 and Theorem 2-93. With this chapter, we will finish our preparations so that we can then develop and analyse the Speech Act Calculus in the next chapters.

**Definition 2-26.** Availability of a proposition in a sentence sequence at a position

$\Gamma$ is available in $\mathcal{S}$ at $i$

iff

$\Gamma \in \text{CFORM}$ and $\mathcal{S} \in \text{SEQ}$ and

(i) $i \in \text{Dom}(\mathcal{S})$,

(ii) $\Gamma = P(\mathcal{S}_i)$, and

(iii) There is no closed segment $\mathfrak{A}$ in $\mathcal{S}$ such that $\min(\text{Dom}(\mathfrak{A})) \leq i < \max(\text{Dom}(\mathfrak{A}))$. 
Definition 2-27. Availability of a proposition in a sentence sequence

Γ is available in ƒₜ
iff
There is an i ∈ Dom(ƒₜ) such that Γ is available in ƒₜ at i.

Note: If it is obvious to which sentence sequence we are referring, we will also use the shorter formulations 'Γ is available at i' or 'Γ is available'.

Definition 2-28. Assignment of the set of available sentences (AVS)

AVS = {(ƒₜ, X) | ƒₜ ∈ SEQ and X = {(Γₜ) | i ∈ Dom(ƒₜ) and P(Γₜ) is available in ƒₜ at i}}.

Definition 2-29. Assignment of the set of available assumption-sentences (AVAS)

AVAS = {(ƒₜ, X) | ƒₜ ∈ SEQ and X = AVS(ƒₜ) ∩ AS(ƒₜ)}.

Note: The titles 'assignment of the set of … sentences' are misleading insofar AVS and AVAS do not assign sets of sentences to sentence sequences but subsets of these sequences, thus sets of ordered pairs, whose second projections are then the respective sentences.

Theorem 2-70. Relation of AVAS, AVS and respective sentence sequence
If ƒₜ ∈ SEQ, then:
(i) AVAS(ƒₜ) = AVS(ƒₜ) ∩ AS(ƒₜ) and
(ii) AVAS(ƒₜ) ⊆ AVS(ƒₜ) ⊆ ƒₜ.

Proof: Follows directly from the definitions. ■

Definition 2-30. Assignment of the set of available propositions (AVP)

AVP = {(ƒₜ, X) | ƒₜ ∈ SEQ and X = {Γ | There is an i ∈ Dom(AVS(ƒₜ)) and Γ = P(Γₜ)}).

Definition 2-31. Assignment of the set of available assumptions (AVAP)

AVAP = {(ƒₜ, X) | ƒₜ ∈ SEQ and X = {Γ | There is an i ∈ Dom(AVAS(ƒₜ)) and Γ = P(Γₜ)}).

Theorem 2-71. Relation of AVAP and AVP
If ƒₜ ∈ SEQ, then AVAP(ƒₜ) ⊆ AVP(ƒₜ).

Proof: Follows with Theorem 2-70 directly from the definitions. ■
Theorem 2-72. AVS-inclusion implies AVAS-inclusion

If \( \bar{f}, \bar{f}' \in \text{SEQ} \) and \( \text{AVS}(\bar{f}) \subseteq \text{AVS}(\bar{f}') \), then \( \text{AVAS}(\bar{f}) \subseteq \text{AVAS}(\bar{f}') \).

Proof: Suppose \( \bar{f}, \bar{f}' \in \text{SEQ} \) and suppose \( \text{AVS}(\bar{f}) \subseteq \text{AVS}(\bar{f}') \). Now, suppose \( (i, \bar{f}_i) \in \text{AVAS}(\bar{f}) \). Then we have \( (i, \bar{f}_i) \in \text{AVS}(\bar{f}) \cap \text{AS}(\bar{f}) \). Then we have \( (i, \bar{f}_i) \in \text{AVS}(\bar{f}) \) and \( \bar{f}_i \in \text{ASENT} \). By hypothesis, we then have \( (i, \bar{f}_i) \in \text{AVS}(\bar{f}') \) and hence also \( (i, \bar{f}_i) \in \bar{f}' \). Since \( \bar{f}_i \in \text{ASENT} \), we then also have \( (i, \bar{f}_i) \in \text{AS}(\bar{f}') \) and thus \( (i, \bar{f}_i) \in \text{AVS}(\bar{f}') \cap \text{AS}(\bar{f}') = \text{AVAS}(\bar{f}') \). ■

Theorem 2-73. AVAS-reduction implies AVS-reduction

If \( \bar{f}, \bar{f}' \in \text{SEQ} \) and \( \text{AVAS}(\bar{f}) \nsubseteq \text{AVAS}(\bar{f}') \), then \( \text{AVS}(\bar{f}) \nsubseteq \text{AVS}(\bar{f}') \).

Proof: Suppose \( \bar{f}, \bar{f}' \in \text{SEQ} \) and suppose \( \text{AVAS}(\bar{f}) \nsubseteq \text{AVAS}(\bar{f}') \). Hence \( \text{AVAS}(\bar{f}) \nsubseteq \text{AVAS}(\bar{f}') \) and with Theorem 2-72 we get \( \text{AVS}(\bar{f}) \nsubseteq \text{AVS}(\bar{f}') \). It follows immediately that \( \text{AVS}(\bar{f}) \nsubseteq \text{AVS}(\bar{f}') \) ■.

Theorem 2-74. AVS-inclusion implies AVP-inclusion

If \( \bar{f}, \bar{f}' \in \text{SEQ} \) and \( \text{AVS}(\bar{f}) \subseteq \text{AVS}(\bar{f}') \), then \( \text{AVP}(\bar{f}) \subseteq \text{AVP}(\bar{f}') \).

Proof: Suppose \( \bar{f}, \bar{f}' \in \text{SEQ} \) and suppose \( \text{AVS}(\bar{f}) \subseteq \text{AVS}(\bar{f}') \). Now, suppose \( \Gamma \in \text{AVP}(\bar{f}) \). Then there is an \( i \in \text{Dom}(\text{AVS}(\bar{f})) \) such that \( \Gamma = \text{P}(\bar{f}_i) \). Then we have \( (i, \bar{f}_i) \in \text{AVS}(\bar{f}) \). By hypothesis, we then have \( (i, \bar{f}_i) \in \text{AVS}(\bar{f}') \). We have \( \text{AVS}(\bar{f}') \subseteq \bar{f}' \) and hence \( (i, \bar{f}_i) \in \bar{f}' \) and therefore \( \bar{f}_i = \bar{f}' \). Hence we have \( \Gamma = \text{P}(\bar{f}_i) = \text{P}(\bar{f}'_i) \). Therefore we have \( i \in \text{Dom}(\text{AVS}(\bar{f}')) \) and \( \Gamma = \text{P}(\bar{f}'_i) \). Therefore we have \( \Gamma \in \text{AVP}(\bar{f}') \). ■

Theorem 2-75. AVAS-inclusion implies AVAP-inclusion

If \( \bar{f}, \bar{f}' \in \text{SEQ} \) and \( \text{AVAS}(\bar{f}) \subseteq \text{AVAS}(\bar{f}') \), then \( \text{AVAP}(\bar{f}) \subseteq \text{AVAP}(\bar{f}') \).

Proof: Suppose \( \bar{f}, \bar{f}' \in \text{SEQ} \) and suppose \( \text{AVAS}(\bar{f}) \subseteq \text{AVAS}(\bar{f}') \). Now, suppose \( \Gamma \in \text{AVAP}(\bar{f}) \). Then there is an \( i \in \text{Dom}(\text{AVAS}(\bar{f})) \) such that \( \Gamma = \text{P}(\bar{f}_i) \). Then we have \( (i, \bar{f}_i) \in \text{AVAS}(\bar{f}) \). By hypothesis, we then have \( (i, \bar{f}_i) \in \text{AVAS}(\bar{f}') \). We have \( \text{AVAS}(\bar{f}') \subseteq \bar{f}' \) and hence \( (i, \bar{f}_i) \in \bar{f}' \) and therefore \( \bar{f}_i = \bar{f}' \). Hence we then have \( \Gamma = \text{P}(\bar{f}_i) = \text{P}(\bar{f}'_i) \). Therefore we have \( i \in \text{Dom}(\text{AVAS}(\bar{f}')) \) and \( \Gamma = \text{P}(\bar{f}'_i) \). Therefore we have \( \Gamma \in \text{AVAP}(\bar{f}') \). ■
Theorem 2-76. **AVAP is at most as great as AVAS**
For all \( \bar{y} \in \text{SEQ} \): \( |\text{AVAP}(\bar{y})| \leq |\text{AVAS}(\bar{y})| \).

*Proof:* Suppose \( \bar{y} \in \text{SEQ} \). According to Definition 2-31, we then have that \( f : \text{AVAP}(\bar{y}) \rightarrow \text{AVAS}(\bar{y}) \), \( f(\Gamma) = (\min\{i \mid i \in \text{Dom}(\text{AVAS}(\bar{y})) \text{ and } P(\bar{y}_i) = \Gamma \}) \), \( \bar{y}_{\min\{i \mid i \in \text{Dom}(\text{AVAS}(\bar{y})) \text{ and } P(\bar{y}_i) = \Gamma \}} \) is an injection of \( \text{AVAP}(\bar{y}) \) into \( \text{AVAS}(\bar{y}) \). ■

Theorem 2-77. **AVAP is empty if and only if AVAS is empty**
For all \( \bar{y} \in \text{SEQ} \): \( |\text{AVAP}(\bar{y})| = 0 \) iff \( |\text{AVAS}(\bar{y})| = 0 \).

*Proof:* Suppose \( \bar{y} \in \text{SEQ} \). Suppose \( |\text{AVAP}(\bar{y})| \neq 0 \). With Theorem 2-76, we then have \( |\text{AVAS}(\bar{y})| \neq 0 \). Now, suppose \( |\text{AVAS}(\bar{y})| \neq 0 \). Then there is \( (i, \bar{y}_i) \in \text{AVAS}(\bar{y}) \). With Definition 2-31, we then have \( P(\bar{y}_i) \in \text{AVAP}(\bar{y}) \) and thus \( |\text{AVAP}(\bar{y})| \neq 0 \). Thus we have \( |\text{AVAP}(\bar{y})| \neq 0 \) iff \( |\text{AVAS}(\bar{y})| \neq 0 \), from which the statement follows immediately. ■

Theorem 2-78. **If AVAS is non-redundant, every assumption is available as an assumption at exactly one position**
If \( \bar{y} \in \text{SEQ} \) and \( |\text{AVAP}(\bar{y})| = |\text{AVAS}(\bar{y})| \), then it holds for all \( \Gamma \in \text{AVAP}(\bar{y}) \) that there is exactly one \( j \in \text{Dom}(\text{AVAS}(\bar{y})) \) such that \( \Gamma = P(\bar{y}_j) \).

*Proof:* Suppose \( \bar{y} \in \text{SEQ} \) and \( |\text{AVAP}(\bar{y})| = |\text{AVAS}(\bar{y})| \). With Theorem 2-70-(ii), we have \( \text{AVAS}(\bar{y}) \subseteq \bar{y} \) and thus, with \( \bar{y} \in \text{SEQ} \) and Definition 1-24 and Definition 1-23, that \( |\text{AVAP}(\bar{y})| = |\text{AVAS}(\bar{y})| = k \) for a \( k \in \mathbb{N} \). Now, suppose \( \Gamma \in \text{AVAP}(\bar{y}) \). Then we have \( k > 0 \). According to Definition 2-31, there is then a \( j \in \text{Dom}(\text{AVAS}(\bar{y})) \) such that \( \Gamma = P(\bar{y}_j) \). Now, suppose \( i \in \text{Dom}(\text{AVAS}(\bar{y})) \) and \( \Gamma = P(\bar{y}_i) \). Suppose for contradiction that \( i \neq j \). Then we would have \( |\text{AVAS}(\bar{y})\backslash\{(j, \bar{y}_j)\}| = k-1 \), while, on the other hand, \( f : \text{AVAP}(\bar{y}) \rightarrow \text{AVAS}(\bar{y})\backslash\{(j, \bar{y}_j)\} \), \( f(B) = (\min\{l \mid l \in \text{Dom}(\text{AVAS}(\bar{y})\backslash\{(j, \bar{y}_j)\}) \text{ and } P(\bar{y}_l) = B \}) \), \( \bar{y}_{\min\{l \mid l \in \text{Dom}(\text{AVAS}(\bar{y})\backslash\{(j, \bar{y}_j)\}) \text{ and } P(\bar{y}_l) = B \}} \) would be an injection of \( \text{AVAP}(\bar{y}) \) into \( \text{AVAS}(\bar{y})\backslash\{(j, \bar{y}_j)\} \) and hence \( k = |\text{AVAP}(\bar{y})| \leq k-1 \). Contradiction! ■
Theorem 2-79. **AVS, AVAS, AVP and AVAP in concatenations with one-member sentence sequences**

If \( \mathcal{S}, \mathcal{S}' \in \text{SEQ} \) and \( \text{Dom}(\mathcal{S}) = 1 \), then:

(i) \( \text{AVS}(\mathcal{S}, \mathcal{S}') \subseteq \text{AVS}(\mathcal{S}) \cup \{(\text{Dom}(\mathcal{S}), \mathcal{S}_0')\} \),

(ii) \( \text{AVAS}(\mathcal{S}, \mathcal{S}') \subseteq \text{AVAS}(\mathcal{S}) \cup \{(\text{Dom}(\mathcal{S}), \mathcal{S}_0')\} \),

(iii) \( \text{AVP}(\mathcal{S}, \mathcal{S}') \subseteq \text{AVP}(\mathcal{S}) \cup \{(\mathcal{S}')\} \),

(iv) \( \text{AVAP}(\mathcal{S}, \mathcal{S}') \subseteq \text{AVAP}(\mathcal{S}) \cup \{(\mathcal{S}')\} \).

**Proof:** Suppose \( \mathcal{S}, \mathcal{S}' \in \text{SEQ} \) and suppose \( \text{Dom}(\mathcal{S}') = 1 \).

**Ad (i):** Suppose \( (i, (\mathcal{S}, \mathcal{S}')) \in \text{AVS}(\mathcal{S}, \mathcal{S}') \). Then we have that \( i \in \text{Dom}(\mathcal{S}, \mathcal{S}') \) and \( \text{P}((\mathcal{S}, \mathcal{S}')) \) is available in \( \mathcal{S}, \mathcal{S}' \) at \( i \). We have \( i \in \text{Dom}(\mathcal{S}) \) or \( i = \text{Dom}(\mathcal{S}) \).

Suppose \( i \in \text{Dom}(\mathcal{S}) \). Then we have \( (\mathcal{S}, \mathcal{S}')) = \mathcal{S}_0 \). Suppose for contradiction that \( \text{P}(\mathcal{S}_0) = \text{P}((\mathcal{S}, \mathcal{S}')) \) is not available in \( \mathcal{S}_0 \) at \( i \). According to Definition 2-26, there would then be an \( \mathcal{A} \) such that \( \mathcal{A} \) is a closed segment in \( \mathcal{S}_0 \) and \( \min(\text{Dom}(\mathcal{A})) \leq i < \max(\text{Dom}(\mathcal{A})) \). Because of \( \mathcal{S} \subseteq \mathcal{S}_0 \), we would then, with Theorem 2-62-(viii), have that \( \mathcal{A} \) is also a closed segment in \( \mathcal{S}_0 \) and \( \min(\text{Dom}(\mathcal{A})) \leq i < \max(\text{Dom}(\mathcal{A})) \). But then \( \text{P}((\mathcal{S}, \mathcal{S}')) \) would not be in \( \mathcal{S}, \mathcal{S}' \) at \( i \). Therefore we have \( i \in \text{Dom}(\mathcal{S}) \) and \( \text{P}((\mathcal{S}, \mathcal{S}')) \) is available in \( \mathcal{S}_0 \) at \( i \) and hence \( (i, (\mathcal{S}, \mathcal{S}')) \in \text{AVS}(\mathcal{S}) \).

Now, suppose \( i = \text{Dom}(\mathcal{S}) \). Then we have \( (\mathcal{S}, \mathcal{S}')) = (\mathcal{S}, \mathcal{S}') \mid \text{Dom}(\mathcal{S}) = \mathcal{S}_0' \) and thus \( (i, (\mathcal{S}, \mathcal{S}')) = (\text{Dom}(\mathcal{S}), \mathcal{S}_0') \in \{(\text{Dom}(\mathcal{S}), \mathcal{S}_0')\} \).

**Ad (ii):** Suppose \( (i, (\mathcal{S}, \mathcal{S}')) \in \text{AVAS}(\mathcal{S}, \mathcal{S}') \). With Theorem 2-70, we then have \( (i, (\mathcal{S}, \mathcal{S}')) \in \text{AVAS}(\mathcal{S}, \mathcal{S}') \) and \( (\mathcal{S}, \mathcal{S}') \), \( \text{ASENT} \). With (i), we then have \( (i, (\mathcal{S}, \mathcal{S}')) \in \text{AVS}(\mathcal{S}) \cup \{(\text{Dom}(\mathcal{S}), \mathcal{S}_0')\} \). Suppose \( (i, (\mathcal{S}, \mathcal{S}')) \not\in \{(\text{Dom}(\mathcal{S}), \mathcal{S}_0')\} \) and thus \( (i, (\mathcal{S}, \mathcal{S}')) \in \text{AVS}(\mathcal{S}) \). Then we have \( (i, (\mathcal{S}, \mathcal{S}')) \in \text{AVS}(\mathcal{S}) \) and \( (\mathcal{S}, \mathcal{S}') \), \( \text{ASENT} \) and thus we have that \( (i, (\mathcal{S}, \mathcal{S}') \), \( \text{AVAS}(\mathcal{S}) \).

**Ad (iii):** Suppose \( \Gamma \in \text{AVP}(\mathcal{S}, \mathcal{S}') \). Then there is an \( i \in \text{Dom}(\mathcal{S}, \mathcal{S}') \) such that \( \Gamma \) is available in \( \mathcal{S}, \mathcal{S}' \) at \( i \). Then we have \( \Gamma = \text{P}((\mathcal{S}, \mathcal{S}')) \) and \( (i, (\mathcal{S}, \mathcal{S}')) \in \text{AVS}(\mathcal{S}, \mathcal{S}') \). With (i), we then have \( (i, (\mathcal{S}, \mathcal{S}')) \in \text{AVS}(\mathcal{S}) \cup \{(\text{Dom}(\mathcal{S}), \mathcal{S}_0')\} \). Now, suppose \( (i, (\mathcal{S}, \mathcal{S}')) \in \text{AVS}(\mathcal{S}) \). Then we have \( i \in \text{Dom}(\mathcal{S}) \) and \( (\mathcal{S}, \mathcal{S}') \), \( \text{ASENT} \) and hence \( \Gamma = \text{P}(\mathcal{S}_0') = \text{C}(\mathcal{S}' \), \( \text{AVAS}(\mathcal{S}) \).

**Ad (iv):** Suppose \( \Gamma \in \text{AVAP}(\mathcal{S}, \mathcal{S}') \). Then there is an \( i \in \text{Dom}(\text{AVAS}(\mathcal{S}, \mathcal{S}')) \) and \( \Gamma = \text{P}((\mathcal{S}, \mathcal{S}')) \). Then we have \( (i, (\mathcal{S}, \mathcal{S}')) \in \text{AVAS}(\mathcal{S}, \mathcal{S}') \). With (ii), we then have \( (i, (\mathcal{S}, \mathcal{S}')) \in \text{AVAS}(\mathcal{S}, \mathcal{S}') \) and \( (\mathcal{S}, \mathcal{S}') \), \( \text{ASENT} \) and hence \( \Gamma = \text{P}(\mathcal{S}_0') = \text{C}(\mathcal{S}', \mathcal{S}) \).
we have \( i \in \text{Dom}(\text{AVAS}(\mathfrak{S})) \) and \( \mathfrak{S}_{i} = (\mathfrak{S} \setminus \mathfrak{S}_{i}) \), and hence \( \Gamma = P(\mathfrak{S}_{i}) \in \text{AVAP}(\mathfrak{S}). \) Now, suppose \( (i, (\mathfrak{S} \setminus \mathfrak{S}_{i})) \in \{(\text{Dom}(\mathfrak{S}), \mathfrak{S}_{i}')\}. \) Then we have \( i = \text{Dom}(\mathfrak{S}) \) and \( (\mathfrak{S} \setminus \mathfrak{S}_{i}) = \mathfrak{S}_{i}' \) and hence \( \Gamma = P(\mathfrak{S}_{i}) = C(\mathfrak{S}) \in \{C(\mathfrak{S})\}. \)

**Theorem 2-80.** *AVS, AVAS, AVP and AVAP in concatenations with sentence sequences*

If \( \mathfrak{S}, \mathfrak{S}' \in \text{SEQ} \), then:

(i) \( \text{AVS}(\mathfrak{S} \setminus \mathfrak{S}') \subseteq \text{AVS}(\mathfrak{S}) \cup \{(\text{Dom}(\mathfrak{S}) + i, \mathfrak{S}'_{i}) \mid i \in \text{Dom}(\mathfrak{S}')\}, \)

(ii) \( \text{AVAS}(\mathfrak{S} \setminus \mathfrak{S}') \subseteq \text{AVAS}(\mathfrak{S}) \cup \{(\text{Dom}(\mathfrak{S}) + i, \mathfrak{S}'_{i}) \mid i \in \text{Dom}(\mathfrak{S}')\}. \)

**Proof:** By induction on \( \text{Dom}(\mathfrak{S}') \). For \( \text{Dom}(\mathfrak{S}') = 0 \), the induction basis follows with \( \mathfrak{S} \setminus \mathfrak{S}' = \mathfrak{S} \). Now, suppose the statement holds for all \( \mathfrak{S}'_{*} \in \text{SEQ} \) with \( \text{Dom}(\mathfrak{S}'_{*}) = j \). For (i), we thus have \( \text{AVS}(\mathfrak{S} \setminus \mathfrak{S}'_{*}) \subseteq \text{AVS}(\mathfrak{S}) \cup \{(\text{Dom}(\mathfrak{S}) + i, \mathfrak{S}'_{*}) \mid i \in \text{Dom}(\mathfrak{S}')\} \) for all \( \mathfrak{S}'_{*} \in \text{SEQ} \) with \( \text{Dom}(\mathfrak{S}'_{*}) = j \). Now, suppose \( \text{Dom}(\mathfrak{S}') = j + 1 \). Then we have \( \text{Dom}(\mathfrak{S} \setminus \mathfrak{S}') = j \). According to the I.H., we thus have \( \text{AVS}(\mathfrak{S} \setminus (\mathfrak{S} \setminus \text{Dom}(\mathfrak{S}))) \subseteq \text{AVS}(\mathfrak{S}) \cup \{(\text{Dom}(\mathfrak{S}) + i, (\mathfrak{S} \setminus \text{Dom}(\mathfrak{S}))_{-1}) \mid i \in \text{Dom}(\mathfrak{S} \setminus \text{Dom}(\mathfrak{S}))_{-1}\} \) = \( \text{AVS}(\mathfrak{S}) \cup \{(\text{Dom}(\mathfrak{S}) + i, \mathfrak{S}'_{i}) \mid i \in \text{Dom}(\mathfrak{S}'_{-1})\} \). We have \( \text{AVS}(\mathfrak{S} \setminus \mathfrak{S}') \subseteq \text{AVS}(\mathfrak{S}) \cup \{(\text{Dom}(\mathfrak{S}) + i, \mathfrak{S}'_{i}) \mid i \in \text{Dom}(\mathfrak{S}'_{-1})\} \). The proof of (ii) is carried out analogously. ■

**Theorem 2-81.** *AVS, AVAS, AVP and AVAP in restrictions on \( \text{Dom}(\mathfrak{S})_{-1} \)*

If \( \mathfrak{S} \in \text{SEQ} \), then:

(i) \( \text{AVS}(\mathfrak{S}) \subseteq \text{AVS}(\mathfrak{S} \setminus \text{Dom}(\mathfrak{S})_{-1}) \cup \{(\text{Dom}(\mathfrak{S})_{-1}, \mathfrak{S}'_{\text{Dom}(\mathfrak{S})_{-1}})\}, \)

(ii) \( \text{AVAS}(\mathfrak{S}) \subseteq \text{AVAS}(\mathfrak{S} \setminus \text{Dom}(\mathfrak{S})_{-1}) \cup \{(\text{Dom}(\mathfrak{S})_{-1}, \mathfrak{S}'_{\text{Dom}(\mathfrak{S})_{-1}})\}, \)

(iii) \( \text{AVP}(\mathfrak{S}) \subseteq \text{AVP}(\mathfrak{S} \setminus \text{Dom}(\mathfrak{S})_{-1}) \cup \{P(\mathfrak{S}_{\text{Dom}(\mathfrak{S})_{-1}})\}, \)

(iv) \( \text{AVAP}(\mathfrak{S}) \subseteq \text{AVAP}(\mathfrak{S} \setminus \text{Dom}(\mathfrak{S})_{-1}) \cup \{P(\mathfrak{S}_{\text{Dom}(\mathfrak{S})_{-1}})\}. \)

**Proof:** Suppose \( \mathfrak{S} \in \text{SEQ} \). For \( \mathfrak{S} = \emptyset \), we have that \( \text{AVS}(\mathfrak{S}) \cup \text{AVAS}(\mathfrak{S}) \cup \text{AVP}(\mathfrak{S}) \cup \text{AVAP}(\mathfrak{S}) = \emptyset \) and thus the theorem holds. Now, suppose \( \mathfrak{S} \neq \emptyset \). Then we have \( \mathfrak{S} = (\mathfrak{S} \setminus \text{Dom}(\mathfrak{S})_{-1}) \setminus \{(0, \mathfrak{S}'_{\text{Dom}(\mathfrak{S})_{-1}})\} \) and the theorem follows with Theorem 2-79. ■
Theorem 2-82. The conclusion is always available
If \( \mathcal{S} \in \text{SEQ}\setminus\{\emptyset\} \), then \( C(\mathcal{S}) \) is available in \( \mathcal{S} \) at \( \text{Dom}(\mathcal{S})^{-1} \).

Proof: Suppose \( \mathcal{S} \in \text{SEQ}\setminus\{\emptyset\} \). Then it holds for all closed segments \( \mathcal{A} \) in \( \mathcal{S} \) that \( \max(\text{Dom}(\mathcal{A})) \leq \text{Dom}(\mathcal{S})^{-1} \) and therefore there is no closed segment \( \mathcal{A} \) in \( \mathcal{S} \) such that \( \min(\text{Dom}(\mathcal{A})) \leq \text{Dom}(\mathcal{S})^{-1} < \max(\text{Dom}(\mathcal{A})) \). Therefore \( P(\text{Dom}(\mathcal{S})^{-1}) = C(\mathcal{S}) \) is available in \( \mathcal{S} \) at \( \text{Dom}(\mathcal{S})^{-1} \). ■

Theorem 2-83. Connections between non-availability and the emergence of a closed segment in the transition from \( \mathcal{S}_1 \upharpoonright \text{Dom}(\mathcal{S}_1)^{-1} \) to \( \mathcal{S}_2 \)
If \( \mathcal{S}_1 \in \text{SEQ} \) and \( \text{AVS}(\mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1}) \setminus \text{AVS}(\mathcal{S}_2) \neq \emptyset \), then:

There is a \( \mathcal{B} \) such that \( \mathcal{B} \) is a closed segment in \( \mathcal{S}_1 \) and

(i) \( \min(\text{Dom}(\mathcal{B})) \leq \text{Dom}(\mathcal{S}_1)^{-2} \) and \( \max(\text{Dom}(\mathcal{B})) = \text{Dom}(\mathcal{S}_1)^{-1} \),

(ii) For all closed segments \( \mathcal{C} \) in \( \mathcal{S}_1 \upharpoonright \text{Dom}(\mathcal{S}_1)^{-1} \) it holds that \( \mathcal{B} \upharpoonright \text{Dom}(\mathcal{S}_1)^{-1} \cap \mathcal{C} = \emptyset \) or \( \min(\text{Dom}(\mathcal{B})) < \min(\text{Dom}(\mathcal{C})) \) and \( \max(\text{Dom}(\mathcal{B})) < \max(\text{Dom}(\mathcal{C})) \),

(iii) For all closed segments \( \mathcal{C}_* \) in \( \mathcal{S}_2 \): If \( \mathcal{C}_* \) is not a closed segment in \( \mathcal{S}_1 \upharpoonright \text{Dom}(\mathcal{S}_1)^{-1} \), then \( \mathcal{C}_* = \mathcal{B} \),

(iv) \( \text{AVS}(\mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1}) \setminus \text{AVS}(\mathcal{S}_2) \subseteq \{ (j, \mathcal{S}_j) \mid \min(\text{Dom}(\mathcal{B})) \leq j < \text{Dom}(\mathcal{S}_1)^{-1} \} \),

(v) \( \text{AVS}(\mathcal{S}_1) = \{ \text{AVS}(\mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1}) \setminus \{ (j, \mathcal{S}_j) \mid \min(\text{Dom}(\mathcal{B})) \leq j < \text{Dom}(\mathcal{S}_1)^{-1} \}, \}

(vi) \( \text{AVAS}(\mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1}) \setminus \text{AVAS}(\mathcal{S}_2) = \{ \min(\text{Dom}(\mathcal{B})), \mathcal{S}_{\min(\text{Dom}(\mathcal{B}))} \} \),

(vii) \( \text{AVAS}(\mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1}) = \text{AVAS}(\mathcal{S}_2) \setminus \{ (\min(\text{Dom}(\mathcal{B})), \mathcal{S}_{\min(\text{Dom}(\mathcal{B}))}) \} \),

(viii) \( \text{AVP}(\mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1}) \setminus \text{AVP}(\mathcal{S}_2) \subseteq \{ \text{P}(\mathcal{S}_j) \mid \min(\text{Dom}(\mathcal{B})) < j < \text{Dom}(\mathcal{S}_1)^{-1} \} \),

(ix) \( \text{AVP}(\mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1}) \subseteq \{ \text{P}(\mathcal{S}_j) \mid j \in \text{Dom}(\text{AVS}(\mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1})) \} \),

(x) \( \text{AVAP}(\mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1}) \setminus \text{AVAP}(\mathcal{S}_2) \subseteq \{ \text{P}(\mathcal{S}_{\min(\text{Dom}(\mathcal{B}))}) \} \), and

(xi) \( \text{AVAP}(\mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1}) = \text{AVAP}(\mathcal{S}_2) \setminus \{ \text{P}(\mathcal{S}_{\min(\text{Dom}(\mathcal{B}))}) \} \).

Proof: Suppose \( \mathcal{S}_1 \in \text{SEQ} \) and suppose \( \text{AVS}(\mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1}) \setminus \text{AVS}(\mathcal{S}_2) \neq \emptyset \). According to Definition 2-28, there is then an \( i \in \text{Dom}(\mathcal{S}_1)^{-1} \) such that \( (i, \mathcal{S}_i) \in \text{AVS}(\mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1}) \setminus \text{AVS}(\mathcal{S}_2) \). Then we have \( \mathcal{S}_1\upharpoonright \text{Dom}(\mathcal{S}_1)^{-1} \neq \emptyset \) and thus \( \mathcal{S}_1 \neq \emptyset \).

According to Definition 2-28 and Definition 2-26, there is then no \( \mathcal{B}' \) such that \( \mathcal{B}' \) is a closed segment in \( \mathcal{S}_1 \upharpoonright \text{Dom}(\mathcal{S}_1)^{-1} \) and \( \min(\text{Dom}(\mathcal{B}'')) \leq i < \max(\text{Dom}(\mathcal{B}'')) \), and that there is a \( \mathcal{B} \) such that \( \mathcal{B} \) is a closed segment in \( \mathcal{S}_1 \) and \( \min(\text{Dom}(\mathcal{B})) \leq i < \max(\text{Dom}(\mathcal{B})) \).

Ad (i): We have \( \max(\text{Dom}(\mathcal{B})) \leq \text{Dom}(\mathcal{S}_1)^{-1} \). Suppose for contradiction that \( \text{Dom}(\mathcal{S}_1)^{-2} < \min(\text{Dom}(\mathcal{B})) \). With Theorem 2-44, we would then have \( \text{Dom}(\mathcal{S}_1)^{-1} \leq \min(\text{Dom}(\mathcal{B})) < \max(\text{Dom}(\mathcal{B})) \leq \text{Dom}(\mathcal{S}_1)^{-1} \). Contradiction! Therefore we have \( \min(\text{Dom}(\mathcal{B})) \leq \text{Dom}(\mathcal{S}_1)^{-1} \).
Dom(ψ)-2. Now, suppose for contradiction that $\max(\text{Dom}(B)) < \text{Dom}(\varphi)-1$. Then we would have $\min(\text{Dom}(B)) < \max(\text{Dom}(B)) < \text{Dom}(\varphi)-1$. With Theorem 2-64-(viii) and Theorem 2-62-(viii), we would then have that $B$ is a closed segment in $\varphi|\text{Dom}(\varphi)-1$ and that $\min(\text{ Dom}(B)) \leq i < \max(\text{Dom}(B))$. But then we would have $(i, \varphi) \not\in \text{AVS}(\varphi)|\text{Dom}(\varphi)-1)$. Therefore we have that $\max(\text{Dom}(B)) = \text{Dom}(\varphi)-1$ and hence that $\min(\text{Dom}(B)) \leq \text{Dom}(\varphi)-2$ and $\max(\text{Dom}(B)) = \text{Dom}(\varphi)-1$.

Ad (ii): Suppose $C$ is a closed segment in $\varphi|\text{Dom}(\varphi)-1$. Now, suppose $B|\text{Dom}(\varphi)-1 \cap C \neq \emptyset$. Then we have $B \cap C \neq \emptyset$. With Theorem 2-57, it then holds that $B \subseteq C$ or $C \subseteq B$. Since $C \subseteq \varphi|\text{Dom}(\varphi)-1$ and $(\text{Dom}(\varphi)-1, \varphi|_{\text{Dom}(\varphi)-1} \in B$, we have $B \not\subseteq C$. Thus we have $C \not\subseteq B$. With Theorem 2-56-(i) and -(iii), we thus have $\min(\text{Dom}(B)) < \text{Dom}(\varphi)|\text{Dom}(\varphi)-1) < \max(\text{Dom}(B)) = \text{Dom}(\varphi)-1$.

Ad (iii): Suppose $C^*$ is a closed segment in $\varphi$, but not a closed segment in $\varphi|\text{Dom}(\varphi)-1$. Then we have $\max(\text{Dom}(C^*)) = \text{Dom}(\varphi)-1$. First, we have $\max(\text{Dom}(C^*)) \leq \text{Dom}(\varphi)-1$. If $\max(\text{Dom}(C^*)) < \text{Dom}(\varphi)-1$, then we would have, with Theorem 2-64-(viii) and Theorem 2-62-(viii), that $C^*$ is a closed segment in $\varphi|\text{Dom}(\varphi)-1$, which contradicts the hypothesis. Therefore we have $\text{Dom}(\varphi)-1 \leq \max(\text{Dom}(C^*))$ and hence $\max(\text{Dom}(C^*)) = \text{Dom}(\varphi)-1 = \max(\text{Dom}(B))$. With Theorem 2-53, it then follows that $C^* = B$.

Ad (iv): Suppose $(i, \varphi) \in \text{AVS}(\varphi|\text{Dom}(\varphi)-1) \setminus \text{AVS}(\varphi)$. Then there is a closed segment $C$ in $\varphi$ such that $\min(\text{Dom}(C)) \leq i < \max(\text{Dom}(C))$ and $C$ is not a closed segment in $\varphi|\text{Dom}(\varphi)-1$. Then it holds with (iii) that $C \subseteq B$ and hence that $\min(\text{Dom}(B)) \leq i < \max(\text{Dom}(B)) = \text{Dom}(\varphi)-1$. It then follows that $(i, \varphi) \not\in \{(j, \varphi) \mid \min(\text{Dom}(B)) \leq j < \text{Dom}(\varphi)-1\}$.

Ad (v): First, suppose $(i, \varphi) \in \text{AVS}(\varphi)$. With Theorem 2-81-(i), we then have $(i, \varphi) \in \text{AVS}(\varphi|\text{Dom}(\varphi)-1) \cup \{(\text{Dom}(\varphi)-1, \varphi|_{\text{Dom}(\varphi)-1})\}$. Also, we have that there is no closed segment $C$ in $\varphi$ such that $\min(\text{Dom}(C)) \leq i < \max(\text{Dom}(C))$. Since $B$ is a closed segment in $\varphi$, it then follows with (i) that $(i, \varphi) \not\in \{(j, \varphi) \mid \min(\text{Dom}(B)) \leq j < \text{Dom}(\varphi)-1\}$. Hence we have $(i, \varphi) \not\in \text{AVS}(\varphi|\text{Dom}(\varphi)-1) \cup \{(\text{Dom}(\varphi)-1, \varphi|_{\text{Dom}(\varphi)-1})\}$.

Now, suppose $(i, \varphi) \in \text{AVS}(\varphi|\text{Dom}(\varphi)-1) \cup \{(j, \varphi) \mid \min(\text{Dom}(B)) \leq j < \text{Dom}(\varphi)-1\}$ or $(\text{Dom}(\varphi)-1, \varphi|_{\text{Dom}(\varphi)-1})$. First, suppose $(i, \varphi) \in \text{AVS}(\varphi|\text{Dom}(\varphi)-1) \cup \{(j, \varphi) \mid \min(\text{Dom}(B)) \leq j < \text{Dom}(\varphi)-1\}$. If $(i, \varphi) \not\in \text{AVS}(\varphi)$, we would have $(i, \varphi) \in \text{AVS}(\varphi|\text{Dom}(\varphi)-1) \cup \text{AVS}(\varphi)$ and $(i, \varphi) \not\in \{(j, \varphi) \mid \min(\text{Dom}(B)) \leq j < \text{Dom}(\varphi)-1\}$,
which contradicts (iv). In the first case, we thus have \((i, \bar{s}_i) \in \text{AVS}(\bar{s})\). Now, suppose \((i, \bar{s}_i) \in \{(\text{Dom}(\bar{s}))-1, \bar{s}_{\text{Dom}(\bar{s})-1}\}\). Then we have \(i = \text{Dom}(\bar{s})-1\) and \(P(\bar{s}_{\text{Dom}(\bar{s})-1}) = C(\bar{s})\) and thus, with Theorem 2-82, that in the second case it holds as well that \((i, \bar{s}_i) \in \text{AVS}(\bar{s})\).

\(Ad \ (vi)\): First, suppose \((i, \bar{s}_i) \in \text{AVAS}(\bar{s} \setminus \text{Dom}(\bar{s}))-1) / \text{AVAS}(\bar{s})\). Then we have \((i, \bar{s}_i) \in (\text{AVS}(\bar{s}) \setminus \text{Dom}(\bar{s}))-1) \cap \text{AS}(\bar{s} \setminus \text{Dom}(\bar{s}))-1) / (\text{AVS}(\bar{s}) \cap \text{AS}(\bar{s}))\). Since \(\text{AS}(\bar{s} \setminus \text{Dom}(\bar{s}))-1) \subset \text{AS}(\bar{s})\), we have \((i, \bar{s}_i) \in \text{AS}(\bar{s})\) and thus \((i, \bar{s}_i) \notin \text{AVS}(\bar{s})\) and hence \((i, \bar{s}_i) \in \text{AVS}(\bar{s}) \setminus \text{Dom}(\bar{s}))-1) / \text{AVAS}(\bar{s})\). With (iv) and (i), it thus holds that \((i, \bar{s}_i) \in \mathcal{B}\). Then we have \((i, \bar{s}_i) \in \text{AS}(\bar{s}) \cap \mathcal{B}\) and hence there is, with Theorem 2-47, a \(\mathcal{C} \subset \mathcal{B}\) such that \(\mathcal{C}\) is a closed segment in \(\bar{s}\) and \(i = \min(\text{Dom}(\mathcal{C}))\). Because of \((i, \bar{s}_i) \in \text{AVS}(\bar{s}) \setminus \text{Dom}(\bar{s}))-1)\), \(\mathcal{C}\) is then not a closed segment in \(\bar{s} \setminus \text{Dom}(\bar{s}))-1\). With (iii), we then have \(\mathcal{C} = \mathcal{B}\) and thus \(i = \min(\text{Dom}(\mathcal{C})) = \min(\text{Dom}(\mathcal{B}))\). Then we have \((i, \bar{s}_i) = (\min(\text{Dom}(\mathcal{B})), \bar{s}_{\min(\text{Dom}(\mathcal{B}))})\).

Now, we have to show that \(\{(\min(\text{Dom}(\mathcal{B})), \bar{s}_{\min(\text{Dom}(\mathcal{B}))})\} \subset \text{AVAS}(\bar{s}) \setminus \text{Dom}(\bar{s}))-1) / \text{AVAS}(\bar{s})\). First, we have \((\min(\text{Dom}(\mathcal{B})), \bar{s}_{\min(\text{Dom}(\mathcal{B}))}) \in \text{AS}(\bar{s})\). Suppose for contradiction that there is a closed segment \(\mathcal{C}\) in \(\bar{s} \setminus \text{Dom}(\bar{s}))-1\) such that \(\min(\text{Dom}(\mathcal{C})) \leq \min(\text{Dom}(\mathcal{B})) < \max(\text{Dom}(\mathcal{C}))\). Then we would have \(\mathcal{C} \cap \mathcal{B} \setminus \text{Dom}(\bar{s}))-1\neq \emptyset\). But with (ii), we would then have \(\min(\text{Dom}(\mathcal{B})) < \min(\text{Dom}(\mathcal{C}))\). Contradiction! Therefore there is no such closed segment \(\mathcal{C}\) in \(\bar{s} \setminus \text{Dom}(\bar{s}))-1\) and hence we have \((\min(\text{Dom}(\mathcal{B})), \bar{s}_{\min(\text{Dom}(\mathcal{B}))}) \in \text{AVS}(\bar{s} \setminus \text{Dom}(\bar{s}))-1\). On the other hand, we have with \(\mathcal{B}\) itself a closed segment \(\mathcal{B}'\) in \(\bar{s}\) such that \(\min(\text{Dom}(\mathcal{B}'))) \leq \min(\text{Dom}(\mathcal{B})) < \max(\text{Dom}(\mathcal{B}')))\) and thus we have \((\min(\text{Dom}(\mathcal{B})), \bar{s}_{\min(\text{Dom}(\mathcal{B}))}) \notin \text{AVS}(\bar{s})\) and hence \((\min(\text{Dom}(\mathcal{B})), \bar{s}_{\min(\text{Dom}(\mathcal{B}))}) \in \text{AVS}(\bar{s} \setminus \text{Dom}(\bar{s}))-1) / \text{AVAS}(\bar{s})\).

\(Ad \ (vii)\): First, suppose \((i, \bar{s}_i) \in \text{AVAS}(\bar{s} \setminus \text{Dom}(\bar{s}))-1)\). Then we have \((i, \bar{s}_i) \in \text{AVAS}(\bar{s})\) or \((i, \bar{s}_i) \notin \text{AVAS}(\bar{s})\). Now, suppose \((i, \bar{s}_i) \notin \text{AVAS}(\bar{s})\). Then we have \((i, \bar{s}_i) \in \text{AVAS}(\bar{s} \setminus \text{Dom}(\bar{s}))-1) / \text{AVAS}(\bar{s})\) and thus, with (vi), \((i, \bar{s}_i) \in \{(\min(\text{Dom}(\mathcal{B})), \bar{s}_{\min(\text{Dom}(\mathcal{B}))})\}\). Therefore we have in both cases \((i, \bar{s}_i) \in \text{AVAS}(\bar{s} \setminus \text{Dom}(\bar{s}))-1) \cup \{(\min(\text{Dom}(\mathcal{B})), \bar{s}_{\min(\text{Dom}(\mathcal{B}))})\}\). Now, suppose \((i, \bar{s}_i) \in \text{AVAS}(\bar{s} \setminus \text{Dom}(\bar{s}))-1) \cup \{(\min(\text{Dom}(\mathcal{B})), \bar{s}_{\min(\text{Dom}(\mathcal{B}))})\}\). First, suppose \((i, \bar{s}_i) \in \text{AVAS}(\bar{s})\). Then we have \((i, \bar{s}_i) \in \text{AS}(\bar{s})\). With Theorem 2-81-(ii), we also have \((i, \bar{s}_i) \in \text{AVAS}(\bar{s} \setminus \text{Dom}(\bar{s}))-1) \cup \{(\min(\text{Dom}(\mathcal{B})), \bar{s}_{\min(\text{Dom}(\mathcal{B}))})\}\). With (i), it holds that \(\max(\text{Dom}(\mathcal{B})) = \text{Dom}(\bar{s})\). Since \(\mathcal{B}\) is a closed segment in \(\bar{s}\) and thus a CdI- or NI- or RA-like segment in \(\bar{s}\), we have, with Theorem 2-29, that \((\text{Dom}(\bar{s}))-1, \bar{s}_{\text{Dom}(\bar{s})-1}) \notin \text{AS}(\bar{s})\) and thus that \((i, \bar{s}_i) \neq \{(\text{Dom}(\bar{s}))-1, \bar{s}_{\text{Dom}(\bar{s})-1})\}\). Thus we have \((i, \bar{s}_i) \in \{(\min(\text{Dom}(\mathcal{B})), \bar{s}_{\min(\text{Dom}(\mathcal{B}))})\}\).
AVAS(\(\mathcal{F}\)\{\text{Dom}(\mathcal{F})\)-1). Now, suppose \((i, \mathcal{F}_i) \in \{(\min(\text{Dom}(\mathcal{B})), \mathcal{F}_{\min(\text{Dom}(\mathcal{B}))})\}\). With (vi), we then have again that \((i, \mathcal{F}_i) \in AVAS(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1).

Ad (viii): Suppose \(\Gamma \in AVS(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1)\)AVP(\(\mathcal{F}\)). Then there is an \(i \in \text{Dom}(AVS(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1))\) and \(\Gamma = P(\mathcal{F}_i)\). Then we have \((i, \mathcal{F}_i) \in AVS(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1)\) and \((i, \mathcal{F}_i) \not\in AVS(\mathcal{F})\), because otherwise we would have \(\Gamma \in AVP(\mathcal{F})\). With (iv), it then holds that \((i, \mathcal{F}_i) \in \{(j, \mathcal{F}_j) \mid \min(\text{Dom}(\mathcal{B})) \leq j < \text{Dom}(\mathcal{F})-1\}\). Then we have \(\Gamma \in \{P(\mathcal{F}_j) \mid \min(\text{Dom}(\mathcal{B})) \leq j < \text{Dom}(\mathcal{F})-1\}\).

Ad (ix): Suppose \(\Gamma \in AVP(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1)\)AVP(\(\mathcal{F}\)). Then there is an \(i \in \text{Dom}(AVS(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1))\) such that \(\Gamma = P(\mathcal{F}_i)\). Then we have \((i, \mathcal{F}_i) \in AVS(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1)\) and thus also \(i < \text{Dom}(\mathcal{F})-1\). We have that \(\Gamma \in \{P(\mathcal{F}_j) \mid \min(\text{Dom}(\mathcal{B})) \leq j < \text{Dom}(\mathcal{F})-1\}\) or \(\Gamma \not\in \{P(\mathcal{F}_j) \mid \min(\text{Dom}(\mathcal{B})) \leq j < \text{Dom}(\mathcal{F})-1\}\). Now, suppose \(\Gamma \not\in \{P(\mathcal{F}_j) \mid \min(\text{Dom}(\mathcal{B})) \leq j < \text{Dom}(\mathcal{F})-1\}\). Then we have \((i, \mathcal{F}_i) \not\in \{(j, \mathcal{F}_j) \mid \min(\text{Dom}(\mathcal{B})) \leq j < \text{Dom}(\mathcal{F})-1\}\) and thus \((i, \mathcal{F}_i) \in AVS(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1)\) \{\{(j, \mathcal{F}_j) \mid \min(\text{Dom}(\mathcal{B})) \leq j < \text{Dom}(\mathcal{F})-1\}\}. With (v), we then have \((i, \mathcal{F}_i) \in AVS(\mathcal{F})\) and, with \(i < \text{Dom}(\mathcal{F})-1\), it then holds that \((i, \mathcal{F}_i) \in AV(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1)\). Therefore we have \(i \in \text{Dom}(AVS(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1))\) and thus \(\Gamma \in \{P(\mathcal{F}_j) \mid j \in \text{Dom}(AVS(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1)\}\}. Therefore we have in both cases \(\Gamma \in \{P(\mathcal{F}_j) \mid j \in \text{Dom}(AVS(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1))\} \cup \{P(\mathcal{F}_j) \mid \min(\text{Dom}(\mathcal{B})) \leq j < \text{Dom}(\mathcal{F})-1\}\).

Ad (x): Suppose \(\Gamma \in AVAP(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1)\)AVP(\(\mathcal{F}\)). Then there is an \(i \in \text{Dom}(AVAP(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1))\) and \(\Gamma = P(\mathcal{F}_i)\). Then we have \((i, \mathcal{F}_i) \in AVAP(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1)\) and \((i, \mathcal{F}_i) \not\in AVAP(\mathcal{F})\), because otherwise we would have \(\Gamma \in AVAP(\mathcal{F})\). With (vi), it then follows that \((i, \mathcal{F}_i) = (\min(\text{Dom}(\mathcal{B})), \mathcal{F}_{\min(\text{Dom}(\mathcal{B}))})\). Then we have \(\Gamma = P(\mathcal{F}_i) = P(\mathcal{F}_{\min(\text{Dom}(\mathcal{B}))}) \in \{P(\mathcal{F}_{\min(\text{Dom}(\mathcal{B}))})\}\).

And last, ad (xi): With (vii) it holds that \(AVS(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1) = AVAS(\mathcal{F}) \cup \{(\min(\text{Dom}(\mathcal{B})), \mathcal{F}_{\min(\text{Dom}(\mathcal{B}))})\}\). We thus have: \(\Gamma \in AVAP(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1)\) iff there is an \(i \in \text{Dom}(AVAP(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1))\) and \(\Gamma = P(\mathcal{F}_i)\) iff there is an \(i \in \text{Dom}(AVAP(\mathcal{F}))\) \{\{\min(\text{Dom}(\mathcal{B}))\}\} and \(\Gamma = P(\mathcal{F}_i)\) iff \(\Gamma \in AVAP(\mathcal{F}) \cup \{P(\mathcal{F}_{\min(\text{Dom}(\mathcal{B}))})\}\). Hence we have \(AVAP(\mathcal{F}\{\text{Dom}(\mathcal{F})\}-1) = AVAP(\mathcal{F}) \cup \{P(\mathcal{F}_{\min(\text{Dom}(\mathcal{B}))})\}\).
Theorem 2-84. AVS-reduction in the transition from \( \mathcal{S} \mid \text{Dom}(\mathcal{S}) \) to \( \mathcal{S} \) if and only if a new closed segment emerges

If \( \mathcal{S} \in \text{SEQ} \), then:

\[
\text{AVS}(\mathcal{S}) \setminus \text{AVS}(\mathcal{S}) \neq \emptyset
\]

iff

There is a \( \mathcal{B} \) such that

(i) \( \mathcal{B} \) is a closed segment in \( \mathcal{S} \), and

(ii) \( \min(\text{Dom}(\mathcal{B})) \leq \text{Dom}(\mathcal{S})-2 \) and \( \max(\text{Dom}(\mathcal{B})) = \text{Dom}(\mathcal{S})-1 \).

Proof: Suppose \( \mathcal{S} \in \text{SEQ} \). The left-right-direction follows immediately with Theorem 2-83. Now, for the right-left-direction, suppose there is a \( \mathcal{B} \) such that \( \mathcal{B} \) is a closed segment in \( \mathcal{S} \) and \( \min(\text{Dom}(\mathcal{B})) \leq \text{Dom}(\mathcal{S})-2 \) and \( \max(\text{Dom}(\mathcal{B})) = \text{Dom}(\mathcal{S})-1 \). Then it holds that \( (\min(\text{Dom}(\mathcal{B})), \text{Dom}(\mathcal{S})-1) \in \text{AVS}(\mathcal{S}) \setminus \text{AVS}(\mathcal{S}) \).

Now, suppose \( \mathcal{C} \) is a closed segment in \( \mathcal{S} \mid \text{Dom}(\mathcal{S}) \). Because of \( \mathcal{C} \subseteq \mathcal{S} \), we then have \( \mathcal{B} \not\subseteq \mathcal{C} \). With Theorem 2-52, we then have \( \min(\text{Dom}(\mathcal{B})) \notin \text{Dom}(\mathcal{C}) \). Thus there is no closed segment \( \mathcal{C} \) in \( \mathcal{S} \) such that \( \min(\text{Dom}(\mathcal{C})) \leq \min(\text{Dom}(\mathcal{B})) \) and thus it holds that \( (\min(\text{Dom}(\mathcal{B})), \text{Dom}(\mathcal{S})-1) \in \text{AVS}(\mathcal{S}) \setminus \text{AVS}(\mathcal{S}) \). Hence we have \( (\min(\text{Dom}(\mathcal{B})), \text{Dom}(\mathcal{S})-1) \in \text{AVS}(\mathcal{S}) \setminus \text{AVS}(\mathcal{S}) \).

Theorem 2-85. AVAS-reduction in the transition from \( \mathcal{S} \mid \text{Dom}(\mathcal{S}) \) to \( \mathcal{S} \) if and only if this involves the emergence of a new closed segment whose first member is exactly the now unavailable assumption-sentence and the maximal member in AVAS(\( \mathcal{S} \mid \text{Dom}(\mathcal{S}) \))

If \( \mathcal{S} \in \text{SEQ} \), then:

\[
\text{AVAS}(\mathcal{S}) \setminus \text{AVAS}(\mathcal{S}) \neq \emptyset
\]

iff

There is a \( \mathcal{B} \) such that

(i) \( \mathcal{B} \) is a closed segment in \( \mathcal{S} \),

(ii) \( \min(\text{Dom}(\mathcal{B})) \leq \text{Dom}(\mathcal{S})-2 \) and \( \max(\text{Dom}(\mathcal{B})) = \text{Dom}(\mathcal{S})-1 \), and

(iii) \( \text{AVAS}(\mathcal{S} \mid \text{Dom}(\mathcal{S})-1) \setminus \text{AVAS}(\mathcal{S}) = \{ (\min(\text{Dom}(\mathcal{B})), \text{Dom}(\mathcal{S})-1) \} \setminus \{ (\max(\text{Dom}(\text{AVAS}(\mathcal{S} \mid \text{Dom}(\mathcal{S})-1)), \text{Dom}(\text{AVAS}(\mathcal{S} \mid \text{Dom}(\mathcal{S})-1))-1)) \} \).

Proof: Suppose \( \mathcal{S} \in \text{SEQ} \). (L-R): Suppose \( \text{AVAS}(\mathcal{S} \mid \text{Dom}(\mathcal{S})-1) \setminus \text{AVAS}(\mathcal{S}) \neq \emptyset \). With Theorem 2-73, we then have that also \( \text{AVS}(\mathcal{S} \mid \text{Dom}(\mathcal{S})-1) \setminus \text{AVS}(\mathcal{S}) \neq \emptyset \). With Theorem 2-83, there is then a \( \mathcal{B} \) such that \( \mathcal{B} \) is a closed segment in \( \mathcal{S} \) and \( \min(\text{Dom}(\mathcal{B})) \leq \text{Dom}(\mathcal{S})-2 \).
Dom(δ)-2 and max(Dom(Ω)) = Dom(δ)-1 and AVAS(δ|Dom(δ)-1)\AVAS(δ) = \{(min(Dom(Ω))), δ_{min(Dom(Ω))}\}.

Then we have min(Dom(Ω)) = max(Dom(AVAS(δ|Dom(δ)-1))). First, we have (min(Dom(Ω)), δ_{min(Dom(Ω))}) ∈ AVAS(δ|Dom(δ)-1) and thus min(Dom(Ω)) ∈ Dom(AVAS(δ|Dom(δ)-1)). Now, suppose k ∈ Dom(AVAS(δ|Dom(δ)-1)) and suppose min(Dom(Ω)) ≤ k. Then we have (k, δ_k) ∈ AVAS(δ|Dom(δ)-1) and thus (k, δ_k) ∈ AS(δ|Dom(δ)-1) and thus also (k, δ_k) ∈ AS(Ω). Also, we have min(Dom(Ω)) ≤ k < Dom(δ)-1 = max(Dom(Ω)). Thus we have k ∈ AS(δ) ∩ Dom(Ω). With Theorem 2-66, we then have k = min(Dom(Ω)) or there is a ¼ such that k = min(Dom(Ω)) and min(Dom(Ω)) < max(Dom(Ω)) < max(Dom(Ω)) = Dom(δ)-1. The second case is, however, excluded, because otherwise there would be, with Theorem 2-64-(viii) and Theorem 2-62-(viii), a closed segment ¼ in ¼ such that AVAS(δ|Dom(δ)-1)) \ AVAS(δ) = \{(min(Dom(Ω))), δ_{min(Dom(Ω))}\}. We have that ¼ is a closed segment in ¼ and, because of max(Dom(Ω)) = Dom(δ)-1, ¼ is not a segment and

\(\text{Dom}(\delta) - 2\) and \(\text{max}(\text{Dom}(\Omega)) = \text{Dom}(\delta) - 1\) and \(\text{AVAS}(\delta|\text{Dom}(\delta) - 1)\setminus\text{AVAS}(\delta) = \{(\text{min}(\text{Dom}(\Omega))), \delta_{\text{min}(\text{Dom}(\Omega))}\}\).

Then we have \(\text{min}(\text{Dom}(\Omega)) = \text{max}(\text{Dom}(\text{AVAS}(\delta|\text{Dom}(\delta) - 1)))\). First, we have \((\text{min}(\text{Dom}(\Omega)), \delta_{\text{min}(\text{Dom}(\Omega))}) \in \text{AVAS}(\delta|\text{Dom}(\delta) - 1)\) and thus \(\text{min}(\text{Dom}(\Omega)) \in \text{Dom}(\text{AVAS}(\delta|\text{Dom}(\delta) - 1))\). Now, suppose \(k \in \text{Dom}(\text{AVAS}(\delta|\text{Dom}(\delta) - 1))\) and suppose \(\text{min}(\text{Dom}(\Omega)) \leq k\). Then we have \((k, \delta_k) \in \text{AVAS}(\delta|\text{Dom}(\delta) - 1)\) and thus \((k, \delta_k) \in \text{AS}(\delta|\text{Dom}(\delta) - 1)\) and thus also \((k, \delta_k) \in \text{AS}(\Omega)\). Also, we have \(\text{min}(\text{Dom}(\Omega)) \leq k < \text{Dom}(\delta) - 1 = \text{max}(\text{Dom}(\Omega))\). Thus we have \(k \in \text{AS}(\delta) \cap \text{Dom}(\Omega)\). With Theorem 2-66, we then have \(k = \text{min}(\text{Dom}(\Omega))\) or there is a \(\frac{1}{n}\) such that \(k = \text{min}(\text{Dom}(\Omega))\) and \(\text{min}(\text{Dom}(\Omega)) < \text{max}(\text{Dom}(\Omega)) < \text{max}(\text{Dom}(\Omega)) = \text{Dom}(\delta) - 1\). The second case is, however, excluded, because otherwise there would be, with Theorem 2-64-(viii) and Theorem 2-62-(viii), a closed segment \(\frac{1}{n}\) in \(\frac{1}{n}\) such that \(\text{AVAS}(\delta|\text{Dom}(\delta) - 1))\setminus\text{AVAS}(\delta) = \{(\text{min}(\text{Dom}(\Omega))), \delta_{\text{min}(\text{Dom}(\Omega))}\}\). Then we have \(\text{AVAS}(\delta|\text{Dom}(\delta) - 1))\setminus\text{AVAS}(\delta) = \emptyset\). ■

**Theorem 2-86.** If the last member of a closed segment \(\Omega\) in \(\Omega\) is identical to the last member of \(\Omega\), then the first member of \(\Omega\) is the maximal member of \(\text{AVAS}(\delta|\text{Dom}(\delta) - 1)\) and is not any more available in \(\delta\).

If \(\Omega\) is a closed segment in \(\Omega\) and \(\text{max}(\text{Dom}(\Omega)) = \text{Dom}(\delta) - 1\), then it holds: \(\text{AVAS}(\delta|\text{Dom}(\delta) - 1)\setminus\text{AVAS}(\delta) = \{(\text{min}(\text{Dom}(\Omega))), \delta_{\text{min}(\text{Dom}(\Omega))}\}\setminus\{(\text{max}(\text{Dom}(\text{AVAS}(\delta|\text{Dom}(\delta) - 1)))), \delta_{\text{max}(\text{Dom}(\text{AVAS}(\delta|\text{Dom}(\delta) - 1)))}\}\).

**Proof:** Suppose \(\Omega\) is a closed segment in \(\Omega\) and \(\text{max}(\text{Dom}(\Omega)) = \text{Dom}(\delta) - 1\). Then \(\Omega\) is a Cdl- or NI- or RA-like segment in \(\Omega\) and \(\delta \in \text{SEQ}\). With Theorem 2-31, we thus have \(\text{min}(\text{Dom}(\Omega)) < \text{max}(\text{Dom}(\Omega)) = \text{Dom}(\delta) - 1\) and hence \(\text{min}(\text{Dom}(\Omega)) \leq \text{Dom}(\delta) - 2\). With Theorem 2-84, we then have \(\text{AVS}(\delta|\text{Dom}(\delta))\setminus\text{AVAS}(\delta) \neq \emptyset\). From this, we get with Theorem 2-83-(vi) that there is a \(\mathcal{C}\) such that \(\mathcal{C}\) is a closed segment in \(\Omega\) and \(\text{AVAS}(\delta|\text{Dom}(\delta) - 1)\setminus\text{AVAS}(\delta) = \{(\text{min}(\text{Dom}(\mathcal{C}))), \delta_{\text{min}(\text{Dom}(\mathcal{C}))}\}\). We have that \(\Omega\) is a closed segment in \(\Omega\) and, because of \(\text{max}(\text{Dom}(\Omega)) = \text{Dom}(\delta) - 1\), \(\Omega\) is not a segment and
thus not a closed segment in $\text{Dom}(\bar{\gamma})$. With Theorem 2-83-(iii), we then have $\mathcal{B} = \mathcal{C}$ and thus $\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \setminus \text{AVAS}(\bar{\gamma}) = \{(\min(\text{Dom}(\mathcal{B})), \, \bar{\gamma}_{\min(\text{Dom}(\mathcal{B}))})\}$. With Theorem 2-85, it follows that $\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \setminus \text{AVAS}(\bar{\gamma}) = \{(\min(\text{Dom}(\mathcal{B})), \, \bar{\gamma}_{\min(\text{Dom}(\mathcal{B}))})\} = \{(\max(\text{Dom}(\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1))), \, \bar{\gamma}_{\max(\text{Dom}(\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1)))}\}$. ■

**Theorem 2-87.** In the transition from $\bar{\gamma}|\text{Dom}(\bar{\gamma})-1$ to $\bar{\gamma}$, the number of available assumption-sentences is reduced at most by one.

If $\bar{\gamma} \in \text{SEQ}$, then $|\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \setminus \text{AVAS}(\bar{\gamma})| \leq 1$.

**Proof:** Suppose $\bar{\gamma} \in \text{SEQ}$. Then we have $\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \setminus \text{AVAS}(\bar{\gamma}) = \emptyset$ or $\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \setminus \text{AVAS}(\bar{\gamma}) \neq \emptyset$. In the first case, we have $|(\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \setminus \text{AVAS}(\bar{\gamma})| = 0$. Now, suppose $\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \setminus \text{AVAS}(\bar{\gamma}) \neq \emptyset$. With Theorem 2-85, there is then a closed segment $\mathcal{B}$ in $\bar{\gamma}$ such that $\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \setminus \text{AVAS}(\bar{\gamma}) = \{(\min(\text{Dom}(\mathcal{B})), \, \bar{\gamma}_{\min(\text{Dom}(\mathcal{B}))})\}$. Then we have $|\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \setminus \text{AVAS}(\bar{\gamma})| = 1$. ■

**Theorem 2-88.** In the transition from $\bar{\gamma}|\text{Dom}(\bar{\gamma})-1$ to $\bar{\gamma}$ proper AVAP-inclusion implies proper AVAS-inclusion.

If $\bar{\gamma} \in \text{SEQ}$ and $\text{AVAP}(\bar{\gamma}) \subseteq \text{AVAP}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1)$, then $\text{AVAS}(\bar{\gamma}) \subseteq \text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1)$.

**Proof:** Suppose $\bar{\gamma} \in \text{SEQ}$ and suppose $\text{AVAP}(\bar{\gamma}) \subseteq \text{AVAP}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1)$. Then there is a $\Gamma \in \text{CFORM}$ such that $\Gamma \in \text{AVAP}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \setminus \text{AVAP}(\bar{\gamma})$. Then there is an $i \in \text{Dom}(\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1))$ such that $\Gamma = \text{P}(\bar{\gamma})$. Then we have $i \not\in \text{Dom}(\text{AVAS}(\bar{\gamma}))$, because otherwise we would have $\Gamma \in \text{AVAP}(\bar{\gamma})$. Thus we have $\text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \setminus \text{AVAS}(\bar{\gamma}) \neq \emptyset$. With Theorem 2-85, there is then a closed segment $\mathcal{B}$ in $\bar{\gamma}$ such that $\max(\text{Dom}(\mathcal{B})) = \text{Dom}(\bar{\gamma})$. Then $\mathcal{B}$ is a CdI- or Ni- or RA-like segment in $\bar{\gamma}$. It then follows, with Theorem 2-29, that $(\text{Dom}(\bar{\gamma})-1, \, \bar{\gamma}_{\text{Dom}(\bar{\gamma})-1}) \not\in \text{AS}(\bar{\gamma})$ and thus $(\text{Dom}(\bar{\gamma})-1, \, \bar{\gamma}_{\text{Dom}(\bar{\gamma})-1}) \not\in \text{AVAS}(\bar{\gamma})$. With Theorem 2-81, we have $\text{AVAS}(\bar{\gamma}) \subseteq \text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \cup \{(\text{Dom}(\bar{\gamma})-1, \, \bar{\gamma}_{\text{Dom}(\bar{\gamma})-1})\}$. Then we have $\text{AVAS}(\bar{\gamma}) \subseteq \text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1)$, and, with $(i, \, \bar{\gamma}) \in \text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1) \setminus \text{AVAS}(\bar{\gamma})$, it follows that $\text{AVAS}(\bar{\gamma}) \subseteq \text{AVAS}(\bar{\gamma}|\text{Dom}(\bar{\gamma})-1)$. ■
Theorem 2-89. Preparatory theorem (a) for Theorem 2-91, Theorem 2-92 and Theorem 2-93
If $\mathfrak{a}$ is a segment in $\mathfrak{a}$ and $l \in \text{Dom}(\mathfrak{a})$, then:
$$(l, \mathfrak{a}_l) \in \text{AVS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$$
iff
For all closed segments $\mathcal{C}$ in $\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a}))$:
$$l < \text{min}(\text{Dom}(\mathcal{C})) \text{ or } \text{max}(\text{Dom}(\mathcal{C})) \leq l.$$  

Proof: Suppose $\mathfrak{a}$ is a segment in $\mathfrak{a}$ and $l \in \text{Dom}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$. (L-R): First, suppose $$(l, \mathfrak{a}_l) \in \text{AVS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$$. Now, suppose $\mathcal{C}$ is a closed segment in $\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a}))$. If $\text{min}(\text{Dom}(\mathcal{C})) \leq l \leq \text{max}(\text{Dom}(\mathcal{C}))$, then we would have $$(l, \mathfrak{a}_l) \notin \text{AVS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$$, which contradicts the hypothesis. Therefore we have $l \neq \text{min}(\text{Dom}(\mathcal{C})) \text{ or } \text{max}(\text{Dom}(\mathcal{C})) \leq l$. (R-L): Now, suppose for all closed segments $\mathcal{C}$ in $\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a}))$:
$$l < \text{min}(\text{Dom}(\mathcal{C})) \text{ or } \text{max}(\text{Dom}(\mathcal{C})) \leq l.$$ By hypothesis, we have $l \in \text{Dom}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$ and thus $P(\mathfrak{a}_l)$ is available in $\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a}))$ at $l$. Hence we have $$(l, \mathfrak{a}_l) \in \text{AVS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$. □

Theorem 2-90. Preparatory theorem (b) for Theorem 2-91, Theorem 2-92 and Theorem 2-93
If $\mathfrak{a}$ is a segment in $\mathfrak{a}$ and $l \in \text{Dom}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$, then:
$$(l, \mathfrak{a}_l) \in \text{AVS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$$
iff
$$(l, \mathfrak{a}_l) \in \text{AS}(\mathfrak{a})$$ and for all closed segments $\mathcal{C}$ in $\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a}))$:
$$l < \text{min}(\text{Dom}(\mathcal{C})) \text{ or } \text{max}(\text{Dom}(\mathcal{C})) \leq l.$$  

Proof: Suppose $\mathfrak{a}$ is a segment in $\mathfrak{a}$ and $l \in \text{Dom}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$. (L-R): First, suppose $$(l, \mathfrak{a}_l) \in \text{AVS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$$. Then we have $$(l, \mathfrak{a}_l) \in \text{AVS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a}))) \cap \text{AS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$$. Because of $\text{AS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a}))) \subseteq \text{AS}(\mathfrak{a})$, we thus have $$(l, \mathfrak{a}_l) \in \text{AS}(\mathfrak{a})$$. With $$(l, \mathfrak{a}_l) \in \text{AVS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$$ and Theorem 2-89, it follows that for all closed segments $\mathcal{C}$ in $\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a}))$:
$$l < \text{min}(\text{Dom}(\mathcal{C})) \text{ or } \text{max}(\text{Dom}(\mathcal{C})) \leq l.$$ (R-L): Now, suppose $$(l, \mathfrak{a}_l) \in \text{AS}(\mathfrak{a})$$ and suppose for all closed segments $\mathcal{C}$ in $\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a}))$:
$$l < \text{min}(\text{Dom}(\mathcal{C})) \text{ or } \text{max}(\text{Dom}(\mathcal{C})) \leq l.$$ By hypothesis, we have $l \in \text{Dom}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$ and thus we have $$(l, \mathfrak{a}_l) \in \text{AS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$$. With Theorem 2-89, it follows that $$(l, \mathfrak{a}_l) \in \text{AVS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$$ and hence we have $$(l, \mathfrak{a}_l) \in \text{AVAS}(\mathfrak{a} \mid \text{max}(\text{Dom}(\mathfrak{a})))$. □
The Availability of Propositions

**Theorem 2-91. Cdl-closes!-Theorem**

\( \mathfrak{A} \) is a segment in \( \mathfrak{f} \) and there are \( \Delta, \Gamma \in \text{CFORM} \) such that

(i) \( P(\mathfrak{f}_{\text{min}(\text{Dom}(\mathfrak{A}))}) = \Delta \) and \( (\text{min}(\text{Dom}(\mathfrak{A})), \mathfrak{f}_{\text{min}(\text{Dom}(\mathfrak{A}))}) \in \text{AVAS}(\mathfrak{f}\upharpoonright\text{max}(\text{Dom}(\mathfrak{A}))) \),

(ii) \( P(\mathfrak{f}_{\text{max}(\text{Dom}(\mathfrak{A}))}^{-1}) = \Gamma \),

(iii) There is no \( r \) such that \( \text{min}(\text{Dom}(\mathfrak{A})) < r \leq \text{max}(\text{Dom}(\mathfrak{A}))-1 \) and \( (r, \mathfrak{f}_{r}) \in \text{AVAS}(\mathfrak{f}\upharpoonright\text{max}(\text{Dom}(\mathfrak{A}))) \), and

(iv) \( \mathfrak{f}_{\text{max}(\text{Dom}(\mathfrak{A}))} = \text{"Therefore } \Delta \rightarrow \Gamma \" \)

iff

\( \mathfrak{A} \) is a \( \text{CdI-closed} \) segment in \( \mathfrak{f} \).

**Proof**: Follows directly from Theorem 2-67, Theorem 2-89 and Theorem 2-90. ■

**Theorem 2-92. NI-closes!-Theorem**

\( \mathfrak{A} \) is a segment in \( \mathfrak{f} \) and there are \( \Delta, \Gamma \in \text{CFORM} \) and \( i \in \text{Dom}(\mathfrak{A}) \) such that

(i) \( \text{min}(\text{Dom}(\mathfrak{A})) \leq i < \text{max}(\text{Dom}(\mathfrak{A})) \),

(ii) \( P(\mathfrak{f}_{\text{min}(\text{Dom}(\mathfrak{A}))}) = \Delta \) and \( (\text{min}(\text{Dom}(\mathfrak{A})), \mathfrak{f}_{\text{min}(\text{Dom}(\mathfrak{A}))}) \in \text{AVAS}(\mathfrak{f}\upharpoonright\text{max}(\text{Dom}(\mathfrak{A}))) \),

(iii) \( P(\mathfrak{f}_{i}) = \Gamma \) and \( P(\mathfrak{f}_{\text{max}(\text{Dom}(\mathfrak{A}))}^{-1}) = \text{"} \neg \Gamma \text{"} \)

or

\( P(\mathfrak{f}_{i}) = \text{"} \neg \Gamma \text{"} \) and \( P(\mathfrak{f}_{\text{max}(\text{Dom}(\mathfrak{A}))}^{-1}) = \Gamma \),

(iv) \( (i, \mathfrak{f}_{i}) \in \text{AVS}(\mathfrak{f}\upharpoonright\text{max}(\text{Dom}(\mathfrak{A}))) \),

(v) There is no \( r \) such that \( \text{min}(\text{Dom}(\mathfrak{A})) < r \leq \text{max}(\text{Dom}(\mathfrak{A}))-1 \) and \( (r, \mathfrak{f}_{r}) \in \text{AVAS}(\mathfrak{f}\upharpoonright\text{max}(\text{Dom}(\mathfrak{A}))) \), and

(vi) \( \mathfrak{f}_{\text{max}(\text{Dom}(\mathfrak{A}))} = \text{"Therefore } \neg \Delta \text{"} \)

iff

\( \mathfrak{A} \) is an NI-closed segment in \( \mathfrak{f} \).

**Proof**: Follows directly from Theorem 2-68, Theorem 2-89 and Theorem 2-90. ■
Theorem 2-93. **PE-closes!-Theorem**

Let $\mathfrak{A}$ be a segment in $\mathcal{H}$ and there are $\xi \in \text{VAR}$, $\beta \in \text{PAR}$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi\}$, $\Gamma \in \text{CFORM}$ and $\mathfrak{B} \in \text{SG}(\mathcal{H})$ such that

(i) $P(\mathfrak{H}_{\min(Dom(\mathfrak{B}))}) = \forall \xi \in \mathfrak{A}$ and $(\min(Dom(\mathfrak{B})), \mathfrak{H}_{\min(Dom(\mathfrak{B}))}) \in \text{AVS}(\mathfrak{A}|\max(Dom(\mathfrak{A})))$,

(ii) $P(\mathfrak{H}_{\min(Dom(\mathfrak{B}))+1}) = [\beta, \xi, \Delta] \text{ and } (\min(Dom(\mathfrak{B}))+1, \mathfrak{H}_{\min(Dom(\mathfrak{B}))+1}) \in \text{AVAS}(\mathfrak{A}|\max(Dom(\mathfrak{A})))$,

(iii) $P(\mathfrak{H}_{\max(Dom(\mathfrak{B}))}) = \text{Therefore } \Gamma$,

(iv) $\mathfrak{H}_{\max(Dom(\mathfrak{B}))} = \mathfrak{A}$,

(v) $\beta \notin \text{STSF} (\{\Delta, \Gamma\})$,

(vi) There is no $j \leq \min(Dom(\mathfrak{B}))$ such that $\beta \in \text{ST}(\mathfrak{H}_j)$,

(vii) $\mathfrak{A} = \mathfrak{B} \setminus \{\min(Dom(\mathfrak{B})), \mathfrak{H}_{\min(Dom(\mathfrak{B}))}\}$ and

(viii) There is no $r$ such that $\min(Dom(\mathfrak{A})) < r \leq \max(Dom(\mathfrak{A}))-1$ and $(r, \mathfrak{H}_r) \in \text{AVAS}(\mathfrak{A}|\max(Dom(\mathfrak{A})))$

iff

$\mathfrak{A}$ is a PE-closed segment in $\mathfrak{H}$.

*Proof: Follows directly from Theorem 2-69, Theorem 2-89 and Theorem 2-90. ■
3 The Speech Act Calculus

The meta-theory of the calculus is now sufficiently developed, so that the calculus can be established (3.1). Then, we will provide a derivation and a consequence concept for the calculus (3.2). The chapter closes with the proof of theorems that describe the working of the calculus and are useful for the further development (3.3).

3.1 The Calculus

With the Speech Act Calculus, the rules for assuming and inferring are established, which ultimately serve to govern the derivation of propositions from sets of propositions. In preparation, we note: An author assumes a proposition \( \Gamma \) by uttering the sentence "Suppose \( \Gamma \)". An author infers a proposition \( \Gamma \) by uttering the sentence "Therefore \( \Gamma \)". An author utters the empty sentence sequence by not uttering anything. An author utters a non-empty sentence sequence \( \xi \) by successively uttering \( \xi_i \) for every \( i \in \text{Dom}(\xi) \). An author extends a sentence sequence \( \xi \) to a sentence sequence \( \xi^* \) if he has uttered \( \xi \) and now utters a sentence sequence \( \xi' \) such that \( \xi^* = \xi \xi' \). An author thus extends an uttered sentence sequence \( \xi \) to the sentence sequence \( \xi \cup \{(\text{Dom}(\xi), "Suppose \Gamma")\} \), by assuming \( \Gamma \), i.e. by uttering "Suppose \( \Gamma \)". An author extends an uttered sentence sequence \( \xi \) to the sentence sequence \( \xi \cup \{(\text{Dom}(\xi), "Therefore \Gamma")\} \) by inferring \( \Gamma \), i.e. by uttering "Therefore \( \Gamma \)".\(^{12}\)

The rules of the calculus – and only these – are to allow one to extend an already uttered sentence sequence \( \xi \) to a sentence sequence \( \xi' \) with \( \text{Dom}(\xi') = \text{Dom}(\xi) + 1 \). After the establishment of the rules, a derivation and a consequence concept can be established, according to which derivations will be exactly those non-empty sentence sequences that can in principle be uttered in accordance with the rules of the calculus (\( \uparrow \) 3.2).

As is usual for pragmatised natural deduction calculi, there is a rule of assumption (Speech-act rule 3-1) and 16 inference rules (Speech-act rule 3-2 to Speech-act rule 3-17). Additionally, the calculus contains an interdiction clause (IDC, Speech-act rule 3-18),

\(^{12}\) For the relation between the performance of speech acts and sequences of speech acts and the uttering of sentences and sequences of sentences, see HINST, P.: Logischer Grundkurs, p. 58–71, SIEGWART, G.: Vorfragen, p. 25–32, Denkwerkzeuge, p. 39–52, and, most recent and in English, Alethic Acts. Here, we obviously assume that the expressions and concatenations thereof stipulated by Postulate 1-1 to Postulate 1-3 are utterable entities.
which forbids all extensions that are not permitted by one of the rules from Speech-act rule 3-1 to Speech-act rule 3-17. Among the rules of inference, there are two for each of the connectives, quantificators (resp. quantifiers) and for the identity predicate. One of the rules regulates the introduction of the respective operator and the other rule regulates its elimination.

A shorthand version of the availability conception may facilitate an easier understanding of the presentation of the calculus: If $\mathcal{S}$ is a sentence sequence, then $(i, \mathcal{S}_i)$ is in $\text{AVS}(\mathcal{S})$ if and only if the proposition of $\mathcal{S}_i$ is available in $\mathcal{S}$ at $i$. Furthermore, $(i, \mathcal{S}_i)$ is in $\text{AVAS}(\mathcal{S})$ if and only if the proposition of $\mathcal{S}_i$ is available in $\mathcal{S}$ at $i$ and $\mathcal{S}_i$ is an assumption-sentence. $\Gamma$ is an element of $\text{AVP}(\mathcal{S})$ if and only if there is $(i, \mathcal{S}_i) \in \text{AVS}(\mathcal{S})$ such that $\Gamma$ is the proposition of $\mathcal{S}_i$, and $\Gamma$ is an element of $\text{AVAP}(\mathcal{S})$ if and only if there is $(i, \mathcal{S}_i) \in \text{AVAS}(\mathcal{S})$ such that $\Gamma$ is the proposition of $\mathcal{S}_i$.

In order to give an intuitively accessible short version of the rules, we stipulate: If one has uttered a sentence sequence $\mathcal{S}$ and $\Gamma$ is available in $\mathcal{S}$ at $i$, then one has gained $\Gamma$ in $\mathcal{S}$ at $i$. If $\Delta$ is the last assumption made in uttering $\mathcal{S}$ that is still available, and if one has gained $\Gamma$ in $\mathcal{S}$ after or with the assumption of $\Delta$, then one has discharged the assumption of $\Delta$ at $i$.

Now the short version of the rules, in which all reference to sentence sequences, positions and all grammatical specifications are neglected: One may assume any proposition $\Gamma$ (AR); if one has last gained $\Gamma$ departing from the assumption of $\Delta$, then one may infer $\neg \Delta \rightarrow \Gamma$ and thus discharge the assumption of $\Delta$ (CdI); if one has gained $\neg \Delta$ and $\neg \Delta \rightarrow \Gamma$, then one may infer $\neg \Delta \rightarrow \Gamma$, then one may infer $\Gamma$ (CdE); if one has gained $\Delta$ and $\Gamma$, then one may infer $\Delta \neg \Gamma$ (CI); if one has gained $\Delta \neg \Gamma$ or gained $\Gamma \Delta \neg \Gamma$, then one may infer $\neg \Delta \rightarrow \Gamma$ (BI); if one has gained $\Delta \rightarrow \Gamma$ and $\Gamma \rightarrow \Delta$, then one may infer $\neg \Delta \leftrightarrow \Gamma$ (BI); if one has gained $\Delta \rightarrow \Gamma$ or gained $\Delta$ and $\neg \Delta \rightarrow \Gamma$, then one may infer $\Gamma$ (BE); if one has gained $\neg \Gamma$ or gained $\Delta$, then one may infer $\neg \Delta \rightarrow \Gamma$ (DI); if one has gained $\neg \Delta \rightarrow \Gamma$, then one may infer $\neg \Gamma$ (DE); if one has gained either $\neg \Gamma$ and last $\neg \Gamma$ or $\neg \neg \Gamma$ and last $\neg \Delta$ departing from the assumption of $\Delta$, then one may infer $\neg \neg \Gamma$ and thus discharge the assumption of $\Delta$ (ND); if one has gained $\neg \neg \Gamma$, then one may infer $\neg \neg \Gamma$ (NI); if one has
gained \([\beta, \xi, \Delta]\), where \(\beta\) is not a subterm of \(\Delta\) or of any available assumption, then one may infer \(\forall \xi \Delta\) (UI); if one has gained \(\forall \xi \Delta\), then one may infer \([\varepsilon, \xi, \Delta]\) (UE); if one has gained \([\varepsilon, \xi, \Delta]\), then one may infer \(\forall \xi \Delta\) (PI); if one has gained \(\forall \xi \Delta\), next assumed \([\beta, \xi, \Delta]\), where \(\beta\) is a new parameter and not a subterm of \(\Delta\), and then, departing from the assumption of \([\beta, \xi, \Delta]\), last gained \(\Gamma\), where \(\beta\) is not a subterm of \(\Gamma\), then one may infer \(\Gamma\) and thus discharge the assumption of \([\beta, \xi, \Delta]\) (PE); one may infer \(\theta = \theta\) (II); if one has gained \(\theta = \theta\), then one may infer \([\theta, \xi, \Delta]\) (IE); that is all one is allowed to do (IDC).

Now follow the rules of the Speech Act Calculus in their authoritative formulation:

**Speech-act rule 3-1. Rule of Assumption (AR)**

If one has uttered \(\delta_j \in \text{SEQ}\) and if \(\Gamma \in \text{CFORM}\), then one may extend \(\delta_j\) to \(\delta_j \cup \{(\text{Dom}(\delta_j), \theta)\} \) (Suppose \(\Gamma\)).

**Speech-act rule 3-2. Rule of Conditional Introduction (CdI)**

If one has uttered \(\delta_j \in \text{SEQ}\) and if \(\Delta, \Gamma \in \text{CFORM}\) and \(i \in \text{Dom}(\delta_j)\), and

(i) \(P(\delta_j) = \Delta\) and \((i, \delta_j) \in \text{AVAS}(\delta_j)\),
(ii) \(P(\delta_j \text{Dom}(\delta_j)-1) = \Gamma\), and
(iii) There is no \(l\) such that \(i < l \leq \text{Dom}(\delta_j)-1\) and \((l, \delta_j) \in \text{AVAS}(\delta_j)\),

then one may extend \(\delta_j\) to \(\delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Delta \rightarrow \Gamma)\}\).

Note that applying the rule of conditional introduction generates CdI-closed segments according to Definition 2-23 (cf. Theorem 2-91). If one extends \(\delta_j\) to \(\delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Delta \rightarrow \Gamma)\}\) by CdI, then none of the propositions that one inferred or assumed by uttering \(\delta_j\) after (and including) the \(i^{th}\) member is available in \(\delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Delta \rightarrow \Gamma)\}\), except for propositions that were available in \(\delta_j\) before the \(i^{th}\) member (cf. Definition 2-26). Of course, this does not apply to the newly available conditional \(\forall \Delta \rightarrow \Gamma\), as it is the proposition of the new last member and thus available in the resulting sentence sequence in any case (cf. Theorem 2-82). Since the proposition of the last member of a sentence sequence \(\delta_j\) is always available in \(\delta_j\) at \(\text{Dom}(\delta_j)-1\), it also suffices in clause (ii) of the rule to demand solely that the consequent of the conditional one wants to infer is the proposition of the last member of \(\delta_j\), without additionally demanding that that proposition is also available there. Similar remarks apply to Speech-act rule 3-10 (NI) and Speech-act rule 3-15 (PE).
Speech-act rule 3-3. Rule of Conditional Elimination (CdE)

If one has uttered $\bar{S}_i \in \text{SEQ}$ and if $\Delta, \Gamma \in \text{CFORM}$ and $\{\Delta, \Gamma \rightarrow \Delta \} \subseteq \text{AVP}(\bar{S}_i)$, then one may extend $\bar{S}_i$ to $\bar{S}_i \cup \{\text{Dom}(\bar{S}_i), \text{Therefore } \Gamma \}$. 

Speech-act rule 3-4. Rule of Conjunction Introduction (CI)

If one has uttered $\bar{S}_i \in \text{SEQ}$ and if $\Delta, \Gamma \in \text{AVP}(\bar{S}_i)$, then one may extend $\bar{S}_i$ to $\bar{S}_i \cup \{\text{Dom}(\bar{S}_i), \text{Therefore } \Delta \land \Gamma \}$. 

Speech-act rule 3-5. Rule of Conjunction Elimination (CE)

If one has uttered $\bar{S}_i \in \text{SEQ}$ and if $\Delta, \Gamma \in \text{CFORM}$ and $\{\Delta \land \Gamma, \Gamma \not\subseteq \text{AVP}(\bar{S}_i)\} \cap \text{AVP}(\bar{S}_i) \neq \emptyset$, then one may extend $\bar{S}_i$ to $\bar{S}_i \cup \{\text{Dom}(\bar{S}_i), \text{Therefore } \Gamma \}$. 

Speech-act rule 3-6. Rule of Biconditional Introduction (BI)

If one has uttered $\bar{S}_i \in \text{SEQ}$ and if $\Delta, \Gamma \in \text{AVP}(\bar{S}_i)$ and $\{\Delta \leftrightarrow \Gamma, \Gamma \rightarrow \Delta \} \subseteq \text{AVP}(\bar{S}_i)$, then one may extend $\bar{S}_i$ to $\bar{S}_i \cup \{\text{Dom}(\bar{S}_i), \text{Therefore } \Delta \leftrightarrow \Gamma \}$. 

Here, the meta-logical requirement of separability, according to which each rule is to regulate only one operator, is violated, because the rule-antecedent demands that certain conditionals are available. The rule of biconditional introduction is thus at the same time a rule for the elimination of conditionals in certain contexts.

Speech-act rule 3-7. Rule of Biconditional Elimination (BE)

If one has uttered $\bar{S}_i \in \text{SEQ}$ and if $\Delta \in \text{AVP}(\bar{S}_i), \Gamma \in \text{CFORM}$ and $\{\Delta \leftrightarrow \Gamma, \Gamma \rightarrow \Delta \} \subseteq \text{AVP}(\bar{S}_i)$, then one may extend $\bar{S}_i$ to $\bar{S}_i \cup \{\text{Dom}(\bar{S}_i), \text{Therefore } \Gamma \}$. 

Speech-act rule 3-8. Rule of Disjunction Introduction (DI)

If one has uttered $\bar{S}_i \in \text{SEQ}$ and if $\Delta, \Gamma \in \text{AVP}(\bar{S}_i)$ and $\{\Delta \lor \Gamma, \Delta \rightarrow \Gamma \} \subseteq \text{AVP}(\bar{S}_i)$, then one may extend $\bar{S}_i$ to $\bar{S}_i \cup \{\text{Dom}(\bar{S}_i), \text{Therefore } \Delta \lor \Gamma \}$. 

Speech-act rule 3-9. Rule of Disjunction Elimination (DE)

If one has uttered $\bar{S}_i \in \text{SEQ}$ and if $\Delta \in \text{AVP}(\bar{S}_i), \Gamma \in \text{CFORM}$ and $\{\Delta \lor \Gamma, \Gamma \rightarrow \Delta \} \subseteq \text{AVP}(\bar{S}_i)$, then one may extend $\bar{S}_i$ to $\bar{S}_i \cup \{\text{Dom}(\bar{S}_i), \text{Therefore } \Gamma \}$. 

Here, the meta-logical requirement of separability is violated a second time, as the rule-antecedent demands that certain conditionals are available. The rule of disjunction elimi-
nation is thus at the same time a rule for the elimination of conditionals in certain contexts.

**Speech-act rule 3-10. Rule of Negation Introduction (NI)**

If one has uttered $\mathfrak{f}_j \in \text{SEQ}$ and if $\Delta, \Gamma \in \text{CFORM}$ and $i, j \in \text{Dom}(\mathfrak{f}_j)$ and

(i) $i \leq j$,

(ii) $P(\mathfrak{f}_i) = \Delta$ and $(i, \mathfrak{f}_i) \in \text{AVAS}(\mathfrak{f}_j)$,

(iii) $P(\mathfrak{f}_j) = \Gamma$ and $P(\mathfrak{f}_{\text{Dom}(\mathfrak{f}_j)-1}) = '\neg\Gamma'$

or

$P(\mathfrak{f}_j) = '\neg\Gamma'$ and $P(\mathfrak{f}_{\text{Dom}(\mathfrak{f}_j)+1}) = \Gamma$,

(iv) $(j, \mathfrak{f}_j) \in \text{AVS}(\mathfrak{f}_j)$, and

(v) There is no $l$, such that $i < l \leq \text{Dom}(\mathfrak{f}_j)-1$ and $(l, \mathfrak{f}_l) \in \text{AVAS}(\mathfrak{f}_j)$,

then one may extend $\mathfrak{f}_j$ to $\mathfrak{f}_j \cup \{(\text{Dom}(\mathfrak{f}_j), '\text{Therefore } \neg\Delta')\}$.

Applying the rule of negation introduction generates NI-closed segments according to Definition 2-24 (cf. Theorem 2-92). Thus, if one extends $\mathfrak{f}_j$ to $\mathfrak{f}_j \cup \{(\text{Dom}(\mathfrak{f}_j), '\text{Therefore } \neg\Delta')\}$ by NI, then none of the propositions that one inferred or assumed by uttering $\mathfrak{f}_j$ after (and including) the $i^{th}$ member is available in $\mathfrak{f}_j \cup \{(\text{Dom}(\mathfrak{f}_j), '\text{Therefore } \neg\Delta')\}$, except for propositions that were available in $\mathfrak{f}_j$ before the $i^{th}$ member (cf. Definition 2-26). Of course, this does not apply to the newly available negation $'\neg\Delta'$. Since the proposition of the last member of a sentence sequence $\mathfrak{f}_j$ is always available in $\mathfrak{f}_j$ at $\text{Dom}(\mathfrak{f}_j)-1$ (cf. Theorem 2-82), it also suffices in clause (iii) of the rule to demand that one of the two contradictory statements is available at $j$ and that the second part of the contradiction is the proposition of the last sentence of $\mathfrak{f}_j$.

**Speech-act rule 3-11. Rule of Negation Elimination (NE)**

If one has uttered $\mathfrak{f}_j \in \text{SEQ}$ and if $\Gamma \in \text{CFORM}$ and $'\neg\neg\Gamma' \in \text{AVP}(\mathfrak{f}_j)$, then one may extend $\mathfrak{f}_j$ to $\mathfrak{f}_j \cup \{(\text{Dom}(\mathfrak{f}_j), '\text{Therefore } \Gamma')\}$.

**Speech-act rule 3-12. Rule of Universal-quantifier Introduction (UI)**

If one has uttered $\mathfrak{f}_j \in \text{SEQ}$ and if $\beta \in \text{PAR}, \xi \in \text{VAR}, \Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi\}, [\beta, \xi, \Delta] \in \text{AVP}(\mathfrak{f}_j)$ and $\beta \not\in \text{STSF}(\{\Delta\} \cup \text{AVAP}(\mathfrak{f}_j))$, then one may extend $\mathfrak{f}_j$ to $\mathfrak{f}_j \cup \{(\text{Dom}(\mathfrak{f}_j), '\text{Therefore } \Lambda \xi \Delta')\}$. 
Speech-act rule 3-13. Rule of Universal-quantifier Elimination (UE)
If one has uttered \( \bar{s} \) \( \in \) SEQ and if \( \theta \) \( \in \) CTERM, \( \xi \) \( \in \) VAR, \( \Delta \) \( \in \) FORM, where \( \text{FV}(\Delta) \subseteq \{\xi\} \), and \( \forall \xi \Delta \in \text{AVP}(\bar{s}) \), then one may extend \( \bar{s} \) to \( \bar{s} \cup \{(\text{Dom}(\bar{s}), \text{Therefore } [\theta, \xi, \Delta])\} \).

Speech-act rule 3-14. Rule of Particular-quantifier Introduction (PI)
If one has uttered \( \bar{s} \) \( \in \) SEQ and if \( \theta \) \( \in \) CTERM, \( \xi \) \( \in \) VAR, \( \Delta \) \( \in \) FORM, where \( \text{FV}(\Delta) \subseteq \{\xi\} \), and \( [\theta, \xi, \Delta] \in \text{AVP}(\bar{s}) \), then one may extend \( \bar{s} \) to \( \bar{s} \cup \{(\text{Dom}(\bar{s}), \text{Therefore } \forall \xi \Delta)\} \).

Speech-act rule 3-15. Rule of Particular-quantifier Elimination (PE)
If one has uttered \( \bar{s} \) \( \in \) SEQ and if \( \beta \) \( \in \) PAR, \( \xi \) \( \in \) VAR, \( \Delta \) \( \in \) FORM, where \( \text{FV}(\Delta) \subseteq \{\xi\} \), \( \Gamma \in \text{CFORM} \) and \( i \in \text{Dom}(\bar{s}) \), and

(i) \( \text{P}(\bar{s}_i) = \forall \xi \Delta \text{ and } (i, \bar{s}_i) \in \text{AVS}(\bar{s}) \),
(ii) \( \text{P}(\bar{s}_{i-1}) = [\beta, \xi, \Delta] \text{ and } (i+1, \bar{s}_{i+1}) \in \text{AVAS}(\bar{s}) \),
(iii) \( \text{P}(\bar{s}_{\text{Dom}(\bar{s})-1}) = \Gamma \),
(iv) \( \beta \notin \text{STSF}([\Delta, \Gamma]) \),
(v) \( \text{There is no } j \leq i \text{ such that } \beta \in \text{ST}(\bar{s}_j) \),
(vi) \( \text{There is no } m \text{ such that } i+1 < m \leq \text{Dom}(\bar{s})-1 \text{ and } (m, \bar{s}_m) \in \text{AVAS}(\bar{s}) \),

then one may extend \( \bar{s} \) to \( \bar{s} \cup \{(\text{Dom}(\bar{s}), \text{Therefore } \Gamma)\} \).

Applying the rule of particular-quantifier elimination generates PE-closed segments according to Definition 2-25 (cf. Theorem 2-93). Thus, if one extends \( \bar{s} \) to \( \bar{s} \cup \{(\text{Dom}(\bar{s}), \text{Therefore } \Gamma)\} \) by PE, then none of the propositions that one inferred or assumed by uttering \( \bar{s} \) after the \( i \)th member is available in \( \bar{s} \) \( \cup \) \( \{(\text{Dom}(\bar{s}), \text{Therefore } \Gamma)\} \), except for propositions that were available in \( \bar{s} \) before the \( i+1 \)th member (cf. Definition 2-26). Of course, this does not apply to the last inferred proposition, i.e. \( \Gamma \), which is in any case available in the resulting sentence sequence. Since the proposition of the last member of a sentence sequence \( \bar{s} \) is always available in \( \bar{s} \) at \( \text{Dom}(\bar{s})-1 \) (cf. Theorem 2-82), it also suffices in clause (iii) of the rule, to demand solely that \( \Gamma \) is the proposition of the last member of \( \bar{s} \).

Speech-act rule 3-16. Rule of Identity Introduction (II)
If one has uttered \( \bar{s} \) \( \in \) SEQ and if \( \theta \) \( \in \) CTERM, then one may extend \( \bar{s} \) to \( \bar{s} \cup \{(\text{Dom}(\bar{s}), \text{Therefore } 0 = \theta)\} \).
Speech-act rule 3.17. Rule of Identity Elimination (IE)
If one has uttered $\delta_j \in \text{SEQ}$ and if $\xi \in \text{VAR}$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi\}$, $\theta_0, \theta_1 \in \text{CTERM}$ and $\{\theta_0 = \theta_1^\prime\}$, then one may extend $\delta_j$ to $\delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } [\theta_1, \xi, \Delta])\}$. 

Last, we formulate a prohibition that makes the interdictory status of the rules explicit. For this, all 17 rule-antecedents for the extension of $\delta_j$ to $\delta_j'$ are required to be unsatisfied. This condition is then sufficient for one not being allowed to extend $\delta_j$ to $\delta_j'$.

Speech-act rule 3.18. Interdiction Clause (IDC)
If $\delta_j \not\in \text{SEQ}$ or if one has not uttered $\delta_j$ or if there are no $B$, $\Gamma$, $\Delta \in \text{FORM}$ and $\theta_0, \theta_1 \in \text{CTERM}$ and $\beta \in \text{PAR}$ and $\xi \in \text{VAR}$ and $\Delta' \in \text{FORM}$, where $\text{FV}(\Delta') \subseteq \{\xi\}$, and $i, j \in \text{Dom}(\delta_j)$ such that

(i) $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{"Suppose } \Gamma\})$ or 
(ii) $P(\delta_j) = \Delta$, $(i, \delta_j) \in \text{AVS}(\delta_j)$, $P(\delta_j_{\text{Dom}(\delta_j)-1}) = \Gamma$, there is no $l$ such that $i < l \leq \text{Dom}(\delta_j)-1$ and $(l, \delta_j) \in \text{AVS}(\delta_j)$, and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Delta \rightarrow \Gamma)\}$ or 
(iii) $\{\Delta, \Delta' \rightarrow \Gamma\} \subseteq \text{AVS}(\delta_j)$ and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Gamma)\}$ or 
(iv) $\{\Delta, \Gamma\} \subseteq \text{AVP}(\delta_j)$ and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Delta \land \Gamma)\}$ or 
(v) $\{\Delta \land \Delta', \Gamma \land \Delta\} \cap \text{AVP}(\delta_j) \neq \emptyset$ and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Gamma)\}$ or 
(vi) $\{\Delta \rightarrow \Gamma\}, \{\Delta \rightarrow \Gamma\} \subseteq \text{AVP}(\delta_j)$ and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Delta \leftrightarrow \Gamma)\}$ or 
(vii) $\Delta \in \text{AVP}(\delta_j)$, $\{\Delta \leftrightarrow \Gamma\}, \{\Gamma \leftrightarrow \Delta\} \cap \text{AVP}(\delta_j) \neq \emptyset$, and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Gamma)\}$ or 
(viii) $\{\Delta, \Gamma\} \cap \text{AVP}(\delta_j) \neq \emptyset$ and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Delta \lor \Gamma)\}$ or 
(ix) $\{\text{B} \lor \Delta\}, \text{B} \rightarrow \Gamma, \{\Delta \rightarrow \Gamma\} \subseteq \text{AVP}(\delta_j)$ and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Gamma)\}$ or 
(x) $i \leq j$, $P(\delta_j) = \Delta$, $(i, \delta_j) \in \text{AVS}(\delta_j)$, $P(\delta_j) = \Gamma$ and $P(\delta_j_{\text{Dom}(\delta_j)-1}) = \text{"Therefore } \Gamma$ or $P(\delta_j) = \text{"Therefore } \Gamma$, and there is no $l$ such that $i < l \leq \text{Dom}(\delta_j)-1$ and $(l, \delta_j) \in \text{AVS}(\delta_j)$, and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Delta)\}$ or 
(xi) $\text{"Therefore } \Gamma$ in $\text{AVP}(\delta_j)$ and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Gamma)\}$ or 
(xii) $[\beta, \xi, \Delta] \in \text{AVP}(\delta_j)$, $\beta \not\in \text{STSF}([\Delta'] \cup \text{AVAP}(\delta_j))$ and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \Delta')\}$ or 
(xiii) $\text{"Therefore } \theta_0, \xi, \Delta)\}$ or 
(xiv) $[\theta_0, \xi, \Delta] \in \text{AVP}(\delta_j)$ and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \text{STSF}(\{\Delta, \Gamma\})\}$ or 
(xv) $P(\delta_j) = \text{"Therefore } \Delta$ or $\{\theta_0, \xi, \Delta\} \in \text{AVS}(\delta_j)$, $P(\delta_j_{i+1}) = [\beta, \xi, \Delta], (i+1, \delta_j_{i+1}) \in \text{AVAS}(\delta_j)$, $P(\delta_j_{\text{Dom}(\delta_j)-1}) = \Gamma$, $\beta \not\in \text{STSF}([\Delta', \Gamma])$, there is no $l$ such that $i < l \leq \text{Dom}(\delta_j)-1$ and $(l, \delta_j) \in \text{AVS}(\delta_j)$, and $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \text{STSF}(\{\Delta', \Gamma\})\}$ or 
(xvi) $\delta'_j = \delta_j \cup \{(\text{Dom}(\delta_j), \text{Therefore } \theta_0 = \theta_0')\}$ or
(xvii) \( \{ \theta_0 = \theta_1 \}, [\theta_0, \xi, \Delta] \} \subseteq \text{AVP}(\emptyset) \) and \( \emptyset' = \emptyset \cup \{(\text{Dom}(\emptyset), \text{Therefore} [\theta_1, \xi, \Delta])\} \), then one may not extend \( \emptyset \) to \( \emptyset' \).

Informally, Speech-act rule 3-18 says: If none of the rules from Speech-act rule 3-1 to Speech-act rule 3-17 allows the extension of \( \emptyset \) to \( \emptyset' \), then one may not extend \( \emptyset \) to \( \emptyset' \).

By setting the 18 rules, the calculus has now been established and can already be used. If one wants to add further rules later, e.g. rules for adducing-as-reason, stating, the positing-as-axiom or defining, one has to adapt Speech-act rule 3-18 accordingly. In the next section, we will now establish a derivation concept and a consequence concept for the calculus (3.2). Then, we will prove some theorems that shed some light on the way in which the calculus works (3.3).
3.2 Derivations and Deductive Consequence Relation

Having established the calculus, we now have to provide a derivation and a consequence concept and to prove the adequacy of the latter. Since the derivation and consequence relations are not to be tied to the actual utterance of sentence sequences, but only to their utterability in accordance with the rules, the derivation concept is not to be established with recourse to the full rules of the calculus – which always demand the utterance of a certain sentence sequence – but only with recourse to those parts of the rules that are specific to sentence sequences and independent of actual utterances.

To do this, we will first define a function for every rule of the calculus that assigns a sentence sequence \( \mathfrak{f} \) the set of sentence sequences to which an author that has uttered \( \mathfrak{f} \) may extend \( \mathfrak{f} \) in compliance with the respective rule (Definition 3-1 to Definition 3-17). Based on these functions, we will then define the function \( \text{RCE} \), which assigns a sentence sequence \( \mathfrak{f} \) the set of rule-compliant extensions of \( \mathfrak{f} \), i.e. the set of sentence sequences to which an author who has uttered \( \mathfrak{f} \) might extend \( \mathfrak{f} \) in accordance with one of the rules of the calculus (Definition 3-18). Then, we will define the set of rule-compliant sentence sequences, \( \text{RCS} \), as the set of sentence sequences for which all non-empty restrictions are rule-compliant extensions of the immediately preceding restriction (Definition 3-19). A derivation of a proposition \( \Gamma \) from a set of propositions \( X \) will then be a non-empty \( \text{RCS} \)-element for which it holds that \( C(\mathfrak{f}) = \Gamma \) and \( \text{AVAP}(\mathfrak{f}) = X \) (Definition 3-20). Then, we will introduce the concept of deductive consequence and related concepts, where a proposition \( \Gamma \) will be a deductive consequence of a set of propositions \( X \) if and only if there is a derivation of \( \Gamma \) from a \( Y \subseteq X \) (Definition 3-21).

As announced, we will first define functions analogous to the rules in 3.1:

**Definition 3-1. Assumption Function (AF)**

\[
\text{AF} = \{(\mathfrak{f}, X) \mid \mathfrak{f} \in \text{SEQ} \text{ and } X = \{\mathfrak{f}' \mid \text{There is } \Gamma \in \text{CFORM such that } \mathfrak{f}' = \mathfrak{f} \cup \{(\text{Dom}(\mathfrak{f}), \text{Suppose } \Gamma^\circ)\}\}\}.
\]

Cf. Speech-act rule 3-1. Since the set of closed formulas is not empty, we have as a corollary that \( \text{AF}(\mathfrak{f}) \) is not empty for any sentence sequence \( \mathfrak{f} \).
**Definition 3-2. Conditional Introduction Function (CdIF)**

CdIF = \{((\bar{s}, X) | \bar{s} \in \text{SEQ} and X = \{\bar{s}' | \text{there are } \Delta, \Gamma \in \text{CFORM and } i \in \text{Dom}(\bar{s}) \text{ such that}

(i) \ P(\bar{s}_i) = \Delta \text{ and } (i, \bar{s}_i) \in \text{AVAS}(\bar{s}),

(ii) \ P(\bar{s}_{\text{Dom}(\bar{s})-1}) = \Gamma,

(iii) \text{there is no } l \text{ such that } i < l \leq \text{Dom}(\bar{s})-1 \text{ and } (l, \bar{s}_l) \in \text{AVAS}(\bar{s}), \text{ and}

(iv) \ \bar{s}' = \bar{s} \cup \{(\text{Dom}(\bar{s}), 'Therefore } \Delta \rightarrow \Gamma')\}\}.

Cf. Speech-act rule 3-2.

**Definition 3-3. Conditional Elimination Function (CdEF)**

CdEF = \{((\bar{s}, X) | \bar{s} \in \text{SEQ} and X = \{\bar{s}' | \text{there are } \Delta, \Gamma \in \text{AVP}(\bar{s}) \text{ such that}

\{\ 'Therefore } \Delta \rightarrow \Gamma'\} \subseteq \text{AVP}(\bar{s}) \text{ and } \bar{s}' = \bar{s} \cup \{(\text{Dom}(\bar{s}), 'Therefore } \Gamma')\}\}.

Cf. Speech-act rule 3-3.

**Definition 3-4. Conjunction Introduction Function (CIF)**

CIF = \{((\bar{s}, X) | \bar{s} \in \text{SEQ} and X = \{\bar{s}' | \text{there are } \Delta, \Gamma \in \text{AVP}(\bar{s}) \text{ such that }

\bar{s}' = \bar{s} \cup \{(\text{Dom}(\bar{s}), 'Therefore } \Delta \land \Gamma')\}\}\}.

Cf. Speech-act rule 3-4.

**Definition 3-5. Conjunction Elimination Function (CEF)**

CEF = \{((\bar{s}, X) | \bar{s} \in \text{SEQ} and X = \{\bar{s}' | \text{there are } \Delta, \Gamma \in \text{AVP}(\bar{s}) \text{ such that }

\\{ 'Therefore } \Delta \land \Gamma' \} \cap \text{AVP}(\bar{s}) \neq \emptyset \text{ and } \bar{s}' = \bar{s} \cup \{(\text{Dom}(\bar{s}), 'Therefore } \Gamma')\}\}\}.

Cf. Speech-act rule 3-5.

**Definition 3-6. Biconditional Introduction Function (BIF)**

BIF = \{((\bar{s}, X) | \bar{s} \in \text{SEQ} and X = \{\bar{s}' | \text{there are } \Delta, \Gamma \in \text{AVP}(\bar{s}) \text{ such that }

\\{ 'Therefore } \Delta \leftrightarrow \Gamma' \} \subseteq \text{AVP}(\bar{s}) \text{ and } \bar{s}' = \bar{s} \cup \{(\text{Dom}(\bar{s}), 'Therefore } \Delta \leftrightarrow \Gamma')\}\}\}.

Cf. Speech-act rule 3-6.

**Definition 3-7. Biconditional Elimination Function (BEF)**

BEF = \{((\bar{s}, X) | \bar{s} \in \text{SEQ} and X = \{\bar{s}' | \text{there are } \Delta \in \text{AVP}(\bar{s}) \text{ and } \Gamma \in \text{CFORM such that }

\{ 'Therefore } \Delta \leftrightarrow \Gamma' \} \cap \text{AVP}(\bar{s}) \neq \emptyset \text{ and } \bar{s}' = \bar{s} \cup \{(\text{Dom}(\bar{s}), 'Therefore } \Gamma')\}\}\}.

Cf. Speech-act rule 3-7.
Definition 3-8. Disjunction Introduction Function (DIF)

\[
\text{DIF} = \{ (\delta, X) \mid \exists \delta' \in \text{SEQ} \text{ and } X = \{\delta' \mid \exists \Delta, \Gamma \in \text{CFORM} \text{ such that } \\
\{\Delta, \Gamma\} \cap \text{AVP}(\delta) \neq \emptyset \text{ and } \delta' = \delta \cup \{(\text{Dom}(\delta), \text{"Therefore } \Delta \lor \Gamma\)}\} \}.
\]

Cf. Speech-act rule 3-8.

Definition 3-9. Disjunction Elimination Function (DEF)

\[
\text{DEF} = \{ (\delta, X) \mid \exists \delta' \in \text{SEQ} \text{ and } X = \{\delta' \mid \exists \text{B}, \Delta, \Gamma \in \text{CFORM} \text{ such that } \{\text{"B } \lor \Delta\}, \text{"B } \rightarrow \Gamma\}, \text{"B } \Delta \rightarrow \Gamma\} \subseteq \text{AVP}(\delta) \text{ and } \delta' = \delta \cup \{(\text{Dom}(\delta), \text{"Therefore } \Gamma\)}\} \}.
\]


Definition 3-10. Negation Introduction Function (NIF)

\[
\text{NIF} = \{ (\delta, X) \mid \exists \delta' \in \text{SEQ} \text{ and } X = \{\delta' \mid \exists \Delta, \Gamma \in \text{CFORM} \text{ and } i, j \in \text{Dom}(\delta) \text{ such that } \\
(i) \quad i \leq j, \\
(ii) \quad \text{P}(\delta_i) = \Delta \text{ and } (i, \delta_i) \in \text{AVAS}(\delta), \\
(iii) \quad \text{P}(\delta_j) = \Gamma \text{ and } \text{P}(\delta_{\text{Dom}(\delta)_-}) = \Gamma \text{ or } \\
\quad \text{P}(\delta_j) = \text{\"\neg \Gamma\" and } \text{P}(\delta_{\text{Dom}(\delta)_-}) = \Gamma, \\
(iv) \quad (j, \delta_j) \in \text{AVS}(\delta), \\
(v) \quad \text{There is no } l \text{ such that } i < l \leq \text{Dom}(\delta)_- 1 \text{ and } (l, \delta_l) \in \text{AVAS}(\delta), \text{ and } \\
(vi) \quad \delta' = \delta \cup \{(\text{Dom}(\delta), \text{"Therefore } \neg \Delta\)}\} \}.
\]

Cf. Speech-act rule 3-10.

Definition 3-11. Negation Elimination Function (NEF)

\[
\text{NEF} = \{ (\delta, X) \mid \exists \delta' \in \text{SEQ} \text{ and } X = \{\delta' \mid \exists \Gamma \in \text{CFORM} \text{ such that } \text{\"\neg \neg \Gamma\"} \in \text{AVP}(\delta), \text{ and } \delta' = \delta \cup \{(\text{Dom}(\delta), \text{"Therefore } \Gamma\)}\} \}.
\]

Cf. Speech-act rule 3-11.

Definition 3-12. Universal-quantifier Introduction Function (UIF)

\[
\text{UIF} = \{ (\delta, X) \mid \exists \delta' \in \text{SEQ} \text{ and } X = \{\delta' \mid \exists \text{\text{\beta \in PAR, } } \xi \in \text{VAR and } \Delta \in \text{FORM, where } \\
\quad \text{FV}(\Delta) \subseteq \{\xi\}, \text{ such that } \\
(i) \quad [\beta, \xi, \Delta] \in \text{AVP}(\delta), \\
(ii) \quad \beta \notin \text{STSF}(\{\Delta\} \cup \text{AVAP}(\delta)), \text{ and } \\
(iii) \quad \delta' = \delta \cup \{(\text{Dom}(\delta), \text{"Therefore } \Delta \} \} \}.
\]

Cf. Speech-act rule 3-12.
Definition 3-13. Universal-quantifier Elimination Function (UEF)

\[ \text{UEF} = \{ (\bar{s}, X) \mid \bar{s} \in \text{SEQ} \text{ and } X = \{ \bar{s}' \mid \text{there are } \theta \in \text{CTERM}, \xi \in \text{VAR}, \Delta \in \text{FORM}, \text{ where } \text{FV}(\Delta) \subseteq \{ \xi \}, \text{ such that } \forall \zeta \Delta \in \text{AVP}(\bar{s}) \text{ and } \bar{s}' = \bar{s} \cup \{ (\text{Dom}(\Delta), \text{"Therefore } [\theta, \xi, \Delta]) \} \} \}. \]


Definition 3-14. Particular-quantifier Introduction Function (PIF)

\[ \text{PIF} = \{ (\bar{s}, X) \mid \bar{s} \in \text{SEQ} \text{ and } X = \{ \bar{s}' \mid \text{there are } \xi \in \text{VAR}, \Delta \in \text{FORM}, \text{ and } \theta \in \text{CTERM} \text{ such that } [\theta, \xi, \Delta] \in \text{AVP}(\bar{s}) \text{ and } \bar{s}' = \bar{s} \cup \{ (\text{Dom}(\Delta), \text{"Therefore } \forall \xi \Delta) \} \} \}. \]

Cf. Speech-act rule 3-14.

Definition 3-15. Particular-quantifier Elimination Function (PEF)

\[ \text{PEF} = \{ (\bar{s}, X) \mid \bar{s} \in \text{SEQ} \text{ and } X = \{ \bar{s}' \mid \text{there are } \beta \in \text{PAR}, \xi \in \text{VAR}, \Delta \in \text{FORM}, \text{ where } \text{FV}(\Delta) \subseteq \{ \xi \}, \Gamma \in \text{CFORM} \text{ and } i \in \text{Dom}(\bar{s}) \text{ such that } \]

\( \begin{align*}
(i) & \quad P(\bar{s}_i) = \forall \xi \Delta \text{ and } (i, \bar{s}_i) \in \text{AVS}(\bar{s}), \\
(ii) & \quad P(\bar{s}_{i+1}) = [\beta, \xi, \Delta] \text{ and } (i+1, \bar{s}_{i+1}) \in \text{AVAS}(\bar{s}), \\
(iii) & \quad P(\bar{s}_{\text{Dom}(\bar{s})-1}) = \Gamma, \\
(iv) & \quad \beta \notin \text{STSF}(\{ \Delta, \Gamma \}), \\
(v) & \quad \text{there is no } j \leq i \text{ such that } \beta \in \text{ST}(\bar{s}_j), \\
(vi) & \quad \text{there is no } m \text{ such that } i+1 < m \leq \text{Dom}(\bar{s})-1 \text{ and } (m, \bar{s}_m) \in \text{AVAS}(\bar{s}), \text{ and } \\
(vii) & \quad \bar{s}' = \bar{s} \cup \{ (\text{Dom}(\bar{s}), \text{"Therefore } \Gamma) \} \}. \]


Definition 3-16. Identity Introduction Function (IIF)

\[ \text{IIF} = \{ (\bar{s}, X) \mid \bar{s} \in \text{SEQ} \text{ and } X = \{ \bar{s}' \mid \text{there is } \theta \in \text{CTERM} \text{ such that } \bar{s}' = \bar{s} \cup \{ (\text{Dom}(\bar{s}), \text{"Therefore } \theta) \} \} \}. \]

Cf. Speech-act rule 3-16. Since the set of closed terms is not empty, it follows as a corollary that, like AF(\(\bar{s}\)), IIF(\(\bar{s}\)) is not empty for any sentence sequence \(\bar{s}\). This state of affairs is reflected in Theorem 3-2.
Definition 3-17. Identity Elimination Function (IEF)

IEF = \{ (\delta', X) \mid \delta' \in \text{SEQ} \text{ and } X = \{ \delta' \} \text{ There are } \theta_0, \theta_1 \in \text{TERM}, \xi \in \text{VAR} \text{ and } \Delta \in \text{FORM}, \text{ where } \text{FV}(\Delta) \subseteq \{ \xi \}, \text{ such that } \{ ^\theta_0 = \theta_1 \}, [ \theta_0, \xi, \Delta ] \subseteq \text{AVP}(\delta) \text{ and } \\
\delta' = \delta \cup \{ ((\text{Dom}(\delta), \Sigma) \} \text{.} \}

Cf. Speech-act rule 3-17.

In the following, we will define the set of rule-compliant sentence sequences, RCS (Definition 3-19), and then the derivation predicate: '.. is a derivation of .. from ..' (Definition 3-20). We will do this in such a way that RCS will contain the empty sentence sequence and all and only those sentence sequences to which one can in principle extend the empty sentence sequence in compliance with the rules of the calculus. Based on the assumption function and the introduction and elimination functions we have just defined, RCS will thus be defined in such a way that RCS is the set of sentence sequences for which all non-empty restrictions are rule-compliant extensions of the immediately preceding restriction. To do this, we first define the function RCE:

Definition 3-18. Assignment of the set of rule-compliant assumption- and inference-extensions of a sentence sequence (RCE)

RCE = \{ (\delta, X) \mid \delta \in \text{SEQ} \text{ and } X = \bigcup \{ \text{AF}(\delta), \text{CdIF}(\delta), \text{CdEF}(\delta), \text{CIF}(\delta), \text{CEF}(\delta), \text{BIF}(\delta), \\
\text{BEF}(\delta), \text{DIF}(\delta), \text{DEF}(\delta), \text{NIF}(\delta), \text{NEF}(\delta), \text{UIF}(\delta), \text{UEF}(\delta), \text{PIF}(\delta), \\
\text{PEF}(\delta), \text{IIF}(\delta), \text{IEF}(\delta) \} \}.

RCE is defined in such a way that an author who has uttered \( \delta \in \text{SEQ} \) may extend \( \delta \) to \( \delta' \) if and only if \( \delta' \in \text{RCE}(\delta) \). Before we defined the set of rule-compliant sentence sequences, RCS, we will prove some theorems about RCE.

Theorem 3-1. RCE-extensions of sentence sequences are non-empty sentence sequences

If \( \delta \in \text{SEQ} \), then \( \text{RCE}(\delta) \subseteq \text{SEQ}\backslash\{\emptyset\} \).

Proof: Suppose \( \delta \in \text{SEQ} \). Suppose \( \delta' \in \text{RCE}(\delta) \). Then we have \( \delta' \in \text{AF}(\delta) \) or \( \delta' \in \text{CdIF}(\delta) \) or \( \delta' \in \text{CdEF}(\delta) \) or \( \delta' \in \text{CIF}(\delta) \) or \( \delta' \in \text{CEF}(\delta) \) or \( \delta' \in \text{BIF}(\delta) \) or \( \delta' \in \text{BEF}(\delta) \) or \( \delta' \in \text{DIF}(\delta) \) or \( \delta' \in \text{DEF}(\delta) \) or \( \delta' \in \text{NIF}(\delta) \) or \( \delta' \in \text{NEF}(\delta) \) or \( \delta' \in \text{UIF}(\delta) \) or \( \delta' \in \text{UEF}(\delta) \) or \( \delta' \in \text{PIF}(\delta) \) or \( \delta' \in \text{PEF}(\delta) \) or \( \delta' \in \text{IIF}(\delta) \) or \( \delta' \in \text{IEF}(\delta) \). It then follows from Definition 3-1 to Definition 3-17 that \( \delta' = \delta \cup \{ ((\text{Dom}(\delta), \Sigma) \} \text{ for a } \Sigma \in \text{SENT} \). In all cases, it then holds with Definition 1-23 and Definition 1-24 that \( \delta' \in \text{SEQ}\backslash\{\emptyset\} \).
Next, we want to show that \(RCE(\xi)\) is not empty for any sentence sequence \(\xi\) and that therefore every sentence sequence can be extended in some way.

**Theorem 3-2.** \(RCE\) is not empty for any sentence sequence

If \(\xi \in SEQ\), then \(RCE(\xi) \neq \emptyset\).

*Proof:* Suppose \(\xi \in SEQ\). We have that \(\{x_0\} \in CTERM\). According to Definition 3-16, we thus have \(\xi \cup \{(\text{Dom}(\xi), "Therefore } x_0 = x_0'\}\) \(\in \text{IIF}(\xi)\). Hence we have \(\xi \cup \{(\text{Dom}(\xi), "Therefore } x_0 = x_0'\}\) \(\in RCE(\xi) \neq \emptyset\). ■

**Theorem 3-3.** The elements of \(RCE(\xi)\) are extensions of \(\xi\) by exactly one sentence

If \(\xi \in SEQ\) and \(\xi' \in RCE(\xi)\), then there are \(\Xi \in \text{PERF}\) and \(\Gamma \in \text{CFORM}\) such that \(\xi' = \xi \cup \{(\text{Dom}(\xi), "\Xi \Gamma\})\).

*Proof:* Suppose \(\xi \in SEQ\) and \(\xi' \in RCE(\xi)\). Then we have \(\xi' \in \text{AF}(\xi)\) or \(\xi' \in \text{CdEf}(\xi)\) or \(\xi' \in \text{CIF}(\xi)\) or \(\xi' \in \text{CEF}(\xi)\) or \(\xi' \in \text{BIF}(\xi)\) or \(\xi' \in \text{BEF}(\xi)\) or \(\xi' \in \text{DIF}(\xi)\) or \(\xi' \in \text{DEF}(\xi)\) or \(\xi' \in \text{NEF}(\xi)\) or \(\xi' \in \text{NEF}(\xi)\) or \(\xi' \in \text{UIF}(\xi)\) or \(\xi' \in \text{UEF}(\xi)\) or \(\xi' \in \text{PIF}(\xi)\) or \(\xi' \in \text{PEF}(\xi)\) or \(\xi' \in \text{IIF}(\xi)\) or \(\xi' \in \text{IEF}(\xi)\).

Suppose \(\xi' \in \text{AF}(\xi)\). According to Definition 3-1, there is then \(\Gamma \in \text{CFORM}\) such that \(\xi' = \xi \cup \{(\text{Dom}(\xi), "Suppose } \Gamma\})\). Then we have \(\xi'_{\text{Dom}(\xi)} = "\text{Suppose } \Gamma\"\) and thus there are \(\Xi \in \text{PERF}\) and \(\Gamma \in \text{CFORM}\) such that \(\xi' = \xi \cup \{(\text{Dom}(\xi), "\Xi \Gamma\})\).

Suppose \(\xi' \in \text{CdEf}(\xi)\) or \(\xi' \in \text{CIF}(\xi)\) or \(\xi' \in \text{CEF}(\xi)\) or \(\xi' \in \text{BIF}(\xi)\) or \(\xi' \in \text{BEF}(\xi)\) or \(\xi' \in \text{DIF}(\xi)\) or \(\xi' \in \text{DEF}(\xi)\) or \(\xi' \in \text{NEF}(\xi)\) or \(\xi' \in \text{NEF}(\xi)\) or \(\xi' \in \text{UIF}(\xi)\) or \(\xi' \in \text{UEF}(\xi)\) or \(\xi' \in \text{PIF}(\xi)\) or \(\xi' \in \text{PEF}(\xi)\) or \(\xi' \in \text{IIF}(\xi)\) or \(\xi' \in \text{IEF}(\xi)\). According to Definition 3-2 to Definition 3-17, there is in each case a \(\Gamma \in \text{CFORM}\) such that \(\xi' = \xi \cup \{(\text{Dom}(\xi), "Therefore } \Gamma\})\). Then we have \(\xi'_{\text{Dom}(\xi)} = "\text{Therefore } \Gamma\"\) and thus there are again \(\Xi \in \text{PERF}\) and \(\Gamma \in \text{CFORM}\) such that \(\xi' = \xi \cup \{(\text{Dom}(\xi), "\Xi \Gamma\})\). ■

**Theorem 3-4.** \(RCE\)-extensions of sentence sequences are greater by exactly one than the initial sentence sequences

If \(\xi \in SEQ\) and \(\xi' \in RCE(\xi)\), then \(\text{Dom}(\xi') = \text{Dom}(\xi)+1\).

*Proof:* Suppose \(\xi \in SEQ\) and \(\xi' \in RCE(\xi)\). With Theorem 3-3, there are \(\Xi \in \text{PERF}\) and \(\Gamma \in \text{CFORM}\) such that \(\xi' = \xi \cup \{(\text{Dom}(\xi), "\Xi \Gamma\})\) and thus we have \(\text{Dom}(\xi') = \text{Dom}(\xi)+1\). ■
**Theorem 3-5.** Unique RCE-predecessors
If $\mathcal{S}_2 \in \text{SEQ}$ and $\mathcal{S}_2' \in \text{RCE}(\mathcal{S}_2)$, then $\mathcal{S}_2' \upharpoonright \text{Dom}(\mathcal{S}_2') - 1 = \mathcal{S}_2$.

**Proof:** Follows immediately from Theorem 3-3 and Theorem 3-4. ■

**Definition 3-19.** The set of rule-compliant sentence sequences (RCS)
$\text{RCS} = \{ \mathcal{S} \mid \mathcal{S} \in \text{SEQ} \text{ and for all } j < \text{Dom}(\mathcal{S}) \text{ it holds that } \mathcal{S} \upharpoonright j+1 \in \text{RCE}(\mathcal{S} \upharpoonright j) \}$.

**Theorem 3-6.** A sentence sequence $\mathcal{S}_2$ is in RCS if and only if $\mathcal{S}_2$ is empty or if $\mathcal{S}_2$ is a rule-compliant extension of $\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1$ and $\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1$ is an RCS-element

$\mathcal{S}_2 \in \text{RCS}$
iff
$\mathcal{S}_2 = \emptyset$ or $\mathcal{S}_2 \in \text{RCE}(\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1)$ and $\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1 \in \text{RCS}$.

**Proof:** (L-R): Suppose $\mathcal{S}_2 \in \text{RCS}$ and $\mathcal{S}_2 \neq \emptyset$. Then we have $\mathcal{S}_2 \in \text{SEQ} \setminus \{ \emptyset \}$. We also have $\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1 \in \text{SEQ}$. It also holds that $\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1 \subseteq \mathcal{S}_2$ and that for all $j < \text{Dom}(\mathcal{S}_2)$: $(\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1) \upharpoonright j = \mathcal{S}_2 \upharpoonright j$. Because of $\mathcal{S}_2 \in \text{RCS}$, we have with Definition 3-19 that for all $j < \text{Dom}(\mathcal{S}_2)$ it holds that $\mathcal{S}_2 \upharpoonright j+1 \in \text{RCE}(\mathcal{S}_2 \upharpoonright j)$. Thus we have, first, that $\mathcal{S}_2 = \mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1 + 1 \in \text{RCE}(\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1)$. Second, it then follows that for all $j < \text{Dom}(\mathcal{S}_2) - 1 = \text{Dom}(\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1)$ it holds that $(\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1) \upharpoonright j+1 = \mathcal{S}_2 \upharpoonright j+1 \in \text{RCE}(\mathcal{S}_2 \upharpoonright j) = \text{RCE}((\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1) \upharpoonright j)$. According to Definition 3-19, we hence have $\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1 \in \text{RCS}$.

(R-L): Suppose $\mathcal{S}_2 = \emptyset$ or $\mathcal{S}_2 \in \text{RCE}(\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1)$ and $\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1 \in \text{RCS}$. If $\mathcal{S}_2 = \emptyset$, then $\mathcal{S}_2 \in \text{SEQ}$ and it holds trivially that $\mathcal{S}_2 \upharpoonright j+1 \in \text{RCE}(\mathcal{S}_2 \upharpoonright j)$ for all $j < \text{Dom}(\mathcal{S}_2)$ and thus we have $\mathcal{S}_2 \in \text{RCS}$. Now, suppose $\mathcal{S}_2 \neq \emptyset$ and $\mathcal{S}_2 \in \text{RCE}(\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1)$ and $\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1 \in \text{RCS}$. According to Definition 3-19, we then have $\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1 \in \text{SEQ}$ and $(\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1) \upharpoonright j+1 \in \text{RCE}((\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1) \upharpoonright j)$ for all $j < \text{Dom}(\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1)$, and, moreover, $\mathcal{S}_2 \in \text{RCE}(\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1)$. According to Theorem 3-1, we then have $\mathcal{S}_2 \in \text{SEQ}$ and thus, with $\mathcal{S}_2 \neq \emptyset$, $\text{Dom}(\mathcal{S}_2) = \text{Dom}(\mathcal{S}_2) - 1 + 1 = \text{Dom}(\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1) + 1$. Then we have for all $j < \text{Dom}(\mathcal{S}_2)$: $\mathcal{S}_2 \upharpoonright j = (\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1) \upharpoonright j$. Thus we have $\mathcal{S}_2 \upharpoonright j+1 = (\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1) \upharpoonright j+1 \in \text{RCE}((\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1) \upharpoonright j) = \text{RCE}(\mathcal{S}_2 \upharpoonright j)$ for all $j < \text{Dom}(\mathcal{S}_2) - 1$. If $j = \text{Dom}(\mathcal{S}_2) - 1$, then we have $\mathcal{S}_2 \upharpoonright j+1 = \mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1 + 1 = \mathcal{S}_2 \in \text{RCE}(\mathcal{S}_2 \upharpoonright \text{Dom}(\mathcal{S}_2) - 1) = \text{RCE}(\mathcal{S}_2 \upharpoonright j)$. Altogether we then have for all $j < \text{Dom}(\mathcal{S}_2)$ that $\mathcal{S}_2 \upharpoonright j+1 \in \text{RCE}(\mathcal{S}_2 \upharpoonright j)$ and hence we have $\mathcal{S}_2 \in \text{RCS}$. ■
The following theorem will often be used in the following chapters, without always being explicitly adduced as a reason:

**Theorem 3-7.** The rule-compliant extension of a RCS-element results in a non-empty RCS-element

If \( \mathcal{S} \in \text{RCS} \) and \( \mathcal{S}' \in \text{AF}(\mathcal{S}) \cup \text{CdIF}(\mathcal{S}) \cup \text{CdEF}(\mathcal{S}) \cup \text{CIF}(\mathcal{S}) \cup \text{CEF}(\mathcal{S}) \cup \text{BIF}(\mathcal{S}) \cup \text{BEF}(\mathcal{S}) \cup \text{DIF}(\mathcal{S}) \cup \text{DEF}(\mathcal{S}) \cup \text{NIF}(\mathcal{S}) \cup \text{NEF}(\mathcal{S}) \cup \text{UIF}(\mathcal{S}) \cup \text{UEF}(\mathcal{S}) \cup \text{PIF}(\mathcal{S}) \cup \text{PEF}(\mathcal{S}) \cup \text{IIF}(\mathcal{S}) \cup \text{IEF}(\mathcal{S}) \), then \( \mathcal{S}' \in \text{RCS}\{\emptyset\} \).

**Proof:** Suppose \( \mathcal{S} \in \text{RCS} \) and \( \mathcal{S}' \in \text{AF}(\mathcal{S}) \cup \text{CdIF}(\mathcal{S}) \cup \text{CdEF}(\mathcal{S}) \cup \text{CIF}(\mathcal{S}) \cup \text{CEF}(\mathcal{S}) \cup \text{BIF}(\mathcal{S}) \cup \text{BEF}(\mathcal{S}) \cup \text{DIF}(\mathcal{S}) \cup \text{DEF}(\mathcal{S}) \cup \text{NIF}(\mathcal{S}) \cup \text{NEF}(\mathcal{S}) \cup \text{UIF}(\mathcal{S}) \cup \text{UEF}(\mathcal{S}) \cup \text{PIF}(\mathcal{S}) \cup \text{PEF}(\mathcal{S}) \cup \text{IIF}(\mathcal{S}) \cup \text{IEF}(\mathcal{S}) \). According to Definition 3-18, we then have \( \mathcal{S}' \in \text{RCE}(\mathcal{S}) \). With Theorem 3-5, we have \( \mathcal{S} = \mathcal{S}' \rceil \text{Dom}(\mathcal{S}')^{-1} \). Because of \( \mathcal{S} \in \text{RCS} \) and with Theorem 3-6, we then have \( \mathcal{S}' \in \text{RCS} \). With Theorem 3-1, we then have \( \mathcal{S}' \neq \emptyset \) and thus \( \mathcal{S}' \in \text{RCS}\{\emptyset\} \).

**Theorem 3-8.** \( \mathcal{S} \) is a non-empty RCS-element if and only if \( \mathcal{S} \) is a non-empty sentence sequence and all non-empty initial segments of \( \mathcal{S} \) are non-empty RCS-elements

\( \mathcal{S} \in \text{RCS}\{\emptyset\} \) iff \( \mathcal{S} \in \text{SEQ}\{\emptyset\} \) and for all \( i \in \text{Dom}(\mathcal{S}) \): \( \mathcal{S}\rceil i+1 \in \text{RCS}\{\emptyset\} \).

**Proof:** (L-R): Suppose \( \mathcal{S} \in \text{RCS}\{\emptyset\} \). According to Definition 3-19, we then have \( \mathcal{S} \in \text{SEQ} \) and for all \( i \in \text{Dom}(\mathcal{S}) \) that \( \mathcal{S}\rceil (i+1) \in \text{RCE}(\mathcal{S}\rceil i) \). With our hypothesis, we then have \( \mathcal{S} \in \text{SEQ}\{\emptyset\} \). Suppose \( 0 \in \text{Dom}(\mathcal{S}) \). Then we have \( \mathcal{S}\rceil 1 \in \text{RCE}(\mathcal{S}\rceil 0) = \text{RCE}(\emptyset) \). With Theorem 3-6, we have \( 0 \in \text{RCS} \) and thus we have, with \( \mathcal{S}\rceil 1 \in \text{RCE}(\emptyset) \) and with Theorem 3-6, that \( \mathcal{S}\rceil 1 \in \text{RCS} \). With \( 0 \in \text{Dom}(\mathcal{S}\rceil 1) \) we then have \( \mathcal{S}\rceil 1 \in \text{RCS}\{\emptyset\} \). Now, suppose for \( i \) it holds that if \( i \in \text{Dom}(\mathcal{S}) \), then \( \mathcal{S}\rceil i+1 \in \text{RCS}\{\emptyset\} \). Now, suppose \( i+1 \in \text{Dom}(\mathcal{S}) \). Then we have \( i \in \text{Dom}(\mathcal{S}) \) and thus, according to the I.H., also \( \mathcal{S}\rceil i+1 \in \text{RCS}\{\emptyset\} \). Also, we have \( \mathcal{S}\rceil i+2 \in \text{RCE}(\mathcal{S}\rceil i+1) \). Because of \( \mathcal{S} \in \text{SEQ} \) and \( i+1 \in \text{Dom}(\mathcal{S}) \), we have \( \mathcal{S}\rceil i+1 = (\mathcal{S}\rceil (i+2))\rceil \text{Dom}(\mathcal{S}\rceil (i+2))^{-1} \). With Theorem 3-6 and Theorem 3-1, we then have \( \mathcal{S}\rceil i+2 \in \text{RCS}\{\emptyset\} \).

(R-L): Now, suppose \( \mathcal{S} \in \text{SEQ}\{\emptyset\} \) for all \( i \in \text{Dom}(\mathcal{S}) \): \( \mathcal{S}\rceil i+1 \in \text{RCS}\{\emptyset\} \). With \( \mathcal{S} \in \text{SEQ}\{\emptyset\} \), we then have \( \text{Dom}(\mathcal{S})^{-1} \in \text{Dom}(\mathcal{S}) \) and hence \( \mathcal{S}\rceil \text{Dom}(\mathcal{S})^{-1}+1 = \mathcal{S} \in \text{RCS}\{\emptyset\} \).
Based on Definition 3-19, we will now introduce a derivation concept. Subsequently, after having proved some theorems and considered an example concerning the derivation concept, we will establish a corresponding consequence concept.

**Definition 3-20. Derivation**

\( \mathfrak{f} \) is a derivation of \( \Gamma \) from \( X \)

iff

(i) \( \mathfrak{f} \in \text{RCS} \setminus \{\emptyset\} \),

(ii) \( \Gamma = C(\mathfrak{f}) \) and

(iii) \( X = \text{AVAP}(\mathfrak{f}) \).

If we take into account Definition 3-19, we now have characterised exatly those non-empty sentence sequences as derivations of a proposition from a set of propositions that can in principle be uttered by successively applying the rules of the Speech Act Calculus.

**Theorem 3-9. Properties of derivations**

If \( \mathfrak{f} \) is a derivation of \( \Gamma \) from \( X \), then:

(i) \( \mathfrak{f} \in \text{SEQ} \setminus \{\emptyset\} \),

(ii) \( \Gamma \in \text{CFORM} \) and

(iii) \( X \subseteq \text{CFORM} \) and \(|X| \in \mathbb{N} \).

*Proof*: Suppose \( \mathfrak{f} \) is a derivation of \( \Gamma \) from \( X \). Then we have \( \mathfrak{f} \in \text{RCS} \setminus \{\emptyset\} \) and \( C(\mathfrak{f}) = \Gamma \) and \( X = \text{AVAP}(\mathfrak{f}) \). With Definition 3-19, we have \( \mathfrak{f} \in \text{SEQ} \setminus \{\emptyset\} \). According to Definition 1-25, Definition 1-24, Definition 1-23, Definition 1-18 and Definition 1-16, we have that \( C(\mathfrak{f}) = \Gamma \in \text{CFORM} \). According to Definition 1-23 and Definition 1-24, we have \( \text{Dom}(\mathfrak{f}) \in \mathbb{N} \). With Definition 2-31, Definition 2-29, Definition 2-28 and Definition 2-26, we thus also have \( X = \text{AVAP}(\mathfrak{f}) \subseteq \text{CFORM} \) and \(|X| = |\text{AVAP}(\mathfrak{f})| \in \mathbb{N} \).

**Theorem 3-10. In non-empty RCS-elements all non-empty initial segments are derivations of their respective conclusions**

If \( \mathfrak{f} \in \text{RCS} \setminus \{\emptyset\} \), then it holds for all \( i \in \text{Dom}(\mathfrak{f}) \) that \( \mathfrak{f}|_i \) is a derivation of \( P(\mathfrak{f}|_i) \) from \( \text{AVAP}(\mathfrak{f}|_i) \).

*Proof*: Suppose \( \mathfrak{f} \in \text{RCS} \setminus \{\emptyset\} \). With Theorem 3-8, it then holds for all \( i \in \text{Dom}(\mathfrak{f}) \) that \( \mathfrak{f}|_i \in \text{RCS} \setminus \{\emptyset\} \). Also, we have for all \( i \in \text{Dom}(\mathfrak{f}) \): \( P(\mathfrak{f}|_i) = C(\mathfrak{f}|_i + 1) \) and \( \text{AVAP}(\mathfrak{f}|_i + 1) = \text{AVAP}(\mathfrak{f}|_i + 1) \).
**Theorem 3-11. Uniqueness-theorem for the Speech Act Calculus**

If $\delta \in \text{SEQ}$, then:

(i) There is no $\Gamma$ and no $X$ such that $\delta$ is a derivation of $\Gamma$ from $X$ or

(ii) There is exactly one $\Gamma$ and exactly one $X$ such that $\delta$ is a derivation of $\Gamma$ from $X$.

*Proof:* Suppose $\delta \in \text{SEQ}$. Then there is no $\Gamma$ and no $X$ such that $\delta$ is a derivation of $\Gamma$ from $X$ or there are a $\Gamma$ and an $X$ such that $\delta$ is a derivation of $\Gamma$ from $X$. In the first case, the statement holds. Now, for the second case, suppose there are a $\Gamma$ and an $X$ such that $\delta$ is a derivation of $\Gamma$ from $X$. According to Definition 3-20, we then have $\delta \in \text{RCS}\setminus\{\emptyset\}$, $\Gamma = C(\delta)$ and $\text{AVAP}(\delta) = X$. We still have to show uniqueness. For this, suppose $\delta$ is a derivation of $\Gamma'$ from $X'$. Then we have $\Gamma' = C(\delta) = \Gamma$ and $X' = \text{AVAP}(\delta) = X$. $\blacksquare$

Now, let us illustrate this result with an example. Suppose $\xi \in \text{VAR}$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi\}$, and suppose $\beta \in \text{PAR}\setminus\text{ST}(\Delta)$. Now, let $\delta^{[3.1]}$ be the following sentence sequence:

**Example [3.1]**

0  Suppose $\land \neg \Delta$
1  Suppose $\lor \xi \Delta$
2  Suppose $[\beta, \xi, \Delta]$
3  Suppose $\lor \xi \Delta$
4  Therefore $\lor \xi \Delta \land [\beta, \xi, \Delta]$
5  Therefore $[\beta, \xi, \Delta]$
6  Therefore $\neg [\beta, \xi, \Delta]$
7  Therefore $\neg \lor \xi \Delta$
8  Therefore $\neg \lor \xi \Delta$
9  Therefore $\neg \lor \xi \Delta$

**Commentary:** According to Theorem 3-11, there should either be no $\Gamma$ and no $X$ such that $\delta^{[3.1]}$ is a derivation of $\Gamma$ from $X$ or we should be able to find unique $\Gamma$ and $X$ such that

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For the formulation of a corresponding theorem for a regulation of the predicate ‘.. is a derivation of .. from ..’ according to which the set of propositions named at the third place has to be a superset of the set of assumptions that actually occur in the respective sentence sequence and are not eliminated there, see footnote 4.
\(\mathfrak{S}^{[3.1]}\) is a derivation of \(\Gamma\) from \(X\). This is actually the case as \(\mathfrak{S}^{[3.1]}\) is a derivation of \("\neg u \vee u\Delta\) from \(\{\neg u \land \neg u\Delta\}\), where both are uniquely determined. This can be made clearer by an informal inspection of the sentence sequence. To do this, we first furnish the sentence sequence with comments that will then be explained.

<table>
<thead>
<tr>
<th>Example</th>
<th>available</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Suppose (\neg u\Delta) (\land \neg u\Delta) (\land \neg u\Delta) (AR) 0</td>
</tr>
<tr>
<td>1</td>
<td>Suppose (\neg u\Delta) (\land \neg u\Delta) (\land \neg u\Delta) (AR) 0, 1</td>
</tr>
<tr>
<td>2</td>
<td>Suppose ([\beta, \xi, \Delta]) (\land \neg u\Delta) (\land \neg u\Delta) (AR) 0, 1, 2</td>
</tr>
<tr>
<td>3</td>
<td>Suppose (\neg u\Delta) (\land \neg u\Delta) (\land \neg u\Delta) (AR) 0, 1, 2, 3</td>
</tr>
<tr>
<td>4</td>
<td>Therefore (\land \neg u\Delta) ([\beta, \xi, \Delta]) (\land \neg u\Delta) (\land \neg u\Delta) (CI); 2, 3 0, 1, 2, 3, 4</td>
</tr>
<tr>
<td>5</td>
<td>Therefore ([\beta, \xi, \Delta]) (\land \neg u\Delta) (\land \neg u\Delta) (CE); 4 0, 1, 2, 3, 4, 5</td>
</tr>
<tr>
<td>6</td>
<td>Therefore (\neg u\Delta) ([\beta, \xi, \Delta]) (\land \neg u\Delta) (\land \neg u\Delta) (UE); 1 0, 1, 2, 3, 4, 5, 6</td>
</tr>
<tr>
<td>7</td>
<td>Therefore (\neg u\Delta) ([\beta, \xi, \Delta]) (\land \neg u\Delta) (\land \neg u\Delta) (NI); 5, 6 0, 1, 2, 7</td>
</tr>
<tr>
<td>8</td>
<td>Therefore (\neg u\Delta) ([\beta, \xi, \Delta]) (\land \neg u\Delta) (\land \neg u\Delta) (PE); 1, 7 0, 1, 8</td>
</tr>
<tr>
<td>9</td>
<td>Therefore (\neg u\Delta) ([\beta, \xi, \Delta]) (\land \neg u\Delta) (\land \neg u\Delta) (NI); 1, 8 0, 9</td>
</tr>
</tbody>
</table>

*Explanation:* In the second column from the right, the rules by which one may extend an already uttered sequence and the respective premise lines are given (cf. ch. 3.1). The uttermost right column displays the line numbers of those lines whose propositions are available in the restriction of \(\mathfrak{S}^{[3.1]}\) on the successor of the current line number. Note that the propositions and assumptions that are available in \(\mathfrak{S}^{[3.1]}\) \(\uparrow i\) \((1 \leq i \leq 10)\) are always uniquely determined.

Also, we have that, for example, the inference in line 8 may only be carried out by PE and the inference in line 9 may only be carried out by NI, in both cases with uniquely determined premise lines. In line 8, NI is not an option, because, on the one hand, the proposition assumed in line 2 is still available in \(\mathfrak{S}^{[3.1]}\) \(\uparrow 8\) so that 1 cannot serve as an initial assumption for NI, while, on the other hand, 3 cannot serve as an initial assumption for NI, because the proposition assumed there is not any more available in \(\mathfrak{S}^{[3.1]}\) \(\uparrow 8\) at this position. Obversly, PE may not be carried out in line 9 (and NI may be carried out), because the representative instance assumption in line 2 is not any more available in \(\mathfrak{S}^{[3.1]}\) \(\uparrow 9\) at this position (and at all).

If one checks all other lines, one can easily convince oneself that \(\mathfrak{S}^{[3.1]} \in \text{RCS}\setminus\{\emptyset\}\). The set of the assumptions that are available in \(\mathfrak{S}^{[3.1]}\) is uniquely determined and determinable,
because, with Definition 2-26, Definition 2-28, Definition 2-29 and Definition 2-31, one can check for every proposition $A$ that has been assumed in $\mathfrak{H}^{3.1}$ whether $A \in AVAP(\mathfrak{H}^{3.1})$. As desired, one can easily convince oneself that $AVAP(\mathfrak{H}^{3.1}) = \{ "\land \xi \rightarrow \Delta" \}$. Obviously, we have $\mathfrak{H}^{3.1}_{Dom(\mathfrak{H}^{3.1})} = \RightarrowTherefore \neg \xi \Delta$ so that Theorem 3-11 is confirmed.

Note that the comments in the right columns do not serve to disambiguate from which set of propositions the proposition in the last line has been derived, but only serve to facilitate an easier traceability and understanding. Note that the rule-commentary to $\mathfrak{H}^{3.1}$ is uniquely determined by coincidence and that there are other sentence sequences for which different rule-commentaries may be produced: There are circumstances under which a transition may be carried out in accordance with different rules, e.g. UE and PE. However, it is not the case that the possibility of alternative rule-commentaries has any effects on the uniqueness of the availability-commentary. Available propositions (or lines) are not determined with recourse to the rule-commentary, but according to the definition of availability and thus, eventually, according to the definition of closed segments. The separate definition of availability excludes that we arrive at different availabilities for one and the same transition, even if that transition can be carried out in accordance with more than one rule. Thus, it is always uniquely determined and determinable if a given sentence sequence is a derivation of a certain proposition from a certain set of propositions.

Closed segments emerge if and only if one may apply CdI, NI or PE (cf. Theorem 3-23 and Theorem 3-24). Thus, if a transition is covered by more than one rule, e.g. UE and PE, availabilities change as they do in a transition by PE. Thus, a user of the Speech Act Calculus is restricted in the performance of certain inferences: For example, one is not free to carry out an assumption-discharging inference by PE as a not assumption-discharging inference by UE.

One may deem that this makes the Speech Act Calculus a bit unhandy, however, this shortcoming, if it is one, comes with the advantage that for every utterance of a sentence sequence by an author, we can uniquely determine if that author has uttered a derivation of a certain proposition from a certain set of propositions: The possibility to describe the utterance of one and the same sentence sequence in different ways so that, for example
3.2 Derivations and Deductive Consequence Relation

the utterance of a sentence sequence $\mathcal{S}$ can be described as an utterance of a derivation of $\Gamma$ from $X$ and can also described as the utterance of a sentence sequence that is not a derivation of $\Gamma$ from $X$, which exists for some calculi, does not exist for the Speech Act Calculus. If one utters derivations in accordance with the rules of the Speech Act Calculus, one does not have to use graphical means for the marking of subderivations nor metatheoretical rule- or dependence-commentaries: In the framework of the Speech Act Calculus utterances of sentence sequences are not up for interpretation.

Now, we will introduce the deductive consequence concept and some other usual meta-logical concepts. In ch. 4, we will then prove some properties of the deductive consequence relation, such as reflexivity, transitivity and closure under introduction and elimination. Subsequently, in ch. 6, we will then provide an adequacy proof for the calculus relative to the classical model-theoretic consequence relation. This relation itself will be established in ch. 5. Now, for the definition of the consequence relation:

**Definition 3-21. Deductive consequence relation**

\[ X \vdash \Gamma \]

iff

\[ X \subseteq \text{CFORM} \text{ and there is an } \mathcal{S} \text{ such that} \]

(i) \[ \mathcal{S} \text{ is a derivation of } \Gamma \text{ from } \text{AVAP}(\mathcal{S}), \]

(ii) \[ \text{AVAP}(\mathcal{S}) \subseteq X. \]

With Theorem 3-9-(iii), it then follows, as usual, that for $X \subseteq \text{CFORM}$ it holds that $X \vdash \Gamma$ if and only if there is a finite $Y \subseteq X$ such that $Y \vdash \Gamma$. From this and Definition 3-23, it then follows that $X$ is consistent if and only if all finite $Y \subseteq X$ are consistent, and, with Definition 3-24, that $X \subseteq \text{CFORM}$ is inconsistent if and only if there is a finite $Y \subseteq X$ such that $Y$ is inconsistent. Under Definition 3-20, the following theorem is equivalent to Definition 3-21:

**Theorem 3-12.** \[ \Gamma \text{ is a deductive consequence of a set of propositions } X \text{ if and only if there is a non-empty RCS-element } \mathcal{S} \text{ such that } \Gamma \text{ is the conclusion of } \mathcal{S} \text{ and } \text{AVAP}(\mathcal{S}) \subseteq X \]

\[ X \vdash \Gamma \text{ iff } X \subseteq \text{CFORM} \text{ and there is } \mathcal{S} \in \text{RCS} \setminus \{\emptyset\} \text{ such that } \Gamma = C(\mathcal{S}) \text{ and } \text{AVAP}(\mathcal{S}) \subseteq X. \]

**Proof:** Follows directly from Definition 3-20 and Definition 3-21. ■
Definition 3-22. Logical provability
\( \vdash \Gamma \iff \emptyset \vdash \Gamma \).

Definition 3-23. Consistency
\( X \) is consistent
\iff
\( X \subseteq \text{CFORM} \) and there is no \( \Gamma \in \text{CFORM} \) such that \( X \vdash \Gamma \) and \( X \vdash \neg \Gamma \).

Definition 3-24. Inconsistency
\( X \) is inconsistent
\iff
\( X \subseteq \text{CFORM} \) and there is a \( \Gamma \in \text{CFORM} \) such that \( X \vdash \Gamma \) and \( X \vdash \neg \Gamma \).

Theorem 3-13. Sets of propositions are inconsistent if and only if they are not consistent
If \( X \subseteq \text{CFORM} \), then: \( X \) is inconsistent iff \( X \) is not consistent.

Proof: Follows directly from Definition 3-23 and Definition 3-24. ■

Definition 3-25. Deductive consequence for sets
\( X \vdash Y \iff X \cup Y \subseteq \text{CFORM} \) and for all \( \Delta \in Y \) it holds that \( X \vdash \Delta \).

Definition 3-26. Logical provability for sets
\( \vdash X \iff \emptyset \vdash X \).

Definition 3-27. The closure of a set of propositions under deductive consequence
\( X^+ = \{ \Delta \mid \Delta \in \text{CFORM} \text{ and } X \vdash \Delta \} \).

Before proving the usual properties for the deductive consequence relation in ch. 4 and ch. 6, we will prove some theorems that illustrate the working of the calculus in the following ch. 3.3.
3.3 AVS, AVAS, AVP and AVAP in Derivations and in Individual Transitions

Now, we will establish some theorems for the rules (cf. ch. 3.1) and operations (cf. ch. 3.2) respectively that describe the working of the Speech Act Calculus. More exactly, we will prove theorems that provide an account of the connections between changes in availabilities (AVS, AVAS, AVP, AVAP) in rule-compliant transitions from a sentence sequence $\mathcal{S}$ to a sentence sequence $\mathcal{S}'$ and the respective rule or operation. At the same time, these theorems provide the basis for the theorems about the deductive consequence relation that are proved in ch. 4 and for the proof of the correctness and the completeness of the Speech Act Calculus in ch. 6. At the end of the chapter, Theorem 3-30 offers an overview of the form of derivations and the availability conditions in derivations in the Speech Act Calculus.

**Theorem 3-14. AVS, AVAS, AVP, AVAP and RCE**

If $\mathcal{S} \in \text{SEQ}$ and $\mathcal{S}' \in \text{RCE}(\mathcal{S})$, then:

(i) $\text{AVS}(\mathcal{S}') \subseteq \text{AVS}(\mathcal{S}) \cup \{(\text{Dom}(\mathcal{S}), \mathcal{S}'_{\text{Dom}(\mathcal{S})})\}$,
(ii) $\text{AVAS}(\mathcal{S}') \subseteq \text{AVAS}(\mathcal{S}) \cup \{(\text{Dom}(\mathcal{S}), \mathcal{S}'_{\text{Dom}(\mathcal{S})})\}$,
(iii) $\text{AVP}(\mathcal{S}') \subseteq \text{AVP}(\mathcal{S}) \cup \{\text{C}(\mathcal{S}')\}$, and
(iv) $\text{AVAP}(\mathcal{S}') \subseteq \text{AVAP}(\mathcal{S}) \cup \{\text{C}(\mathcal{S}')\}$.

**Proof:** Suppose $\mathcal{S} \in \text{SEQ}$ and $\mathcal{S}' \in \text{RCE}(\mathcal{S})$. With Theorem 3-3, there are then $\Xi \in \text{PERF}$ and $\Gamma \in \text{CFORM}$ such that $\mathcal{S}' = \mathcal{S} \cup \{(\text{Dom}(\mathcal{S}), \Xi \Gamma\}) = \mathcal{S} \setminus \{(0, \Xi \Gamma\})$; and the statement follows with Theorem 2-79. ■

**Theorem 3-15. AVS, AVAS, AVP, AVAP and AR**

If $\mathcal{S} \in \text{SEQ}$ and $\mathcal{S}' \in \text{AF}(\mathcal{S})$, then:

(i) $\text{AVS}(\mathcal{S}') \setminus \text{AVS}(\mathcal{S}) = \{(\text{Dom}(\mathcal{S}), \mathcal{S}'_{\text{Dom}(\mathcal{S})})\}$,
(ii) $\text{AVS}(\mathcal{S}') = \text{AVS}(\mathcal{S}) \cup \{(\text{Dom}(\mathcal{S}), \mathcal{S}'_{\text{Dom}(\mathcal{S})})\}$,
(iii) $\text{AVAS}(\mathcal{S}') \setminus \text{AVAS}(\mathcal{S}) = \{(\text{Dom}(\mathcal{S}), \mathcal{S}'_{\text{Dom}(\mathcal{S})})\}$,
(iv) $\text{AVAS}(\mathcal{S}') = \text{AVAS}(\mathcal{S}) \cup \{(\text{Dom}(\mathcal{S}), \mathcal{S}'_{\text{Dom}(\mathcal{S})})\}$,
(v) $\text{AVP}(\mathcal{S}') \setminus \text{AVP}(\mathcal{S}) \subseteq \{\text{C}(\mathcal{S}')\}$,
(vi) $\text{AVP}(\mathcal{S}') = \text{AVP}(\mathcal{S}) \cup \{\text{C}(\mathcal{S}')\}$,
(vii) $\text{AVAP}(\mathfrak{A}') \setminus \text{AVAP}(\mathfrak{A}) \subseteq \{C(\mathfrak{A}')\}$, and
(viii) $\text{AVAP}(\mathfrak{A}') = \text{AVAP}(\mathfrak{A}) \cup \{C(\mathfrak{A}')\}$.

**Proof:** Suppose $\mathfrak{A} \in \text{SEQ}$ and $\mathfrak{A}' \in \text{AF}(\mathfrak{A})$. With Definition 3-18, it then holds that $\mathfrak{A}' \in \text{RCE}(\mathfrak{A})$. With Definition 3-1, we have that there is $\Gamma \in \text{CFORM}$ such that $\mathfrak{A}' = \mathfrak{A} \cup \{(\text{Dom}(\mathfrak{A}), \text{Suppose } \Gamma)\}$. Thus we have $\mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})-1 = \mathfrak{A} \upharpoonright \text{Dom}(\mathfrak{A}) = \mathfrak{A}$.

**Ad (i):** Suppose $(i, \mathfrak{A}') \in \text{AVS}(\mathfrak{A}') \setminus \text{AVS}(\mathfrak{A})$. With Theorem 3-14-(i), we then have $(i, \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})) \in \text{AVS}(\mathfrak{A}')$. With Theorem 2-82, we have $(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})) \in \text{AS}(\mathfrak{A}')$ and we have $(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})) \notin \text{AVS}(\mathfrak{A}) \subseteq \mathfrak{A}$. Hence we have $(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})) \in \text{AVS}(\mathfrak{A}') \setminus \text{AVS}(\mathfrak{A})$.

**Ad (ii):** With Theorem 3-14-(i), it holds that $\text{AVS}(\mathfrak{A}') \subseteq \text{AVS}(\mathfrak{A}) \cup \{(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A}))\}$. Also, we have that $(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})) = (\text{Dom}(\mathfrak{A}), \text{Suppose } \Gamma) \in \text{AS}(\mathfrak{A}')$. It then holds, with Theorem 2-30, that there is no CdI- or NI- or RA-like and thus no closed segment $\mathfrak{B}$ in $\mathfrak{A}'$ such that $\min(\text{Dom}(\mathfrak{B})) \leq \text{Dom}(\mathfrak{A})-1 = \text{Dom}(\mathfrak{A}')-2$ and $\max(\text{Dom}(\mathfrak{B})) = \text{Dom}(\mathfrak{A}) = \text{Dom}(\mathfrak{A}')-1$. With Theorem 2-84, we then have $\text{AVS}(\mathfrak{A}) \setminus \text{AVS}(\mathfrak{A}') = \emptyset$ and thus $\text{AVS}(\mathfrak{A}) \subseteq \text{AVS}(\mathfrak{A}')$. With (i), we have $(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})) \in \text{AVS}(\mathfrak{A}')$ and hence we have $\text{AVS}(\mathfrak{A}) \cup \{(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A}))\} \subseteq \text{AVS}(\mathfrak{A}')$.

**Ad (iii):** Suppose $(i, \mathfrak{A}') \in \text{AVAS}(\mathfrak{A}') \setminus \text{AVAS}(\mathfrak{A})$. With Theorem 3-14-(ii), it then follows that $(i, \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})) \in \text{AVS}(\mathfrak{A}')$. Also, we have that $(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})) = (\text{Dom}(\mathfrak{A}), \text{Suppose } \Gamma) \in \text{AS}(\mathfrak{A}')$ and thus we have $(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})) \in \text{AVAS}(\mathfrak{A}')$ and $(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})) \notin \text{AVAS}(\mathfrak{A}) \subseteq \mathfrak{A}$.

**Ad (iv):** With (iii), we have $(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})) \in \text{AVAS}(\mathfrak{A}') = \text{AVS}(\mathfrak{A}') \cap \text{AS}(\mathfrak{A}')$. With (ii), we thus have $\text{AVAS}(\mathfrak{A}) \cup \{(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A}))\} = (\text{AVS}(\mathfrak{A}) \cap \text{AS}(\mathfrak{A})) \cup ((\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A})) \cap \text{AS}(\mathfrak{A}')) = (\text{AVS}(\mathfrak{A}) \cup \{(\text{Dom}(\mathfrak{A}), \mathfrak{A}' \upharpoonright \text{Dom}(\mathfrak{A}))\}) \cap \text{AS}(\mathfrak{A}') = \text{AVS}(\mathfrak{A}) \cap \text{AS}(\mathfrak{A}) = \text{AVAS}(\mathfrak{A})$.

**Ad (v), (vi), (vii), (viii):** (v) follows with Theorem 3-14-(iii), and (vii) follows with Theorem 3-14-(iv). (vi) follows with Definition 2-30 and (ii). (viii) follows with Definition 2-31 and (iv).
Theorem 3-16. AVAS-increase only for AR
If $\bar{\mathfrak{f}} \in \text{SEQ}$ and $\bar{\mathfrak{f}}' \in \text{RCE}(\bar{\mathfrak{f}})$, then:

(i) If $\text{AVAS}(\bar{\mathfrak{f}}) \subset \text{AVAS}(\bar{\mathfrak{f}}')$, then $\bar{\mathfrak{f}}' \in \text{AF}(\bar{\mathfrak{f}})$, and

(ii) If $\text{AVAP}(\bar{\mathfrak{f}}) \subset \text{AVAP}(\bar{\mathfrak{f}}')$, then $\bar{\mathfrak{f}}' \in \text{AF}(\bar{\mathfrak{f}})$.

Proof: Suppose $\bar{\mathfrak{f}} \in \text{SEQ}$ and $\bar{\mathfrak{f}}' \in \text{RCE}(\bar{\mathfrak{f}})$. Ad (i): Suppose $\text{AVAS}(\bar{\mathfrak{f}}) \subset \text{AVAS}(\bar{\mathfrak{f}}')$. Then there is $(i, \bar{\mathfrak{f}}_i' \bar{\mathfrak{f}}_i)$ $\in$ $\text{AVAS}(\bar{\mathfrak{f}}') \setminus \text{AVAS}(\bar{\mathfrak{f}})$. Then we have $(i, \bar{\mathfrak{f}}_i') \in \text{AS}(\bar{\mathfrak{f}}')$. With Theorem 3-14-(ii), we also have $(i, \bar{\mathfrak{f}}_i') = (\text{Dom}(\bar{\mathfrak{f}}), \bar{\mathfrak{f}}_i' \text{Dom}(\bar{\mathfrak{f}}))$ and hence $(\text{Dom}(\bar{\mathfrak{f}}), \bar{\mathfrak{f}}_i' \text{Dom}(\bar{\mathfrak{f}})) \in \text{AS}(\bar{\mathfrak{f}}')$. With Definition 3-1, we then have $\bar{\mathfrak{f}}' \in \text{AF}(\bar{\mathfrak{f}})$.

Ad (ii): Suppose $\text{AVAP}(\bar{\mathfrak{f}}) \subset \text{AVAP}(\bar{\mathfrak{f}}')$. With Theorem 2-75, we then have $\text{AVAS}(\bar{\mathfrak{f}}') \subset \text{AVAS}(\bar{\mathfrak{f}})$ and thus there is $(i, \bar{\mathfrak{f}}_i') \in \text{AVAS}(\bar{\mathfrak{f}}') \setminus \text{AVAS}(\bar{\mathfrak{f}})$. Then the statement follows in the same way as (i). ■

Theorem 3-17. AVS, AVAS, AVP and AVAP in transitions without AR
If $\bar{\mathfrak{f}} \in \text{SEQ}$ and $\bar{\mathfrak{f}}' \in \text{RCE}(\bar{\mathfrak{f}}) \setminus \text{AF}(\bar{\mathfrak{f}})$, then:

(i) $\text{AVS}(\bar{\mathfrak{f}}') \subseteq \text{AVS}(\bar{\mathfrak{f}}) \cup \{(\text{Dom}(\bar{\mathfrak{f}}), \bar{\mathfrak{f}}_i' \text{Dom}(\bar{\mathfrak{f}}))\}$,

(ii) $\text{AVAS}(\bar{\mathfrak{f}}') \subseteq \text{AVAS}(\bar{\mathfrak{f}})$,

(iii) $\text{AVP}(\bar{\mathfrak{f}}') \subseteq \text{AVP}(\bar{\mathfrak{f}}) \cup \{C(\bar{\mathfrak{f}}')\}$, and

(iv) $\text{AVAP}(\bar{\mathfrak{f}}') \subseteq \text{AVAP}(\bar{\mathfrak{f}})$.

Proof: Suppose $\bar{\mathfrak{f}}' \in \text{RCE}(\bar{\mathfrak{f}}) \setminus \text{AF}(\bar{\mathfrak{f}})$. (i) and (iii) follow with Theorem 3-14-(i) and -(iii).

Ad (ii): With $\bar{\mathfrak{f}}' \in \text{RCE}(\bar{\mathfrak{f}}) \setminus \text{AF}(\bar{\mathfrak{f}})$ and Definition 3-1 to Definition 3-18, we have that $(\text{Dom}(\bar{\mathfrak{f}}), \bar{\mathfrak{f}}_i' \text{Dom}(\bar{\mathfrak{f}})) = (\text{Dom}(\bar{\mathfrak{f}}), \text{'Therefore P(\bar{\mathfrak{f}}_i' \text{Dom}(\bar{\mathfrak{f}}))') \notin \text{AS}(\bar{\mathfrak{f}}')$ and hence $(\text{Dom}(\bar{\mathfrak{f}}), \bar{\mathfrak{f}}_i' \text{Dom}(\bar{\mathfrak{f}})) \notin \text{AVAS}(\bar{\mathfrak{f}}')$. With Theorem 3-14-(ii), we then have $\text{AVAS}(\bar{\mathfrak{f}}') \subseteq \text{AVAS}(\bar{\mathfrak{f}})$. Ad (iv): (iv) follows with Theorem 2-75 from (ii). ■

Theorem 3-18. Non-empty AVAS is sufficient for CdI
If $\bar{\mathfrak{f}} \in \text{SEQ}$ and $\text{AVAS}(\bar{\mathfrak{f}}) \neq \emptyset$, then $\bar{\mathfrak{f}} \cup \{(\text{Dom}(\bar{\mathfrak{f}}), \text{'Therefore P(\text{max}(\text{Dom}(\text{AVAS}(\bar{\mathfrak{f}}))) \rightarrow C(\bar{\mathfrak{f}}'))}) \} \in \text{CdIF}(\bar{\mathfrak{f}})$.

Proof: Suppose $\bar{\mathfrak{f}} \in \text{SEQ}$ and $\text{AVAS}(\bar{\mathfrak{f}}) \neq \emptyset$. Then we have $(\text{max}(\text{Dom}(\text{AVAS}(\bar{\mathfrak{f}}))), \bar{\mathfrak{f}}_{\text{max}(\text{Dom}(\text{AVAS}(\bar{\mathfrak{f}})))}) \in \text{AVAS}(\bar{\mathfrak{f}})$ and $P(\bar{\mathfrak{f}}_{\text{Dom}(\bar{\mathfrak{f}})-1}) = C(\bar{\mathfrak{f}})$ and there is no $l$ with $\max(\text{Dom}(\text{AVAS}(\bar{\mathfrak{f}}))) < l \leq \text{Dom}(\bar{\mathfrak{f}})-1$ such that $(l, \bar{\mathfrak{f}}_l) \in \text{AVAS}(\bar{\mathfrak{f}})$. With Definition 3-2, we then have $\bar{\mathfrak{f}} \cup \{(\text{Dom}(\bar{\mathfrak{f}}), \text{'Therefore P(\bar{\mathfrak{f}}_{\text{max}(\text{Dom}(\text{AVAS}(\bar{\mathfrak{f}}))) \rightarrow C(\bar{\mathfrak{f}}'))}) \} \in \text{CdIF}(\bar{\mathfrak{f}})$. ■
**Theorem 3-19. AVS, AVAS, AVP, AVAP and Cdl**

If \( \mathfrak{s} \in \text{SEQ} \) and \( \mathfrak{s}' \in \text{CdlI}(\mathfrak{s}) \), then:

(i) \( \{ (j, \mathfrak{s}_j') \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{s}))) \leq j \leq \text{Dom}(\mathfrak{s}) \} \) is a Cdl-closed segment in \( \mathfrak{s}' \),

(ii) \( \text{AVS}(\mathfrak{s}) \setminus \text{AVS}(\mathfrak{s}') \subseteq \{ (j, \mathfrak{s}_j') \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{s}))) \leq j < \text{Dom}(\mathfrak{s}) \} \),

(iii) \( \text{AVS}(\mathfrak{s}') = (\text{AVS}(\mathfrak{s}) \cup \{ (j, \mathfrak{s}_j') \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{s}))) \leq j < \text{Dom}(\mathfrak{s}) \}) \cup \{ (\text{Dom}(\mathfrak{s}), \mathfrak{s}'_{\text{Dom}(\mathfrak{s})}) \} \),

(iv) \( \text{AVAS}(\mathfrak{s}) \setminus \text{AVAS}(\mathfrak{s}') = \{ (\max(\text{Dom}(\text{AVAS}(\mathfrak{s}))), \mathfrak{s}'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{s})))}) \} \),

(v) \( \text{AVAS}(\mathfrak{s}) = \text{AVAS}(\mathfrak{s}') \cup \{ (\max(\text{Dom}(\text{AVAS}(\mathfrak{s}))), \mathfrak{s}'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{s})))}) \} \),

(vi) \( \text{AVP}(\mathfrak{s}) \setminus \text{AVP}(\mathfrak{s}') \subseteq \{ (\mathfrak{s}_i'), \mathfrak{s}'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{s})))} \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{s}))) \leq j < \text{Dom}(\mathfrak{s}) \} \),

(vii) \( \text{AVP}(\mathfrak{s}) \subseteq \{ (\mathfrak{s}_i'), j \in \text{Dom}(\text{AVS}(\mathfrak{s}') \cup \text{Dom}(\mathfrak{s})) \} \cup \{ (\mathfrak{s}_i'), \mathfrak{s}'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{s})))} \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{s}))) \leq j < \text{Dom}(\mathfrak{s}) \} \),

(viii) \( \text{AVAP}(\mathfrak{s}) \setminus \text{AVAP}(\mathfrak{s}') \subseteq \{ (\mathfrak{s}_i'), \mathfrak{s}'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{s})))} \} \),

(ix) \( \text{AVAP}(\mathfrak{s}) = \text{AVAP}(\mathfrak{s}') \cup \{ (\mathfrak{s}_i'), \mathfrak{s}'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{s})))} \} \), and

(x) \( \text{C}(\mathfrak{s}') = \{ \mathfrak{s}'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{s})))} \rightarrow \text{C}(\mathfrak{s}) \} \).

**Proof:** Suppose \( \mathfrak{s} \in \text{SEQ} \) and \( \mathfrak{s}' \in \text{CdlI}(\mathfrak{s}) \). With Definition 3-18, it then holds that \( \mathfrak{s}' \in \text{RCE}(\mathfrak{s}) \). With Definition 3-2, we have that there are \( \Delta, \Gamma \in \text{CFORM} \) and \( i \in \text{Dom}(\mathfrak{s}) \) such that \( P(\mathfrak{s}_i) = \Delta \) and \( (i, \mathfrak{s}_i) \in \text{AVAS}(\mathfrak{s}) \) and \( P(\mathfrak{s}_{\text{Dom}(\mathfrak{s})-1}) = \Gamma \) and there is no \( l \) such that \( i < l \leq \text{Dom}(\mathfrak{s})-1 \) and \( (l, \mathfrak{s}_l) \in \text{AVAS}(\mathfrak{s}) \), and \( \mathfrak{s}' = \mathfrak{s} \cup \{ (\text{Dom}(\mathfrak{s}), \text{\"Therefore } \Delta \rightarrow \Gamma\text{\")} \} \). Then we have \( \mathfrak{s}' \in \text{SEQ} \) and \( \mathfrak{s}'_{\text{Dom}(\mathfrak{s})-1} = \mathfrak{s}'_{\text{Dom}(\mathfrak{s})} \).

We thus have that \( \mathfrak{B} = \{ (i, \mathfrak{s}_i') \mid i \leq j \leq \text{Dom}(\mathfrak{s}) \} \) is a segment in \( \mathfrak{s}' \) and that \( P(\mathfrak{s}_i') = \Delta \) and \( (i, \mathfrak{s}_i') \in \text{AVAS}(\mathfrak{s}'_{\text{Dom}(\mathfrak{s})}) \) and \( P(\mathfrak{s}_{\text{Dom}(\mathfrak{s})-1}') = \Gamma \) and that there is no \( l \) such that \( i < l \leq \text{Dom}(\mathfrak{s})-1 \) and \( (l, \mathfrak{s}_l') \in \text{AVAS}(\mathfrak{s}'_{\text{Dom}(\mathfrak{s})}) \), and \( P(\mathfrak{s}_{\text{Dom}(\mathfrak{s})}') = \text{\"Therefore } \Delta \rightarrow \Gamma\text{\")} \). With Theorem 2-91, we then have that \( \mathfrak{B} \) is a Cdl-closed segment and thus a closed segment in \( \mathfrak{s}' \).

Since \( \max(\text{Dom}(\mathfrak{B})) = \text{Dom}(\mathfrak{s}) = \text{Dom}(\mathfrak{s}')-1 \), it follows, with Theorem 2-86, that

\[
\text{AVAS}(\mathfrak{s}')_{\text{Dom}(\mathfrak{s}')-1} \setminus \text{AVAS}(\mathfrak{s'}) = \{ (\min(\text{Dom}(\mathfrak{B})), \mathfrak{s}'_{\min(\text{Dom}(\mathfrak{B}))}) \} = \{ (\max(\text{Dom}(\text{AVAS}(\mathfrak{s}')_{\text{Dom}(\mathfrak{s}')-1})), \mathfrak{s}'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{s}')_{\text{Dom}(\mathfrak{s}')-1}))}) \}
\]

Since \( \mathfrak{s}' = \mathfrak{s}'_{\text{Dom}(\mathfrak{s})}-1 \), we thus have \( \text{AVAS}(\mathfrak{s}) \setminus \text{AVAS}(\mathfrak{s}') = \{ (\min(\text{Dom}(\mathfrak{B})), \mathfrak{s}'_{\min(\text{Dom}(\mathfrak{B}))}) \} = \{ (\max(\text{Dom}(\text{AVAS}(\mathfrak{s}))), \mathfrak{s}'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{s})))}) \} \). Thus we have \( i = \min(\text{Dom}(\mathfrak{B})) = \max(\text{Dom}(\text{AVAS}(\mathfrak{s}))) \) and it holds that \( \mathfrak{B} = \{ (i, \mathfrak{s}_i') \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{s}))) \leq j \leq \text{Dom}(\mathfrak{s}) \} \). Thus we have (i). We then also have that \( P(\mathfrak{s}'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{s})))}) = P(\mathfrak{s}_i) = \Delta \). Because of \( \text{C}(\mathfrak{s}) = \Gamma \) and \( \text{C}(\mathfrak{s}') = \text{\"Therefore } \Delta \rightarrow \Gamma\text{\")} \), then it follows that (x) holds. With \( \text{AVAS}(\mathfrak{s}) \setminus \text{AVAS}(\mathfrak{s}') \neq 0 \) and Theorem 2-73, we also have \( \text{AVS}(\mathfrak{s}) \setminus \text{AVS}(\mathfrak{s}') \neq 0 \). With this and with \( \mathfrak{s}' = \mathfrak{s}'_{\text{Dom}(\mathfrak{s}')-1} \) and \( \mathfrak{B} = \{ (j, \mathfrak{s}_j') \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{s}))) \leq j \leq \text{Dom}(\mathfrak{s}) \} \).
Theorem 3-20. AVS, AVAS, AVP and AVAP and NI

If \( \mathcal{S} \in \text{SEQ} \) and \( \mathcal{S}' \in \text{NIF}(\mathcal{S}) \), then:

(i) \( \{ (j, \mathcal{S}') \mid \max(\text{Dom}(\text{AVAS}(\mathcal{S}))) \leq j \leq \text{Dom}(\mathcal{S}) \} \) is an NI-closed segment in \( \mathcal{S}' \),

(ii) \( \text{AVS}((\mathcal{S})) \setminus \text{AVS}(\mathcal{S}') \subseteq \{ (j, \mathcal{S}') \mid \max(\text{Dom}(\text{AVAS}(\mathcal{S}))) \leq j < \text{Dom}(\mathcal{S}) \} \),

(iii) \( \text{AVS}(\mathcal{S}') = (\text{AVS}(\mathcal{S}) \setminus \{ (j, \mathcal{S}') \mid \max(\text{Dom}(\text{AVAS}(\mathcal{S}))) \leq j < \text{Dom}(\mathcal{S}) \}) \cup \{ \text{Dom}(\mathcal{S}), \mathcal{S}' \text{Dom}(\mathcal{S}) \} \),

(iv) \( \text{AVAS}(\mathcal{S}) \setminus \text{AVAS}(\mathcal{S}') = \{ \max(\text{Dom}(\text{AVAS}(\mathcal{S}))), \mathcal{S}' \max(\text{Dom}(\text{AVAS}(\mathcal{S}))) \} \),

(v) \( \text{AVAS}(\mathcal{S}) = \text{AVAS}(\mathcal{S}') \cup \{ \max(\text{Dom}(\text{AVAS}(\mathcal{S}))), \mathcal{S}' \max(\text{Dom}(\text{AVAS}(\mathcal{S}))) \} \),

(vi) \( \text{AVP}(\mathcal{S}) \setminus \text{AVP}(\mathcal{S}') \subseteq \{ P(\mathcal{S}') \mid \max(\text{Dom}(\text{AVAS}(\mathcal{S}))) \leq j < \text{Dom}(\mathcal{S}) \} \),

(vii) \( \text{AVP}(\mathcal{S}) \subseteq \{ P(\mathcal{S}'), \mathcal{S} \mid \text{Dom}(\text{AVS}(\mathcal{S}')) \} \cup \{ P(\mathcal{S}'), \mathcal{S} \mid \max(\text{Dom}(\text{AVAS}(\mathcal{S}))) \leq j < \text{Dom}(\mathcal{S}) \} \),

(viii) \( \text{AVAP}(\mathcal{S}) \setminus \text{AVAP}(\mathcal{S}') \subseteq \{ \mathcal{S}' \max(\text{Dom}(\text{AVAS}(\mathcal{S}))) \} \),

(ix) \( \text{AVAP}(\mathcal{S}) = \text{AVAP}(\mathcal{S}') \cup \{ \mathcal{S}' \max(\text{Dom}(\text{AVAS}(\mathcal{S}))) \} \), and

(x) \( \text{C}(\mathcal{S}) = \lnot \text{P}(\mathcal{S} \max(\text{Dom}(\text{AVAS}(\mathcal{S})))) \).
Theorem 3-21. AVS, AVAS, AVP, AVAP and PE

If \( \mathfrak{F} \in \text{SEQ} \) and \( \mathfrak{F}' \in \text{PEF}(\mathfrak{F}) \), then:

(i) \( \{(j, \mathfrak{F}_j') \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{F}_j))) \leq j \leq \text{Dom}(\mathfrak{F}_j)\} \) is a PE-closed segment in \( \mathfrak{F}' \),

(ii) \( \text{AVS}(\mathfrak{F}) \cap \text{AVAS}(\mathfrak{F}') \subseteq \{(j, \mathfrak{F}_j') \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{F}_j))) \leq j < \text{Dom}(\mathfrak{F}_j)\} \),

(iii) \( \text{AVS}(\mathfrak{F}') = (\text{AVS}(\mathfrak{F}) \cap \{(j, \mathfrak{F}_j') \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{F}_j))) \leq j < \text{Dom}(\mathfrak{F}_j)\}) \cup \{(\text{Dom}(\mathfrak{F}_j), \mathfrak{F}_j')\} \),

(iv) \( \text{AVS}(\mathfrak{F}) \cap \text{AVAS}(\mathfrak{F}') = \{(\max(\text{Dom}(\text{AVAS}(\mathfrak{F}_j))), \mathfrak{F}_j'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{F}_j)))})\} \),

(v) \( \text{AVS}(\mathfrak{F}) = \text{AVS}(\mathfrak{F}') \cup \{(\max(\text{Dom}(\text{AVAS}(\mathfrak{F}_j))), \mathfrak{F}_j'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{F}_j)))})\} \),

(vi) \( \text{AVP}(\mathfrak{F}) \cap \text{AVP}(\mathfrak{F}') \subseteq \{\mathfrak{F}_j' \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{F}_j))) \leq j < \text{Dom}(\mathfrak{F}_j)\} \),

(vii) \( \text{AVP}(\mathfrak{F}) = \{\mathfrak{F}_j' \mid j \in \text{Dom}(\text{AVS}(\mathfrak{F}')) \cup \{\mathfrak{F}_j' \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{F}_j))) \leq j < \text{Dom}(\mathfrak{F}_j)\} \),

(viii) \( \text{AVAP}(\mathfrak{F}) \cap \text{AVAP}(\mathfrak{F}') \subseteq \{\mathfrak{F}_j'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{F}_j)))}\} \),

(ix) \( \text{AVAP}(\mathfrak{F}) = \text{AVAP}(\mathfrak{F}') \cup \{\mathfrak{F}_j'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{F}_j)))}\} \), and

(x) \( \text{C}(\mathfrak{F}_j') = \text{C}(\mathfrak{F}_j) \).

Proof: Suppose \( \mathfrak{F} \in \text{SEQ} \) and \( \mathfrak{F}' \in \text{PEF}(\mathfrak{F}) \). With Definition 3-18, we then have \( \mathfrak{F}' \in \text{RCE}(\mathfrak{F}) \). With Definition 3-15, we have that there are \( \beta \in \text{PAR}, \xi \in \text{VAR}, \Delta \in \text{FORM} \), where \( \text{FV}(\Delta) \subseteq \{\xi\}, \Gamma \in \text{CFORM} \) and \( i \in \text{Dom}(\mathfrak{F}) \) such that \( \text{P}(\mathfrak{F}_i) = \bigvee \xi \Delta \) and \( (i, \mathfrak{F}_i) \in \text{AVS}(\mathfrak{F}), \text{P}(\mathfrak{F}_{i_1}) = [\beta, \xi, \Delta] \) and \( (i_1, \mathfrak{F}_{i_1}) \in \text{AVAS}(\mathfrak{F}), \) and \( \text{P}(\mathfrak{F}_{\text{Dom}(\mathfrak{F}_i)}) = \Gamma, \beta \notin \text{STSF}(\Delta, \Gamma') \), and that there is no \( j \leq i \) such that \( \beta \in \text{ST}(\mathfrak{F}_j) \) and that there is no \( m \) such that \( i_1 < m \leq \text{Dom}(\mathfrak{F}_i)-1 \) and \( (m, \mathfrak{F}_m) \in \text{AVAS}(\mathfrak{F}), \mathfrak{F}_i = \mathfrak{F}' \cup \{(\text{Dom}(\mathfrak{F}_j), \mathfrak{F}_j')\} \).

We thus have that \( \mathfrak{B} = \{(j, \mathfrak{F}_j') \mid i+1 \leq j \leq \text{Dom}(\mathfrak{F})\} \) is a segment in \( \mathfrak{F}' \) and that \( \beta \in \text{PAR}, \xi \in \text{VAR}, \Delta \in \text{FORM} \), where \( \text{FV}(\Delta) \subseteq \{\xi\}, \Gamma \in \text{CFORM} \) and \( \text{P}(\mathfrak{F}_i') = \bigvee \xi \Delta \) and \( (i, \mathfrak{F}_i') \in \text{AVS}(\mathfrak{F}'_{\text{Dom}(\mathfrak{F}_i)}) \), \( \text{P}(\mathfrak{F}_{i_1'}) = [\beta, \xi, \Delta] \) and \( (i+1, \mathfrak{F}_{i_1'}) \in \text{AVAS}(\mathfrak{F}'_{\text{Dom}(\mathfrak{F}_i)-1}) \), and \( \text{P}(\mathfrak{F}_{\text{Dom}(\mathfrak{F}_i)}) = \Gamma, \beta \notin \text{STSF}(\Delta, \Gamma') \) and that there is no \( j \leq i \) such that \( \beta \in \text{ST}(\mathfrak{F}_j') \) and that there is no \( m \) such that \( i+1 < m \leq \text{Dom}(\mathfrak{F})-1 \) and \( (m, \mathfrak{F}_m') \in \text{AVAS}(\mathfrak{F}'_{\text{Dom}(\mathfrak{F}_i)}) \).
and that $P(\delta'_{\text{dom}(\delta)}) = \Gamma$. With Theorem 2-93, it then holds that $\mathcal{B}$ is a PE-closed segment and thus a closed segment in $\delta'$.

Since $\max(\text{Dom}(\mathcal{B})) = \text{Dom}(\delta) = \delta' - 1$, it follows, with Theorem 2-86, that $\text{AVAS}(\delta')^\uparrow \text{Dom}(\delta') - 1 \cap \text{AVAS}(\delta') = \{(\min(\text{Dom}(\mathcal{B})), \delta'_{\min(\text{Dom}(\mathcal{B}))})\} = \{(\max(\text{Dom}(\text{AVAS}(\delta'))^\uparrow \text{Dom}(\delta') - 1)), \delta'_{\max(\text{Dom}(\text{AVAS}(\delta')))}\}$. Since $\delta = \delta'_{\text{Dom}(\delta') - 1}$, we thus have $\text{AVAS}(\delta) \setminus \text{AVAS}(\delta') = \{(\min(\text{Dom}(\mathcal{B})), \delta'_{\min(\text{Dom}(\mathcal{B}))})\} = \{(\max(\text{Dom}(\text{AVAS}(\delta')))), \delta'_{\max(\text{Dom}(\text{AVAS}(\delta')))}\}$. Thus we have $i = \min(\text{Dom}(\mathcal{B})) = \max(\text{Dom}(\text{AVAS}(\delta')))$ and it holds that $\mathcal{B} = \{(j, \delta') \mid \max(\text{Dom}(\text{AVAS}(\delta'))) \leq j \leq \text{Dom}(\delta')\}$. Thus we have (i). We then also have that $C(\delta) = P(\delta'_{\text{Dom}(\delta') - 1}) = \Gamma = C(\delta')$ and thus we have (x). With AVAS(\delta') \cap \text{AVAS}(\delta')) \neq 0$ and Theorem 2-73, we also have $\text{AVS}(\delta') \subseteq \text{AVS}(\delta)$. With this and with $\delta = \delta'_{\text{Dom}(\delta') - 1}$ and $\mathcal{B} = \{(j, \delta') \mid \max(\text{Dom}(\text{AVAS}(\delta'))) \leq j \leq \text{Dom}(\delta')\}$, the remaining clauses ((ii) to (ix)) follow with Theorem 2-83-(iv) to -(xi) and with the fact that closed segments with the same end are identical (Theorem 2-53).

\textbf{Theorem 3-22. If the proposition assumed last is only once available as an assumption, then it is discharged by CdI, NI and PE}

If $\delta \in \text{SEQ}$, $\Delta \in \text{CFORM}$ and for all $i \in \text{Dom}(\text{AVAS}(\delta))$: If $P(\delta_i) = \Delta$, then $i = \max(\text{Dom}(\text{AVAS}(\delta)))$, then it holds for all $\delta' \in \text{CdI}(\delta) \cup \text{NIF}(\delta) \cup \text{PEF}(\delta)$ that $\text{AVAP}(\delta') \subseteq \text{AVAP}(\delta) \setminus \{\Delta\}$.

\textbf{Proof:} Suppose $\delta \in \text{SEQ}$, $\Delta \in \text{CFORM}$ and suppose it holds for all $i \in \text{Dom}(\text{AVAS}(\delta))$ that if $P(\delta_i) = \Delta$, then $i = \max(\text{Dom}(\text{AVAS}(\delta)))$. Now, suppose $\delta' \in \text{CdI}(\delta) \cup \text{NIF}(\delta) \cup \text{PEF}(\delta)$. With Theorem 3-19-(iv), - (v), Theorem 3-20-(iv), - (v) and Theorem 3-21-(iv), - (v), we then have that $\text{AVAS}(\delta) \cap \text{AVAS}(\delta') = \{(\max(\text{Dom}(\text{AVAS}(\delta))), \delta'_{\max(\text{Dom}(\text{AVAS}(\delta)))})\}$ and $\text{AVAS}(\delta') \subseteq \text{AVAS}(\delta)$. With Theorem 2-75, we then have $\text{AVAP}(\delta') \subseteq \text{AVAP}(\delta)$.

Then it holds that $\Delta \notin \text{AVAP}(\delta')$. To see this, suppose for contradiction that $\Delta \in \text{AVAP}(\delta')$. According to Definition 2-31, there would then be an $i \in \text{Dom}(\text{AVAS}(\delta'))$ such that $\Delta = P(\delta'_i)$. With $\text{AVAS}(\delta') \subseteq \text{AVAS}(\delta)$, we would then have that $i \in \text{Dom}(\text{AVAS}(\delta))$ and that $\Delta = P(\delta_i)$. Since, by hypothesis, it holds for all $i \in \text{Dom}(\text{AVAS}(\delta))$ that if $P(\delta_i) = \Delta$, then $i = \max(\text{Dom}(\text{AVAS}(\delta)))$, we would thus have $\max(\text{Dom}(\text{AVAS}(\delta))) = i \in \text{Dom}(\text{AVAS}(\delta'))$. But with $\text{AVAS}(\delta) \cap \text{AVAS}(\delta') = \{(\max(\text{Dom}(\text{AVAS}(\delta))), \delta'_{\max(\text{Dom}(\text{AVAS}(\delta)))})\}$, we have $\max(\text{Dom}(\text{AVAS}(\delta))) \notin \text{AVAP}(\delta')$.■
Dom(AVAS(\(\delta')\)). Contradiction! Therefore we have \(\Delta \not\in AVAP(\delta')\) and thus \(AVAP(\delta') \subseteq AVAP(\delta)\). ■

**Theorem 3-23.** *AVAS-reduction by and only by CdI, NI and PE*

If \(\delta \in SEQ\) and \(\delta' \in RCE(\delta)\), then:

\[ AVAS(\delta') \subseteq AVAS(\delta) \quad \text{iff} \quad AVAS(\delta') \setminus AVAS(\delta) = \{(\max(Dom(AVAS(\delta))), \delta_{\max(Dom(AVAS(\delta)))})\} \quad \text{and} \quad \delta' \in CdIF(\delta) \cup NIF(\delta) \cup PEF(\delta). \]

**Proof:** Suppose \(\delta \in SEQ\) and \(\delta' \in RCE(\delta)\). The right-left-direction follows with clauses (iv) and (v) of Theorem 3-19, Theorem 3-20 and Theorem 3-21.

Now, for the left-right-direction, suppose \(AVAS(\delta') \subseteq AVAS(\delta)\) and with Theorem 3-1, we have \(\delta' \in SEQ\). With Theorem 3-5, we have \(\delta' \subseteq Dom(\delta)\) and thus \(Dom(\delta) = Dom(\delta')\). Because of \(AVAS(\delta') \subseteq AVAS(\delta)\) and with Theorem 2-85, we thus have that there is a closed segment \(\mathfrak{A}\) in \(\delta'\) such that \(\min(Dom(\mathfrak{A})) \leq \max(Dom(\mathfrak{A})) = \delta\) and thus \(Dom(\delta) = Dom(\delta')\).

Because of \(AVAS(\delta') \subseteq AVAS(\delta)\) and with Theorem 2-85, we thus have that there is a closed segment \(\mathfrak{A}\) in \(\delta'\) such that \(\min(Dom(\mathfrak{A})) \leq \max(Dom(\mathfrak{A})) = \delta\) and thus \(Dom(\delta) = Dom(\delta')\).

With Theorem 2-61, we have that \(\mathfrak{A}\) is a CdI- or NI- or PE-closed segment in \(\delta'\). Now, suppose \(\mathfrak{A}\) is a CdI-closed segment in \(\delta'\). With Theorem 2-91, it then holds that

a) \((\min(Dom(\mathfrak{A})), \delta_{\min(Dom(\mathfrak{A}))}) = (\min(Dom(\mathfrak{A})), \delta_{\min(Dom(\mathfrak{A}))}) \in AVAS(\delta),\)

b) \(P(\delta_{\min(Dom(\mathfrak{A}))}) = C(\delta),\)

c) There is no \(r\) such that \(\min(Dom(\mathfrak{A})) < r \leq Dom(\delta)\) and \(\delta' = (r, \delta_r)\) \(\in AVAS(\delta),\) and

d) \(\delta'_{\min(Dom(\mathfrak{A}))} = \text{"Therefore } P(\delta_{\min(Dom(\mathfrak{A}))}) \rightarrow C(\delta)\".\)

According to Definition 3-2, we then have \(\delta' \in CdIF(\delta)\). Now, suppose \(\mathfrak{A}\) is an NI-closed segment in \(\delta'\). With Theorem 2-92, it then holds that there are \(i \in Dom(\delta')\) and \(\Gamma \in CFORM\) such that

a) \(\min(Dom(\mathfrak{A})) \leq i \leq Dom(\delta)\),

b) \((\min(Dom(\mathfrak{A})), \delta_{\min(Dom(\mathfrak{A}))}) = (\min(Dom(\mathfrak{A})), \delta_{\min(Dom(\mathfrak{A}))}) \in AVAS(\delta),\)
According to Definition 3-10, we then have \( s'_f \in \text{NIF}(s_f) \). Now, suppose \( \mathcal{A} \) is a PE-closed segment in \( s'_f \). With Theorem 2-93, it then holds that there are \( \xi \in \text{VAR}, \beta \in \text{PAR}, \Delta \in \text{FORM}, \) where \( \text{FV}(\Delta) \subseteq \{\xi\}, \Gamma \in \text{CFORM} \) and \( \mathcal{B} \in \text{SG}(s'_f) \) such that:

- P(\( s'_{\text{min}(\text{Dom}(\mathcal{B}))} \)) = \( \lnot \forall \xi \Delta \) and (min(\( \text{Dom}(\mathcal{B}) \)), \( s'_{\text{min}(\text{Dom}(\mathcal{B}))} \)) \( \in \text{AVS}(s_f) \),
- P(\( s'_{\text{min}(\text{Dom}(\mathcal{B}))+1} \)) = [\( \beta, \xi, \Delta \) and (min(\( \text{Dom}(\mathcal{B}) \))+1, \( s'_{\text{min}(\text{Dom}(\mathcal{B}))+1} \)) \( \in \text{AVAS}(s_f) \),
- P(\( s'_{\text{max}(\text{Dom}(\mathcal{B}))} \)) = \( \Gamma \),
- \( s'_{\text{max}(\text{Dom}(\mathcal{B}))} \) = \( \lnot \therefore \Gamma \)
- \( \beta \notin \text{STSF} \{\{\Delta, \Gamma\}\}, \)
- There is no \( j \leq \text{min}(\text{Dom}(\mathcal{B})) \) such that \( \beta \in \text{ST}(s'_f) \),
- \( \mathcal{A} = \mathcal{B} \setminus \{\text{min}(\text{Dom}(\mathcal{B})), \( s'_{\text{min}(\text{Dom}(\mathcal{B}))} \)\} \) and
- There is no \( r \) such that min(\( \text{Dom}(\mathcal{A}) \)) < \( r \leq \text{Dom}(s'_f) \)-1 and (\( r, s'_f \) \( \in \text{AVAS}(s_f) \).

With g), we have min(\( \text{Dom}(\mathcal{A}) \)) = min(\( \text{Dom}(\mathcal{B}) \))+1 and \( \text{Dom}(s_f) = \text{max}(\text{Dom}(\mathcal{A})) = \text{max}(\text{Dom}(\mathcal{B})) \). It then follows that min(\( \text{Dom}(\mathcal{B}) \)) < min(\( \text{Dom}(\mathcal{A}) \)) \leq \text{Dom}(s_f)-1 and therefore we have min(\( \text{Dom}(\mathcal{B}) \)), min(\( \text{Dom}(\mathcal{B}) \))+1 \( \in \text{Dom}(s_f) \) and max(\( \text{Dom}(\mathcal{B}) \))-1 = \( \text{Dom}(s_f)-1 \). It then follows that

- P(\( s'_{\text{min}(\text{Dom}(\mathcal{B}))} \)) = \( \lnot \forall \xi \Delta \) and (min(\( \text{Dom}(\mathcal{B}) \)), \( s'_{\text{min}(\text{Dom}(\mathcal{B}))} \)) \( \in \text{AVS}(s_f) \),
- P(\( s'_{\text{min}(\text{Dom}(\mathcal{B}))+1} \)) = [\( \beta, \xi, \Delta \) and (min(\( \text{Dom}(\mathcal{B}) \))+1, \( s'_{\text{min}(\text{Dom}(\mathcal{B}))+1} \)) \( \in \text{AVAS}(s_f) \),
- P(\( s'_{\text{Dom}(s_f)} \)) = \( \Gamma \),
- \( s'_{\text{Dom}(s_f)} \) = \( \lnot \therefore \Gamma \)
- \( \beta \notin \text{STSF} \{\{\Delta, \Gamma\}\}, \)
- There is no \( j \leq \text{min}(\text{Dom}(\mathcal{B})) \) such that \( \beta \in \text{ST}(s_f) \),
- There is no \( r \) such that min(\( \text{Dom}(\mathcal{B}) \))+1 < \( r \leq \text{Dom}(s_f)-1 \) and (\( r, s_f \) \( \in \text{AVAS}(s_f) \).

According to Definition 3-15, we then have \( s'_f \in \text{PEF}(s_f) \). Hence we have in all three cases that \( s'_f \in \text{CDIF}(s_f) \cup \text{NIF}(s_f) \cup \text{PEF}(s_f) \).
Theorem 3-24. AVS-reduction by and only by CdI, NI and PE
If \( \mathfrak{f} \in \text{SEQ} \) and \( \mathfrak{f}' \in \text{RCE}(\mathfrak{f}) \), then:

\[
\text{AVS}(\mathfrak{f}) \nsubseteq \text{AVS}(\mathfrak{f}')
\]

iff

\[
\{ (j, \mathfrak{f}'_j) \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{f}))) \leq j \leq \text{Dom}(\mathfrak{f}) \} \text{ is a CdI- or NI- or PE-closed segment in } \mathfrak{f}'
\]

and \( \mathfrak{f}' \in \text{CdIF}(\mathfrak{f}) \cup \text{NIF}(\mathfrak{f}) \cup \text{PEF}(\mathfrak{f}). \)

Proof: Suppose \( \mathfrak{f} \in \text{SEQ} \) and \( \mathfrak{f}' \in \text{RCE}(\mathfrak{f}) \). The right-left-direction follows with clause (iv) of Theorem 3-19, Theorem 3-20 and Theorem 3-21, and with Theorem 2-72. Now, for the left-right-direction, suppose \( \text{AVS}(\mathfrak{f}) \nsubseteq \text{AVS}(\mathfrak{f}'). \) Then we have \( \text{AVS}(\mathfrak{f}) \nsubseteq \text{AVS}(\mathfrak{f}'). \) with \( \mathfrak{f}' \in \text{RCE}(\mathfrak{f}) \) and Theorem 3-1, we have \( \mathfrak{f}' \in \text{SEQ} \) and, with Theorem 3-5, \( \mathfrak{f}'_j \mid \text{Dom}(\mathfrak{f}') \) \( = \mathfrak{f}. \) With Theorem 2-83-(vi) and -(vii), it then follows that \( \text{AVAS}(\mathfrak{f}') \subseteq \text{AVAS}(\mathfrak{f}). \) With Theorem 3-23, it then holds that \( \mathfrak{f}' \in \text{CdIF}(\mathfrak{f}) \cup \text{NIF}(\mathfrak{f}) \cup \text{PEF}(\mathfrak{f}). \) With Theorem 3-19-(i), Theorem 3-20-(i) and Theorem 3-21-(i), it then follows that \( \{ (j, \mathfrak{f}'_j) \mid \max(\text{Dom}(\text{AVAS}(\mathfrak{f}))) \leq j \leq \text{Dom}(\mathfrak{f}) \} \) is a CdI- or NI- or PE-closed segment in \( \mathfrak{f}'. \) ■

Theorem 3-25. AVS if CdI, NI and PE are excluded
If \( \mathfrak{f} \in \text{SEQ} \) and \( \mathfrak{f}' \in \text{RCE}(\mathfrak{f}) \), then:

\[
\text{AVS}(\mathfrak{f}') = \text{AVS}(\mathfrak{f}) \cup \{ (\text{Dom}(\mathfrak{f}), \mathfrak{f}'_\text{Dom}(\mathfrak{f}')) \}.
\]

Proof: Let \( \mathfrak{f} \in \text{SEQ} \) and \( \mathfrak{f}' \in \text{RCE}(\mathfrak{f}) \). Because of Theorem 3-14-(i), we have \( \text{AVS}(\mathfrak{f}') \subseteq \text{AVS}(\mathfrak{f}) \cup \{ (\text{Dom}(\mathfrak{f}), \mathfrak{f}'_\text{Dom}(\mathfrak{f}')) \} \). With Theorem 2-82, we have that \( \text{C}(\mathfrak{f}') = \text{P}(\mathfrak{f}'_\text{Dom}(\mathfrak{f}')) \) is available in \( \mathfrak{f}' \) at \( \text{Dom}(\mathfrak{f}) \). Therefore \( \text{Dom}(\mathfrak{f}) \subseteq \text{AVS}(\mathfrak{f}') \). If \( \text{AVS}(\mathfrak{f}) \nsubseteq \text{AVS}(\mathfrak{f}') \), then we would have, with Theorem 3-24, that \( \mathfrak{f}' \in \text{CdIF}(\mathfrak{f}) \cup \text{NIF}(\mathfrak{f}) \cup \text{PEF}(\mathfrak{f}), \) which contradicts the hypothesis. Therefore we have \( \text{AVS}(\mathfrak{f}) \subseteq \text{AVS}(\mathfrak{f}'). \) Hence we also have \( \text{AVS}(\mathfrak{f}) \cup \{ (\text{Dom}(\mathfrak{f}), \mathfrak{f}'_\text{Dom}(\mathfrak{f}')) \} \subseteq \text{AVS}(\mathfrak{f}'). \) ■

Theorem 3-26. AVS, AVAS, AVP, AVAP and CI, BI, DI, UI, PI, II
If \( \mathfrak{f} \in \text{SEQ} \) and \( \mathfrak{f}' \in \text{CIF}(\mathfrak{f}) \cup \text{BIF}(\mathfrak{f}) \cup \text{DIF}(\mathfrak{f}) \cup \text{UIF}(\mathfrak{f}) \cup \text{PIF}(\mathfrak{f}) \cup \text{IIF}(\mathfrak{f}), \) then:

(i) \( \text{AVS}(\mathfrak{f}') \subseteq \text{AVS}(\mathfrak{f}) \cup \{ (\text{Dom}(\mathfrak{f}), \mathfrak{f}'_\text{Dom}(\mathfrak{f})) \}, \)

(ii) \( \text{AVAS}(\mathfrak{f}') \subseteq \text{AVAS}(\mathfrak{f}), \)

(iii) If \( \text{AVAS}(\mathfrak{f}') \subseteq \text{AVAS}(\mathfrak{f}), \) then \( \mathfrak{f}' \in \text{PEF}(\mathfrak{f}), \)

(iv) \( \text{AVP}(\mathfrak{f}') \subseteq \text{AVP}(\mathfrak{f}) \cup \{ \text{C}(\mathfrak{f}') \}, \)
(v) If AVAP(\(\mathcal{S}'\)) \(\subset\) AVAP(\(\mathcal{S}\)), and
(vi) If AVAP(\(\mathcal{S}'\)) \(\subset\) AVAP(\(\mathcal{S}\)), then \(\mathcal{S}' \in\) PEF(\(\mathcal{S}\)).

Proof: Suppose \(\mathcal{S} \in \text{SEQ}\) and \(\mathcal{S}' \in \text{CIF}(\mathcal{S}) \cup \text{BIF}(\mathcal{S}) \cup \text{DIF}(\mathcal{S}) \cup \text{UIF}(\mathcal{S}) \cup \text{PIF}(\mathcal{S}) \cup \text{IIF}(\mathcal{S})\). With Definition 3-18, we then have \(\mathcal{S}' \in \text{RCE}(\mathcal{S})\). With Definition 3-4, Definition 3-6, Definition 3-8, Definition 3-12, Definition 3-14 and Definition 3-16, we have that there are \(A, B \in \text{CFORM}\) and \(\theta \in \text{CTERM}\) and \(\beta \in \text{PAR}\) and \(\xi \in \text{VAR}\) and \(\Delta \in \text{FORM}\), where \(\text{FV}(\Delta) \subset \{\xi\}\) such that \(\mathcal{S}' = \mathcal{S} \cup \{\text{Dom}(\mathcal{S}), \text{Therefore } A \land B^*\}\) or \(\mathcal{S}' = \mathcal{S} \cup \{\text{Dom}(\mathcal{S}), \text{Therefore } A \leftrightarrow B^*\}\) or \(\mathcal{S}' = \mathcal{S} \cup \{\text{Dom}(\mathcal{S}), \text{Therefore } A \lor B^*\}\) or \(\mathcal{S}' = \mathcal{S} \cup \{\text{Dom}(\mathcal{S}), \text{Therefore } \lambda \xi \Delta^*\}\) or \(\mathcal{S}' = \mathcal{S} \cup \{\text{Dom}(\mathcal{S}), \text{Therefore } \forall \xi \Delta^*\}\) or \(\mathcal{S}' = \mathcal{S} \cup \{\text{Dom}(\mathcal{S}), \text{Therefore } 0 = \theta^*\}\). With the theorems on unique readability (Theorem 1-10, Theorem 1-11 and Theorem 1-12), we then have (\(\text{Dom}(\mathcal{S}), \mathcal{S}'_{\text{Dom}(\mathcal{S})}\) \(\not\in\) AS(\(\mathcal{S}'\)) and thus, with Definition 3-1, that \(\mathcal{S}' \not\in\) AF(\(\mathcal{S}\)). Then (i), (ii), (iv) and (v) follow with Theorem 3-17-(i), -(ii), -(iii) and -(iv). With Theorem 3-19-(x), Theorem 3-20-(x) and unique readability, it follows that \(\mathcal{S}' \not\in\) CdIF(\(\mathcal{S}\)) \(\cup\) NIF(\(\mathcal{S}\)). With Theorem 3-23, it then follows that if AVAS(\(\mathcal{S}'\)) \(\subset\) AVAS(\(\mathcal{S}\)), then \(\mathcal{S}' \in\) PEF(\(\mathcal{S}\)) and hence we have (iii). Now, suppose for (vi) that AVAP(\(\mathcal{S}'\)) \(\subset\) AVAP(\(\mathcal{S}\)). Then we have AVAP(\(\mathcal{S}\)) \(\not\subset\) AVAP(\(\mathcal{S}'\)) and thus, with Theorem 2-75, AVAS(\(\mathcal{S}\)) \(\not\subset\) AVAS(\(\mathcal{S}'\)). With (ii), we then have AVAS(\(\mathcal{S}'\)) \(\subset\) AVAS(\(\mathcal{S}\)) and thus, with (iii), that \(\mathcal{S}' \in\) PEF(\(\mathcal{S}\)).

Theorem 3-27. AVS, AVAS, AVP, AVAP and CdE, CE, BE, DE, NE, UE, IE

If \(\mathcal{S} \in \text{SEQ}\) and \(\mathcal{S}' \in \text{CDEF}(\mathcal{S}) \cup \text{CEF}(\mathcal{S}) \cup \text{BEF}(\mathcal{S}) \cup \text{DEF}(\mathcal{S}) \cup \text{NEF}(\mathcal{S}) \cup \text{UEF}(\mathcal{S}) \cup \text{IEF}(\mathcal{S})\), then:

(i) AVS(\(\mathcal{S}'\)) \(\subset\) AVS(\(\mathcal{S}\)) \(\cup\) \{\text{Dom}(\mathcal{S}), \mathcal{S}'_{\text{Dom}(\mathcal{S})}\},
(ii) AVS(\(\mathcal{S}\)) \(\subset\) AVAS(\(\mathcal{S}\)),
(iii) If AVAS(\(\mathcal{S}'\)) \(\subset\) AVAS(\(\mathcal{S}\)), then \(\mathcal{S}' \in\) CDEF(\(\mathcal{S}\)) \(\cup\) NIF(\(\mathcal{S}\)) \(\cup\) PEF(\(\mathcal{S}\)),
(iv) AVP(\(\mathcal{S}'\)) \(\subset\) AVP(\(\mathcal{S}\)) \(\cup\) \{C(\(\mathcal{S}'\))\},
(v) AVAP(\(\mathcal{S}'\)) \(\subset\) AVAP(\(\mathcal{S}\)), and
(vi) If AVAP(\(\mathcal{S}'\)) \(\subset\) AVAP(\(\mathcal{S}\)), then \(\mathcal{S}' \in\) CDEF(\(\mathcal{S}\)) \(\cup\) NIF(\(\mathcal{S}\)) \(\cup\) PEF(\(\mathcal{S}\)).

Proof: Suppose \(\mathcal{S} \in \text{SEQ}\) and \(\mathcal{S}' \in \text{CDEF}(\mathcal{S}) \cup \text{CEF}(\mathcal{S}) \cup \text{BEF}(\mathcal{S}) \cup \text{DEF}(\mathcal{S}) \cup \text{NEF}(\mathcal{S}) \cup \text{UEF}(\mathcal{S}) \cup \text{IEF}(\mathcal{S})\). With Definition 3-18, we then have \(\mathcal{S}' \in \text{RCE}(\mathcal{S})\). With Definition 3-3, Definition 3-5, Definition 3-7, Definition 3-9, Definition 3-11, Definition 3-13 and Definition 3-17, we have \(\mathcal{S}' = \mathcal{S} \cup \{\text{Dom}(\mathcal{S}), \text{Therefore } P(\mathcal{S}'_{\text{Dom}(\mathcal{S})})\}\). Then we have (\(\text{Dom}(\mathcal{S}), \mathcal{S}'_{\text{Dom}(\mathcal{S})}\) \(\not\in\) AS(\(\mathcal{S}'\)) and thus (\(\text{Dom}(\mathcal{S}), \mathcal{S}'_{\text{Dom}(\mathcal{S})}\) \(\not\in\) AVAS(\(\mathcal{S}'\)) and \(\mathcal{S}' \not\in\) AF(\(\mathcal{S}\)).
Then, with Theorem 3-14-(i), -(ii) and -(iii), we have (i), (ii), (iv) and (v). Clause (iii) follows with Theorem 3-23. Now, suppose for (vi) that AVAP(§') ⊂ AVAP(§). Then we have AVAP(§) ⊄ AVAP(§') and thus, with Theorem 2-75, AVAS(§) ⊄ AVAS(§'). With (ii), we then have AVAS(§') ⊂ AVAS(§) and thus, with (iii), that §' ∈ CdIF(§) ∪ NIF(§) ∪ PEF(§).

Theorem 3-28. Without AR, CdI, NI or PE there is no AVAP-change
If § ∈ RCS and § ∉ AF(§|Dom(§)-1) ∪ CdIF(§|Dom(§)-1) ∪ NIF(§|Dom(§)-1) ∪ PEF(§|Dom(§)-1), then AVAP(§) = AVAP(§|Dom(§)-1).

Proof: Suppose § ∈ RCS and § ∉ AF(§|Dom(§)-1) ∪ CdIF(§|Dom(§)-1) ∪ NIF(§|Dom(§)-1) ∪ PEF(§|Dom(§)-1). We have § = ∅ or § ≠ ∅. In the first case, we have §|Dom(§)-1 ⊂ § = ∅ and the theorem holds. Now, suppose § ≠ ∅. According to Theorem 3-6 and Definition 3-18, it then follows that first § ∈ CIF(§|Dom(§)-1) or § ∈ BIF(§|Dom(§)-1) or § ∈ DIF(§|Dom(§)-1) or § ∈ UIF(§|Dom(§)-1) or § ∈ PIF(§|Dom(§)-1) or § ∈ PIF(§|Dom(§)-1) or § ∈ IIF(§|Dom(§)-1) or second § ∈ CdEF(§|Dom(§)-1) or § ∈ DEF(§|Dom(§)-1) or § ∈ NEF(§|Dom(§)-1) or § ∈ UEF(§|Dom(§)-1) or § ∈ IEF(§|Dom(§)-1). In the first six cases, AVAP(§) = AVAP(§|Dom(§)-1) follows from Theorem 3-26-(v) and -(vi). In the remaining cases AVAP(§) = AVAP(§|Dom(§)-1) follows from Theorem 3-27-(v) and -(vi). ■

Theorem 3-29. AVS, AVAS, AVP and AVAP of restrictions whose conclusion stays available remain intact in the unrestricted sentence sequence.
If § ∈ RCS and Γ is available in § at i, then:
(i) AVS(§|i+1) ⊂ AVS(§),
(ii) AVAS(§|i+1) ⊂ AVAS(§),
(iii) AVP(§|i+1) ⊂ AVP(§), and
(iv) AVAP(§|i+1) ⊂ AVAP(§).

Proof: Suppose § ∈ RCS and Γ is available in § at i. According to Definition 2-26, we then have i ∈ Dom(§) and Γ = P(§), and there is no closed segment Ξ in § such that min(Dom(Ξ)) ≤ i < max(Dom(Ξ)).

Ad (i): To show AVS(§|i+1) ⊂ AVS(§), suppose (j, Σ) ∈ AVS(§|i+1). With Definition 2-28, we then have j ∈ Dom(§|i+1) and (§|i+1)j = Σ and P(Σ) is available in
3.3 AVS, AVAS, AVP and AVAP in Derivations and in Individual Transitions

According to Definition 2-26, there is thus no closed segment \( \mathfrak{A} \) in \( \mathfrak{H} \) such that \( \min(\text{Dom}(\mathfrak{A})) \leq j < \max(\text{Dom}(\mathfrak{A})) \). Now, suppose for contradiction, that \((j, \Sigma) \notin AVS(\mathfrak{H})\). Then we would have \( j \notin \text{Dom}(\mathfrak{H}) \) or \( \mathfrak{H}_j \neq \Sigma \) or \( P(\Sigma) \) is not available in \( \mathfrak{H} \) at \( j \). Since \( \mathfrak{H}|^j+i+1 \) is a restriction of \( \mathfrak{H} \) and \( j \in \text{Dom}(\mathfrak{H}|^j+i+1) \), the first two cases are excluded. Thus, we would have \( j \in \text{Dom}(\mathfrak{H}) \) and \( \mathfrak{H}_j = \Sigma \) and \( P(\Sigma) \) is not available in \( \mathfrak{H} \) at \( j \). According to Definition 2-26, there is thus a closed segment \( \mathfrak{A} \) in \( \mathfrak{H} \) such that \( \min(\text{Dom}(\mathfrak{A})) \leq j < \max(\text{Dom}(\mathfrak{A})) \). According to Theorem 2-64-(viii), \( \mathfrak{A} \) is also a closed segment in \( \mathfrak{H}|^\text{max(Dom(\mathfrak{A}))}+1 \). If \( i < \text{max(Dom(\mathfrak{A}))} \), then we would have, because of \( j \in \text{Dom}(\mathfrak{H}) \) and thus \( j \leq i \), that also \( \min(\text{Dom}(\mathfrak{A})) \leq i < \text{max(Dom(\mathfrak{A}))} \). Thus we would have that \( P(\mathfrak{H}_i) = \Gamma \) is not available in \( \mathfrak{H} \) at \( i \), which contradicts the hypothesis. Therefore we have \( \text{max(Dom(\mathfrak{A}))} \leq i \) and thus \( \text{max(Dom(\mathfrak{A}))}+1 \leq i+1 \). Therefore we have \( \mathfrak{H}|^\text{max(Dom(\mathfrak{A}))}+1 \subset \mathfrak{H}|^j+i+1 \). With Theorem 2-62-(viii), \( \mathfrak{A} \) is then also a closed segment in \( \mathfrak{H}|^j+i+1 \). Therefore there is a closed segment \( \mathfrak{A} \) in \( \mathfrak{H}|^j+i+1 \) such that \( \min(\text{Dom}(\mathfrak{A})) \leq j < \text{max(Dom(\mathfrak{A}))} \). Contradiction! Therefore \((j, \Sigma) \in AVS(\mathfrak{H})\).

Ad (ii), (iii) and (iv): With Theorem 2-72, (ii) follows from (i). With Theorem 2-74, (iii) follows from (i). With Theorem 2-75, (iv) follows from (ii). ■

Theorem 3-30. AVS, AVAS, AVP and AVAP in derivations

If \( \mathfrak{H} \in \text{SEQ} \), then:

\( \mathfrak{H} \in \text{RCS} \) if and only if for all \( i \in \text{Dom}(\mathfrak{H}) \):

(i) \( \mathfrak{H}|^i+i+1 \in \text{AF}(\mathfrak{H}|^i+i) \) and

\( a) \ AVS(\mathfrak{H}|^i+i+1) \setminus \text{AVS}(\mathfrak{H}|^i+i) = \{(i, \mathfrak{H}_i),\} \),
(\( b) \ AVS(\mathfrak{H}|^i+i+1) = \text{AVS}(\mathfrak{H}|^i+i) \cup \{(i, \mathfrak{H}_i),\} \),
(\( c) \ AVAS(\mathfrak{H}|^i+i+1) \setminus \text{AVAS}(\mathfrak{H}|^i+i) = \{(i, \mathfrak{H}_i),\} \),
(\( d) \ AVAS(\mathfrak{H}|^i+i+1) = \text{AVAS}(\mathfrak{H}|^i+i) \cup \{(i, \mathfrak{H}_i),\} \),
(\( e) \ AVP(\mathfrak{H}|^i+i+1) \setminus \text{AVP}(\mathfrak{H}|^i+i) \subseteq \{P(\mathfrak{H}_i)\} \),
(\( f) \ AVP(\mathfrak{H}|^i+i+1) = \text{AVP}(\mathfrak{H}|^i+i) \cup \{P(\mathfrak{H}_i)\} \),
(\( g) \ AVAP(\mathfrak{H}|^i+i+1) \setminus \text{AVAP}(\mathfrak{H}|^i+i) \subseteq \{P(\mathfrak{H}_i)\} \), and
(\( h) \ AVAP(\mathfrak{H}|^i+i+1) = \text{AVAP}(\mathfrak{H}|^i+i) \cup \{P(\mathfrak{H}_i)\} \) or

or

(ii) \( \mathfrak{H}|^j+i+1 \in \text{CdIF}(\mathfrak{H}|^j+i) \) and

\( a) \ \{(j, \mathfrak{H}_j) | \max(\text{Dom}(\text{AVAS}(\mathfrak{H}|^j+i))) \leq j \leq i \} \) is a CdI-closed segment in \( \mathfrak{H}|^j+i+1 \),
(\( b) \ AVS(\mathfrak{H}|^j+i) \setminus \text{AVS}(\mathfrak{H}|^j+i+1) \subseteq \{(j, \mathfrak{H}_j) | \max(\text{Dom}(\text{AVAS}(\mathfrak{H}|^j+i))) \leq j < i \},

c) \( \text{AVS}(\delta^i)_{i+1} = \)
\( \{ (j, \delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_j))) \leq j < i \} \cup \{(i, \delta_i)\}, \)

d) \( \text{AVAS}(\delta^i_i) \cup \text{AVAS}(\delta^i_{i+1}) = \{(\max(\text{Dom}(\text{AVAS}(\delta^i_i))), \delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))}\}, \)

e) \( \text{AVAS}(\delta^i_i) = \)
\( \text{AVAS}(\delta^i_{i+1}) \cup \{(\max(\text{Dom}(\text{AVAS}(\delta^i_i))), \delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))}\}, \)

f) \( \text{AVP}(\delta^i_i) \cup \text{AVP}(\delta^i_{i+1}) \subset \{P(\delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_j))) \leq j < i\}, \)

g) \( \text{AVP}(\delta^i_i) \cup \{P(\delta_j) \mid j \in \text{Dom}(\text{AVS}(\delta^i_{i+1}))) \cup \{P(\delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_i))) \leq j < i\}, \)

h) \( \text{AVAP}(\delta^i_i) \cup \text{AVAP}(\delta^i_{i+1}) \subset \{P(\delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))})\}, \)

i) \( \text{AVAP}(\delta^i_i) = \text{AVAP}(\delta^i_{i+1}) \cup \{P(\delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))})\}, \) and

j) \( P(\delta_i) = \lnot P(\delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))}) \to P(\delta_{i+1}) \)

or

(iii) \( \delta^i_{i+1} \in \text{NIF}(\delta^i_i) \) and

a) \( \{(j, \delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_j))) \leq j < i\} \) is an NI-closed segment in \( \delta^i_{i+1} \),

b) \( \text{AVS}(\delta^i_i) \cup \text{AVS}(\delta^i_{i+1}) = \{(j, \delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_j))) \leq j < i\}, \)

c) \( \text{AVS}(\delta^i_i)_{i+1} = \)
\( \text{AVS}(\delta^i_{i+1}) \cup \{(j, \delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_j))) \leq j < i\} \cup \{(i, \delta_i)\}, \)

d) \( \text{AVS}(\delta^i_i) \cup \text{AVS}(\delta^i_{i+1}) = \{(\max(\text{Dom}(\text{AVAS}(\delta^i_i))), \delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))}\}, \)

e) \( \text{AVS}(\delta^i_i) = \)
\( \text{AVS}(\delta^i_{i+1}) \cup \{(\max(\text{Dom}(\text{AVAS}(\delta^i_i))), \delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))}\}, \)

f) \( \text{AVP}(\delta^i_i) \cup \text{AVP}(\delta^i_{i+1}) \subset \{P(\delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_j))) \leq j < i\}, \)

g) \( \text{AVP}(\delta^i_i) \cup \{P(\delta_j) \mid j \in \text{Dom}(\text{AVS}(\delta^i_{i+1}))) \cup \{P(\delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_i))) \leq j < i\}, \)

h) \( \text{AVAP}(\delta^i_i) \cup \text{AVAP}(\delta^i_{i+1}) \subset \{P(\delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))})\}, \)

i) \( \text{AVAP}(\delta^i_i) = \text{AVAP}(\delta^i_{i+1}) \cup \{P(\delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))})\}, \) and

j) \( P(\delta_i) = \lnot P(\delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))}) \)

or

(iv) \( \delta^i_{i+1} \in \text{PEF}(\delta^i_i) \) and

a) \( \{(j, \delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_j))) \leq j < i\} \) is a PE-closed segment in \( \delta^i_{i+1} \),

b) \( \text{AVS}(\delta^i_i) \cup \text{AVS}(\delta^i_{i+1}) = \{(j, \delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_j))) \leq j < i\}, \)

c) \( \text{AVS}(\delta^i_i)_{i+1} = \)
\( \text{AVS}(\delta^i_{i+1}) \cup \{(j, \delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_j))) \leq j < i\} \cup \{(i, \delta_i)\}, \)

d) \( \text{AVS}(\delta^i_i) \cup \text{AVS}(\delta^i_{i+1}) = \{(\max(\text{Dom}(\text{AVAS}(\delta^i_i))), \delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))}\}, \)

e) \( \text{AVS}(\delta^i_i) = \)
\( \text{AVS}(\delta^i_{i+1}) \cup \{(\max(\text{Dom}(\text{AVAS}(\delta^i_i))), \delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))}\}, \)

f) \( \text{AVP}(\delta^i_i) \cup \text{AVP}(\delta^i_{i+1}) \subset \{P(\delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_j))) \leq j < i\}, \)

g) \( \text{AVP}(\delta^i_i) \cup \{P(\delta_j) \mid j \in \text{Dom}(\text{AVS}(\delta^i_{i+1}))) \cup \{P(\delta_j) \mid \max(\text{Dom}(\text{AVAS}(\delta^i_i))) \leq j < i\}, \)

h) \( \text{AVAP}(\delta^i_i) \cup \text{AVAP}(\delta^i_{i+1}) \subset \{P(\delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))})\}, \)

i) \( \text{AVAP}(\delta^i_i) = \text{AVAP}(\delta^i_{i+1}) \cup \{P(\delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))})\}, \) and

j) \( P(\delta_i) = \lnot P(\delta_{\max(\text{Dom}(\text{AVAS}(\delta^i_i)))}) \)
h) $\text{AVAP}(\mathcal{A}|i) \cap \text{AVAP}(\mathcal{A}|i+1) \subseteq \{P(\mathcal{A}_{\max(\text{Dom}(\text{AVAS}(\mathcal{A}|i)))})\},$

i) $\text{AVAP}(\mathcal{A}|i) = \text{AVAP}(\mathcal{A}|i+1) \cup \{P(\mathcal{A}_{\max(\text{Dom}(\text{AVAS}(\mathcal{A}|i)))})\},$ and

j) $P(\mathcal{A}_i) = \text{"}P(\mathcal{A}_{i+1}))$

or

(v) $\mathcal{A}|i+1 \in \text{CIF}(\mathcal{A}|i) \cup \text{BIF}(\mathcal{A}|i) \cup \text{DIF}(\mathcal{A}|i) \cup \text{UIF}(\mathcal{A}|i) \cup \text{PIF}(\mathcal{A}|i) \cup \text{IIF}(\mathcal{A}|i)$ and

a) $\text{AVS}(\mathcal{A}|i+1) \subseteq \text{AVS}(\mathcal{A}|i) \cup \{(i, \mathcal{A}_i)\},$

b) $\text{AVAS}(\mathcal{A}|i+1) \subseteq \text{AVAS}(\mathcal{A}|i),$

c) If $\text{AVAS}(\mathcal{A}|i+1) \subseteq \text{AVAS}(\mathcal{A}|i),$ then $\mathcal{A}|i+1 \in \text{PEF}(\mathcal{A}|i),$

d) $\text{AVP}(\mathcal{A}|i+1) \subseteq \text{AVP}(\mathcal{A}|i) \cup \{P(\mathcal{A}_i)\},$

e) $\text{AVAP}(\mathcal{A}|i+1) \subseteq \text{AVAP}(\mathcal{A}|i),$ and

f) If $\text{AVAP}(\mathcal{A}|i+1) \subseteq \text{AVAP}(\mathcal{A}|i),$ then $\mathcal{A}|i+1 \in \text{PEF}(\mathcal{A}|i)$

or

(vi) $\mathcal{A}|i+1 \in \text{CdEF}(\mathcal{A}|i) \cup \text{CEF}(\mathcal{A}|i) \cup \text{BEF}(\mathcal{A}|i) \cup \text{DEF}(\mathcal{A}|i) \cup \text{NEF}(\mathcal{A}|i) \cup \text{UEF}(\mathcal{A}|i) \cup \text{IEF}(\mathcal{A}|i)$ and

a) $\text{AVS}(\mathcal{A}|i+1) \subseteq \text{AVS}(\mathcal{A}|i) \cup \{(i, \mathcal{A}_i)\},$

b) $\text{AVAS}(\mathcal{A}|i+1) \subseteq \text{AVAS}(\mathcal{A}|i),$

c) If $\text{AVAS}(\mathcal{A}|i+1) \subseteq \text{AVAS}(\mathcal{A}|i),$ then $\mathcal{A}|i+1 \in \text{CdIF}(\mathcal{A}|i) \cup \text{NIF}(\mathcal{A}|i) \cup \text{PEF}(\mathcal{A}|i),$

d) $\text{AVP}(\mathcal{A}|i+1) \subseteq \text{AVP}(\mathcal{A}|i) \cup \{P(\mathcal{A}_i)\},$

e) $\text{AVAP}(\mathcal{A}|i+1) \subseteq \text{AVAP}(\mathcal{A}|i),$ and

f) If $\text{AVAP}(\mathcal{A}|i+1) \subseteq \text{AVAP}(\mathcal{A}|i),$ then $\mathcal{A}|i+1 \in \text{CdIF}(\mathcal{A}|i) \cup \text{NIF}(\mathcal{A}|i) \cup \text{PEF}(\mathcal{A}|i).$

**Proof:** Suppose $\mathcal{A} \in \text{SEQ}.$ (L-R): Suppose $\mathcal{A} \in \text{RCS}.$ Then it holds, with Definition 3-19, for all $i \in \text{Dom}(\mathcal{A}): \mathcal{A}|i+1 \in \text{RCE}(\mathcal{A}|i).$ With Definition 3-18, it then holds for all $i \in \text{Dom}(\mathcal{A})$ that $\mathcal{A}|i+1 \in \text{AF}(\mathcal{A}|i) \cup \text{CdIF}(\mathcal{A}|i) \cup \text{NIF}(\mathcal{A}|i) \cup \text{PEF}(\mathcal{A}|i) \cup \text{CIF}(\mathcal{A}|i) \cup \text{BIF}(\mathcal{A}|i) \cup \text{DIF}(\mathcal{A}|i) \cup \text{UIF}(\mathcal{A}|i) \cup \text{PIF}(\mathcal{A}|i) \cup \text{IIF}(\mathcal{A}|i) \cup \text{CdEf}(\mathcal{A}|i) \cup \text{CEF}(\mathcal{A}|i) \cup \text{BEF}(\mathcal{A}|i) \cup \text{DEF}(\mathcal{A}|i) \cup \text{NEF}(\mathcal{A}|i) \cup \text{UEF}(\mathcal{A}|i) \cup \text{IEF}(\mathcal{A}|i).$ It then follows for $\mathcal{A}|i+1 \in \text{AF}(\mathcal{A}|i),$ with Theorem 3-15, that (i) holds, for $\mathcal{A}|i+1 \in \text{CdIF}(\mathcal{A}|i),$ with Theorem 3-19, that (ii) holds, for $\mathcal{A}|i+1 \in \text{NIF}(\mathcal{A}|i),$ with Theorem 3-20, that (iii) holds, for $\mathcal{A}|i+1 \in \text{PEF}(\mathcal{A}|i),$ with Theorem 3-21, that (iv) holds, for $\mathcal{A}|i+1 \in \text{CIF}(\mathcal{A}|i) \cup \text{BIF}(\mathcal{A}|i) \cup \text{DIF}(\mathcal{A}|i) \cup \text{UIF}(\mathcal{A}|i) \cup \text{PIF}(\mathcal{A}|i) \cup \text{IIF}(\mathcal{A}|i) \cup \text{CdEf}(\mathcal{A}|i) \cup \text{CEF}(\mathcal{A}|i) \cup \text{BEF}(\mathcal{A}|i) \cup \text{DEF}(\mathcal{A}|i) \cup \text{NEF}(\mathcal{A}|i) \cup \text{UEF}(\mathcal{A}|i) \cup \text{IEF}(\mathcal{A}|i),$ with Theorem 3-26, that (v) holds, and, last, for $\mathcal{A}|i+1 \in \text{CdEf}(\mathcal{A}|i) \cup \text{CEF}(\mathcal{A}|i) \cup \text{BEF}(\mathcal{A}|i) \cup \text{DEF}(\mathcal{A}|i) \cup \text{NEF}(\mathcal{A}|i) \cup \text{UEF}(\mathcal{A}|i) \cup \text{IEF}(\mathcal{A}|i),$ with Theorem 3-27, that (v) holds.
(R-L): Now, suppose for all $i \in \text{Dom}(\mathfrak{f})$ holds one of the cases (i) to (vi). With Definition 3-18, it then holds for all $i \in \text{Dom}(\mathfrak{f})$ that $\mathfrak{f} \upharpoonright i + 1 \in \text{RCE}(\mathfrak{f} \upharpoonright i)$. With Definition 3-19, we have $\mathfrak{f} \in \text{RCS}$. ■
4 Theorems about the Deductive Consequence Relation

In the following, we will prove theorems about the deductive consequence relation that show that usual properties such as reflexivity, monotony, closure under introduction and elimination of the logical operators and transitivity hold for this relation, and that serve at the same time to prepare the proof of completeness in ch. 6.2. To do this, we first have to do some preparatory work (4.1). Subsequently, we will show that the deductive consequence relation has the desired properties (4.2).

4.1 Preparations

First, we will pave the way for showing that the deductive consequence relation is closed under CdI. To do this, we first show that for every derivation \( \xi \) there is a derivation \( \xi^* \) with \( \text{AVAP}(\xi^*) \subseteq \text{AVAP}(\xi) \) and \( C(\xi^*) = C(\xi) \) in which none of the assumed propositions is available at two positions (Theorem 4-1). Theorem 4-2 then shows that for every derivation \( \xi \) and every \( \Gamma \in \text{CFORM} \) there is a derivation \( \xi^* \) with \( \text{AVAP}(\xi^*) \subseteq \text{AVAP}(\xi) \) and \( C(\xi^*) = C(\xi) \) such that \( \Gamma \) is available as an assumption only if it is available as the last assumption. This theorem provides the basis for the closure under CdI.

The remaining theorems aim at the closure under introductions and eliminations for which the antecedents of the closure clauses (cf. Theorem 4-18) have the form \( X_0 \vdash A_0, \ldots, X_{n-1} \vdash A_{n-1} \). Here, we cannot simply concatenate derivations because the emergence of closed segments or the violation of parameter conditions can cause problems. Therefore, we have to show that derivations can be manipulated by adding blocking members, substitution of parameters and the multiple application of UI and UE, so that the desired concatenations can be carried out.

To do this, we first show that derivations that do not have common parameters can be concatenated (Theorem 4-4) if we interpose an assumption that blocks the emergence of closed segments (Theorem 4-3) and that can then be eliminated (Theorem 4-7). Then, we will show that the substitution of a new parameter for a parameter (that may already be used) is RCS-preserving (Theorem 4-8). The proof of this theorem serves as a model for the proof of the next theorem (Theorem 4-9), which on its part prepares the generalisation theorem (Theorem 4-24). Then, we show that the simultaneous substitution of several new
Theorems about the Deductive Consequence Relation

and pairwise different parameters for pairwise different parameters is also RCS-

preserving (Theorem 4-10). Then, we establish some properties of UI- and UE-extensions

derivations, until, eventually, we prove Theorem 4-14, which assures us that two arbi-

trary derivations can be joined in such a way that, on the one hand, no further available

assumptions have to be added, and that, on the other hand, the conclusions of both deriva-

tions are still available.

**Theorem 4-1. Non-redundant AVAS**

If \( \Delta \in \text{RCS}\setminus\{\emptyset\} \) then there is an \( \Delta^* \in \text{RCS}\setminus\{\emptyset\} \) such that

(i) \( \text{AVAP}(\Delta^*) \subseteq \text{AVAP}(\Delta) \)

(ii) \( C(\Delta^*) = C(\Delta) \), and

(iii) \( |\text{AVAS}(\Delta^*)| = |\text{AVAP}(\Delta^*)| \).

**Proof:** Suppose \( \Delta \in \text{RCS}\setminus\{\emptyset\} \). The proof is carried out by induction on \( |\text{AVAS}(\Delta)| \). Sup-

pose \( |\text{AVAS}(\Delta)| = 0 \). Obviously, we have \( \text{AVAP}(\Delta) \subseteq \text{AVAP}(\Delta) \) and \( C(\Delta) = C(\Delta) \) and,

with Theorem 2-77, we also have \( |\text{AVAP}(\Delta)| = 0 \).

Now, suppose \( |\text{AVAS}(\Delta)| = k \neq 0 \). Suppose the statement holds for all \( \Delta' \in \text{RCS}\setminus\{\emptyset\} \) with \( |\text{AVAS}(\Delta')| < k \). With Theorem 2-76, we then have \( |\text{AVAP}(\Delta)| \leq |\text{AVAS}(\Delta)| \). Now,

suppose \( |\text{AVAP}(\Delta)| \neq |\text{AVAS}(\Delta)| \). Then we have \( |\text{AVAP}(\Delta)| < |\text{AVAS}(\Delta)| \). Also, it holds

that \( \text{AVAS}(\Delta) \neq \emptyset \). With Theorem 3-18, we thus have \( \Delta_1 = \Delta \cup \{(\text{Dom}(\Delta), \therefore P(\Delta_{\max(\text{Dom}(\text{AVAS}(\Delta)))) \rightarrow C(\Delta)) \} \in \text{CdIF}(\Delta) \). With Theorem 3-19-(ix), we then have

\( \text{AVAP}(\Delta_1) \subseteq \text{AVAP}(\Delta) \) and with Theorem 3-19-(iv) and -(v) follows \( |\text{AVAS}(\Delta_1)| < k \).

According to the I.H., there is then \( \Delta_2 \in \text{RCS}\setminus\{\emptyset\} \) such that \( \text{AVAP}(\Delta_2) \subseteq \text{AVAP}(\Delta) \), C(\( \Delta_2 \)) = C(\( \Delta \)) and \( |\text{AVAS}(\Delta_2)| = |\text{AVAP}(\Delta_2)| \). Then we have \( \text{AVAP}(\Delta_2) \subseteq \text{AVAP}(\Delta) \) and \( C(\Delta_2) = C(\Delta) = P(\Delta_{\max(\text{Dom}(\text{AVAS}(\Delta)))}) \rightarrow C(\Delta) \). We have

\( \text{P}(\Delta_{\max(\text{Dom}(\text{AVAS}(\Delta_1))))}) \in \text{AVAP}(\Delta_2) \) or \( \text{P}(\Delta_{\max(\text{Dom}(\text{AVAS}(\Delta_1))))}) \notin \text{AVAP}(\Delta_2) \).

Suppose \( \text{P}(\Delta_{\max(\text{Dom}(\text{AVAS}(\Delta_1))))}) \in \text{AVAP}(\Delta_2) \). Then we have \( \Delta_3 = \{0, \therefore P(\Delta_{\max(\text{Dom}(\text{AVAS}(\Delta_1))))}) \in \text{CdEF}(\Delta_2) \) and, with Theorem 3-27-(v), it holds that \( \text{AVAP}(\Delta_3) \subseteq \text{AVAP}(\Delta_2) \) and \( |\text{AVAS}(\Delta_3)| = |\text{AVAP}(\Delta_3)| \). Since, with Theorem 3-27-(ii), we have \( \text{AVAS}(\Delta_3) \subseteq \text{AVAS}(\Delta_2) \) there would thus be \( i, j \in \text{Dom}(\Delta_3) \) with \( i \neq j \) and \( A \in \text{CFORM} \) such that \( (i, \therefore Suppose for contradiction that \( |\text{AVAS}(\Delta_3)| > |\text{AVAP}(\Delta_3)| \). Then there would be \( i, j \in \text{Dom}(\Delta_3) \) with \( i \neq j \) and \( A \in \text{CFORM} \) such that \( (i, \therefore Suppose A') \in \text{AVAS}(\Delta_3) \) and (j, \( \therefore Suppose A') \in \text{AVAS}(\Delta_3) \). Since, with Theorem 3-27-(ii), we have \( \text{AVAS}(\Delta_3) \subseteq \text{AVAS}(\Delta_2) \) there would thus be \( i, j \in \text{Dom}(\Delta_2) \) with \( i \neq j \) and \( A \in \text{CFORM} \) such that \( (i, \therefore Therefore \( P(\Delta_{\max(\text{Dom}(\text{AVAS}(\Delta)))}) \rightarrow C(\Delta) \).
"Suppose $A^\gamma \in AVAS(\tilde{\gamma})$ and $(j, \supset A^\gamma) \in AVAS(\tilde{\gamma})$. But then we would also have $|AVAS(\tilde{\gamma})| > |AVAP(\tilde{\gamma})|$. Therefore we have $|AVAS(\tilde{\gamma})| \leq |AVAP(\tilde{\gamma})|$ and thus, with Theorem 2-76, $|AVAS(\tilde{\gamma})| = |AVAP(\tilde{\gamma})|$.

Now, suppose $P(\tilde{\gamma} \max(Dom(AVAS(\tilde{\gamma})))) \notin AVAP(\tilde{\gamma})$. Now, let $\tilde{\gamma}^4 = \tilde{\gamma} \setminus \{(0, \supset P(\tilde{\gamma} \max(Dom(AVAS(\tilde{\gamma}))))\}$. Then we have $\tilde{\gamma}^4 \in AF(\tilde{\gamma})$. With Theorem 3-15-(viii), we then have $AVAP(\tilde{\gamma}^4) = AVAP(\tilde{\gamma}) \cup \{P(\tilde{\gamma} \max(Dom(AVAS(\tilde{\gamma}))))\} \subseteq AVAP(\tilde{\gamma})$, and we have $C(\tilde{\gamma}) = P(\tilde{\gamma} \max(Dom(AVAS(\tilde{\gamma}))))$ and $|AVAS(\tilde{\gamma}^4)| = |AVAP(\tilde{\gamma}^4)|$. The latter is shown as follows:

First, we have $|AVAP(\tilde{\gamma})| = |AVAS(\tilde{\gamma})|$, and $\{P(\tilde{\gamma} \max(Dom(AVAS(\tilde{\gamma}))))\}$. Furthermore, we have $AVAS(\tilde{\gamma}) \cap \{P(\tilde{\gamma} \max(Dom(AVAS(\tilde{\gamma}))))\} = \emptyset$ and $AVAP(\tilde{\gamma}) \cap \{P(\tilde{\gamma} \max(Dom(AVAS(\tilde{\gamma}))))\} = \emptyset$. With Theorem 3-15-(iv) and -(viii), we thus have:

$|AVAS(\tilde{\gamma})| = |AVAS(\tilde{\gamma})| \cup \{P(\tilde{\gamma} \max(Dom(AVAS(\tilde{\gamma}))))\}$

$= |AVAP(\tilde{\gamma})| + |\{P(\tilde{\gamma} \max(Dom(AVAS(\tilde{\gamma}))))\}|$

$= |AVAP(\tilde{\gamma})| + |\{P(\tilde{\gamma} \max(Dom(AVAS(\tilde{\gamma}))))\}|$

$= |AVAP(\tilde{\gamma})| + |\{P(\tilde{\gamma} \max(Dom(AVAS(\tilde{\gamma}))))\}|$

$= |AVAP(\tilde{\gamma})|$.}

With Theorem 3-15-(vi), we also have that $\{P(\tilde{\gamma} \max(Dom(AVAS(\tilde{\gamma}))))\}$, $P(\tilde{\gamma} \ max(Dom(AVAS(\tilde{\gamma})))) \rightarrow C(\tilde{\gamma}) \in AVAP(\tilde{\gamma})$. Thus we have $\tilde{\gamma}^5 = \tilde{\gamma}^4 \setminus \{(0, \supset C(\tilde{\gamma}))\} \in CdEF(\tilde{\gamma})$ and, with Theorem 3-27-(v), we then have $AVAP(\tilde{\gamma}^5) \subseteq AVAP(\tilde{\gamma}^4) \subseteq AVAP(\tilde{\gamma})$ and $C(\tilde{\gamma}^5) = C(\tilde{\gamma})$ and $|AVAS(\tilde{\gamma}^5)| = |AVAP(\tilde{\gamma}^5)|$. The latter results from $|AVAS(\tilde{\gamma}^5)| = |AVAP(\tilde{\gamma}^5)|$ in the same way in which we have shown above that $|AVAS(\tilde{\gamma})| = |AVAP(\tilde{\gamma})|$. ■

The following theorem serves especially to prepare the closure under CdI (Theorem 4-18-(i)).

**Theorem 4-2. CdI-preparation theorem**

If $\tilde{\gamma} \in RCS \setminus \{\emptyset\}$ and $\Gamma \in CFORM$, then there is an $\tilde{\gamma}^* \in RCS \setminus \{\emptyset\}$ such that

(i) $AVAP(\tilde{\gamma}^*) \subseteq AVAP(\tilde{\gamma})$,

(ii) $C(\tilde{\gamma}^*) = C(\tilde{\gamma})$, and

(iii) For all $i \in Dom(AVAS(\tilde{\gamma}^*))$: If $P(\tilde{\gamma}^*) = \Gamma$, then $i = \max(Dom(AVAS(\tilde{\gamma}^*)))$.

**Proof:** Suppose $\tilde{\gamma} \in RCS \setminus \{\emptyset\}$ and $\Gamma \in CFORM$. Then we have $\Gamma \notin AVAP(\tilde{\gamma})$ or $\Gamma \in AVAP(\tilde{\gamma})$. If $\Gamma \notin AVAP(\tilde{\gamma})$, then $\tilde{\gamma}$ itself is an $\tilde{\gamma}^* \in RCS \setminus \{\emptyset\}$ such that (i), (ii) and (iii) hold trivially. Now, suppose $\Gamma \in AVAP(\tilde{\gamma})$. The proof is carried out by induction on
|AVAS(\(\delta\))|. Suppose |AVAS(\(\delta\))| = 0. With Theorem 2-77, it follows that |AVAP(\(\delta\))| = 0, whereas, according to the hypothesis, |AVAS(\(\delta\))| \(\neq\) 0. Thus the statement holds trivially for |AVAS(\(\delta\))| = 0.

Now, suppose |AVAS(\(\delta\))| = \(k \neq 0\). Suppose the statement holds for all \(\delta' \in \text{RCS}\{\emptyset\}\) with |AVAS(\(\delta'\))| < \(k\). With Theorem 4-1, there is an \(\delta^1 \in \text{RCS}\{\emptyset\}\) such that AVAP(\(\delta^1\)) \(\subseteq\) AVAP(\(\delta\)), \(C(\delta^1) = C(\delta)\) and |AVAS(\(\delta^1\))| = |AVAP(\(\delta^1\))| \(\leq\) |AVAP(\(\delta\))| \(\leq\) |AVAS(\(\delta\))|.

We also have, with |AVAS(\(\delta^1\))| = |AVAP(\(\delta^1\))|, that it holds for all \(\in \text{AVAP}(\delta^1)\) that there is exactly one \(i \in \text{Dom}(\text{AVAS}(\delta^1))\) such that \(\text{B} = \text{P}(\delta_i)\). Suppose, for all \(i \in \text{Dom}(\text{AVAS}(\delta^1))\): If \(\text{P}(\delta_i) = \Gamma\), then \(i = \max(\text{Dom}(\text{AVAS}(\delta^1)))\). Then we have that \(\delta^1\) is the desired element of RCS\{\emptyset\}.

Now, suppose not for all \(i \in \text{Dom}(\text{AVAS}(\delta^1))\): If \(\text{P}(\delta_i) = \Gamma\), then \(i = \max(\text{Dom}(\text{AVAS}(\delta^1)))\). Then there is an \(i \in \text{Dom}(\text{AVAS}(\delta^1))\) such that \(\text{P}(\delta_i) = \Gamma\) and \(i \neq \max(\text{Dom}(\text{AVAS}(\delta^1)))\). Then we have AVAS(\(\delta^1\)) \(\neq\) \(\emptyset\) and \(\Gamma \in \text{AVAP}(\delta^1)\), and it holds for all \(j \in \text{Dom}(\text{AVAS}(\delta^1))\): If \(\text{P}(\delta_j) = \Gamma\), then \(j = i\) and thus also \(j \neq \max(\text{Dom}(\text{AVAS}(\delta^1)))\). Thus we have \(\text{P}(\delta^1_{\max(\text{Dom}(\text{AVAS}(\delta^1)))}) \neq \Gamma\). We also have, with AVAS(\(\delta^1\)) \(\neq\) \(\emptyset\), Theorem 3-18 and \(C(\delta^1) = C(\delta)\): \(\delta^2 = \delta^1-\{0, \text{therefore} \text{P}(\delta^1_{\max(\text{Dom}(\text{AVAS}(\delta^1)))}) \subseteq \text{CdIF}(\delta^1)\). Then it holds, with Theorem 3-22, that AVAP(\(\delta^2\)) \(\subseteq\) AVAP(\(\delta^1\))\(\{P(\delta^1_{\max(\text{Dom}(\text{AVAS}(\delta^1)))}) \subseteq \text{AVAP}(\delta)\). With Theorem 3-19-(iv) and -(v), it holds that |AVAS(\(\delta^2\))| < |AVAS(\(\delta^1\))| \(\leq\) |AVAS(\(\delta\))| and that |AVAS(\(\delta^2\))| = |AVAP(\(\delta^2\))|. The latter is shown as follows:

Suppose for contradiction that |AVAS(\(\delta^2\))| > |AVAP(\(\delta^2\))|. Then there would be \(i, j \in \text{Dom}(\delta^2)\) with \(i \neq j\) and \(A \in \text{CFORM}\) such that \(\text{P}(\delta^2) \cap A \in \text{AVAS}(\delta^2)\) and \(j, \text{Suppose A}\) \(\in\) AVAS(\(\delta^2\)). Since, with Theorem 3-19-(v), AVAS(\(\delta^2\)) \(\subseteq\) AVAS(\(\delta^1\)), there would thus be \(i, j \in \text{Dom}(\delta^1)\) with \(i \neq j\) and \(A \in \text{CFORM}\) such that \(\text{P}(\delta^2) \cap A \in \text{AVAS}(\delta^1)\) and \(j, \text{Suppose A}\) \(\in\) AVAS(\(\delta^1\)). But then we would also have |AVAS(\(\delta^1\))| > |AVAP(\(\delta^1\))|. Therefore we have |AVAS(\(\delta^2\))| \(\leq\) |AVAP(\(\delta^2\))| and thus, with Theorem 2-76, that |AVAS(\(\delta^2\))| = |AVAP(\(\delta^2\))|.

We have |AVAS(\(\delta^2\))| < |AVAS(\(\delta^1\))| \(\leq\) |AVAS(\(\delta\))| = \(k\). According to the I.H., there is thus an \(\delta^3 \in \text{RCS}\{\emptyset\}\) such that AVAP(\(\delta^3\)) \(\subseteq\) AVAP(\(\delta^2\)) and \(C(\delta^3) = C(\delta^2)\) and for all \(i \in \text{Dom}(\text{AVAS}(\delta^3))\): If \(\text{P}(\delta^3) = \Gamma\), then \(i = \max(\text{Dom}(\text{AVAS}(\delta^3)))\). Then we have AVAP(\(\delta^3\)) \(\subseteq\) AVAP(\(\delta^2\)) \(\subseteq\) AVAP(\(\delta^1\)) \(\subseteq\) AVAP(\(\delta\)), \(\text{P}(\delta^3_{\max(\text{Dom}(\text{AVAS}(\delta^3)))}) \notin\) AVAP(\(\delta^3\)) and \(C(\delta^3) = \text{P}(\delta^3_{\max(\text{Dom}(\text{AVAS}(\delta^3)))}) \to C(\delta^3)\). With \(\Gamma \in \text{AVAP}(\delta^3)\) or \(\Gamma \notin\) AVAP(\(\delta^3\)), we can then distinguish two cases.
First case: $\Gamma \in \text{AVAP}(\delta^3)$. Then we have $\Gamma = P(\delta^3_{\text{max}(\text{Dom}(\text{AVAS}(\delta^3)))})$ and for all $i \in \text{Dom}(\text{AVAS}(\delta^3))$: If $\Gamma = P(\delta_i)$, then $i = \text{max}(\text{Dom}(\text{AVAS}(\delta^3)))$. With Theorem 3-18, we then have that $\delta^4 = \delta^3 \sim \{0, \Gamma\}$. Therefore $\Gamma \rightarrow (P(\delta^1_{\text{max}(\text{Dom}(\text{AVAS}(\delta^3))))}) \rightarrow C(\delta_3) \in \text{CdIF}(\delta^3)$. With Theorem 3-22, it then follows that $\text{AVAP}(\delta^3) \subseteq \text{AVAP}(\delta^3) \setminus \{\Gamma\} \subseteq \text{AVAP}(\delta)$. Thus we have $\Gamma \notin \text{AVAP}(\delta^3)$ and thus that for all $i \in \text{Dom}(\text{AVAS}(\delta^3))$: $P(\delta_i^3) \neq \Gamma$.

Now, let $\delta^5 = \delta^4 \sim \{0, \delta^3 \text{Suppose } P(\delta^1_{\text{max}(\text{Dom}(\text{AVAS}(\delta^3))))) \rightarrow C(\delta_3)\} \in \text{CdEF}(\delta^5)$, and, with Theorem 3-27-(v), it holds that $\text{AVAP}(\delta^3) \subseteq \text{AVAP}(\delta^3) \setminus \{\Gamma\} \subseteq \text{AVAP}(\delta)$. Thus we have $\Gamma \notin \text{AVAP}(\delta^3)$ and thus that for all $i \in \text{Dom}(\text{AVAS}(\delta^3))$: $P(\delta_i^3) \neq \Gamma$.

Suppose for contradiction that there is an $i \in \text{Dom}(\text{AVAS}(\delta^3))$ such that $P(\delta_i^3) = \Gamma$ and $i \neq \text{max}(\text{Dom}(\text{AVAS}(\delta^3))))$. With Theorem 3-27-(ii), it then follows that $i \in \text{Dom}(\text{AVAS}(\delta^3))$. Then we have $i = \text{max}(\text{Dom}(\text{AVAS}(\delta^3))) = \text{Dom}(\delta^3)+1$. However, according to the construction of $\delta^6$, we have $\text{max}(\text{Dom}(\text{AVAS}(\delta^6))) \leq \text{Dom}(\delta^4)+1 = i$.

With $i \neq \text{max}(\text{Dom}(\text{AVAS}(\delta^6)))$, we would thus have $\text{max}(\text{Dom}(\text{AVAS}(\delta^6))) < i$. But, with $i \in \text{Dom}(\text{AVAS}(\delta^6))$, we have $i \leq \text{max}(\text{Dom}(\text{AVAS}(\delta^6)))$. Contradiction!

We have $\delta^3 \sim \{0, \Gamma\}$. Therefore $P(\delta^1_{\text{max}(\text{Dom}(\text{AVAS}(\delta^3))))) \rightarrow C(\delta_3) \in \text{AVP}(\delta^5)$. Now, suppose for contradiction that $P(\delta^1_{\text{max}(\text{Dom}(\text{AVAS}(\delta^3))))) \notin \text{AVP}(\delta^5)$. Then we would have $(\text{Dom}(\delta^3) \sim \delta^3 \supseteq \text{AVAS}(\delta^3)) \neq \text{AVS}(\delta^5)$ and thus $(\text{Dom}(\delta^5) \sim \delta^5 \supseteq \text{AVAS}(\delta^5)) \notin \text{AVS}(\delta^5)$. With Theorem 2-85, we would then have $\text{AVAS}(\delta^5) \subseteq \text{AVAS}(\delta^5) \setminus \{\text{max}(\text{Dom}(\text{AVAS}(\delta^3)))\} = \{\text{max}(\text{Dom}(\text{AVAS}(\delta^3)))\} = \{\text{Dom}(\delta^3)+1, \text{Suppose } \Gamma\}$ and therefore $\text{Dom}(\delta^5) = \text{Dom}(\delta^4)+1$. Contradiction!

Thus we have that $\delta^7 = \delta^6 \sim \{0, \Gamma\} \in \text{CdEF}(\delta^5)$ and, with Theorem 3-27-(v), it holds that $\text{AVAP}(\delta^5) \subseteq \text{AVAP}(\delta^6) \subseteq \text{AVAP}(\delta)$. We also have, with
Theorem 3-27-(ii), for all $i \in \text{Dom}(\text{AVAS}(\mathcal{A}))$: If $P(\mathcal{A}_i) = \Gamma$, then $i = \max(\text{Dom}(\text{AVAS}(\mathcal{A})))$. Thus we have that $\mathcal{A}_i$ is the desired element of $\text{RCS\setminus\{\emptyset\}}$.

Second case: $\Gamma \not\in \text{AVAP}(\mathcal{A}_3)$. Now, let $\mathcal{A}_8 = \mathcal{A}_3 \setminus \{(0, \) Suppose $P(\mathcal{A}_{\max(\text{Dom}(\text{AVAS}(\mathcal{A}_3)))}) = \Gamma$. Then we have $\mathcal{A}_8 \in \text{AF}(\mathcal{A}_3)$. Then we have $\mathcal{A}_8 \in \text{AVAP}(\mathcal{A}_8) = \text{AVAP}(\mathcal{A}_3) \cup \{P(\mathcal{A}_{\max(\text{Dom}(\text{AVAS}(\mathcal{A}_3)))})\} \subseteq \text{AVAP}(\mathcal{A}_8)$. With Theorem 3-15-(vi), we have $\mathcal{A}_8 \in \text{AVP}(\mathcal{A}_8)$. Therefore, $\mathcal{A}_9 = \mathcal{A}_8 \setminus \{(0, \beta)\}$, $\text{CdEF}(\mathcal{A}_8)$, and with Theorem 3-27-(v), it holds that $\text{AVAP}(\mathcal{A}_9) \subseteq \text{AVAP}(\mathcal{A}_8) \subseteq \text{AVAP}(\mathcal{A}_3)$. Moreover, for all $i \in \text{Dom}(\text{AVAS}(\mathcal{A}_9))$: If $P(\mathcal{A}_9) = \Gamma$, then $i = \max(\text{Dom}(\text{AVAS}(\mathcal{A}_9)))$. Then we have $\mathcal{A}_9$ is the desired element of $\text{RCS\setminus\{\emptyset\}}$. ■

Theorem 4-3. Blocking assumptions

If $\mathfrak{A}$ is a closed segment in $\mathcal{A}$, $i \in \text{Dom}(\mathfrak{A}) \cap \text{Dom}(\text{AS}(\mathcal{A}))$, $\Delta = P(\mathcal{A}_i)$ and $\text{PAR} \cap \text{ST}(\Delta) = \emptyset$, then there is a $j \in \text{Dom}(\mathfrak{A})$ such that $i \neq j$ and $\Delta \in \text{SE}(\mathcal{A})$.

Proof: Suppose $\mathfrak{A}$ is a closed segment in $\mathcal{A}$, $i \in \text{Dom}(\mathfrak{A}) \cap \text{Dom}(\text{AS}(\mathcal{A}))$, $\Delta = P(\mathcal{A}_i)$ and $\text{PAR} \cap \text{ST}(\Delta) = \emptyset$. With Theorem 2-47, it then follows that there is a closed segment $\mathcal{B}$ in $\mathcal{A}$ with $\mathcal{B} \subseteq \mathfrak{A}$ such that $i = \min(\text{Dom}(\mathcal{B}))$. With Theorem 2-42, $\mathcal{B}$ is then a CdI- or Ni- or RA-like segment in $\mathcal{A}$. Suppose $\mathcal{B}$ is a CdI- or an Ni-like segment in $\mathcal{A}$. Then it holds, with Definition 2-11 and Definition 2-12, that $\max(\text{Dom}(\mathcal{B})) \in \text{Dom}(\mathcal{A})$, $\max(\text{Dom}(\mathcal{B})) \neq i$ and $\Delta \in \text{SE}(\mathcal{A}_{\max(\text{Dom}(\mathcal{B}))})$. Now, suppose $\mathcal{B}$ is an RA-like segment in $\mathcal{A}$. With Definition 2-13, it then holds that $\min(\text{Dom}(\mathcal{B})) - 1 \in \text{Dom}(\mathcal{A})$ and $\min(\text{Dom}(\mathcal{B})) = i$. Moreover, there are then $\xi \in \text{VAR}$, $\Delta^+ \in \text{FORM}$, where $\text{FV}(\Delta^+) \subseteq \{\xi\}$ and $\beta \in \text{PAR}$ such that $P(\mathcal{A}_{\min(\text{Dom}(\mathcal{B}))}) = \{\vee \xi \Delta^+ \land \Delta = P(\mathcal{A}_{\min(\text{Dom}(\mathcal{B}))}) = [\beta, \xi, \Delta^+]\}$. By hypothesis, we have $\text{PAR} \cap \text{ST}(\Delta) = \emptyset$, and thus we have $\beta \not\in \text{ST}(\{\beta, \xi, \Delta^+\})$. With Theorem 1-14-(ii), we then have $\Delta = [\beta, \xi, \Delta^+] = \Delta^+$. Thus we have $P(\mathcal{A}_{\min(\text{Dom}(\mathcal{B}))}) = \{\vee \xi \Delta^+ \land \text{hence } \Delta \in \text{SE}(\mathcal{A}_{\min(\text{Dom}(\mathcal{B}))})\}$ and the statement holds. ■
Theorem 4-4. Concatenation of RCS-elements that do not have any parameters in common, where the concatenation includes an interposed blocking assumption

If \( \delta, \delta' \in \text{RCS}, \text{PAR} \cap \text{STSEQ}(\delta) \cap \text{STSEQ}(\delta') = \emptyset \) and \( \alpha \in \text{CONST}(\text{STSEQ}(\delta) \cup \text{STSEQ}(\delta')) \), then there is an \( \delta^* \in \text{RCS} \setminus \{ \emptyset \} \) such that

(i) \( \text{Dom}(\delta^*) = \text{Dom}(\delta) + 1 + \text{Dom}(\delta') \),

(ii) \( \delta^*\mid_{\text{Dom}(\delta)} = \delta \),

(iii) \( \delta^*\mid_{\text{Dom}(\delta')} = \{ \text{Suppose } \alpha = \alpha' \} \),

(iv) For all \( i \in \text{Dom}(\delta') \) it holds that \( \delta'_i = \delta^*\mid_{\text{Dom}(\delta)+1+i} \),

(v) \( \text{Dom}(\text{AVS}(\delta^*)) = \text{Dom}(\text{AVS}(\delta)) \cup \{ \text{Dom}(\delta) \} \cup \{ \text{Dom}(\delta) + 1 + l \mid l \in \text{Dom}(\text{AVS}(\delta')) \} \),

(vi) \( \text{AVP}(\delta^*) = \text{AVP}(\delta) \cup \{ \text{Suppose } \alpha = \alpha' \} \cup \text{AVP}(\delta') \), and

(vii) \( \text{AVAP}(\delta^*) = \text{AVAP}(\delta) \cup \{ \text{Suppose } \alpha = \alpha' \} \cup \text{AVAP}(\delta') \).

Proof: We show by induction on \( \text{Dom}(\delta') \) that under the specified conditions there is always an \( \delta^* \) such that clauses (i) to (v) are satisfied. (vi) and (vii) then follow from the preceding clauses. First, we have from (i) to (v) and Definition 2-30:

\[
B \in \text{AVP}(\delta^*)
\]

iff

there is an \( i \in \text{Dom}(\text{AVS}(\delta^*)) \) such that \( B = P(\delta^*_i) \)

iff

there is an \( i \in \text{Dom}(\text{AVS}(\delta)) \cup \{ \text{Dom}(\delta) \} \cup \{ \text{Dom}(\delta) + 1 + l \mid l \in \text{Dom}(\text{AVS}(\delta')) \} \) such that \( B = P(\delta^*_i) \)

iff

\( B \in \text{AVP}(\delta) \cup \{ \text{Suppose } \alpha = \alpha' \} \cup \text{AVP}(\delta') \).

Second, (vii) results from (i) to (v) and Definition 2-31 as follows:

\[
B \in \text{AVAP}(\delta^*)
\]

iff

there is an \( i \in \text{Dom}(\text{AVAS}(\delta^*)) \) such that \( B = P(\delta^*_i) \)

iff

there is an \( i \in \text{Dom}(\text{AVS}(\delta^*)) \cap \text{Dom}(\text{AS}(\delta^*)) \) such that \( B = P(\delta^*_i) \)

iff

there is an \( i \in (\text{Dom}(\text{AVS}(\delta)) \cup \{ \text{Dom}(\delta) \} \cup \{ \text{Dom}(\delta) + 1 + l \mid l \in \text{Dom}(\text{AVS}(\delta')) \}) \cap \text{Dom}(\text{AS}(\delta^*)) \) such that \( B = P(\delta^*_i) \)

iff

there is an \( i \in (\text{Dom}(\text{AVS}(\delta)) \cap \text{Dom}(\text{AS}(\delta^*))) \cup (\{ \text{Dom}(\delta) \} \cap \text{Dom}(\text{AS}(\delta^*))) \cup (\{ \text{Dom}(\delta) + 1 + l \mid l \in \text{Dom}(\text{AVS}(\delta')) \}) \cap \text{Dom}(\text{AS}(\delta^*)) \) such that \( B = P(\delta^*_i) \)

iff
there is an \( i \in \{\text{Dom}(\text{AVS}(\hat{\beta})) \cap \text{Dom}(\text{AS}(\hat{\beta}))) \cup \{\text{Dom}(\hat{\beta})\} \cap \text{Dom}(\text{AS}(\hat{\beta}^*))) \cup (\{\text{Dom}(\hat{\beta})\}+1+l \mid l \in \text{Dom}(\text{AVS}(\hat{\beta}^*))) \cap (\{\text{Dom}(\hat{\beta})\}+1+l \mid l \in \text{Dom}(\text{AS}(\hat{\beta}))) \) such that \( B = P(\hat{\beta}^*) \)

iff

there is an \( i \in \text{Dom}(\text{AVS}(\hat{\beta})) \cup \{\text{Dom}(\hat{\beta})\} \cup (\{\text{Dom}(\hat{\beta})\}+1+l \mid l \in \text{Dom}(\text{AVS}(\hat{\beta}^*))) \) such that \( B = P(\hat{\beta}^*) \)

iff

\( B \in \text{AVAP}(\hat{\beta}) \cup \{\alpha = \alpha^2\} \cup \text{AVAP}(\hat{\beta}) \).

**Now for the proof by induction:** Suppose the statement holds for \( k < \text{Dom}(\hat{\beta}') \) and suppose \( \hat{\beta}, \hat{\beta}' \) are as required and suppose \( \alpha \in \text{CONST}(\text{STSEQ}(\hat{\beta}) \cup \text{STSEQ}(\hat{\beta}')) \). Suppose \( \text{Dom}(\hat{\beta}') = 0 \). Then we have \( \hat{\beta}' = 0 \) and with \( \hat{\beta}^* = \hat{\beta}' \{0, \{\text{Suppose } \alpha = \alpha^2\}\} \) and Theorem 3-15-(ii) the statement holds. Now, suppose \( \text{Dom}(\hat{\beta}') > 0 \). Then we have \( \hat{\beta}' \in \text{RCS}\{\emptyset\} \). With Theorem 3-6, we then have \( \hat{\beta}' \in \text{RCE}(\hat{\beta}') \cap \text{RCS}(\hat{\beta}^*) \) and \( \hat{\beta}^* \cap \text{STSEQ}(\hat{\beta}') \) is an \( \hat{\beta}^* \in \text{RCS} \{\emptyset\} \). With Theorem 3-6, we then have \( \hat{\beta}' \in \text{RCE}(\hat{\beta}') \cap \text{RCS}(\hat{\beta}^*) \) and \( \hat{\beta}^* \cap \text{STSEQ}(\hat{\beta}') \) is an \( \hat{\beta}^* \in \text{RCS} \{\emptyset\} \). According to the I.H., there is then for \( \hat{\beta}, \hat{\beta}' \cap \text{STSEQ}(\hat{\beta}') \) and \( \alpha \) an \( \hat{\beta}^* \in \text{RCS} \{\emptyset\} \) for which (i) to (v) hold. Then it holds that:

\[ \begin{align*}
\text{i') } \text{Dom}(\hat{\beta}^*) &= \text{Dom}(\hat{\beta})+1+\text{Dom}(\hat{\beta}')-1 = \text{Dom}(\hat{\beta})+\text{Dom}(\hat{\beta}'), \\
\text{ii') } \hat{\beta}^* \cap \text{Dom}(\hat{\beta}) &= \hat{\beta}, \\
\text{iii') } \hat{\beta}^* \cap \text{Dom}(\hat{\beta}) &= \{\text{Suppose } \alpha = \alpha^2\}, \\
\text{iv') } \forall i \in \text{Dom}(\hat{\beta}') \text{-1 it holds that } \hat{\beta}' = (\hat{\beta}' \cap \text{Dom}(\hat{\beta}')-1)_i = \hat{\beta}^* \cap \text{Dom}(\hat{\beta})+1+i, \\
\text{v') } \text{Dom}(\text{AVS}(\hat{\beta}^*)) &= \\
\text{Dom}(\text{AVS}(\hat{\beta})) \cup \{\text{Dom}(\hat{\beta})\} \cup \{\text{Dom}(\hat{\beta})+1+l \mid l \in \text{Dom}(\text{AVS}(\hat{\beta}^*)-1))\}.
\end{align*} \]

From \( \hat{\beta}' \in \text{RCE}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \) it follows, with Definition 3-18, that \( \hat{\beta}' \in \text{AF}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \) or \( \hat{\beta}' \in \text{CdIF}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \) or \( \hat{\beta}' \in \text{BIF}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \) or \( \hat{\beta}' \in \text{IIF}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \) or \( \hat{\beta}' \in \text{DEF}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \) or \( \hat{\beta}' \in \text{NIF}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \) or \( \hat{\beta}' \in \text{NEF}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \) or \( \hat{\beta}' \in \text{PIF}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \) or \( \hat{\beta}' \in \text{PEF}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \) or \( \hat{\beta}' \in \text{UEF}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \) or \( \hat{\beta}' \in \text{PIF}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \) or \( \hat{\beta}' \in \text{EIF}(\hat{\beta}') \cap \text{Dom}(\hat{\beta}') \). Now let

\[ \text{vi') } \hat{\beta}' = \hat{\beta}^* \cup \{\text{Dom}(\hat{\beta})+1+\text{Dom}(\hat{\beta}')-1, \hat{\beta}' \cap \text{Dom}(\hat{\beta}') \} \].
Then we already have that $\delta^+ \neq \emptyset$ and clauses (i) to (iv) hold for $\delta^+$. Now, we will show that for each of the cases AF … IEF we have that $\delta^+ \in \text{RCS}\setminus\{\emptyset\}$ and that (v) holds, with which we have that $\delta^+$ is in each case the desired RCS-element. First, we note that, because of $\alpha \in \text{CONST}\setminus(\text{STSEQ}(\delta) \cup \text{STSEQ}(\delta^*))$, there is no $l \in \text{Dom}(\delta^+) \subset \text{Dom}(\delta^+)$ such that $l \neq \text{Dom}(\delta)$ and $\alpha = \alpha^* \in \text{SE}(\delta^+)$. With $\delta^* \text{Dom}(\delta) = \delta^+ \text{Dom}(\delta) = \text{Suppose } \alpha = \alpha^*$ and Theorem 4-3, it thus holds:

\[
\text{vii'} \text{ There is no closed segment } \mathfrak{A} \text{ in } \delta^+ \text{ and there is no closed segment } \mathfrak{A} \text{ in } \delta^* \text{ such that } \min(\text{Dom}(\mathfrak{A})) \leq \text{Dom}(\delta) < \max(\text{Dom}(\mathfrak{A})).
\]

Thus it also follows that:

\[
\text{viii'} \text{ Dom}(\delta) \in \text{Dom}(\text{AVAS}(\delta^*)), \text{ Dom}(\delta) \in \text{Dom}(\text{AVAS}(\delta^*)) \text{ and } \text{Dom}(\delta) \leq \max(\text{Dom}(\text{AVAS}(\delta^*))).
\]

To simplify the treatment of CdEF, CIF, CEF, BIF, BEF, DIF, DEF, NEF, UIF, UEF, PIF, IIF and IEF, we will now show in preparation of the main part of the proof that:

\[
\text{ix'} \text{ If } \delta^+ \in \text{CdIF}(\delta^+) \cup \text{NIF}(\delta^+) \cup \text{PEF}(\delta^*), \text{ then } \delta^+ \in \text{CdIF}(\delta^+1) \cup \text{NIF}(\delta^+1) \cup \text{PEF}(\delta^+1).
\]

\[\text{Preparatory part: First, suppose } \delta^+ \in \text{CdIF}(\delta^*). \text{ According to Definition 3-2, Theorem 3-19-(i) and viii'), there is then } \text{Dom}(\delta) + 1 + i \in \text{Dom}(\text{AVAS}(\delta^*)) \text{ such that, with i') and iv'), } P(\delta^* \text{Dom}(\delta) + 1 + i) = P(\delta^+) \text{ and } C(\delta^*) = P(\delta^* \text{Dom}(\delta) + 1 + i) = P(\delta^* \text{Dom}(\delta) - 2) = C(\delta^+). \text{ and there is no } l \text{ such that } \text{Dom}(\delta) + 1 + i < l \leq \text{Dom}(\delta) + 1 + i + 1 = \text{Dom}(\delta^*) - 2 \text{ and } l \in \text{Dom}(\text{AVAS}(\delta^*)). \text{ and } \delta^+ = \delta^* \cup \{(\text{Dom}(\delta) + 1 + i, \text{Dom}(\delta^*) - 1), \} \text{ Therefore } P(\delta^* \cup \{(\text{Dom}(\delta) + 1 + i, \text{Dom}(\delta^*) - 1), \} \text{ Therefore } P(\delta^+) \rightarrow C(\delta^+). \text{ and there is no } l \text{ such that } i < l \leq \text{Dom}(\delta^*) - 2 \text{ and } l \in \text{Dom}(\text{AVAS}(\delta^*) \text{Dom}(\delta^*) - 1)). \text{ Also, with vi'), we have } \delta^+ = \delta^+ \text{Dom}(\delta^*) \cup \{(\text{Dom}(\delta^*) - 1, \) \text{ Therefore } P(\delta^+) \rightarrow C(\delta^+). \text{ Hence we have } \delta^+ \in \text{CdIF}(\delta^+ \text{Dom}(\delta^*) - 1). \text{ In the case that } \delta^+ \in \text{NIF}(\delta^*), \text{ one shows analogously that then also } \delta^+ \in \text{NIF}(\delta^+ \text{Dom}(\delta^*) - 1). \text{ Now, suppose } \delta^+ \in \text{PEF}(\delta^*). \text{ With Definition 3-15, Theorem 3-21-(i), } P(\delta^* \text{Dom}(\delta)) = \text{Suppose } \alpha = \alpha^* \text{ and vii') and viii'), there are then } \beta \in \text{PAR}, \xi \in \text{VAR}, \Delta \in \text{FORM}, \text{ where } FV(\Delta) \subseteq \{\xi\}, \text{ and } \text{Dom}(\delta) + 1 + i \in \text{Dom}(\text{AVAS}(\delta^*)) \text{ such that, with i') and iv'), } \nabla \xi \Delta = P(\delta^* \text{Dom}(\delta) + 1 + i) = P(\delta^+) \text{ and } [\beta, \xi, \Delta] = P(\delta^* \text{Dom}(\delta) + 2 + i) = P(\delta^+2 + i), \text{ where } \text{Dom}(\delta) + 2 + i \in \text{Dom}(\text{AVAS}(\delta^*)) \text{ and } C(\delta^*) = P(\delta^* \text{Dom}(\delta) + 1 + i) = P(\delta^+1) = C(\delta^+). \text{ and there is no } l \text{ such that } i < l \leq \text{Dom}(\delta^*) - 2 \text{ and } l \in \text{Dom}(\text{AVAS}(\delta^*) \text{Dom}(\delta^*) - 1)) \]
and \( \mathcal{A}^+ = \mathcal{A}^* \cup \{(\text{Dom}(\mathcal{A}^*))+1+\text{Dom}(\mathcal{A}^*)\} \), with \( \text{therefore } C(\mathcal{A}^*) \} = \mathcal{A}^* \cup \{(\text{Dom}(\mathcal{A}^*))+1+\text{Dom}(\mathcal{A}^*)\}, \text{"therefore } C(\mathcal{A}^*) \} \) and \( \beta \not\in \text{STSF}(\{\Delta, C(\mathcal{A}^*)\}) \) and there is no \( l \leq \text{Dom}(\mathcal{A}^*)+1+\text{Dom}(\mathcal{A}^*) \) such that \( \mathcal{A}^* \not\in \text{ST}(\mathcal{A}^*_j) \) and there is no \( l \) such that \( \text{Dom}(\mathcal{A}^*)+2+i < l \leq \text{Dom}(\mathcal{A}^*)+1+\text{Dom}(\mathcal{A}^*)-2 \) and \( l \in \text{Dom}(\text{AVS}(\mathcal{A}^*)) \). It then holds with \( i' \), \( iv' \) and \( v' \): \( i \in \text{Dom}(\text{AVS}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1)) \) and \( i+1 \in \text{Dom}(\text{AVS}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1)) \) and \( \beta \not\in \text{STSF}(\{\Delta, C(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1)\}) \) and there is no \( l \) such that \( i+1 < l \leq \text{Dom}(\mathcal{A}_i^*)-2 \) and \( l \in \text{Dom}(\text{AVS}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1)) \). Also, with \( vi' \), we have \( \mathcal{A}_i^* = \mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1 \cup \{(\text{Dom}(\mathcal{A}_i^*))+1+\text{Dom}(\mathcal{A}_i^*)\}, \text{"therefore } C(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1) \} \) and hence we have \( \mathcal{A}_i^* \not\in \text{PEF}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1) \).

**Main part:** Now, we will show that for each of the cases AF ... IEF it holds that \( \mathcal{A}_i^* \in \text{RCS}\{\emptyset\} \) and that \( v \) holds:

\( (AF) \): Suppose \( \mathcal{A}_i^* \in \text{AF}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1) \). According to Definition 3-1, we then have \( \mathcal{A}_i^* = \mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1 \cup \{(\text{Dom}(\mathcal{A}_i^*))+1+\text{Dom}(\mathcal{A}_i^*)\} \). With \( vi' \), we then have \( \mathcal{A}_i^* = \mathcal{A}_i^* \cup \{(\text{Dom}(\mathcal{A}_i^*))+1+\text{Dom}(\mathcal{A}_i^*)\} \) \( \in \text{AF}(\mathcal{A}_i^*) \subset \text{RCS}\{\emptyset\} \). With Theorem 3-15(ii), it then follows that \( \text{AVS}(\mathcal{A}_i^*) = \text{AVS}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1) \cup \{(\text{Dom}(\mathcal{A}_i^*)+1+\text{Dom}(\mathcal{A}_i^*)\} \). Also, with \( vi' \), we have \( \mathcal{A}_i^* = \mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1 \cup \{(\text{Dom}(\mathcal{A}_i^*))+1+\text{Dom}(\mathcal{A}_i^*)\}, \text{"therefore } C(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1) \} \), and hence we have \( \mathcal{A}_i^* \not\in \text{PEF}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1) \).

\( (CdIF, NIF) \): Now, suppose \( \mathcal{A}_i^* \in \text{CdIF}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1) \). According to Definition 3-2, there is then an \( i \in \text{Dom}(\mathcal{A}_i^*)-1 \) such that, with \( iv' \), \( \text{P}(\mathcal{A}_i^*) = \text{P}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1) \) and \( i \in \text{Dom}(\text{AVS}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1)) \) and \( \text{C}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1) = \text{P}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1) = \text{C}(\mathcal{A}_i^*) \) and there is no \( l \) such that \( i < l \leq \text{Dom}(\mathcal{A}_i^*)-2 \) and \( l \in \text{Dom}(\text{AVS}(\mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1)) \) and \( \mathcal{A}_i^* = \mathcal{A}_i^*|\text{Dom}(\mathcal{A}_i^*)-1 \cup \{(\text{Dom}(\mathcal{A}_i^*))+1+\text{Dom}(\mathcal{A}_i^*)\}, \text{"therefore } \text{P}(\mathcal{A}_i^*) \rightarrow \text{C}(\mathcal{A}_i^*) \). With \( vi' \), we then have \( \mathcal{A}_i^* = \mathcal{A}_i^* \cup \{(\text{Dom}(\mathcal{A}_i^*))+1+\text{Dom}(\mathcal{A}_i^*)\}, \text{"therefore } \text{P}(\mathcal{A}_i^*) \rightarrow \text{C}(\mathcal{A}_i^*) \). \) With \( iv' \) and \( v' \), we then have \( \text{Dom}(\mathcal{A}_i^*)+1+i \in \text{Dom}(\text{AVS}(\mathcal{A}_i^*)) \) and there is no \( l \) such that \( \text{Dom}(\mathcal{A}_i^*)+1+i < l \leq \)
and there is no 

{Dom(AVAS(\(\tilde{\Sigma}^j\))) \quad if \quad k \in Dom(AVS(\tilde{\Sigma}^j))} and thus \(\tilde{\Sigma}^j \in NIF(\tilde{\Sigma}^j \upharpoonright \text{Dom}(\tilde{\Sigma}^j) - 1)\), one shows analogously that then also \(\tilde{\Sigma}^j \in NIF(\tilde{\Sigma}^j \upharpoonright \text{Dom}(\tilde{\Sigma}^j) - 1)\), and thus \(v\) holds. In the case that \(\tilde{\Sigma}^j \in NIF(\tilde{\Sigma}^j \upharpoonright \text{Dom}(\tilde{\Sigma}^j) - 1)\), one shows analogously that then also \(\tilde{\Sigma}^j \in NIF(\tilde{\Sigma}^j \upharpoonright \text{Dom}(\tilde{\Sigma}^j) - 1)\), and thus \(v\) holds.

(PEF): Now, suppose \(\tilde{\Sigma}^j \in PEF(\tilde{\Sigma}^j \upharpoonright \text{Dom}(\tilde{\Sigma}^j) - 1)\). According to Definition 3-15, there are then \(\beta \in \text{PAR}, \xi \in \text{VAR}, \Delta \in \text{FORM}, \text{where } FV(\Delta) \subseteq \{\xi\}, \text{and } i \in \text{Dom}(AVS(\tilde{\Sigma}^j \upharpoonright \text{Dom}(\tilde{\Sigma}^j) - 1)) \text{ such that, with } v', \forall \xi\Delta^\alpha = P(\tilde{\Sigma}^j) \in P(\text{Dom}(\tilde{\Sigma}^j) + 1) \text{ and } [\beta, \xi, \Delta] = P(\tilde{\Sigma}^j, i) = P(\text{Dom}(\tilde{\Sigma}^j) + 1) \text{ and } C(\tilde{\Sigma}^j, \text{Dom}(\tilde{\Sigma}^j) \upharpoonright \text{Dom}(\tilde{\Sigma}^j) - 1) \text{ and } C(\tilde{\Sigma}^j, \text{Dom}(\tilde{\Sigma}^j) \upharpoonright \text{Dom}(\tilde{\Sigma}^j) - 1) \text{ and } \beta \not\in \text{STSF}(\{\Delta, C(\tilde{\Sigma}^j, \text{Dom}(\tilde{\Sigma}^j) \upharpoonright \text{Dom}(\tilde{\Sigma}^j) - 1))\} \text{ and there is no } j \leq i \text{ such that } \beta \in \text{ST}(\tilde{\Sigma}^j), \text{ and there is no } l \text{ such that } i + 1 < l \leq \text{Dom}(\tilde{\Sigma}^j) - 2 \text{ and } l \in \text{Dom}(AVS(\tilde{\Sigma}^j \upharpoonright \text{Dom}(\tilde{\Sigma}^j) - 1)).

With \(v'\) and \(v'\), we then have: \(\text{Dom}(\tilde{\Sigma}^j) + 1 + i \in \text{Dom}(AVS(\tilde{\Sigma}^j))\) and \(\text{Dom}(\tilde{\Sigma}^j) + 2 + i \in \text{Dom}(AVS(\tilde{\Sigma}^j))\) and there is no \(l\) such that \(\text{Dom}(\tilde{\Sigma}^j) + 2 + i < l \leq \text{Dom}(\tilde{\Sigma}^j) + 1 + \text{Dom}(\tilde{\Sigma}^j) - 2\).
and \( l \in \text{Dom}(\text{AVS}(\delta^*)) \). With \( \text{vi}' \), we also have that \( \delta^+ = \delta^* \cup \{(\text{Dom}(\delta^*))+1+\text{Dom}(\delta')-1, \) "Therefore \( C(\delta^* \cup \text{Dom}(\delta')-1) \} = \delta^* \cup \{(\text{Dom}(\delta^*))+1+\text{Dom}(\delta')-1, \) "Therefore \( C(\delta^*) \}\}.

We have that \( \xi \in \text{FV}(\Delta) \) or \( \xi \notin \text{FV}(\Delta) \). Suppose \( \xi \in \text{FV}(\Delta) \). Then we have \( \beta \in \text{ST}([\beta, \xi, \Delta]) \subseteq \text{STSEQ}(\delta^*) \). Since, according to the hypothesis, \( \text{PAR} \cap \text{STSEQ}(\delta^*) \cap \text{STSEQ}(\delta^*') = \emptyset \), we thus have \( \beta \notin \text{STSEQ}(\delta^*) \). With \( \text{ii}' \) to \( \text{iv}' \), \( \beta \notin \text{STSF}([\Delta, \text{STSEQ}(\delta^*)]) \) and that there is no \( j \leq i \) such that \( \beta \in \text{ST}(\delta^*_j) \), it then follows that \( \beta \notin \text{STSF}([\Delta, \text{STSEQ}(\delta^*')]) \) and that there is no \( j \leq \text{Dom}(\delta^*)+1+i \) such that \( \beta \in \text{ST}(\delta^*') \). Thus we have \( \delta^+ \in \text{PEF}(\delta^*') \). Now, suppose \( \xi \notin \text{FV}(\Delta) \). Then we have \( \beta \notin \text{ST}([\beta, \xi, \Delta]) \). We have that there is a \( \beta^* \in \text{PAR}\text{STSEQ}(\delta^*) \cup \text{STSEQ}(\delta^*) \). With Theorem 1-14-(ii), we then have \( \{[\beta^*, \xi, \Delta] = \Delta = [\beta, \xi, \Delta] = P(\delta^*_i) = P(\delta^*\text{Dom}(\delta^*+1+i)) \}. Also, we have that \( \beta^* \notin \text{STSF}([\Delta, \text{STSEQ}(\delta^*')]) \) and that there is no \( j \leq \text{Dom}(\delta^*)+1+i \) such that \( \beta^* \in \text{ST}(\delta^*) \). Thus we then have again \( \delta^+ \in \text{PEF}(\delta^*) \). Hence we have in both cases that \( \delta^+ \in \text{PEF}(\delta^*) \subseteq \text{RCS}\{\emptyset\} \). That \( \text{v} \) holds, then follows, with \( \text{v}' \) and Theorem 3-21-(iii), in the same way as it did for \( \text{DIF} \) and \( \text{NIF} \).

(CDIF, CIF, CEF, BIF, BEF, DIF, DEF, NEF, UEF, PIF, IIF, IEF): Now, suppose \( \delta^* \in \text{CDIF}(\delta^* \cup \text{Dom}(\delta^*')) \). According to Definition 3-3, there are then \( \Delta, \Gamma \in \text{CFORM} \) such that \( \Delta, [\Delta \rightarrow \Gamma^*] \in \text{AVP}(\delta^* \cup \text{Dom}(\delta^*)) \) and \( \delta^* = \delta^* \cup \{(\text{Dom}(\delta^*)-1, \) "Therefore \( \Gamma^* \}\} \). With \( \text{vi}' \), it then holds that \( \delta^+ = \delta^* \cup \{(\text{Dom}(\delta^*))+1+\text{Dom}(\delta^*)-1, \) "Therefore \( \Gamma^* \}\}. With \( \Delta, [\Delta \rightarrow \Gamma^*] \in \text{AVP}(\delta^* \cup \text{Dom}(\delta^*)) \), Definition 2-30 and \( \text{iv}' \), we have that there are \( i, j \in \text{Dom}(\text{AVS}(\delta^* \cup \text{Dom}(\delta^*')-1)) \) such that \( \Delta = P(\delta^*_i) = P(\delta^*\text{Dom}(\delta^*+1+i)) \) and \( \Delta \rightarrow \Gamma^* = P(\delta^*_j) = P(\delta^*\text{Dom}(\delta^*+1+i)) \). With \( \text{v} \), we then have that \( \text{Dom}(\delta^*)+1+i, \text{Dom}(\delta^*)+1+j \in \text{Dom}(\text{AVS}(\delta^*)) \). Hence we have \( \delta^+ \in \text{CDIF}(\delta^*) \subseteq \text{RCS}\{\emptyset\} \).

We have \( \delta^* \in \text{CDIF}(\delta^* \cup \text{Dom}(\delta^*')-1) \cup \text{NIF}(\delta^* \cup \text{Dom}(\delta^*')-1) \cup \text{PEF}(\delta^* \cup \text{Dom}(\delta^*')-1) \) or \( \delta^* \notin \text{CDIF}(\delta^* \cup \text{Dom}(\delta^*')-1) \cup \text{NIF}(\delta^* \cup \text{Dom}(\delta^*')-1) \cup \text{PEF}(\delta^* \cup \text{Dom}(\delta^*')-1) \). In the first case, \( \text{v} \) is shown in the same way as for the respective subcases. Now, suppose \( \delta^* \notin \text{CDIF}(\delta^* \cup \text{Dom}(\delta^*')-1) \cup \text{NIF}(\delta^* \cup \text{Dom}(\delta^*')-1) \cup \text{PEF}(\delta^* \cup \text{Dom}(\delta^*')-1) \). With \( \text{ix}' \), it then holds that \( \delta^+ \notin \text{CDIF}(\delta^*) \cup \text{NIF}(\delta^*) \cup \text{PEF}(\delta^*) \). With Theorem 3-25, it then holds that \( \text{AVS}(\delta^*) = \text{AVS}(\delta^* \cup \text{Dom}(\delta^*')-1) \cup \{(\text{Dom}(\delta^*)-1, \) "Therefore \( \Gamma^* \}\} \) and \( \text{AVS}(\delta^*) = \text{AVS}(\delta^*) \cup \{(\text{Dom}(\delta^*)+1+\text{Dom}(\delta^*)-1, \) "Therefore \( \Gamma^* \}\} \). With \( \text{v} \), it then follows in the same way as for \( \text{AF} \) that \( \text{AVS}(\delta^*) = \text{Dom}(\text{AVS}(\delta^*)) \cup \{\text{Dom}(\delta^*) \} \cup \{(\text{Dom}(\delta^*)+1+i: l \in \text{Dom}(\text{AVS}(\delta^*)) \) and thus that \( \text{v} \) holds.
If $s' \in \text{CIF}(s) \cup \text{CEF}(s) \cup \text{BIF}(s) \cup \text{DIF}(s) \cup \text{DEF}(s) \cup \text{NEF}(s) \cup \text{UEF}(s) \cup \text{PIF}(s) \cup \text{IF}(s) \cup \text{IEF}(s)$, then $s' \in \text{IF}(s)$ showing analogously that then also $s^+ \in \text{CIF}(s) \cup \text{CEF}(s) \cup \text{BIF}(s) \cup \text{DIF}(s) \cup \text{DEF}(s) \cup \text{NEF}(s) \cup \text{UEF}(s) \cup \text{PIF}(s) \cup \text{IF}(s) \cup \text{EIF}(s) \subseteq \text{RCS}\{\emptyset\}$ and that $v$ holds in each case.

$(UIF)$: Now, suppose $s^+ \in \text{UIF}(s)$. According to Definition 3-12, there are then $\beta \in \text{PAR}$, $\xi \in \text{VAR}$ and $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi\}$, such that $[\beta, \xi, \Delta] \in \text{AVP}(s) \cup \text{AVAP}(s) \cup \text{STSEQ}(s)$ and $s' = s^+$. Then $s' = s^+ \cup \text{STSEQ}(s)$ such that $v' \rightarrow v$ holds in each case. Hence we have that $s^+ \in \text{UIF}(s)$. Now, suppose $\xi \notin \text{FV}(\Delta)$. Then we have $s' \notin \text{ST}(\Delta)$ and $s' \notin \text{STSEQ}(s)$. Since, according to the hypothesis, $\text{PAR} \cap \text{STSEQ}(s) \subseteq \text{STSEQ}(s)$, we then have $s' \notin \text{ST}(\Delta)$. It follows with $i') \rightarrow v'$ and $\beta \notin \text{STSF}(\Delta) \cup \text{AVAP}(s) \cup \text{STSEQ}(s)$ that $s' \notin \text{ST}(\Delta) \cup \text{AVAP}(s) = \emptyset$. Thus we have $s^+ \in \text{UIF}(s)$. Now, suppose $\xi \notin \text{FV}(\Delta)$. Then we have $\beta \notin \text{ST}(\Delta)$ and $s' \notin \text{STSEQ}(s) \cup \text{STSEQ}(s)$. With Theorem 1-14-(ii), we then have $[\beta, \xi, \Delta] = P(s') \cup \text{STSEQ}(s)$. Also, we have that $\beta \notin \text{STSF}(\Delta) \cup \text{AVAP}(s)$. Thus we have again $s^+ \in \text{UIF}(s)$. Hence we have that $s^+ \in \text{UIF}(s) \subseteq \text{RCS}\{\emptyset\}$.

Theorem 4-5. Successful CE-extension

If $s \in \text{RCS}\{\emptyset\}$ and $A \land B \in \text{AVP}(s)$, then there is then an $s^* \in \text{RCS}\{\emptyset\}$ such that

(i) $\text{AVAP}(s) = \text{AVAP}(s^*)$

(ii) $A, B \in \text{AVP}(s^*)$, and

(iii) $C(s^*) = B$.

Proof: Suppose $s \in \text{RCS}\{\emptyset\}$ and $A \land B \in \text{AVP}(s)$. Then there is then an $i \in \text{Dom}(s)$ such that $P(s_i) = A \land B$ and $(s_i, s_i) \in \text{AV}(s)$. Let the following sentence sequences be defined, where $a \in \text{CONST}\\text{STSEQ}(s)$:
\[
\begin{align*}
\mathcal{H}_1 & = \mathcal{H} \cup \{(\text{Dom}(\mathcal{H}), \text{Therefore } \alpha = \alpha')\} \\
\mathcal{H}_2 & = \mathcal{H}_1 \cup \{(\text{Dom}(\mathcal{H}_1), \text{Therefore } A')\} \\
\mathcal{H}_3 & = \mathcal{H}_2 \cup \{(\text{Dom}(\mathcal{H}_2), \text{Therefore } \alpha = \alpha')\} \\
\mathcal{H}_4 & = \mathcal{H}_3 \cup \{(\text{Dom}(\mathcal{H}_3), \text{Therefore } B')\}.
\end{align*}
\]

With Theorem 1-10 and Theorem 1-11, we have that \(C(\mathcal{H}_1)\) and \(C(\mathcal{H}_3)\) are neither negations nor conditionals, and neither identical to \(C(\mathcal{H})\) nor to \(C(\mathcal{H}_2)\), because otherwise \(\alpha \in \text{STSEQ}(\mathcal{H})\) or \(\alpha \in \text{ST}(\mathcal{H}_2) \subseteq \text{STSEQ}(\mathcal{H})\). Therefore \(\mathcal{H}_1 \notin \text{CdIF}(\mathcal{H}) \cup \text{NIF}(\mathcal{H}) \cup \text{PEF}(\mathcal{H})\) and \(\mathcal{H}_3 \notin \text{CdIF}(\mathcal{H}_2) \cup \text{NIF}(\mathcal{H}_2) \cup \text{PEF}(\mathcal{H}_2)\). If \(\alpha = \alpha' \in \text{SF}(A) \cup \text{SF}(B)\), then we would have \(\alpha \in \text{ST}(\mathcal{H}_2) \subseteq \text{STSEQ}(\mathcal{H})\). Therefore we have \(\alpha = \alpha' \notin \text{SF}(A)\) and \(\alpha = \alpha' \notin \text{SF}(B)\) and thus \(\mathcal{H}_2 \notin \text{CdIF}(\mathcal{H}_1) \cup \text{PEF}(\mathcal{H}_1)\) and \(\mathcal{H}_4 \notin \text{CdIF}(\mathcal{H}_3) \cup \text{PEF}(\mathcal{H}_3)\). Suppose for contradiction that \(\mathcal{H}_2 \in \text{NIF}(\mathcal{H}_1)\) or \(\mathcal{H}_4 \in \text{NIF}(\mathcal{H}_3)\). Then there would be a \(j \in \text{Dom}(\mathcal{H}_3)\) such that \(P(\mathcal{H}_j) = \lnot \alpha = \alpha'\). With Theorem 1-10 and Theorem 1-11, we have \(j \notin \{\text{Dom}(\mathcal{H}_3)-1, \text{Dom}(\mathcal{H}_3)-3\}\). Because of \(\alpha = \alpha' \notin \text{SF}(A)\), we have \(j \neq \text{Dom}(\mathcal{H}_3)-2\). Therefore we would have \(j \in \text{Dom}(\mathcal{H}_3) \backslash \{\text{Dom}(\mathcal{H}_3)-1, \text{Dom}(\mathcal{H}_3)-2, \text{Dom}(\mathcal{H}_3)-3\} = \text{Dom}(\mathcal{H})\). With \(\alpha \in \text{ST}(\mathcal{H}_j) = \text{ST}(\mathcal{H}_j)\), we would then have \(\alpha \in \text{STSEQ}(\mathcal{H})\). Contradiction! Therefore \(\mathcal{H}_2 \notin \text{NIF}(\mathcal{H}_1)\) and \(\mathcal{H}_4 \notin \text{NIF}(\mathcal{H}_3)\).

On the other hand, we have, first, with Definition 3-16, that \(\mathcal{H}_1 \in \text{IIF}(\mathcal{H})\), thus \(\mathcal{H}_3 \in \text{RCS}\backslash\{\emptyset\}\), and with Theorem 3-25, \(\text{AVS}(\mathcal{H}_1) = \text{AVS}(\mathcal{H}) \cup \{(\text{Dom}(\mathcal{H}), \text{Therefore } \alpha = \alpha')\}\). Thus we have \(\text{AVAS}(\mathcal{H}_1) = \text{AVAS}(\mathcal{H})\) and \(\text{AVP}(\mathcal{H}_1) \subseteq \text{AVP}(\mathcal{H}_1)\). Therefore we have, second, with Definition 3-5, that \(\mathcal{H}_2 \in \text{CEF}(\mathcal{H}_1) \subseteq \text{RCS}\backslash\{\emptyset\}\) and, with Theorem 3-25, \(\text{AVS}(\mathcal{H}_2) = \text{AVS}(\mathcal{H}_1) \cup \{(\text{Dom}(\mathcal{H}_1), \text{Therefore } A')\}\). Thus we have \(\text{AVAS}(\mathcal{H}_2) = \text{AVAS}(\mathcal{H}_1), \text{AVP}(\mathcal{H}_1) \subseteq \text{AVP}(\mathcal{H}_2)\) and \(A \in \text{AVP}(\mathcal{H}_2)\). Third, with Definition 3-16, we have \(\mathcal{H}_3 \in \text{IIF}(\mathcal{H}_2), \mathcal{H}_3 \in \text{RCS}\backslash\{\emptyset\}\) and, with Theorem 3-25, \(\text{AVS}(\mathcal{H}_3) = \text{AVS}(\mathcal{H}_2) \cup \{(\text{Dom}(\mathcal{H}_2), \text{Therefore } \alpha = \alpha')\}\). Thus we have \(\text{AVAS}(\mathcal{H}_3) = \text{AVAS}(\mathcal{H}_2)\) and \(A, \text{AVP}(\mathcal{H}_2) \subseteq \text{AVP}(\mathcal{H}_3)\). Fourth, with Definition 3-5, we then have \(\mathcal{H}_4 \in \text{CEF}(\mathcal{H}_3) \subseteq \text{RCS}\backslash\{\emptyset\}\) and, with Theorem 3-25, \(\text{AVS}(\mathcal{H}_4) = \text{AVS}(\mathcal{H}_3) \cup \{(\text{Dom}(\mathcal{H}_3), \text{Therefore } B')\}\). Thus we have \(\text{AVAS}(\mathcal{H}_4) = \text{AVAS}(\mathcal{H}_3), A \in \text{AVP}(\mathcal{H}_3) \subseteq \text{AVP}(\mathcal{H}_4)\) and \(B \in \text{AVP}(\mathcal{H}_4)\). Hence we have \(\mathcal{H}_4 \in \text{RCS}\backslash\{\emptyset\}, \text{AVAP}(\mathcal{H}_4) = \text{AVAP}(\mathcal{H}_3) = \text{AVAP}(\mathcal{H}_2) = \text{AVAP}(\mathcal{H}_1) = \text{AVAP}(\mathcal{H}_0), A, B \in \text{AVP}(\mathcal{H}_0)\) and \(C(\mathcal{H}_4) = B\).
Theorem 4-6. Available propositions as conclusions

If $\mathcal{S} \in \text{RCS}\{\emptyset\}$ and $A \in \text{AVP}(\mathcal{S})$, then there is an $\mathcal{S}^* \in \text{RCS}\{\emptyset\}$ such that

(i) $\text{AVAP}(\mathcal{S}^*) = \text{AVAP}(\mathcal{S})$,

(ii) $\text{AVP}(\mathcal{S}) \subseteq \text{AVP}(\mathcal{S}^*)$, and

(iii) $C(\mathcal{S}^*) = A$.

Proof: Suppose $\mathcal{S} \in \text{RCS}\{\emptyset\}$ and $A \in \text{AVP}(\mathcal{S})$. Then there is an $i \in \text{Dom}(\mathcal{S})$ such that $P(\mathcal{S}_i) = A$ and $(i, \mathcal{S}_i) \in \text{AVS}(\mathcal{S})$. Let the following sentence sequences be defined, where $\alpha \in \text{CONST} \cup \text{STSEQ}(\mathcal{S})$:

\[
\begin{align*}
\mathcal{S}_1^1 & = \mathcal{S} \cup \{(\text{Dom}(\mathcal{S}), \text{Therefore } \alpha = \alpha^n)\} \\
\mathcal{S}_1^2 & = \mathcal{S}_1^1 \cup \{(\text{Dom}(\mathcal{S}^1), \text{Therefore } A \land A^n)\} \\
\mathcal{S}_3^1 & = \mathcal{S}_2^1 \cup \{(\text{Dom}(\mathcal{S}^3), \text{Therefore } \alpha = \alpha^n)\} \\
\mathcal{S}_4^1 & = \mathcal{S}_3^1 \cup \{(\text{Dom}(\mathcal{S}^3), \text{Therefore } A^n)\}.
\end{align*}
\]

With Theorem 1-10 and Theorem 1-11, $C(\mathcal{S}_1^1)$, $C(\mathcal{S}_2^1)$ and $C(\mathcal{S}_3^1)$ are neither negations nor conditionals. Moreover, $C(\mathcal{S}_1^1)$ and $C(\mathcal{S}_2^1)$ are neither identical to $C(\mathcal{S})$ nor to $C(\mathcal{S}_3^1)$. With Theorem 1-10-(vi) $C(\mathcal{S})$ is not identical to $C(\mathcal{S}_1^1)$. Therefore $\mathcal{S}_1^1 \notin \text{CdIF}(\mathcal{S}) \cup \text{NIF}(\mathcal{S}) \cup \text{PEF}(\mathcal{S})$, $\mathcal{S}_2^1 \notin \text{CdIF}(\mathcal{S}^1) \cup \text{NIF}(\mathcal{S}^1) \cup \text{PEF}(\mathcal{S}^1)$, and $\mathcal{S}_3^1 \notin \text{CdIF}(\mathcal{S}^3) \cup \text{NIF}(\mathcal{S}^3) \cup \text{PEF}(\mathcal{S}^3)$. If $\alpha = \alpha^n \in \text{SF}(A)$, then we would have $\alpha \in \text{ST}(\mathcal{S}_i) \subseteq \text{STSEQ}(\mathcal{S})$. Therefore we have $\alpha = \alpha^n \in \text{SF}(A)$, and thus $\mathcal{S}_4^1 \notin \text{CdIF}(\mathcal{S}^3) \cup \text{PEF}(\mathcal{S}^3)$. Now, suppose for contradiction that $\mathcal{S}_4^1 \in \text{NIF}(\mathcal{S}^3)$. Then there would be a $j \in \text{Dom}(\mathcal{S}_3^3)$ such that $P(\mathcal{S}_j) = \neg \neg \alpha = \alpha^n$. With Theorem 1-10 and Theorem 1-11, we have $j \notin \{\text{Dom}(\mathcal{S}_3^3)-1, \text{Dom}(\mathcal{S}_3^3)-2, \text{Dom}(\mathcal{S}_3^3)-3\}$. Therefore $j \in \text{Dom}(\mathcal{S}_3^3) \cap \{\text{Dom}(\mathcal{S}_3^3)-1, \text{Dom}(\mathcal{S}^3)-2, \text{Dom}(\mathcal{S}^3)-3\} = \text{Dom}(\mathcal{S})$. With $\alpha \in \text{ST}(\mathcal{S}_3^3) = \text{ST}(\mathcal{S}_i)$, we would then have $\alpha \in \text{STSEQ}(\mathcal{S})$. Contradiction! Therefore $\mathcal{S}_4^1 \notin \text{NIF}(\mathcal{S}^3)$.

On the other hand, we have, first, with Definition 3-16, that $\mathcal{S}_1^1 \in \text{IIF}(\mathcal{S})$, thus $\mathcal{S}_1^1 \in \text{RCS}\{\emptyset\}$ and, with Theorem 3-25, $\text{AVS}(\mathcal{S}_1^1) = \text{AVS}(\mathcal{S}) \cup \{(\text{Dom}(\mathcal{S}), \text{Therefore } \alpha = \alpha^n)\}$. Thus we have $\text{AVAS}(\mathcal{S}_1^1) = \text{AVAS}(\mathcal{S})$ and $A \in \text{AVP}(\mathcal{S}) \subseteq \text{AVP}(\mathcal{S}_1^1)$. Therefore we have, second, with Definition 3-4, $\mathcal{S}_2^1 \in \text{CIF}(\mathcal{S}_1^1) \subseteq \text{RCS}\{\emptyset\}$ and, with Theorem 3-25, $\text{AVS}(\mathcal{S}_2^1) = \text{AVS}(\mathcal{S}_1^1) \cup \{(\text{Dom}(\mathcal{S}^1), \text{Therefore } A \land A^n)\}$. Thus we have $\text{AVAS}(\mathcal{S}_2^1) = \text{AVAS}(\mathcal{S}_1^1)$, $\text{AVP}(\mathcal{S}_2^1) \subseteq \text{AVP}(\mathcal{S}_2^1)$ and $A \land A^n \in \text{AVP}(\mathcal{S}_2^1)$. Then we have, third, with Definition 3-16, $\mathcal{S}_3^1 \in \text{IIF}(\mathcal{S}_2^1) \subseteq \text{RCS}\{\emptyset\}$ and, with Theorem 3-25, $\text{AVS}(\mathcal{S}_3^1) = \text{AVS}(\mathcal{S}_2^1) \cup \{(\text{Dom}(\mathcal{S}^3), \text{Therefore } \alpha = \alpha^n)\}$. Thus we have $\text{AVAS}(\mathcal{S}_3^1) = \text{AVAS}(\mathcal{S}_2^1)$ and $A \land A^n \in \text{AVP}(\mathcal{S}_3^1) \subseteq \text{AVP}(\mathcal{S}_3^1)$. Fourth, with Definition 3-5, we thus have $\mathcal{S}_4^1 \in \text{CEF}(\mathcal{S}_3^1) \subseteq \text{CEF}(\mathcal{S}_3^1)$.
RCS\{\emptyset\} and, with Theorem 3-25, AVS(\delta^4) = AVS(\delta^3) \cup \{\text{Dom}(\delta^3), \text{"Therefore } A\} \}.

Thus we have AVAS(\delta^4) = AVAS(\delta^3) and AVP(\delta^3) \subseteq AVP(\delta^4). Hence we have \delta^4 \in RCS\{\emptyset\}, AVAP(\delta^4) = AVAP(\delta^3) = AVAP(\delta^2) = AVAP(\delta^1) = AVAP(\delta), AVP(\delta) \subseteq AVP(\delta^3) and C(\delta^4) = A. \blacksquare

**Theorem 4-7. Eliminability of an assumption of \"a = a\"**

If \delta \in RCS\{\emptyset\}, \alpha \in \text{CONST} and A, B \in AVP(\delta), then there is a \delta^* \in RCS\{\emptyset\} such that

(i) AVP(\delta^*) \subseteq AVAP(\delta)\{\"a = a\"\},

(ii) A, B \in AVP(\delta^*), and

(iii) C(\delta^*) = B.

**Proof:** Let \delta \in RCS\{\emptyset\}, \alpha \in \text{CONST} and A, B \in AVP(\delta). Suppose \"a = a\" \not\in AVAP(\delta). Then we have AVAP(\delta) \subseteq AVAP(\delta)\{\"a = a\"\}. With Theorem 4-6, there is then an \delta^* \in RCS\{\emptyset\} such that AVAP(\delta^*) = AVAP(\delta) \subseteq AVAP(\delta)\{\"a = a\"\}, A, B \in AVP(\delta) \subseteq AVP(\delta^*) and C(\delta^*) = B.

Now, suppose \"a = a\" \in AVAP(\delta). Then we have \delta^1 = \delta \cup \{\text{Dom}(\delta), \text{"Therefore A} \wedge B\}\} \in \text{CIF}(\delta). Then we have \delta^1 \in RCS\{\emptyset\} and \"A \wedge B\" \in AVP(\delta^1) and, with Theorem 3-26-(v), AVAP(\delta^1) \subseteq AVAP(\delta). According to Theorem 4-2, there is then an \delta^3 \in RCS\{\emptyset\} such that AVAP(\delta^3) \subseteq AVAP(\delta) \subseteq AVAP(\delta)\{\"a = a\"\}, C(\delta^3) = C(\delta^1) = \"A \wedge B\" and for all \ k \in \text{Dom}(AVAS(\delta^3))\}: If P(\delta^1_k) = \"a = a\", then \ k = \max(\text{Dom}(AVAS(\delta^3)))). Then we have \"a = a\" \in AVAP(\delta^3) or \"a = a\" \not\in AVAP(\delta^3).

**First case:** Suppose \"a = a\" \in AVAP(\delta^3). Then we have P(\delta^3_{\max(\text{Dom}(AVAS(\delta^3))})) = \"a = a\" and for all \ k \in \text{Dom}(AVAS(\delta^3))\}: If P(\delta^3_k) = \"a = a\", then \ k = \max(\text{Dom}(AVAS(\delta^3)))). Now, let the following sentence sequences be defined:

\[
\begin{align*}
\delta^2 &= \delta^1 \cup \{\text{Dom}(\delta^1), \text{"Therefore a = a \rightarrow (A \wedge B)\}}
\delta^3 &= \delta^2 \cup \{\text{Dom}(\delta^2), \text{"Therefore a = a\"}\} \\
\delta^4 &= \delta^3 \cup \{\text{Dom}(\delta^3), \text{"Therefore A} \wedge B\}\}.
\end{align*}
\]

According to Definition 3-2, we have \delta^2 \in \text{CdIF}(\delta^3), thus \delta^2 \in RCS\{\emptyset\} and, with Theorem 3-19-(ix), AVAP(\delta^7) \subseteq AVAP(\delta^5) \subseteq AVAP(\delta). With Theorem 3-22, we have that \"a = a\" \not\in AVAP(\delta^5) and thus AVAP(\delta^5) \subseteq AVAP(\delta)\{\"a = a\"\}. We also have \"a = a \rightarrow (A \wedge B)\" \in AVP(\delta^5).

With Theorem 1-10 and Theorem 1-11, C(\delta^5) and C(\delta^4) are neither negations nor conditionals and also C(\delta^5) is not identical to C(\delta^5) and C(\delta^5) is not identical to C(\delta^5).
Therefore we have $\tilde{y}^3 \in \text{CdIF}(\tilde{y}^3) \cup \text{NIF}(\tilde{y}^3) \cup \text{PEF}(\tilde{y}^3)$ and $\tilde{y}^4 \notin \text{CdIF}(\tilde{y}^3) \cup \text{NIF}(\tilde{y}^3) \cup \text{PEF}(\tilde{y}^3)$. According to Definition 3-16, we have $\tilde{y}^3 \in \text{IIF}(\tilde{y}^3) \subseteq \text{RCS}\{\emptyset\}$ and, with Theorem 3-25, $\text{AVS}(\tilde{y}^3) = \text{AVS}(\tilde{y}^3) \cup \{(\text{Dom}(\tilde{y}^3), \{\text{Therefore } \alpha = \alpha^*\}\}$. Thus we have $\text{AVAS}(\tilde{y}^3) = \text{AVAS}(\tilde{y}^3)$, $\{\alpha = \alpha \rightarrow (A \wedge B^*) \in \text{AV}(\tilde{y}^3) \subseteq \text{AV}(\tilde{y}^3)\}$ and $\{\alpha = \alpha^* \in \text{AV}(\tilde{y}^3)\}$. According to Definition 3-3, we therefore have $\tilde{y}^4 \in \text{CdEF}(\tilde{y}^3) \subseteq \text{RCS}\{\emptyset\}$ and, with Theorem 3-25, $\text{AVS}(\tilde{y}^4) = \text{AVS}(\tilde{y}^3) \cup \{\text{(Dom}(\tilde{y}^3), \{\text{Therefore } A \wedge B^*\}\}$. Thus we have $\text{AVAS}(\tilde{y}^4) = \text{AVS}(\tilde{y}^3)$. Thus we have $\tilde{y}^4 \in \text{RCS}\{\emptyset\}$, $\text{AVAP}(\tilde{y}^4) = \text{AVAP}(\tilde{y}^3) = \text{AVAP}(\tilde{y}^2) \subseteq \text{AVAP}(\tilde{y})\{\{\alpha = \alpha^*\}\} \text{ and } \{\alpha \wedge B^* \in \text{AVP}(\tilde{y})\}$. With Theorem 4-5, there is then an $\tilde{y}^* \in \text{RCS}\{\emptyset\}$ such that $\text{AVAP}(\tilde{y}^*) = \text{AVAP}(\tilde{y}^4) \subseteq \text{AVAP}(\tilde{y})\{\{\alpha = \alpha^*\}\}$ and $A, B \in \text{AVP}(\tilde{y}^*)$ and $C(\tilde{y}^*) = B$.

Second case: Suppose $\alpha = \alpha^* \notin \text{AVP}(\tilde{y})$ and thus $\text{AVP}(\tilde{y}) \subseteq \text{AVAP}(\tilde{y})\{\{\alpha = \alpha^*\}\}$. We have $\{A \wedge B^* = C(\tilde{y}) \in \text{AVP}(\tilde{y})\}$. With Theorem 4-5 there is then an $\tilde{y}^* \in \text{RCS}\{\emptyset\}$ such that $\text{AVAP}(\tilde{y}^*) = \text{AVAP}(\tilde{y}^4) \subseteq \text{AVAP}(\tilde{y})\{\{\alpha = \alpha^*\}\}$ and $A, B \in \text{AVP}(\tilde{y}^*)$ and $C(\tilde{y}^*) = B$. ■

**Theorem 4-8. Substitution of a new parameter for a parameter is RCS-preserving**

If $\tilde{y} \in \text{RCS}$, and $\beta^* \in \text{PAR}\backslash \text{STSEQ}(\tilde{y})$ and $\beta \in \text{PAR}\backslash \{\beta^*\}$, then $[\beta^*, \beta, \tilde{y}] \in \text{RCS}$ and $\text{Dom}(\text{AVS}([\beta^*, \beta, \tilde{y}])) = \text{Dom}(\text{AVS}(\tilde{y}))$.

**Proof:** By induction on $\text{Dom}(\tilde{y})$. Suppose $\tilde{y} \in \text{RCS}$, and $\beta^* \in \text{PAR}\backslash \text{STSEQ}(\tilde{y})$ and $\beta \in \text{PAR}\backslash \{\beta^*\}$ and that the statement holds for all $k < \text{Dom}(\tilde{y})$. Suppose $\text{Dom}(\tilde{y}) = 0$. Then we have $\tilde{y} = 0 = [\beta^*, \beta, \tilde{y}]$ and thus $[\beta^*, \beta, \tilde{y}] \in \text{RCS}$ and $\text{Dom}(\text{AVS}([\beta^*, \beta, \tilde{y}])) = \emptyset = \text{Dom}(\text{AVS}(\tilde{y}))$. Now, suppose $0 < \text{Dom}(\tilde{y})$. Then we have $\tilde{y} \in \text{RCS}\{\emptyset\}$. With Theorem 3-6, we then have $\tilde{y} \in \text{RCE}(\tilde{y})|\text{Dom}(\tilde{y})-1)$. According to the I.H., we then have:

a) $\tilde{y}^* = [\beta^*, \beta, \tilde{y}][\text{Dom}(\tilde{y})-1] \in \text{RCS}$ and $\text{Dom}(\text{AVS}(\tilde{y}^*)) = \text{Dom}(\text{AVS}(\tilde{y})|\text{Dom}(\tilde{y})-1))$.

With $\tilde{y} \in \text{RCE}(\tilde{y})|\text{Dom}(\tilde{y})-1)\text{ and Definition } 3-18, we have that $\tilde{y} \in \text{AF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{CdIF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{CdEF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{CEF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{CIF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{CEF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{BI}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{BEF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{DIF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{DEF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{NIF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{NEF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{UF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{UEF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{PIF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{PEF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{IIF}(\tilde{y})|\text{Dom}(\tilde{y})-1) \text{ or } \tilde{y} \in \text{IEF}(\tilde{y})|\text{Dom}(\tilde{y})-1)$. Since operators are not affected by substitution, we first have:
b) For all $i \in \text{Dom}(\mathcal{S})$: $P(\mathcal{S}^*) = [\beta^*, \beta, P(\mathcal{S})]$ and $\mathcal{S}^* = \langle \Xi \mathcal{S} \rangle$, where $\mathcal{S} = \langle \Xi P(\mathcal{S}) \rangle$ for a $\Xi \in \text{PERF}$.

With $\beta^* \in \text{PAR} \setminus \text{STSEQ}(\mathcal{S})$ and $\beta \in \text{PAR} \setminus \{\beta^*\}$, we have:

c) For every $i \in \text{Dom}(\mathcal{S})$: $\beta^* \not\in \text{ST}(P(\mathcal{S}))$ and $\beta \not\in \text{ST}([\beta^*, \beta, P(\mathcal{S})])$.

if not, we would have $\beta^* \in \text{STSEQ}(\mathcal{S})$ or $\beta = \beta^*$, which both contradict the hypothesis.

Now, let:

d) $\mathcal{S}^+ = \mathcal{S}^* \cup \{\text{Dom}(\mathcal{S})-1, [\beta^*, \beta, \mathcal{S}^*]\}$.

Then we have that $\mathcal{S}^+ = [\beta^*, \beta, \mathcal{S}]$. Now we will show that in each of the cases AF … IEF we have that $\mathcal{S}^+ \in \text{RCS}$ and $\text{Dom}(\text{AVS}(\mathcal{S})) = \text{Dom}(\text{AVS}(\mathcal{S}))$, with which we prove that the statement holds for $[\beta^*, \beta, \mathcal{S}]$.

To simplify the treatment of CdEF, CIF, CEF, BIF, BEF, DIF, DEF, NEF, UIF, UEF, PIF, IIF and IEF, we will now show in preparation of the main part of the proof that

e) If $\mathcal{S}^+ \in \text{CdIF}(\mathcal{S}^*) \cup \text{NIF}(\mathcal{S}^*) \cup \text{PEF}(\mathcal{S}^*)$, then $\mathcal{S} \in \text{CdIF}(\mathcal{S}) \cup \text{NIF}(\mathcal{S}) \cup \text{PEF}(\mathcal{S})$.

**Preparatory part:** Suppose $\mathcal{S}^+ \in \text{CdIF}(\mathcal{S}^*)$. According to Definition 3-2, there is then an $i \in \text{Dom}(\text{AVS}(\mathcal{S}))$ such that, with b) and d), $P(\mathcal{S}^*) = [\beta^*, \beta, P(\mathcal{S})]$ and $\mathcal{C}(\mathcal{S}^*) = [\beta^*, \beta, P(\mathcal{S}^*)]$ and there is no $l$ such that $i < l \leq \text{Dom}(\mathcal{S})-2$ and $l \in \text{Dom}(\text{AVS}(\mathcal{S}^*))$, and $\mathcal{S}^+ = \mathcal{S}^* \cup \{(\text{Dom}(\mathcal{S})-1, '\text{Therefore } P(\mathcal{S}^*) \rightarrow P(\mathcal{S}^*)')\} = \mathcal{S}^* \cup \{(\text{Dom}(\mathcal{S})-1, '\text{Therefore } [\beta^*, \beta, P(\mathcal{S})] \rightarrow [\beta^*, \beta, P(\mathcal{S}^*)])\}$. With d), we have $'\text{Therefore } [\beta^*, \beta, P(\mathcal{S})] \rightarrow [\beta^*, \beta, P(\mathcal{S}^*)]\}' = [\beta^*, \beta, \mathcal{S}^*] \rightarrow [\beta^*, \beta, \mathcal{S}^*] = [\beta^*, \beta, \mathcal{S}^*] \rightarrow [\beta^*, \beta, \mathcal{S}^*] = [\beta^*, \beta, \mathcal{S}^*]$. With Theorem 1-21, we then have $'\text{Therefore } P(\mathcal{S}) \rightarrow P(\mathcal{S}^*)\}' = \mathcal{S}^* \cup \{(\text{Dom}(\mathcal{S})-1, '\text{Therefore } P(\mathcal{S}) \rightarrow P(\mathcal{S}^*)')\}$. We also have with a) and b): $i \in \text{Dom}(\text{AVS}(\mathcal{S}) \cup \{(\text{Dom}(\mathcal{S})-1, '\text{Therefore } P(\mathcal{S}) \rightarrow P(\mathcal{S}^*)')\}$. Hence we have $\mathcal{S} \in \text{CdIF}(\mathcal{S}) \cup \text{NIF}(\mathcal{S}) \cup \text{PEF}(\mathcal{S})$. In the case that $\mathcal{S}^+ \in \text{NIF}(\mathcal{S}^*)$, one shows analogously that then also $\mathcal{S} \in \text{NIF}(\mathcal{S}) \cup \text{PEF}(\mathcal{S})$.

Now, suppose $\mathcal{S}^+ \in \text{PEF}(\mathcal{S}^*)$. According to Definition 3-15 and with b) and d), there are then $\beta^* \in \text{PAR}$, $\zeta \in \text{VAR}$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\zeta\}$, and $i \in \text{Dom}(\text{AVS}(\mathcal{S}^*))$ such that $P(\mathcal{S}^*) = \nabla_{\zeta} \Delta$ and $P(\mathcal{S}^*_{i+1}) = [\beta^*, \zeta, \Delta] = [\beta^*, \beta, P(\mathcal{S})]$, where $i+1 \in \text{Dom}(\text{AVS}(\mathcal{S}^*))$, $[\beta^*, \beta, P(\mathcal{S}^*_{i+1})] = C(\mathcal{S}^*)$, $\beta^* \not\in \text{STSF}(\Delta, [\beta^*, \beta,$
there is no $j \leq i$ such that $\beta^+ \in ST(\Delta_j)$, there is no $l$ such that $i+1 < l \leq \text{Dom}(\Delta)-2$ and $l \in \text{Dom}(\text{AVAS}(\Delta^*))$, and $\Delta_j = \Delta_j^* \cup \{(\text{Dom}(\Delta)-1, \therefore \text{Then } \text{C}(\Delta_j^*)) \}

= \Delta_j^* \cup \{(\text{Dom}(\Delta)-1, \therefore \text{Then } [\beta^+, \beta, P(\Delta_j^*))] = \Delta_j^* \cup \{(\text{Dom}(\Delta)-1, [\beta^+, \beta, \therefore \text{Then } P(\Delta_j^*))}\}. \text{ With d), we have } [\beta^+, \beta, \therefore \text{Then } P(\Delta_j^*)) = [\beta^+, \beta, \Delta_j^*]. \text{ With Theorem 1-21, we then have } \therefore \text{Then } P(\Delta_j^*)) = \Delta_j^* \text{ and thus } \Delta_j = \Delta_j^* \cup \{(\text{Dom}(\Delta)-1, \therefore \text{Then } P(\Delta_j^*))\}

Then we have, with a) and b): $i \in \text{Dom}(\text{AVS}(\Delta^*)\text{Dom}(\Delta)-1))$, $i+1 \in \text{Dom}(\text{AVAS}(\Delta^*)\text{Dom}(\Delta)-1))$ and there is no $l$ such that $i+1 < l \leq \text{Dom}(\Delta)-2$ such that $l \in \text{Dom}(\text{AVAS}(\Delta^*)\text{Dom}(\Delta)-1))$. Now, we have to show that $P(\Delta_j)$, $P(\Delta_{j+1})$ and $P(\Delta_{j+2})$ satisfy the conditions for $\Delta_j^* \in \text{PEF}(\Delta_j^*)\text{Dom}(\Delta)-1))$.

We have $[\beta^+, \beta, P(\Delta_j)] = P(\Delta_j^*) = \text{Form}(\Delta^*)$ and $[\beta^+, \beta, P(\Delta_{j+1})] = P(\Delta_{j+1}) = [\beta^+, \zeta, \Delta]$. Since operators are not affected by substitution, we thus have, because of $[\beta^+, \beta, P(\Delta_j)] = \text{Form}(\Delta^*)$, that $P(\Delta_j^*) = \text{Form}(\Delta^*)$ for a $\Delta^+ \in \text{Form}$, where $\beta^+ \notin \text{ST}(\Delta^*)$ and $\text{FV}(\Delta^*) \subseteq \{\zeta\}$. Thus we have $\text{Form}(\Delta^*) = [\beta^+, \beta, P(\Delta_j^*)] = [\beta^+, \beta, \text{Form}(\Delta^*) = [\beta^+, \beta, \Delta^*]$ and hence $\Delta = [\beta^+, \beta, \Delta^*]$. Thus we have: $[\beta^+, \beta, P(\Delta_{j+1})] = [\beta^+, \zeta, \Delta] = [\beta^+, \zeta, \beta^+, \beta, \Delta^*]$ and $\beta^+ \notin \text{ST}(\beta^+, \beta, \Delta^*)]$. Also, we have $\beta^+ = \beta^+$ or $\beta^+ \neq \beta^+$.

First case: Suppose $\beta^+ = \beta^+$. Then we have $\beta^+ \notin \text{ST}(\beta^+, \beta, \Delta^*)] and thus $\beta^+ \notin \text{ST}(\Delta^*)]$. Then we have $\Delta = [\beta^+, \beta, \Delta^+] = \Delta^+$ and, because of $\beta^+ = \beta^+$, we then have $[\beta^+, \beta, P(\Delta_{j+1})] = [\beta^+, \zeta, \Delta] = [\beta^+, \zeta, \Delta^*]$. We have $\beta^+ \notin \text{ST}(\Delta^*)] and $\beta^+ \notin \text{ST}(\beta^+, \beta, \Delta^*]$. It thus holds with Theorem 1-23, because of $[\beta^+, \beta, P(\Delta_{j+1})] = [\beta^+, \zeta, \Delta^*], that $P(\Delta_{j+1}) = [\beta^+, \zeta, \Delta^*]. Now, suppose for contradiction that $\beta \in \text{STSF}(\{\Delta^*, P(\Delta_{j+2})\})$ or that there is a $j \leq i$ such that $\beta \in \text{ST}(\Delta_j)$. Then we would have, with b) and $\beta^+ = \beta^+$, that $\beta^+ \in \text{STSF}(\{\beta^+, \beta, \Delta^*, [\beta^+, \beta, P(\Delta_{j+2})\}) or that there is $j \leq i$ such that $\beta^+ \in \text{ST}(\Delta_j^*))$. Contradiction! Hence we have $P(\Delta_j) = \text{Form}(\Delta^*) and $P(\Delta_{j+1}) = [\beta^+, \zeta, \Delta^*] and $\beta^+ \notin \text{STSF}(\{\Delta^*, P(\Delta_{j+2})\}) and there is no $j \leq i$ such that $\beta \in \text{ST}(\Delta_j)$ and thus we have $\beta \in \text{ST}(\Delta_j)$.

Second case: Suppose $\beta^+ \neq \beta^+$. With $\beta^+ \in \text{ST}(\{\beta^+, \beta, P(\Delta_{j+1})\}) and $\beta^+ \notin \text{ST}(\{\beta^+, \beta, P(\Delta_{j+1})\})$, we can distinguish two subcases. First subcase: Suppose $\beta^+ \in \text{ST}(\{\beta^+, \beta, P(\Delta_{j+1})\})$. Then we have $\beta^+ \neq \beta$ and thus $\beta \notin \text{ST}(\beta^+)$. Then, with $\Delta = [\beta^+, \beta, \Delta^*] and Theorem 1-25-(ii): [\beta^+, \beta, P(\Delta_{j+1})] = [\beta^+, \zeta, \Delta] = [\beta^+, \zeta, \beta^+, \beta, \Delta^*] = [\beta^+, \beta, [\beta^+, \beta, \Delta^*]]. We also have $\beta^+ \notin \text{ST}(P(\Delta_{j+1})) and, because of $\beta^+ \neq \beta^+$ and $\beta^+ \notin \text{ST}(\Delta^*)$, we also have $\beta^+ \notin \text{ST}(\beta^+, \zeta, \Delta^*]. With Theorem 1-20, we thus have $P(\Delta_{j+1}) = [\beta^+, \zeta, \Delta^*]. Now, suppose for contradiction that $\beta^+ \in \text{STSF}(\{\Delta^+, P(\Delta_{j+2})\}) or that there is a $j \leq i$ such that $\beta^+ \in \text{ST}(\Delta_j$). Because of $\beta^+ \neq \beta$ and with b), we would then also have $\beta^+ \in \text{STSF}(\{\beta^+, \beta, \Delta^*, }
[β*, β, P(\(\text{Dom}(\delta)\))] or there would be a \(j \leq i\) such that \(\beta^+ \in \text{ST}(\delta)\). Contradiction!

Hence the parameter condition for \(\beta^+\) is satisfied in \(\delta||\text{Dom}(\delta)\) and thus we have for the first subcase again that \(\delta \in \text{PEF}(\delta||\text{Dom}(\delta))\).

Second subcase: Now, suppose \(\beta^+ \notin \text{ST}(\{\beta^*, \beta, P(\text{Dom}(\delta))\})\). Then it holds, with \([\beta^*, \beta, P(\text{Dom}(\delta))]= [\beta^+, \xi, [\beta^*, \beta, \Delta^+]]\), that \(\xi \notin \text{FV}(\{\beta^*, \beta, \Delta^+\})\). Then we have \([\beta^*, \beta, P(\text{Dom}(\delta))]= [\beta^+, \xi, [\beta^*, \beta, \Delta^+]]\) and thus, with \(\beta^+ \notin \text{ST}(P(\text{Dom}(\delta))) \cup \text{ST}(\Delta^+)\) and with Theorem 1-20, \(P(\text{Dom}(\delta)) = \Delta^+, \text{where, with} \xi \notin \text{FV}(\{\beta^*, \beta, \Delta^+\}), \text{also} \xi \notin \text{FV}(\Delta^+)\). Now, let \(\beta^+ \in \text{PAR}: \text{STSEQ}(\delta||\text{Dom}(\delta))\). Then it holds, with \(\xi \notin \text{FV}(\Delta^+)\), that \(P(\text{Dom}(\delta)) = \Delta^+ = [\beta^+, \xi, \Delta^+]\) and we have that \(\beta^+ \notin \text{STSF}(\Delta^+, \{\beta^*, \beta, \Delta^+\})\) and there is no \(j \leq i\) such that \(\beta^+ \in \text{ST}(\delta_j)\). Thus we then also have \(\delta \in \text{PEF}(\delta||\text{Dom}(\delta))\). Hence we have in both subcases and thus in both cases that \(\delta \in \text{PEF}(\delta||\text{Dom}(\delta))\).

Main part: Now we will show that for each of the cases AF ... IEF it holds that \(\delta^+ \in \text{RCS and Dom(\text{AVS}(\delta^+))} = \text{Dom(\text{AVS}(\delta))}\). First, we will deal with CdIF, NIF and PEF. Then we can make an exclusion assumption that allows us to determine \(\text{Dom(\text{AVS}(\delta^+))}\) for all other cases just with \(a, b, c\) and Theorem 3-25.

(CdIF, NIF): Suppose \(\delta \in \text{CdIF}(\delta||\text{Dom}(\delta))\). According to Definition 3-2, there is then an \(i \in \text{Dom(\text{AVS}(\delta||\text{Dom}(\delta))})\) such that there is no \(l \in \text{Dom(\text{AVS}(\delta||\text{Dom}(\delta))})\) with \(i < l \leq \text{Dom}(\delta)\), and \(\delta = \delta||\text{Dom}(\delta)\) \(\cup \{\text{Dom}(\delta), \text{Therefore P(\delta) \rightarrow C(\delta||\text{Dom}(\delta))}\}\). Then it holds with \(a, b, c\) and \(d\): \(i \in \text{Dom(\text{AVS}(\delta^+))}\) and there is no \(l\) such that \(i < l \leq \text{Dom}(\delta)\) and \(l \in \text{Dom(\text{AVS}(\delta^+))}\), and \(P(\delta^+)= [\beta^*, \beta, P(\delta)]\) and \(C(\delta^+) = [\beta^*, \beta, C(\text{Dom}(\delta))\}]) and \(\delta^+ = \delta^+ \cup \{\text{Dom}(\delta), \text{Therefore P(\delta) \rightarrow C(\delta||\text{Dom}(\delta))}\}\} = \delta^+ \cup \{\text{Dom}(\delta), \text{Therefore P(\delta) \rightarrow C(\delta^+)}\}\}. Thus we have \(\delta^+ \in \text{CdIF}(\delta^+)\) and thus \(\delta^+ \in \text{RCS}\).

With Theorem 3-19-(iii), we then have \(\text{AVS}(\delta) = \text{AVS}(\text{Dom}(\delta)) \cup \{j, \delta_j\} | i < j < \text{Dom}(\delta)\} \cup \{\text{Dom}(\delta), \text{Therefore P(\delta) \rightarrow C(\delta||\text{Dom}(\delta))}\}\) and that \(\text{AVS}(\delta^+) = \text{AVS}(\delta^+) \cup \{j, \delta_j\} | i < j < \text{Dom}(\delta)\} \cup \{\text{Dom}(\delta), \text{Therefore [\beta^*, \beta, P(\delta)] \rightarrow [\beta^*, \beta, C(\text{Dom}(\delta))\}])\}. With \(\text{Dom(\text{AVS}(\delta^+))} = \text{Dom(\text{AVS}(\delta||\text{Dom}(\delta)))}\), it then follows that also \(\text{Dom(\text{AVS}(\delta^+))} = \text{Dom(\text{AVS}(\delta))}\). In the case that \(\delta \in \text{NIF}(\delta||\text{Dom}(\delta))\), one shows analogously that then also \(\delta^+ \in \text{NIF}(\delta^+) \subset \text{RCS and Dom(\text{AVS}(\delta^+))} = \text{Dom(\text{AVS}(\delta))}\).

(PEF): Now, suppose \(\delta \in \text{PEF}(\delta||\text{Dom}(\delta))\). According to Definition 3-15, there are then \(\beta^+ \in \text{PAR}, \xi \in \text{VAR}, \Delta \in \text{FORM}, \text{where FV}(\Delta) \subset \{\xi\}\), and \(i \in
Dom(AVS(\(\xi\)!Dom(\(\xi\))-1)) such that P(\(\xi\)) = \(\forall \zeta \Delta \), P(\(\xi_{i+1}\)) = [\(\beta^*, \zeta, \Delta\)], where \(i+1 \in \text{Dom(AVS(\(\xi\)!Dom(\(\xi\))-1))}\), \(\beta^* \not\in \text{STSF({\(\Delta, P(\(\xi\)!Dom(\(\xi\))-2))})}\), there is no \(j \leq i\) such that \(\beta^* \in \text{ST}(\(\xi_j\))\), there is no \(l\) such that \(i+1 < l \leq \text{Dom(\(\xi\))-2}\) and \(l \in \text{Dom(AVS(\(\xi\)!Dom(\(\xi\))-1))}\), and \(\delta = \delta|\text{Dom(\(\xi\))-1} \cup \{(\text{Dom(\(\xi\))-1, } \text{"Therefore } P(\(\xi\)!Dom(\(\xi\))-2))\}}\).

Then it follows, with a), b) and d), that \(i \in \text{Dom(AVS(\(\xi\)*))}\) and \(P(\(\xi\)*) = [\(\beta^*, \beta, P(\(\xi\))\]) = [\(\beta^*, \beta, [\(\beta^*, \zeta, \Delta]\), \(i+1 \in \text{Dom(AVS(\(\xi\)*))}\) and \(P(\(\xi\)_{i+1}) = [\(\beta^*, \beta, P(\(\xi\)_{i+1})\]) = [\(\beta^*, \beta, [\(\beta^*, \zeta, \Delta]\), \(C(\(\xi\)*) = P(\(\xi\)_{Dom(\(\xi\))-2}) = [\(\beta^*, \beta, P(\(\xi\)_{Dom(\(\xi\))-2})\) and \(\delta^* = \delta|\text{Dom(\(\xi\))-1} \cup \{(\text{Dom(\(\xi\))-1, } \text{"Therefore } C(\(\xi\)!Dom(\(\xi\))-1))\}) = \delta| \cup \{(\text{Dom(\(\xi\))-1, } \text{"Therefore } C(\(\xi\)*))\} \)} and there is no \(l\) such that \(i+1 < l \leq \text{Dom(\(\xi\))-2}\) and \(l \in \text{Dom(AVS(\(\xi\)*))}\). With \(\beta^* = \beta\) and \(\beta^* \neq \beta\), we can distinguish two cases.

**First case:** Suppose \(\beta^* = \beta\). Then we have \(P(\(\xi\)_{i+1}) = [\(\beta^*, \beta, [\(\beta^*, \zeta, \Delta]\), \(i+1 \in \text{Dom(AVS(\(\xi\)*))}\) and, with \(\beta^* \not\in \text{ST}(\Delta)\), also \(\beta \not\in \text{ST}(\Delta)\) and hence, with Theorem 1-24-(ii), \(P(\(\xi\)_{i+1}) = [\(\beta^*, \beta, [\(\beta^*, \zeta, \Delta]\), \(\Delta = \Delta\). With \(\beta \not\in \text{ST}(\Delta)\), we then have \([\(\beta^*, \beta, \Delta]\) = \(\Delta\) and thus \(P(\(\xi\)*) = \(\forall \zeta[\(\beta^*, \beta, \Delta]\) = \(\forall \zeta[\(\beta^*, \beta, \Delta]\). With \(\beta = \beta^*\) and \(\beta^* \not\in \text{STSEQ}(\(\xi\))\), we also have \(\beta^* \not\in \text{STSF}({\(\Delta, P(\(\xi\)!Dom(\(\xi\))-2))})\) and thus also \(\beta^* \not\in \text{STSF}({\(\Delta, P(\(\xi\)!Dom(\(\xi\))-2))})\). Now, suppose for contradiction that there is a \(j \leq i\) such that \(\beta^* \in \text{ST}(\(\xi_j\)). With b), we would then have \(\beta^* \in \text{ST}(\(\xi_j\)) = [\(\beta^*, \beta, \delta\)]. With \(\beta^* \not\in \text{STSEQ}(\(\delta\)), it also holds that \(\beta^* \not\in \text{ST}(\(\delta\)). But then we have, with \(\beta^* \in \text{ST}(\(\xi_j\)), that \(\beta \in \text{ST}(\(\delta\)), while, on the other hand, we have, by hypothesis, that \(\beta = \beta^* \not\in \text{ST}(\(\delta\)). Contradiction! Therefore we have that there is no \(j \leq i\) such that \(\beta^* \in \text{ST}(\(\xi_j\)). Hence, altogether, we have \(\delta^* \in \text{PEF}(\(\xi\)*).\)

**Second case:** Now, suppose \(\beta^* \neq \beta\). With \(\beta^* \neq \beta^*\) and \(\beta^* = \beta^*\), we can then distinguish two subcases. **First subcase:** Suppose \(\beta^* \neq \beta^*\). With Theorem 1-25-(ii) and \(\beta^* \neq \beta^*\), we then have \(P(\(\xi\)*_{i+1}) = [\(\beta^*, \beta, [\(\beta^*, \zeta, \Delta]\), \(\Delta = \Delta\). We also have \(P(\(\xi\)*) = \(\forall \zeta[\(\beta^*, \beta, \Delta]\) = \(\forall \zeta[\(\beta^*, \beta, \Delta]\). If \(\beta^* \in \text{STSF}({\(\beta^*, \beta, \Delta\)}, \(\Delta = \Delta\). Then it would hold, because of \(\beta^* \neq \beta\) and with b), that \(\beta^* \not\in \text{STSEQ}({\(\Delta, P(\(\xi\)!Dom(\(\xi\))-2))})\) or that there is a \(j \leq i\) such that \(\beta^* \not\in \text{ST}(\(\delta\)), which contradicts the assumption about \(\beta^*\). Therefore we have \(\beta^* \not\in \text{STSF}({\(\beta^*, \beta, \Delta\)}, \(\beta \not\in \text{ST}(\(\xi_j\))\) and there is no \(j \leq i\) such that \(\beta^* \in \text{ST}(\(\delta\)); hence we have again \(\delta^* \in \text{PEF}(\(\xi\)*).\)

**Second subcase:** Now, suppose \(\beta^* = \beta^*\). Then we have \(\zeta \not\in \text{FV}(\(\Delta\)), because, if not, we would have \(\beta^* \in \text{ST}([\(\beta^*, \zeta, \Delta\)]) \subseteq \text{STSEQ}(\(\delta\)). We then have \([\(\beta^*, \zeta, \Delta]\) = \(\Delta\) and thus \(P(\(\xi\)*_{i+1}) = [\(\beta^*, \beta, [\(\beta^*, \zeta, \Delta]\), \(\Delta = \Delta\) and we have \(P(\(\xi\)*) = \(\forall \zeta[\(\beta^*, \beta, \Delta]\). Now, let \(\beta^* \in \text{PAR}\!\text{STSEQ}(\(\xi\)). With \(\zeta \not\in \text{FV}(\(\Delta\)), we also have \(\zeta \not\in \text{FV}(\(\beta^*, \beta, \Delta\)) and thus \(P(\(\xi\)*_{i+1}) = [\(\beta^*, \beta, \Delta\) and it holds that \(\beta^* \not\in \text{STSF}({\(\beta^*, \beta, \Delta\}}, \(\beta^*, \beta, \Delta\), \([\(\beta^*, \beta, \Delta\]}\).
P(\Omega_{Dom(\overline{\delta})}) \} and there is no \( j \leq i \) such that \( \beta^j \in ST(\overline{\delta}^*) \). Thus we have again \( \delta^+ \in PEF(\overline{\delta}^*) \). Thus we have in both subcases and hence in both cases that \( \delta^+ \in PEF(\overline{\delta}^*) \) and thus \( \delta^+ \in RCS \).

It then follows, with Theorem 3-21-(iii), that AVS(\overline{\delta}) = AVS(\overline{\delta}|Dom(\overline{\delta})-1) \cup \{(j, \overline{\delta}) | i+1 \leq j < Dom(\overline{\delta})-1 \} \cup \{(Dom(\overline{\delta})-1, \text{"Therefore P(\overline{\delta}|Dom(\overline{\delta})-2)"})\} and that AVS(\overline{\delta}^+) = AVS(\overline{\delta}^*) \cup \{(j, \overline{\delta}) | i+1 \leq j < Dom(\overline{\delta})-1 \} \cup \{(Dom(\overline{\delta})-1, \text{"Therefore [\beta^*, \beta, P(\overline{\delta}|Dom(\overline{\delta})-2)"})\}. With Dom(AVS(\overline{\delta}^*)) = Dom(AVS(\overline{\delta}|Dom(\overline{\delta})-1)), it then follows that Dom(AVS(\overline{\delta}^*)) = Dom(AVS(\overline{\delta})).

**Exclusion assumption:** For the remaining steps, suppose \( \overline{\delta} \notin CdIF(\overline{\delta}|Dom(\overline{\delta})-1) \cup NIF(\overline{\delta}|Dom(\overline{\delta})-1) \cup PEF(\overline{\delta}|Dom(\overline{\delta})-1) \). With c), we then have \( \overline{\delta}^+ \notin CdIF(\overline{\delta}^*) \cup NIF(\overline{\delta}^*) \cup PEF(\overline{\delta}^*) \). With Theorem 3-25, we then have for all of the following cases that AVS(\overline{\delta}) = AVS(\overline{\delta}|Dom(\overline{\delta})-1) \cup \{(Dom(\overline{\delta})-1, C(\overline{\delta}))\} and that AVS(\overline{\delta}^+) = AVS(\overline{\delta}^*) \cup \{(Dom(\overline{\delta})-1, C(\overline{\delta}^*))\}. With Dom(AVS(\overline{\delta}^*)) = Dom(AVS(\overline{\delta}|Dom(\overline{\delta})-1)), it then follows that Dom(AVS(\overline{\delta}^*)) = Dom(AVS(\overline{\delta})) for all remaining cases.

(AF): Suppose \( \overline{\delta} \in AF(\overline{\delta}|Dom(\overline{\delta})-1) \). With Definition 3-1, we then have \( \overline{\delta} = \overline{\delta}|Dom(\overline{\delta})-1 \cup \{(Dom(\overline{\delta})-1, \text{"Suppose P(\overline{\delta}|Dom(\overline{\delta})-1)"})\} \). With d), we then have \( \overline{\delta}^+ = \overline{\delta}^* \cup \{(Dom(\overline{\delta})-1, \text{"Suppose [\beta^*, \beta, P(\overline{\delta}|Dom(\overline{\delta})-1)"})\} \in AF(\overline{\delta}^*) \) and thus \( \delta^+ \in RCS \).

(CdEF, CIF, CEF, BIF, BEF, DIF, DEF, NEF): Now, suppose \( \overline{\delta} \in CdEF(\overline{\delta}|Dom(\overline{\delta})-1) \). With Definition 3-3, there are then \( A, B \in CFORM \) such that \( A \rightarrow B \in AVP(\overline{\delta}|Dom(\overline{\delta})-1) \) and \( \overline{\delta} = \overline{\delta}|Dom(\overline{\delta})-1 \cup \{(Dom(\overline{\delta})-1, \text{"Therefore B"})\} \). With d), it then follows that \( \overline{\delta}^+ = \overline{\delta}^* \cup \{(Dom(\overline{\delta})-1, \text{"Therefore [\beta^*, \beta, B"})\} \). Since A, \( A \rightarrow B \in AVP(\overline{\delta}|Dom(\overline{\delta})-1) \), we then have, with Definition 2-30, that there are \( i, j \in Dom(AVS(\overline{\delta}|Dom(\overline{\delta})-1)) \) such that \( P(\overline{\delta}_i) = A \) and \( P(\overline{\delta}_j) = A \rightarrow B \). With a) and b), it then follows that \( i, j \in Dom(AVS(\overline{\delta}^*)) \) and \( P(\overline{\delta}^*) = [\beta^*, \beta, A] \) and \( P(\overline{\delta}^*) = [\beta^*, \beta, A] \rightarrow [\beta^*, \beta, B] \). With d), we then have \( \overline{\delta}^+ = \overline{\delta}^* \cup \{(Dom(\overline{\delta})-1, \text{"Therefore [\beta^*, \beta, B"})\} \in CdEF(\overline{\delta}^*) \) and thus \( \delta^+ \in RCS \). For CIF, CEF, BIF, BEF, DIF, DEF and NEF the proof is carried out analogously.

(UIF): Now, suppose \( \overline{\delta} \in UIF(\overline{\delta}|Dom(\overline{\delta})-1) \). According to Definition 3-12, there are then \( \beta^* \in PAR, \zeta \in VAR, \Delta \in FORM, \) where \( \text{FV}(\Delta) \subseteq \{\zeta\} \), such that \( [\beta^+, \zeta, \Delta] \in AVP(\overline{\delta}|Dom(\overline{\delta})-1) \), \( \beta^+ \notin STSF(\{\Delta\} \cup \text{AVAP}(\overline{\delta}|Dom(\overline{\delta})-1)) \), and \( \overline{\delta} = \overline{\delta}|Dom(\overline{\delta})-1 \cup \{(Dom(\overline{\delta})-1, \text{"Therefore } \Delta \in \overline{\delta}\}) \). With d), we then have \( \overline{\delta}^+ = \overline{\delta}^* \cup \{(Dom(\overline{\delta})-1, [\beta^*, \beta, \text{"Therefore } \Delta \in \overline{\delta}\})\} \). With \( [\beta^+, \zeta, \Delta] \in AVP(\overline{\delta}|Dom(\overline{\delta})-1) \) and Definition 2-30, we then have that there is an \( i \in \)
Dom(Δ) such that [β*, ζ, Δ] = P(Δ). With a) and b), it then follows that.

First case: Suppose β* = β. Then we have P(Δ) = [β*, β, [β*, ζ, Δ] = [β*, β, [β*, ζ, Δ]] and, with β* \not\in ST(Δ), we also have β \not\in ST(Δ) and thus we have, with Theorem 1-24-(ii), that P(Δ) = [β*, β, [β*, ζ, Δ] = [β*, β, ζ, Δ]. With β \not\in ST(Δ), we then have [β*, β, ζ, Δ] = Δ and thus C(Δ) = 7\zeta[β*, β, ζ, Δ] = 7\zeta[β*, β, ζ, Δ. With β* = β and β* \not\in STSEQ(Δ), we also have β, β* \not\in STSF(Δ, ∪ AVAP(Δ)|Dom(Δ)-1)) and thus, with a) and b), also β* \not\in STSF(Δ, ∪ AVAP(Δ)). To see this, suppose for contradiction that β* \in STSF(Δ, ∪ AVAP(Δ)). Then we have β* \not\in ST(Δ), because, if not, we would have β* \in ST(Δ) \subset ST(7\zeta[β*, β, ζ, Δ] = ST(C(Δ)) \subset STSEQ(Δ), which contradicts β* \not\in STSEQ(Δ). Therefore there would be a B \in AVAP(Δ) such that β* \in ST(B). With Definition 2-31, there would then be a j \in Dom(Δ) such that β* \in ST(Δ). Then we have β* \not\in STSEQ(Δ), we also have β* \not\in ST(Δ). But then we have, with β* \in ST(Δ) and P(Δ) = [β*, β, P(Δ)], that β \in ST(Δ). Moreover, with a) and b), it follows from j \in Dom(Δ) such that j \in Dom(Δ) and hence that P(Δ) \in AVAP(Δ)|Dom(Δ)-1). But then we would have β* \in STSF(Δ, ∪ AVAP(Δ)|Dom(Δ)-1)) whereas, by hypothesis, we have β = β* \not\in STSEQ(Δ, ∪ AVAP(Δ)|Dom(Δ)-1). Contradiction! Therefore we have β* \not\in STSF(Δ, ∪ AVAP(Δ)). Since we have P(Δ) = [β*, ζ, Δ, i \in Dom(Δ) and C(Δ) = 7\zeta[β*, ζ, Δ], we thus have Δ* \in UIF(Δ).

Second case: Now, suppose β* \not\in β*. With β* \not\in β* and β* = β*, we can then distinguish two subcases. First subcase: Suppose β* \not\in β*. With Theorem 1-25-(ii) and β* \not\in β, we then have P(Δ) = [β*, β, [β*, ζ, Δ] = [β*, β, ζ, Δ]. Also, we have C(Δ) = 7\zeta[β*, β, ζ, Δ]. Now, suppose for contradiction that β* \in STSF(Δ, ∪ AVAP(Δ)). Since β* \not\in ST(Δ), we have β* \not\in ST([β*, β, ζ, Δ]). Therefore we would have β* \in STSF(Δ, ∪ AVAP(Δ)) and thus there would be, with Definition 2-31, a j \in Dom(Δ) such that β* \in ST(Δ). Since, with b), P(Δ) = [β*, β, P(Δ)] and since β* \not\in β*, we would thus have that β* \in ST(Δ). With a) and b), it follows from j \in Dom(Δ) that j \in Dom(Δ) and thus we would have P(Δ) \in AVAP(Δ)|Dom(Δ)-1) and thus β* \in STSEQ(Δ)|Dom(Δ)-1) whereas, by hypothesis, we have β* \not\in STSEQ(Δ)|Dom(Δ)-1). Contradiction! Therefore we have β* \not\in STSEQ(Δ) and hence again Δ* \in UIF(Δ).
Second subcase: Now, suppose $\beta^+ = \beta^*$. Then we have $\zeta \notin \text{FV}(\Delta)$, because, if not, we would have $\beta^* \in \text{ST}([\beta^+, \zeta, \Delta]) \subseteq \text{STSEQ}(\delta)$. Thus we then have $[\beta^+, \zeta, \Delta] = \Delta$ and thus $P(\delta^*) = [\beta^*, \beta, [\beta^+, \zeta, \Delta]] = [\beta^*, \beta, \Delta]$, and we have $C(\delta^*) = \lnot \lnot \zeta \Delta \beta$. Now, let $\beta^\delta \in \text{PAR}\setminus \text{STSEQ}(\delta^*)$. With $\zeta \notin \text{FV}(\Delta)$, we also have $\zeta \notin \text{FV}([\beta^*, \beta, \Delta])$, and thus $P(\delta^*) = [\beta^*, \beta, \Delta] = [\beta^\delta, \zeta, [\beta^*, \beta, \Delta]]$, and it holds that $\beta^\delta \notin \text{STSF}([\beta^*, \beta, \Delta] \cup \text{AVAP}(\delta^*))$ and thus again $\delta^+ \in \text{UIF}(\delta^*)$. Thus we have in both subcases and hence in both cases that $\delta^+ \in \text{UIF}(\delta^*) \subseteq \text{RCS}$.

(UEF): Now, suppose $\delta \in \text{UEF}(\delta)\setminus \text{Dom}(\delta)-1$. According to Definition 3-13, there are then $\theta \in \text{CTERM}$, $\zeta \in \text{VAR}$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\zeta\}$, such that $\lnot \lnot \zeta \Delta \beta \in \text{AVP}(\delta)\setminus \text{Dom}(\delta)-1$, and $\delta = \delta\setminus \text{Dom}(\delta)-1 \cup \{(\text{Dom}(\delta)-1, \theta\therefore \{\theta, \zeta, \Delta\}\})$. With d), we then have $\delta^+ = \delta^* \cup \{(\text{Dom}(\delta)-1, \beta^*, \theta, \theta\therefore \{\theta, \zeta, \Delta\}\}) = \delta^* \cup \{(\text{Dom}(\delta)-1, \theta\therefore \{\theta, \zeta, \Delta\}\})$. With $\lnot \lnot \zeta \Delta \beta \in \text{AVP}(\delta)\setminus \text{Dom}(\delta)-1$ and Definition 2-30, there is then an $i \in \text{Dom}(\text{AVS}(\delta)\setminus \text{Dom}(\delta)-1)$ such that $P(\delta_i) = \lnot \lnot \zeta \Delta \beta$. With a) and b), we then have $i \in \text{Dom}(\text{AVS}(\delta^*))$ and $P(\delta^*) = [\beta^*, \beta, \lnot \lnot \zeta \Delta \beta] = \lnot \lnot \zeta \Delta \beta$. With Theorem 1-26-(ii), we have $C(\delta^*) = [\beta^*, \beta, \zeta, \Delta] = [[\beta^*, \beta, \theta, \zeta, \Delta], \beta^*, \beta, \Delta]$, where, with $\theta \in \text{CTERM}$, also $[\beta^*, \beta, \theta] \in \text{CTERM}$, and, with $\text{FV}(\Delta) \subseteq \{\zeta\}$, also $\text{FV}([\beta^*, \beta, \Delta]) \subseteq \{\zeta\}$. Hence we have $\delta^+ \in \text{UIF}(\delta^*) \subseteq \text{RCS}$.

(PIF): Now, suppose $\delta \in \text{PIF}(\delta)\setminus \text{Dom}(\delta)-1$. According to Definition 3-14, there are then $\theta \in \text{CTERM}$, $\zeta \in \text{VAR}$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\zeta\}$, such that $[\theta, \zeta, \Delta] \in \text{AVP}(\delta)\setminus \text{Dom}(\delta)-1$, and $\delta = \delta\setminus \text{Dom}(\delta)-1 \cup \{(\text{Dom}(\delta)-1, \zeta\therefore \{\theta, \zeta, \Delta\}\})$. With d), we then have $\delta^+ = \delta^* \cup \{(\text{Dom}(\delta)-1, \beta^*, \theta, \theta\therefore \{\theta, \zeta, \Delta\}\}) = \delta^* \cup \{(\text{Dom}(\delta)-1, \zeta\therefore \{\theta, \zeta, \Delta\}\})$. With $[\theta, \zeta, \Delta] \in \text{AVP}(\delta)\setminus \text{Dom}(\delta)-1$ and Definition 2-30, there is an $i \in \text{Dom}(\text{AVS}(\delta)\setminus \text{Dom}(\delta)-1)$ such that $P(\delta_i) = [\theta, \zeta, \Delta]$. With a) and b), we then have $i \in \text{Dom}(\text{AVS}(\delta^*))$ and $P(\delta^*) = [\beta^*, \beta, \text{FV}(\delta_i)] = [\beta^*, \beta, \theta, \zeta, \Delta]$, where, with $\theta \in \text{CTERM}$, also $[\beta^*, \beta, \theta] \in \text{CTERM}$, and, with $\text{FV}(\Delta) \subseteq \{\zeta\}$, also $\text{FV}([\beta^*, \beta, \Delta]) \subseteq \{\zeta\}$. Hence we have $\delta^+ \in \text{PIF}(\delta^*) \subseteq \text{RCS}$.

(III): Now, suppose $\delta \in \text{IIIF}(\delta)\setminus \text{Dom}(\delta)-1$. With Definition 3-16, there is then $\theta \in \text{CTERM}$ such that $\delta = \delta\setminus \text{Dom}(\delta)-1 \cup \{(\text{Dom}(\delta)-1, \theta\therefore \{\theta\}\})$. With d), we then have $\delta^+ = \delta^* \cup \{(\text{Dom}(\delta)-1, \beta^*, \beta, \theta\therefore \{\theta\}\}) = \delta^* \cup \{(\text{Dom}(\delta)-1, \theta\therefore \{\theta\}\})$, where, with $\theta \in \text{CTERM}$, also $[\beta^*, \beta, \theta] \in \text{CTERM}$. Hence we have $\delta^+ \in \text{IIIF}(\delta^*) \subseteq \text{RCS}$. 

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\((\text{IEF})\): Now, suppose \(\delta^+ \in \text{IEF}(\delta)|\text{Dom}(\delta)-1\). With Definition 3-17, there are then \(\theta_0, \theta_1 \in \text{CTERM}, \zeta \in \text{VAR} \) and \(\Delta \in \text{FORM}\), where \(\text{FV}(\Delta) \subseteq \{\zeta\}\), such that \(\theta_0 = \theta_1\), \([\theta_0, \zeta, \Delta] \in \text{AVP}(\delta)|\text{Dom}(\delta)-1\), and \(\delta = \delta^+|\text{Dom}(\delta)-1 \cup \{\text{Dom}(\delta)-1, \text{Therefore } [\theta_1, \zeta, \Delta]\}\).

With \(\delta\), we then have \(\delta^+ = \delta^+ \cup \{\text{Dom}(\delta)-1, \{[\beta^*, \beta, \text{Therefore } [\theta_1, \zeta, \Delta]\}\}\} = \delta^+ \cup \{\text{Dom}(\delta)-1, \text{Therefore } [\beta^*, \beta, [\theta_0, \zeta, \Delta]\}\}\). With \(\theta_0 = \theta_1\), \([\theta_0, \zeta, \Delta] \in \text{AVP}(\delta)|\text{Dom}(\delta)-1\) and Definition 2-30, there are then \(i, j \in \text{Dom}(\text{AVS}(\delta)|\text{Dom}(\delta)-1)\) such that \(P(\delta_i) = [\theta_0 = \theta_1]\) and \(P(\delta_j) = [\theta_0, \zeta, \Delta] \in \text{CTERM}, \text{also } \{\theta_0, \zeta, \Delta\} \in \text{CTERM}\), and, with \(\text{FV}(\Delta) \subseteq \{\zeta\}\), also \(\text{FV}([\beta^*, \beta, \Delta]) \subseteq \{\zeta\}\). Hence it follows that \(\delta^+ \in \text{IEF}(\delta^*) \subseteq \text{RCS}\). ■

The following theorem prepares the generalisation theorem (Theorem 4-24). The proof resembles the proof of Theorem 4-8.

**Theorem 4-9. Substitution of a new parameter for an individual constant is RCS-preserving**

If \(\delta \in \text{RCS}\), \(\alpha \in \text{CONST}\) and \(\beta \in \text{PAR}\setminus \text{STSEQ}(\delta)\), then there is an \(\delta^+ \in \text{RCS}\setminus \{\emptyset\}\) such that

(i) \(\alpha \notin \text{STSEQ}(\delta)\),

(ii) \(\text{STSEQ}(\delta^+) \subseteq \text{STSEQ}(\delta) \cup \{\beta\}\),

(iii) \(\text{AVAP}(\delta) = \{[\alpha, \beta, B] \mid B \in \text{AVP}(\delta^+)\}\), and

(iv) If \(\delta \neq \emptyset\), then \(\text{C}(\delta) = [\alpha, \beta, \text{C}(\delta^+)\}\).

**Proof:** Suppose \(\delta \in \text{RCS}\), \(\alpha \in \text{CONST}\) and \(\beta \in \text{PAR}\setminus \text{STSEQ}(\delta)\). Let \(\delta^+\) be defined as follows:

a) \(\delta^+ = \{(0, \text{Therefore } \beta = \beta)\} \setminus \beta, \alpha, \delta\).

Then clauses (i) and (ii) already hold and we also have \(\delta^+ \neq \emptyset\). For \(\delta^+\), we will will now show by induction on \(\text{Dom}(\delta)\) that \(\delta^+ \in \text{RCS}\) and

b) \(\text{Dom}(\text{AVS}(\delta^+)) = \{l + 1 \mid l \in \text{Dom}(\text{AVS}(\delta))\} \cup \{\emptyset\}\).

Clauses (iii) and (iv) then follow with a) and b). Ad (iii): Suppose \(\Delta \in \text{AVAP}(\delta)\). Then there is an \(i \in \text{Dom}(\text{AVS}(\delta))\) such that \(\delta_i = \text{Suppose } \Delta\). Therefore, with b), \(i + 1 \in \text{Dom}(\text{AVS}(\delta^+))\) and, with a), \(\delta^+_{i+1} = \text{Suppose } [\beta, \alpha, \Delta]\). Therefore we have \([\beta, \alpha, \Delta] \in \text{AVP}(\delta)\setminus \text{STSEQ}(\delta)\)
AVAP(Δ) and thus \([α, β, [β, α, Δ]] \in \{[α, β, B] | B \in AVAP(Δ)\}\). We have β \∉ STSEQ(Δ) and thus β \∉ ST(Δ) and thus, with Theorem 1-24-(ii), \([α, β, [β, α, Δ]] = [α, α, Δ] = Δ\). Therefore Δ \in \{[α, β, B] | B \in AVAP(Δ)\}. Now, suppose Δ \in \{[α, β, B] | B \in AVAP(Δ)\}. Then there is a \(Δ∗ \in AVAP(Δ)\) such that \(Δ = [α, β, Δ∗]\). Because of \(Δ∗ \in AVAP(Δ)\), there is then, with a), an \(i+1 \in \text{Dom}(AVS(Δ))\) with \(Δ^{i+1} = \text{Suppose } Δ^∗\). With b), we then have \(i \in \text{Dom}(AVS(Δ))\) and, with a), \(Δ^{i+1} = [β, α, Δ]\). Thus we have \([β, α, Δ]\) = \(\text{Suppose } Δ^∗\), and thus \([α, β, [β, α, Δ]] = [α, β, \text{Suppose } Δ^∗]\) = \(\text{Suppose } [α, β, Δ^∗]\) = \(\text{Suppose } Δ\). With Theorem 1-24-(iii) and β \∉ STSEQ(Δ), we then have \([α, β, [β, α, Δ]] = [α, Δ, Δ]\) and thus \(Δ = \text{Suppose } Δ\) and \(P(Δ) = Δ\). Thus we have \(Δ \in AVAP(Δ)\). Hence we have (iii).

Ad (iv): Suppose \(Δ \neq 0\). Because of β \∉ STSEQ(Δ) and a) and Theorem 1-24-(ii), we have \([α, β, C(Δ)] = [α, β, P(Δ)]\) = \([α, β, [β, α, P(Δ)]\) = \([α, α, P(Δ)]\) = \([α, α, P(Δ)]\). We have \(\text{Dom}(Δ) = \text{Dom}(Δ)\). Hence we have \([α, β, C(Δ)] = P(Δ)\). Then we have \(Δ = \text{RCS}\) and \(P(Δ) = Δ\). Now, suppose 0 < \(\text{Dom}(Δ)\). Then we have \(Δ \in \text{RCS} \setminus \{0\}\). With Theorem 3-6, we then have \(Δ \in \text{RCE}(Δ)\). According to the I.H., we then have

c) \(Δ^∗ = \{(l+1 | l \in \text{Dom}(AVS(Δ^∗)) = \{1+1 | l \in \text{Dom}(AVS(Δ^∗)) = \{0\}\} \cup \{\} .\)

With \(Δ \in \text{RCE}(Δ)\) and Definition 3-18, we have that \(|Δ \in \text{AF}(Δ)\) or \(|Δ \in \text{CIF}(Δ)\) or \(|Δ \in \text{CEF}(Δ)\) or \(|Δ \in \text{BIF}(Δ)\) or \(|Δ \in \text{BIF}(Δ)\) or \(|Δ \in \text{NIF}(Δ)\) or \(|Δ \in \text{NEF}(Δ)\) or \(|Δ \in \text{UIF}(Δ)\) or \(|Δ \in \text{PIF}(Δ)\) or \(|Δ \in \text{PEF}(Δ)\) or \(|Δ \in \text{IEF}(Δ)\) or \(|Δ \in \text{IEF}(Δ)\) or \(|Δ \in \text{IEF}(Δ)\).

Since operators are not affected by substitution, we have

d) For all \(l \in \text{Dom}(Δ)\): \(P(Δ^∗_{l+1}) = [β, α, P(Δ)]\) and \(Δ^∗_{l+1} = \text{Suppose } Δ\) for a \(Δ \in \text{PERF}\).
With $\beta \in \text{PAR} \cup \text{STSEQ}(\tilde{\delta})$ and $\alpha \in \text{CONST}$, we also have

e) For all $i \in \text{Dom}(\tilde{\delta})$: $\beta \not\in \text{ST}(P(\tilde{\delta}_i))$ and $\alpha \not\in \text{ST}([\beta, \alpha, P(\tilde{\delta}_i)])$,

because, if not, we would have $\beta \in \text{STSEQ}(\tilde{\delta})$ or $\alpha = \beta$, which contradicts the hypothesis and Postulate 1-1 respectively. With a), it holds that

$$\tilde{\delta}^+ = \tilde{\delta}^* \cup \{(\text{Dom}(\tilde{\delta}^*), \tilde{\delta}^+_{\text{Dom}(\tilde{\delta}^*)})\} = \tilde{\delta}^* \cup \{(\text{Dom}(\tilde{\delta}), [\beta, \alpha, \tilde{\delta}_{\text{Dom}(\tilde{\delta})-1}])\}.$$

Now, we will show that in each of the cases $\text{AF ... IEF}$ it holds that $\tilde{\delta}^+ \in \text{RCS}$ and that b), with which $\tilde{\delta}^+$ is then in each case the desired RCS-element. In order to ease the treatment of $\text{CdEF}, \text{CIF}, \text{CEF}, \text{BIF}, \text{BEF}, \text{DIF}, \text{DEF}, \text{NEF}, \text{UFE}, \text{UEF}, \text{PIF}, \text{IIF}$ and IEF, we will now first show that

$$\text{g) If } \tilde{\delta}^+ \in \text{CdIF}(\tilde{\delta}^*) \cup \text{NIF}(\tilde{\delta}^*) \cup \text{PEF}(\tilde{\delta}^*), \text{ then } \tilde{\delta} \in \text{CdIF}(\tilde{\delta} \cap \text{Dom}(\tilde{\delta})-1) \cup \text{NIF}(\tilde{\delta} \cap \text{Dom}(\tilde{\delta})-1) \cup \text{PEF}(\tilde{\delta} \cap \text{Dom}(\tilde{\delta})-1).$$

4.1 Preparations

Preparatory part: Suppose $\tilde{\delta}^+ \in \text{CdIF}(\tilde{\delta}^*)$. According to Definition 3-2 and with c) and f), there is then an $i \in \text{Dom}(\text{AVAS}(\tilde{\delta}^*))$ such that there is no $l$ such that $i < l \leq \text{Dom}(\tilde{\delta})$ and $l \in \text{Dom}(\text{AVAS}(\tilde{\delta}^*))$, and $\tilde{\delta}^+ = \tilde{\delta}^* \cup \{(\text{Dom}(\tilde{\delta}^*), \tilde{\delta}^+_{\text{Dom}(\tilde{\delta}^*)})\}$. Therefore we have $\tilde{\delta}^+ = \tilde{\delta}^* \cup \{(\text{Dom}(\tilde{\delta}^*), \tilde{\delta}^+_{\text{Dom}(\tilde{\delta}^*)})\}$. With d), we have $\text{P}(\tilde{\delta}^*) = [\beta, \alpha, P(\tilde{\delta}_{l-1})]$ and $C(\tilde{\delta}^*) = [\beta, \alpha, P(\tilde{\delta}_{\text{Dom}(\tilde{\delta})-1})]$. Therefore we have $\tilde{\delta}^+ = \tilde{\delta}^* \cup \{(\text{Dom}(\tilde{\delta}), \text{Therefore } P(\tilde{\delta}_{l-1}) \rightarrow [\beta, \alpha, P(\tilde{\delta}_{\text{Dom}(\tilde{\delta})-1})]\}$. With f), it holds that $\text{Therefore } [\beta, \alpha, P(\tilde{\delta}_{l-1})] \rightarrow [\beta, \alpha, P(\tilde{\delta}_{\text{Dom}(\tilde{\delta})-1})] \rightarrow [\beta, \alpha, P(\tilde{\delta}_{\text{Dom}(\tilde{\delta})-2})] \rightarrow \text{Therefore } P(\tilde{\delta}_{\text{Dom}(\tilde{\delta})-2})].$ Theorem 1-21 then yields $\text{Therefore } P(\tilde{\delta}_{l-1}) \rightarrow P(\tilde{\delta}_{\text{Dom}(\tilde{\delta})-2})] = \tilde{\delta}_{\text{Dom}(\tilde{\delta})-1}$ and thus we have $\tilde{\delta} = \tilde{\delta}^* \cap \text{Dom}(\tilde{\delta})-1 \cup \{(\text{Dom}(\tilde{\delta}-1), \text{Therefore } P(\tilde{\delta}_{l-1}) \rightarrow P(\tilde{\delta}_{\text{Dom}(\tilde{\delta})-2})]\}. With c), d) and $i \not= 0$, we also have $i-1 \in \text{Dom}(\text{AVAS}(\tilde{\delta} \cap \text{Dom}(\tilde{\delta})-1)$ and there is no $l$ such that $i-1 < l \leq \text{Dom}(\tilde{\delta})$ and $l \in \text{Dom}(\text{AVAS}(\tilde{\delta} \cap \text{Dom}(\tilde{\delta})-1)). Hence we have $\tilde{\delta} \in \text{CdIF}(\tilde{\delta} \cap \text{Dom}(\tilde{\delta})-1).$ In the case that $\tilde{\delta}^+ \in \text{NIF}(\tilde{\delta}^*)$, one shows analogously that then also $\tilde{\delta} \in \text{NIF}(\tilde{\delta} \cap \text{Dom}(\tilde{\delta})-1).$

Now, suppose $\tilde{\delta}^+ \in \text{PEF}(\tilde{\delta}^*)$. According to Definition 3-15 and with c), d) and f), there are then $\beta^* \in \text{PAR}, \zeta \in \text{VAR}, \Delta \in \text{FORM}, \text{ where } FV(\Delta) \subseteq \{\zeta\},$ and $i \in \text{Dom}(\text{AVS}(\tilde{\delta}^*))$ such that $P(\tilde{\delta}^*) = \text{Therefore } P(\tilde{\delta}^*) \rightarrow [\beta^*, \zeta, \Delta] = [\beta, \alpha, P(\tilde{\delta})], \text{ where } i+1 \in \text{Dom}(\text{AVS}(\tilde{\delta}^*)), [\beta, \alpha, P(\tilde{\delta}_{\text{Dom}(\tilde{\delta})-2})] = C(\tilde{\delta}^*)$, $\beta^* \not\in \text{ST}(\tilde{\delta}^*_j)$, there is no $j \leq i$ such that $\beta^* \in \text{ST}(\tilde{\delta}^*_j)$, there is no $l$ such that $i+1 < l \leq \text{Dom}(\tilde{\delta})$ and $l \in \text{Dom}(\text{AVAS}(\tilde{\delta}^*)), \text{ and } \tilde{\delta}^+ = \tilde{\delta}^* \cup \{(\text{Dom}(\tilde{\delta}), \text{Therefore } C(\tilde{\delta}^*)\} = \tilde{\delta}^* \cup \{(\text{Dom}(\tilde{\delta}), \text{Therefore } [\beta, \alpha, P(\tilde{\delta}_{\text{Dom}(\tilde{\delta})-2})]\} = \tilde{\delta}^* \cup \{(\text{Dom}(\tilde{\delta}), [\beta, \alpha, P(\tilde{\delta}_{\text{Dom}(\tilde{\delta})-2})]\}.$
P(ΔDom(§i)\{2\})]. With i), we have [β, α, Therefore P(ΔDom(§i)\{2\}) = [β, α, ΔDom(§i)\{1\}]. Theorem 1-21 then yields [Therefore P(ΔDom(§i)\{2\}) = ΔDom(§i)\{1\} and thus ﹃ = 1\{Dom(§i)\{1\} ∪ ((Dom(§i)\{1\}, [Therefore P(ΔDom(§i)\{2\}))]. With P(§i*), = νζ* = [β = β = P(§i*0), it holds that i ≠ 0 and thus that P(§i*) = νζ* = [β, α, P(§i\{1\})].

With c), d) and i ≠ 0, we have i-1 ∈ Dom(AVS(§\{1\}Dom(§i)\{1\})), i ∈ Dom(AVAS(§\{1\}Dom(§i)\{1\})) and there is no l such that i < l ≤ Dom(§\{1\})-2 and l ∈ Dom(AVAS(§\{1\}Dom(§i)\{1\})). Now, we have to show that P(§\{i\}-1), P(§\{i\}) and P(ΔDom(§i)\{2\}) satisfy the requirements for ﹃ ∈ PEF(§\{1\}Dom(§i)\{1\}).

We have [β, α, P(§\{i\}-1)] = P(§\{i\}0) = νζ* and [β, α, P(§\{i\})] = P(§\{i\}0) = [β*, ζ, Δ]. Since operators are not affected by substitution, we thus have because of [β, α, P(§\{i\}-1)] = νζ* for a Δ* ∈ FORM, where β ∉ ST(Δ*) and FV(Δ*) ⊆ ζ. Thus we have [β, [β, α, νζ*0]] = [β, α, νζ*0] = [β*, ζ, β, α, Δ*] and hence Δ = [β, α, Δ*]. Thus we have [β, α, P(§\{i\})] = [β*, ζ, Δ] = [β*, ζ, β, α, Δ*] and β* ∉ ST([β, α, Δ*]). Also, we have β = β* or β ≠ β*. If β = β*, then there would be no j ≤ i such that β ∈ ST(§i*). However, we have β ∈ ST([Therefore β = β*]) = ST(§i*0) and 0 ≤ i. Therefore we have β ≠ β*. With β* ∈ ST([β, α, P(§\{i\})]) and β* ∉ ST([β, α, P(§\{i\})]), we can then distinguish two cases.

**First case:** Suppose β* ∈ ST([β, α, P(§\{i\})]). With Δ = [β, α, Δ*] and Theorem 1-25-(ii), we have [β, α, P(§\{i\})] = [β*, ζ, Δ*] = [β*, ζ, [β, α, Δ*]] = [β, α, [β*, ζ, Δ*]]. We have that β ∉ ST(P(§\{i\})) and, because of β ≠ β* and β ∉ ST(Δ*), also β ∉ ST([β*, ζ, Δ*]) and thus, with Theorem 1-20, P(§\{i\}) = [β*, ζ, Δ*]. Now, suppose for contradiction that β* ∈ STSF({Δ*, P(ΔDom(§i)\{2\})) or that there is a j ≤ i-1 such that β* ∈ ST(§\{i\}). Because of β* ≠ α and with d), we would then also have β* ∈ STSF({[β, α, Δ*], β, α, P(§Dom(§i)\{2\})}) or there would be a j ≤ i such that β* ∈ ST(§\{i\}). Contradiction! Thus the parameter conditions for β* are also satisfied in §\{i\}Dom(§i)\{1\} and hence we have §\{i\} ∈ PEF(§\{1\}Dom(§i)\{1\}).

**Second case:** Now, suppose β* ∉ ST([β, α, P(§\{i\})]). With [β, α, P(§\{i\})] = [β*, ζ, [β, α, Δ*]], we then have ζ ∉ FV([β, α, Δ*]). Then we have [β, α, P(§\{i\})] = [β*, ζ, [β, α, Δ*]] = [β, α, Δ*] and thus, with β ∉ ST(P(§\{i\})) ∪ ST(Δ*) and Theorem 1-20, P(§\{i\}) = Δ*, where, with ζ ∉ FV([β, α, Δ*]), also ζ ∉ FV(Δ*). Now, let β* ∈ PAR\STSEQ(§\{1\}Dom(§i)\{1\}). With ζ ∉ FV(Δ*), we then have P(§\{i\}) = Δ* = [β*, ζ, Δ*] and it holds that β* ∉ STSF({Δ*, P(ΔDom(§i)\{2\})) and that there is no j ≤ i such that β* ∈ ST(§\{i\}). Hence we have again §\{i\} ∈ PEF(§\{1\}Dom(§i)\{1\}). Therefore we have in both cases §\{i\} ∈ PEF(§\{1\}Dom(§i)\{1\}).
Main part: Now we will show that in each of the cases AF ... IEF it holds that $\mathcal{F}^+ \in RCS$ and $\text{Dom}(\text{AVS}(\mathcal{F}^+)) = \{l \in \mathbb{N} | l \in \text{Dom}(\text{AVS}(\mathcal{F}))\} \cup \{0\}$. First we will deal with CdIF, NIF and PEF. Then we can make an exclusion assumption that allows us to determine $\text{Dom}(\text{AVS}(\mathcal{F}^+))$ for all other cases just with c), g) and Theorem 3-25.

(CdIF, NIF): Suppose $\mathcal{F} \in \text{CdIF}(\mathcal{F}|\text{Dom}(\mathcal{F})-1)$. According to Definition 3-2, there is then an $i \in \text{Dom}(\text{AVS}(\mathcal{F}_i|\text{Dom}(\mathcal{F})-1))$ such that there is no $l \in \text{Dom}(\text{AVS}(\mathcal{F}_i|\text{Dom}(\mathcal{F})-1))$ such that $i < l \in \text{Dom}(\mathcal{F})-2$, and $\mathcal{F}_i = \mathcal{F}_i|\text{Dom}(\mathcal{F})-1 \cup \{(\text{Dom}(\mathcal{F})-1, \text{Therefore } P(\mathcal{F}_i) \rightarrow C(\mathcal{F}_i|\text{Dom}(\mathcal{F})-1))\}$. With a), d) and f), it then holds that $i+1 \in \text{Dom}(\text{AVS}(\mathcal{F}^+))$ and that there is no $l$ such that $i+1 < l \in \text{Dom}(\mathcal{F})-1 = \text{Dom}(\mathcal{F}^+)-1$ and $l \in \text{Dom}(\text{AVS}(\mathcal{F}^+))$, and $P(\mathcal{F}^+_{i+1}) = [\beta, \alpha, P(\mathcal{F}_i)]$ and $C(\mathcal{F}^+_{i+1}) = [\beta, \alpha, C(\mathcal{F}_i|\text{Dom}(\mathcal{F})-1)]$ and $\mathcal{F}^+_i = \mathcal{F}^+_i \cup \{(\text{Dom}(\mathcal{F}_i), [\beta, \alpha, \text{Therefore } P(\mathcal{F}_i) \rightarrow C(\mathcal{F}_i|\text{Dom}(\mathcal{F})-1))\}$). Hence we have $\mathcal{F}_i^+ \in \text{CdIF}(\mathcal{F}^+_i)$ and thus $\mathcal{F}_i^+ \in RCS$.

With Theorem 3-19-(iii), we then have $\text{AVS}(\mathcal{F}) = AVS(\mathcal{F}_i|\text{Dom}(\mathcal{F})-1) \cup \{(j, \mathcal{F}_j) \mid i < j < \text{Dom}(\mathcal{F})-1 \cup \{(\text{Dom}(\mathcal{F})-1, \text{Therefore } P(\mathcal{F}_i) \rightarrow C(\mathcal{F}_i|\text{Dom}(\mathcal{F})-1))\}$ and $\text{AVS}(\mathcal{F}^+_{i+1}) = AVS(\mathcal{F}^+_{i+1}) \cup \{(\text{Dom}(\mathcal{F}_i), \text{Therefore } P(\mathcal{F}_i) \rightarrow C(\mathcal{F}_i|\text{Dom}(\mathcal{F})-1))\}$. Hence we have $\mathcal{F}_i^+ \in NIF(\mathcal{F}|\text{Dom}(\mathcal{F})-1)$, one shows analogously that then also $\mathcal{F}_i^+ \in NIF(\mathcal{F}^+_i) \subseteq RCS$ and $\text{Dom}(\text{AVS}(\mathcal{F}^+_i)) = \{l+1 \mid l \in \text{Dom}(\text{AVS}(\mathcal{F}))\} \cup \{0\}$.

(PEF): Now, suppose $\mathcal{F} \in \text{PEF}(\mathcal{F}|\text{Dom}(\mathcal{F})-1)$. According to Definition 3-15, there are then $\beta^* \in \text{PAR}, \zeta \in \text{VAR}, \Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\zeta\}$, and $i \in \text{Dom}(\text{AVS}(\mathcal{F}^*_i|\text{Dom}(\mathcal{F})-1))$ such that $P(\mathcal{F}_i) = \forall \zeta \Delta, P(\mathcal{F}_i+1) = [\beta^*, \zeta, \Delta]$, where $i+1 \in \text{Dom}(\text{AVS}(\mathcal{F}^*_i|\text{Dom}(\mathcal{F})-1))$, $\beta^* \not\in \text{STSF}(\{\Delta, P(\mathcal{F}|\text{Dom}(\mathcal{F})-2))\}$, there is no $j \leq i$ such that $\beta^* \in \text{ST}(\mathcal{F}_j)$, there is no $j$ such that $i+1 < l \leq \text{Dom}(\mathcal{F})-2$ and $l \in \text{Dom}(\text{AVS}(\mathcal{F}^*_i|\text{Dom}(\mathcal{F})-1))$, and $\mathcal{F}_i = \mathcal{F}_i|\text{Dom}(\mathcal{F})-1 \cup \{(\text{Dom}(\mathcal{F}_i), \text{Therefore } P(\mathcal{F}_i) \rightarrow C(\mathcal{F}_i|\text{Dom}(\mathcal{F})-1))\}$.

With c), d) and f), it then follows that $i+1 \in \text{Dom}(\text{AVS}(\mathcal{F}^*_i))$ and $P(\mathcal{F}^*_{i+1}) = [\beta, \alpha, P(\mathcal{F}_i)] = [\beta, \alpha, P(\mathcal{F}_i)] = [\beta^*, \zeta, \Delta], i+2 \in \text{Dom}(\text{AVS}(\mathcal{F}^*_i))$ and $P(\mathcal{F}^*_{i+2}) = [\beta, \alpha, P(\mathcal{F}_i)] = [\beta, \alpha, P(\mathcal{F}_i)] = [\beta^*, \zeta, \Delta]$, $C(\mathcal{F}^*_{i+1}) = P(\mathcal{F}^*_{i+2}) = [\beta, \alpha, C(\mathcal{F}_i)]$ and $\mathcal{F}^*_i = \mathcal{F}^*_i \cup \{(\text{Dom}(\mathcal{F}_i), \text{Therefore } C(\mathcal{F}_i|\text{Dom}(\mathcal{F})-1))\} = \mathcal{F}^*_i \cup \{(\text{Dom}(\mathcal{F}_i), \text{Therefore } C(\mathcal{F}_i))\}$, and that there is no $l$ such
that \( i+2 < l \leq \text{Dom}(\delta)-1 = \text{Dom}(\delta^*)-1 \) and \( l \in \text{Dom}(\text{AVAS}(\delta^*)) \). With \( \beta^* \neq \beta \) and \( \beta^* = \beta \), we can distinguish two cases.

**First case:** Suppose \( \beta^* \neq \beta \). With Theorem 1-25-(ii), we have \( \text{P}(\delta^*_{i+2}) = [\beta, \alpha, [\beta^*, \zeta, \Delta]] = [\beta^*, \zeta, [\beta, \alpha, \Delta]] \). Also, we have \( \text{P}(\delta^*_{i+1}) = \eta \). If \( \beta^* \in \text{STSF}([[\beta, \alpha, \Delta], [\beta, \alpha, \text{P}(\delta_{\text{Dom}(\delta)-2})]]) \) or if there was a \( j \leq i+1 \) such that \( \beta^* \in \text{ST}(\delta_j) \), then we would have, because of \( \beta^* \neq \beta \) and with d), also \( \beta^* \in \text{STSF}([\Delta, \text{P}(\delta_{\text{Dom}(\delta)-2})]) \) or there would be a \( j \leq i \) such that \( \beta^* \in \text{ST}(\delta_j) \). Contradiction! Therefore we have \( \beta^* \notin \text{STSF}([[\beta, \alpha, \Delta], [\beta, \alpha, \text{P}(\delta_{\text{Dom}(\delta)-2})]]) \) and there is no \( j \leq i+1 \) such that \( \beta^* \in \text{ST}(\delta_j) \) and hence we have that \( \delta^+ \in \text{PEF}(\delta^*) \) and thus \( \delta^+ \in \text{RCS} \).

**Second case:** Now, suppose \( \beta^* = \beta \). Then we have \( \zeta \notin \text{FV}(\Delta) \), because, if not, we would have \( \beta \in \text{ST}([\beta^*, \zeta, \Delta]) \subseteq \text{STSEQ}^{(\delta)} \). Then we have \( [\beta^*, \zeta, \Delta] = \Delta \) and thus \( \text{P}(\delta^*_{i+2}) = [\beta, \alpha, [\beta^*, \zeta, \Delta]] = [\beta, \alpha, \Delta] \). Now, let \( \beta^* \in \text{PAR} \backslash \text{STSEQ}^{(\delta^*)} \). Then with \( \zeta \notin \text{FV}(\Delta) \) also \( \zeta \notin \text{FV}([\beta, \alpha, \Delta]) \) and thus \( \text{P}(\delta^*_{i+2}) = [\beta, \alpha, \Delta] = [\beta^*, \zeta, [\beta, \alpha, \Delta]] \) and it holds that \( \beta^+ \notin \text{STSF}([\Delta, \text{P}(\delta_{\text{Dom}(\delta)-2})]) \) and there is no \( j \leq i+1 \) such that \( \beta^+ \in \text{ST}(\delta_j) \). Hence we have again \( \delta^+ \in \text{PEF}(\delta^*) \) and thus \( \delta^+ \in \text{RCS} \).

With Theorem 3-21-(iii), we have that \( \text{AVS}(\delta) = \text{AVS}(\delta \upharpoonright \text{Dom}(\delta)-1), \{j, \delta_j\} | i+1 \leq j < \text{Dom}(\delta)-1 \} \cup \{ \text{Dom}(\delta)-1, \text{Therefore } \text{P}(\delta_{\text{Dom}(\delta)-2}) \} \) and that \( \text{AVS}(\delta^*) = \text{AVS}(\delta^*) \cup \{ \text{Dom}(\delta), \text{Therefore } \beta, \alpha, \text{P}(\delta_{\text{Dom}(\delta)-2}) \} \}. \) With \( \text{Dom}(\text{AVS}(\delta^*)) = \{ i+1 | l \in \text{Dom}(\text{AVS}(\delta^*) \text{Dom}(\delta)-1) \} \cup \{ 0 \}, \) it then follows that \( \text{Dom}(\text{AVS}(\delta^*)) = \{ i+1 | l \in \text{Dom}(\text{AVS}(\delta^*)) \} \cup \{ 0 \} \).

**Exclusion assumption:** For the remaining cases suppose \( \delta \notin \text{CdIF}(\delta \upharpoonright \text{Dom}(\delta)-1) \cup \text{NIF}(\delta \upharpoonright \text{Dom}(\delta)-1) \cup \text{PEF}(\delta \upharpoonright \text{Dom}(\delta)-1) \). With g), we then have \( \delta^+ \notin \text{CdIF}(\delta^*) \cup \text{NIF}(\delta^*) \cup \text{PEF}(\delta^*) \). With Theorem 3-25, we thus have for all of the following cases that \( \text{AVS}(\delta) = \text{AVS}(\delta \upharpoonright \text{Dom}(\delta)-1) \cup \{ \text{Dom}(\delta)-1, \text{C}(\delta) \} \) and that \( \text{AVS}(\delta^*) = \text{AVS}(\delta^*) \cup \{ \text{Dom}(\delta), \text{C}(\delta^*) \} \}. \) With \( \text{Dom}(\text{AVS}(\delta^*)) = \{ i+1 | l \in \text{Dom}(\text{AVS}(\delta \upharpoonright \text{Dom}(\delta)-1)) \} \cup \{ 0 \} \), it then holds for all remaining cases that \( \text{Dom}(\text{AVS}(\delta^*)) = \{ i+1 | l \in \text{Dom}(\text{AVS}(\delta^*)) \} \cup \{ 0 \} \).

**(AF):** Suppose \( \delta \in \text{AF}(\delta \upharpoonright \text{Dom}(\delta)-1) \). According to Definition 3-1, we then have \( \delta = \delta \upharpoonright \text{Dom}(\delta)-1 \cup \{ \text{Dom}(\delta)-1, \text{Suppose } \text{P}(\delta_{\text{Dom}(\delta)-1}) \}. \) With f), we then have \( \delta^+ = \delta^* \cup \{ \text{Dom}(\delta), \text{Suppose } \beta, \alpha, \text{P}(\delta_{\text{Dom}(\delta)-1}) \} \in \text{AF}(\delta^*) \) and thus \( \delta^+ \in \text{RCS} \).

**(CdEF, CIF, CEF, BIF, BEF, DIF, DEF, NEF):** Now, suppose \( \delta \in \text{CdEF}(\delta \upharpoonright \text{Dom}(\delta)-1) \). According to Definition 3-3, there are then A, B \in \text{CFORM} such
that A, "A → B" ∈ AVP(δ|Dom(δ)-1) and δ = δ|Dom(δ)-1 ∪ {(Dom(δ)-1, "Therefore B")}. With f), it then follows that: δ+ = δ* ∪ {(Dom(δ), "Therefore [β, α, B]")}. With A, "A → B" ∈ AVP(δ|Dom(δ)-1) and Definition 2-30, there are i, j ∈ Dom(AVS(δ|Dom(δ)-1)) such that P(δi) = A and P(δj) = "A → B". With c) and d), it then follows that: i+1, j+1 ∈ Dom(AVS(δ*) and P(δ*+1) = [β, α, A] and P(δj+1) = [β, α, A] → [β, α, B]. Thus we have δ+ = δ* ∪ {(Dom(δ), "Therefore [β, α, B]")} ∈ CDef(δ*) and thus δ+ ∈ RCS. CIF, CEF, BIF, BEF, DIF, DEF and NEF are treated analogously.

(UIF): Now, suppose δ ∈ UIF(δ|Dom(δ)-1). According to Definition 3-12, there are then β* ∈ PAR, ζ ∈ VAR, Δ ∈ FORM, where FV(Δ) ⊆ {ζ}, such that [β*, ζ, Δ] ∈ AVP(δ|Dom(δ)-1), β* ∉ STSF({Δ} ∪ AVP(δ|Dom(δ)-1)) and δ = δ|Dom(δ)-1 ∪ {(Dom(δ)-1, "Therefore ΛζΔ")}. With f), we then have δ+ = δ* ∪ {(Dom(δ), [β, α, "Therefore ΛζΔ"])}. With [β*, ζ, Δ] ∈ AVP(δ|Dom(δ)-1) and Definition 2-30, there is an i ∈ Dom(AVS(δ|Dom(δ)-1)) such that [β*, ζ, Δ] = P(δi). With a) and d), it then follows that i+1 ∈ Dom(AVS(δ*)) and that P(δ*+1) = [β, α, P(δj)] = [β, α, [β*, ζ, Δ]]. With β* ≠ β and β* = β, we can distinguish two cases.

First case: Suppose β* ≠ β. With Theorem 1-25-(ii), we have P(δ*+1) = [β, α, [β*, ζ, Δ]] = [β*, ζ, [β, α, Δ]]. We have C(δ*) = "Λζ[β, α, Δ]". Now, suppose for contradiction that β* ∈ STSF({[β, α, Δ]} ∪ AVP(δ*)), Since β* ≠ β and β* ∈ ST(Δ), we have β* ∉ ST([β, α, Δ]). Thus we would have β* ∉ STSF(AVP(δ*)). With Definition 2-31, there would then be a j ∈ Dom(AVAS(δ*)) such that β* ∈ ST(P(δj)). With δj, j ∈ ISENT, we have j ≠ 0. But with d), we would then have P(δj) = [β, α, P(δj+1)] and since β* ≠ β, we would then have β* ∈ ST(P(δj+1)). With c) and d) and j ∈ Dom(AVAS(δ*)), we would also have that j+1 ∈ Dom(AVAS(δ|Dom(δ)-1)). Thus we would have P(δj+1) ∈ AVP(δ|Dom(δ)-1) and β* ∈ STSF(AVP(δ|Dom(δ)-1)), whereas, by hypothesis, we have β* ∉ STSF(AVP(δ|Dom(δ)-1))). Contradiction! Therefore we have β* ∉ STSF([β, α, Δ] ∪ AVP(δ*)) and hence δ+ ∈ UIF(δ*)

Second case: Now, suppose β* = β. Then we have ζ ∉ FV(Δ), because, if not, we would have β ∈ ST([β*, ζ, Δ]) ⊂ STSEQ(δ). Thus we have [β*, ζ, Δ] = Δ and thus P(δ*+1) = [β, α, [β*, ζ, Δ]] = [β, α, Δ] and we have C(δ*) = "Λζ[β, α, Δ]". Now, let β* ∈ PAR\STSEQ(δ*). Then with ζ ∉ FV(Δ) also ζ ∉ FV([β, α, Δ]) and thus P(δ*+1) = [β, α,
Δ = [β, ζ, [β, α, Δ]] and it holds that β' ∉ STSF( {{β, α, Δ}} ∪ AVAP(δ*)). Hence we have again δ* ∈ UEF(δ*). Thus we have in both cases that δ* ∈ UEF(δ*) ⊆ RCS.

(UEF): Now, suppose δ ∈ UEF(δ|Dom(δ)-1). According to Definition 3-13, there are then θ ∈ CTERM, ζ ∈ VAR, Δ ∈ FORM, where FV(Δ) ⊆ {ζ}, such that "∧ζΔ" ∈ AVP(δ|Dom(δ)-1) and δ = δ|Dom(δ)-1 ∪ \{((Dom(δ)-1, "Therefore [θ, ζ, Δ]"))}. With \(i\), we then have δ* = δ* ∪ {((Dom(δ), [β, α, "Therefore [θ, ζ, Δ]"]))} = δ* ∪ {((Dom(δ), "Therefore [β, α, [θ, ζ, Δ]]")}. With "∧ζΔ" ∈ AVP(δ|Dom(δ)-1) and Definition 2-30, there is an i ∈ Dom(AVS(δ|Dom(δ)-1)) such that P(δi) = "∧ζΔ". With c) and d), we then have \(i+1 \in Dom(AVS(δ*))\) and P(δ* \(i+1\)) = [β, α, "∧ζΔ"] = "∧ζ[β, α, Δ]". With Theorem 1-26-(ii), we have C(δ*) = [β, α, [θ, ζ, Δ]] = [[β, α, θ], ζ, [β, α, Δ]], where, with \(θ \in CTERM\), also \([β, α, θ] \in CTERM\) and, with FV(Δ) ⊆ {ζ}, also FV([β, α, Δ]) ⊆ {ζ}. Hence we have δ* ∈ UEF(δ*) ⊆ RCS.

(PIF): Now, suppose δ ∈ PIF(δ|Dom(δ)-1). According to Definition 3-14, there are then θ ∈ CTERM, ζ ∈ VAR, Δ ∈ FORM, where FV(Δ) ⊆ {ζ}, such that [θ, ζ, Δ] ∈ AVP(δ|Dom(δ)-1) and δ = δ|Dom(δ)-1 ∪ \{((Dom(δ)-1, "Therefore \∨ζΔ")}. With \(i\), we then have δ* = δ* ∪ {((Dom(δ), [β, α, "Therefore \∨ζΔ")])} = δ* ∪ {((Dom(δ), "Therefore \∨ζ[β, α, Δ]"))}. With [θ, ζ, Δ] ∈ AVP(δ|Dom(δ)-1) and Definition 2-30, there is an i ∈ Dom(AVS(δ|Dom(δ)-1)) such that P(δi) = [θ, ζ, Δ]. With c) and d), we then have \(i+1 \in Dom(AVS(δ*))\) and P(δ* \(i+1\)) = [β, α, P(δi)]. With Theorem 1-26-(ii), we then have P(δ* \(i+1\)) = [β, α, P(δi)] = [β, α, [θ, ζ, Δ]] = [[β, α, θ], ζ, [β, α, Δ]], where, with \(θ \in CTERM\), also \([β, α, θ] \in CTERM\) and, with FV(Δ) ⊆ {ζ}, also FV([β, α, Δ]) ⊆ {ζ}. Hence we have δ* ∈ PIF(δ*) ⊆ RCS.

(IIF): Now, suppose δ ∈ IIF(δ|Dom(δ)-1). According to Definition 3-16, there is then \(θ \in CTERM\) such that δ = δ|Dom(δ)-1 ∪ \{((Dom(δ)-1, "Therefore \θ = \θ")}. With \(i\), we then have δ* = δ* ∪ {((Dom(δ), [β, α, "Therefore \θ = \θ")])} = δ* ∪ {((Dom(δ), "Therefore [β, α, [θ, ζ, Δ]]")}, where with \(θ \in CTERM\) also \([β, α, θ] \in CTERM\). Hence we have δ* ∈ IIF(δ*) ⊆ RCS.

(IEF): Now, suppose δ ∈ IEF(δ|Dom(δ)-1). According to Definition 3-17, there are then \(θ_0, θ_1 \in CTERM\), ζ ∈ VAR and Δ ∈ FORM, where FV(Δ) ⊆ {ζ}, such that "θ₀ = θ₁", [θ₀, ζ, Δ] ∈ AVP(δ|Dom(δ)-1) and δ = δ|Dom(δ)-1 ∪ \{((Dom(δ)-1, "Therefore [θ₀, ζ, Δ])"}). With \(i\), we then have δ* = δ* ∪ {((Dom(δ), [β, α, "Therefore [θ₀, ζ, Δ]"]})) = δ* ∪ {((Dom(δ), "Therefore [β, α, [θ₀, ζ, Δ]]")}. With "θ₀ = θ₁", [θ₀, ζ, Δ] ∈ AVP(δ|Dom(δ)-1) and Definition 2-30, there are i, j ∈ Dom(AVS(δ|Dom(δ)-1)) such
that $P(\delta_i) = \{\theta_0 = \theta_i\}$ and $P(\delta_j) = [\theta_0, \zeta, \Delta]$. With c) and d), it then holds that $i+1, j+1 \in \text{Dom}(\text{AVS}(\delta^*))$ and $P(\delta^*_{i+1}) = [\beta, \alpha, P(\delta_i)] = [\beta, \alpha, \{\theta_0 = \theta_i\}] = [\beta, \alpha, \{\theta_0, \theta_1\}]$ and $P(\delta^*_{j+1}) = [\beta, \alpha, P(\delta_j)]$. With Theorem 1-26(ii), we then have $P(\delta^*_{j+1}) = [\beta, \alpha, P(\delta_j)] = [\beta, \alpha, [\theta_0, \zeta, \Delta]] = [[\beta, \alpha, \theta_0], \zeta, [\beta, \alpha, \Delta]]$ and $C(\delta^*) = [\beta, \alpha, [\theta_1, \zeta, \Delta]] = [[\beta, \alpha, \theta_1], \zeta, [\beta, \alpha, \Delta]]$, where with $\theta_0, \theta_1 \in \text{CTERM}$ also $[\beta, \alpha, \theta_0], [\beta, \alpha, \theta_1] \in \text{CTERM}$ and with $\text{FV}(\Delta) \subseteq \{\zeta\}$ also $\text{FV}([\beta, \alpha, \Delta]) \subseteq \{\zeta\}$. Hence we have $\delta^+ \in \text{IEF}(\delta^*) \subseteq \text{RCS}$. ■

In the proof of the following theorem, Theorem 4-8 provides the induction basis and is used in the induction step. The theorem prepares the RCS-preserving concatenation of two RCS-elements that share common parameters.

**Theorem 4-10.** Multiple substitution of new and pairwise different parameters for pairwise different parameters is RCS-preserving

If $\delta \in \text{RCS}$, $k \in \mathbb{N} \setminus \{0\}$ and $\{\beta_0^*, \ldots, \beta_{k-1}^*\} \subseteq \text{PAR} \setminus \text{STSEQ}(\delta)$, where for all $i, j < k$ with $i \neq j$ it holds that $\beta_i^* \neq \beta_j^*$ and $\{\beta_0^*, \ldots, \beta_{k-1}^*\} \subseteq \text{PAR} \setminus \{\beta_0^*, \ldots, \beta_{k-1}^*\}$, where for all $i, j < k$ with $i \neq j$ it holds that $\beta_i \neq \beta_j$, then $[\beta_0^*, \ldots, \beta_{k-1}^*], (\beta_0, \ldots, \beta_{k-1}, \delta) \in \text{RCS}$ and $\text{Dom}(\text{AVS}([\beta_0^*, \ldots, \beta_{k-1}^*], (\beta_0, \ldots, \beta_{k-1}, \delta)]) = \text{Dom}(\text{AVS}(\delta))$.

**Proof:** By induction on $k$. With Theorem 4-8, the statement holds for $k = 1$. Now, suppose the statement holds for $k$. Now, suppose $\delta \in \text{RCS}$, $k+1 \in \mathbb{N} \setminus \{0\}$ and $\{\beta_0^*, \ldots, \beta_k^*\} \subseteq \text{PAR} \setminus \text{STSEQ}(\delta)$, where for all $i, j < k+1$ with $i \neq j$ it holds that $\beta_i^* \neq \beta_j^*$ and $\{\beta_0^*, \ldots, \beta_k^*\} \subseteq \text{PAR} \setminus \{\beta_0^*, \ldots, \beta_k^*\}$, where for all $i, j < k+1$ with $i \neq j$ it holds that $\beta_i \neq \beta_j$. According to the I.H., we then have $[\beta_0^*, \ldots, \beta_{k-1}^*], (\beta_0, \ldots, \beta_{k-1}, \delta) \in \text{RCS}$ and $\text{Dom}(\text{AVS}([\beta_0^*, \ldots, \beta_{k-1}^*], (\beta_0, \ldots, \beta_{k-1}, \delta)]) = \text{Dom}(\text{AVS}(\delta))$. With Theorem 1-27(iv), we have $[\beta_k^*, \beta_{k+1}^*, [\beta_0^*, \ldots, \beta_{k-1}^*], (\beta_0, \ldots, \beta_{k-1}, \delta)] = [\beta_0^*, \ldots, \beta_k^*, (\beta_0, \ldots, \beta_{k-1}, \delta)]$. With Theorem 4-8, we thus have $[\beta_0^*, \ldots, \beta_k^*, (\beta_0, \ldots, \beta_{k-1}, \delta)] \in \text{RCS}$ and $\text{Dom}(\text{AVS}([\beta_0^*, \ldots, \beta_k^*], (\beta_0, \ldots, \beta_{k-1}, \delta)]) = \text{Dom}(\text{AVS}(\delta))$. ■
Theorem 4-11. UI-extension of a sentence sequence

If $\mathfrak{F}\in\text{RCS}\setminus\{\emptyset\}, k\in\mathbb{N}\setminus\{0\}, \{\xi_0, \ldots, \xi_k\} \subseteq \text{VAR}$, where for all $i, j < k$ with $i \neq j$ it holds that $\xi_i \neq \xi_j$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi_0, \ldots, \xi_k\}$, and $\{\beta_0, \ldots, \beta_k\} \subseteq \text{PAR}\setminus\text{STSF}(\{\Delta\} \cup \text{AVAP}(\mathfrak{F}))$, where for all $i, j < k$ with $i \neq j$ it holds that $\beta_i \neq \beta_j$, and $C(\mathfrak{F}) = [\{\beta_0, \ldots, \beta_k\}, \{\xi_0, \ldots, \xi_k\}, \Delta]$, then there is an $\mathfrak{F}^* \in \text{RCS}\setminus\{\emptyset\}$ such that

(i) $\text{PAR} \cap \text{STSEQ}(\mathfrak{F}^*) = \text{PAR} \cap \text{STSEQ}(\mathfrak{F})$, 
(ii) $\text{AVAP}(\mathfrak{F}^*) \subseteq \text{AVAP}(\mathfrak{F})$, and 
(iii) $C(\mathfrak{F}^*) = \{\land\xi_0 \ldots \land\xi_k \Delta\}$.

Proof: By induction on $k$. Suppose $k = 1$ and $\mathfrak{F} \in \text{RCS}\setminus\{\emptyset\}$, suppose $\xi \in \text{VAR}$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi\}$, and $\beta \in \text{PAR}\setminus\text{STSF}(\{\Delta\} \cup \text{AVAP}(\mathfrak{F}))$ and $C(\mathfrak{F}) = [\beta, \xi, \Delta]$. With Theorem 2-82, we have $[\beta, \xi, \Delta] = C(\mathfrak{F}) \in \text{AVP}(\mathfrak{F})$, and thus, according to Definition 3-12, $\mathfrak{F}^* = \mathfrak{F} \cup \{(\text{Dom}(\mathfrak{F}), \text{Therefore} \land\xi \Delta') \} \in \text{UIF}(\mathfrak{F}) \subseteq \text{RCS}\setminus\{\emptyset\}$ and $C(\mathfrak{F}^*) = \{\land\xi \Delta\}$. We also have that $\text{PAR} \cap \text{STSEQ}(\mathfrak{F}^*) = (\text{PAR} \cap \text{STSEQ}(\mathfrak{F})) \cup (\text{PAR} \cap \text{ST}(\land\xi \Delta)) = \text{PAR} \cap \text{STSEQ}(\mathfrak{F})$, and, with Theorem 3-26-(v), we have $\text{AVP}(\mathfrak{F}^*) \subseteq \text{AVAP}(\mathfrak{F})$.

Now, suppose the statement holds for $k$ and suppose $\mathfrak{F} \in \text{RCS}\setminus\{\emptyset\}, \{\xi_0, \ldots, \xi_k\} \subseteq \text{VAR}$, where for all $i, j < k+1$ with $i \neq j$ it holds that $\xi_i \neq \xi_j$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi_0, \ldots, \xi_k\}$, and $\{\beta_0, \ldots, \beta_k\} \subseteq \text{PAR}\setminus\text{STSF}(\{\Delta\} \cup \text{AVAP}(\mathfrak{F}))$, where for all $i, j < k+1$ with $i \neq j$ it holds that $\beta_i \neq \beta_j$, and $C(\mathfrak{F}) = [\{\beta_0, \ldots, \beta_k\}, \{\xi_0, \ldots, \xi_k\}, \Delta]$. With Theorem 1-28-(ii), we then have $C(\mathfrak{F}) = [\{\beta_0, \ldots, \beta_k\}, \{\xi_0, \ldots, \xi_k\}, \Delta] = [\beta_k, \xi_k, [\{\beta_0, \ldots, \beta_{k-1}\}, \{\xi_0, \ldots, \xi_{k-1}\}, \Delta]]$. With $\text{FV}(\Delta) \subseteq \{\xi_0, \ldots, \xi_k\}$ we then have $\text{FV}([\{\beta_0, \ldots, \beta_{k-1}\}, \{\xi_0, \ldots, \xi_{k-1}\}, \Delta]) \subseteq \{\xi_k\}$. Since $\beta_i$ are pairwise different and $\{\beta_0, \ldots, \beta_k\} \subseteq \text{PAR}\setminus\text{STSF}(\{\Delta\} \cup \text{AVAP}(\mathfrak{F}))$, we then have $\beta_k \in \text{PAR}\setminus\text{STSF}(\{[\{\beta_0, \ldots, \beta_{k-1}\}, \{\xi_0, \ldots, \xi_{k-1}\}, \Delta]\} \cup \text{AVAP}(\mathfrak{F}))$. Since $[\beta_k, \xi_k, \{[\{\beta_0, \ldots, \beta_{k-1}\}, \{\xi_0, \ldots, \xi_{k-1}\}, \Delta]\} = C(\mathfrak{F}) \in \text{AVP}(\mathfrak{F})$, we then have, according to Definition 3-12, $\mathfrak{F}^* = \mathfrak{F} \cup \{(\text{Dom}(\mathfrak{F}), \text{Therefore} \land\xi_k \{[\{\beta_0, \ldots, \beta_{k-1}\}, \{\xi_0, \ldots, \xi_{k-1}\}, \Delta]\}) \} \in \text{UIF}(\mathfrak{F}) \subseteq \text{RCS}\setminus\{\emptyset\}$ and $C(\mathfrak{F}^*) = \{\land\xi_k \{[\{\beta_0, \ldots, \beta_{k-1}\}, \{\xi_0, \ldots, \xi_{k-1}\}, \Delta]\}\} \in \text{PAR} \cap \text{STSEQ}(\mathfrak{F})$ and, with Theorem 3-26-(v), we have $\text{AVP}(\mathfrak{F}^*) \subseteq \text{AVAP}(\mathfrak{F})$. Since the $\xi_i$ are pairwise different, we have for all $i < k$: $\xi_i \neq \xi_k$. Thus we then have $C(\mathfrak{F}) = \{\land\xi_k \{[\{\beta_0, \ldots, \beta_{k-1}\}, \{\xi_0, \ldots, \xi_{k-1}\}, \Delta]\}\} \subseteq \{\xi_0, \ldots, \xi_k\}$, where the $\xi_i$ with $i < k$ are pairwise different. With $\{\beta_0, \ldots, \beta_k\} \subseteq \text{PAR}\setminus\text{STSF}(\{\Delta\} \cup \text{AVAP}(\mathfrak{F}))$, we have $\{\beta_0, \ldots, \beta_{k-1}\} \subseteq \text{PAR}\setminus\text{STSF}(\{\land\xi_k \Delta\} \cup \text{AVAP}(\mathfrak{F}))$, where the $\beta_i$ with $i < k$ are also pairwise different. According to the I.H.,
there is thus, with $C(\delta') = \{[\beta_0, \ldots, \beta_{k-1}], (\xi_0, \ldots, \xi_{k-1}), \Lambda \xi_{k}\Delta\}$, an $\delta^* \in \text{RCS}\{\emptyset\}$ such that $\text{PAR} \cap \text{STSEQ}(\delta^*) = \text{PAR} \cap \text{STSEQ}(\delta') = \text{PAR} \cap \text{STSEQ}(\delta)$, $\text{AVAP}(\delta^*) \subseteq \text{AVAP}(\delta') \subseteq \text{AVAP}(\delta)$ and $C(\delta^*) = \Lambda \xi_{0, \ldots, \xi_{k}}\Delta$. ■

**Theorem 4.12. UE-extension of a sentence sequence**

If $\delta \in \text{RCS}\{\emptyset\}, k \in \mathbb{N}\{\emptyset\}, \{\theta_0, \ldots, \theta_{k-1}\} \subseteq \text{CTERM}, \{\xi_0, \ldots, \xi_{k-1}\} \subseteq \text{VAR}$, where for all $i, j < k$ with $i \neq j$ it holds that $\xi_i \neq \xi_j$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi_0, \ldots, \xi_{k-1}\}$, and $\Lambda \xi_{0, \ldots, \xi_{k-1}}\Delta \in \text{AVP}(\delta)$, then there is an $\delta^* \in \text{RCS}\{\emptyset\}$ such that

(i) $\text{Dom}(\delta^*) = \text{Dom}(\delta) + k$,

(ii) $\delta^*]\text{Dom}(\delta^*) = \delta$,

(iii) $\text{AVAP}(\delta^*) \subseteq \text{AVAP}(\delta)$,

(iv) For all $i < k$: $C(\delta^*]\text{Dom}(\delta^*)+i+1) = \Lambda \xi_{i+1}\ldots \Lambda \xi_{k}\Delta \{[\theta_0, \ldots, \theta_i], (\xi_0, \ldots, \xi_i), \Delta\}$, and

(v) $C(\delta^*) = \{[\theta_0, \ldots, \theta_{k-1}], (\xi_0, \ldots, \xi_{k-1}), \Delta\}$.

**Proof:** By induction on $k$: Suppose $k = 1$. Suppose $\delta \in \text{RCS}\{\emptyset\}, \theta \in \text{CTERM}, \xi \in \text{VAR}, \Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi\}$, and $\Lambda \xi\Delta \in \text{AVP}(\delta)$. With Definition 3-13, it then holds that $\delta^* = \delta^\sim \{0, \text{Therefore} [0, \xi, \Delta]\} \in \text{UEF}(\delta) \subseteq \text{RCS}\{\emptyset\}$, and it holds that $\text{Dom}(\delta^*) = \text{Dom}(\delta) + 1$ and $\delta^*]\text{Dom}(\delta^*) = \delta^*$ and, with Theorem 3-27-(v), that $\text{AVAP}(\delta^*) \subseteq \text{AVAP}(\delta)$. Because of $k = 1$, clause (iv) is satisfied trivially and we have $C(\delta^*) = \{[0, \xi, \Delta]\}$.

Now, suppose the statement holds for $k$ and suppose $\delta \in \text{RCS}\{\emptyset\}, \{\theta_0, \ldots, \theta_k\} \subseteq \text{CTERM}, \{\xi_0, \ldots, \xi_k\} \subseteq \text{VAR}$, where for all $i, j < k+1$ with $i \neq j$ it holds that $\xi_i \neq \xi_j$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi_0, \ldots, \xi_{k-1}\}$, and $\Lambda \xi_{0, \ldots, \xi_{k-1}}\Delta \in \text{AVP}(\delta)$. With $\text{FV}(\Delta) \subseteq \{\xi_0, \ldots, \xi_{k-1}\}$, we then have $\text{FV}(\Lambda \xi_1\ldots \Lambda \xi_{k}\Delta) \subseteq \{\xi_0\}$ and, with $\theta_0 \in \text{CTERM}$ and $\Lambda \xi_{0, \ldots, \xi_{k-1}}\Delta \in \text{AVP}(\delta)$ and Definition 3-13, we have $\delta^* = \delta^\sim \{0, \text{Therefore} [\theta_0, \xi_0, \Lambda \xi_1\ldots \Lambda \xi_{k}\Delta]\} \in \text{UEF}(\delta) \subseteq \text{RCS}\{\emptyset\}$. Then we have $\text{Dom}(\delta^*) = \text{Dom}(\delta) + 1$ and $\delta^*]\text{Dom}(\delta^*) = \delta^*$ and, with Theorem 3-27-(v), we have $\text{AVAP}(\delta') \subseteq \text{AVAP}(\delta)$. Since the $\xi_i$ are pairwise different, we have for all $i$ with $0 < i \leq k$: $\xi_0 \neq \xi_i$. Thus we then have $C(\delta') = [\theta_0, \xi_0, \Lambda \xi_1\ldots \Lambda \xi_{k}\Delta]$.

Now, let $\xi_i = \xi_{i+1}$ and $\theta_i = \theta_{i+1}$ for all $i \in k$. Then we have $\{\theta_0', \ldots, \theta_{k-1}'\} \subseteq \text{CTERM}, \{\xi_0, \ldots, \xi_{k-1}\} \subseteq \text{VAR}$, where for all $i, j < k$ with $i \neq j$ and $\xi_i, \xi_j, [\theta_0, \xi_0, \Delta]$ \text{FORM}$, where, with $\text{FV}(\Delta) \subseteq \{\xi_0, \ldots, \xi_{k-1}\}$ and $\theta_0 \in \text{CTERM}$, it holds that $\text{FV}(\{[\theta_0, \xi_0, \Delta]\} \subseteq \{\xi_1, \ldots, \xi_{k-1}\} = \{\xi_0, \ldots, \xi_{k-1}\}$, and, with Theorem 2-82, it holds that $\Lambda \xi_{0, \ldots, \xi_{k-1}}\Delta = \Lambda \xi_{1,\ldots, \xi_{k-1}}\Delta] = C(\delta') \in \text{AVP}(\delta')$. According to the I.H., there is then an $\delta^* \in \text{RCS}\{\emptyset\}$ such that:
a) \( \text{Dom}(\mathcal{S}^*) = \text{Dom}(\mathcal{S}) + k \),

b) \( \mathcal{S}^* \uparrow \text{Dom}(\mathcal{S}) = \mathcal{S} \)

c) \( \text{AVAP}(\mathcal{S}^*) \subseteq \text{AVAP}(\mathcal{S}) \),

d) For all \( i < k: \ C(\mathcal{S}^* \uparrow \text{Dom}(\mathcal{S}) + i + 1) = \langle \land_{\xi_{i+1}} \ldots \land_{\xi_0} [\langle 0_i, \ldots, 0, \xi_0, \ldots, \xi_i \rangle], [\theta_0, \xi_0, \Delta] \rangle \), and

e) \( C(\mathcal{S}^*) = [\langle 0_i, \ldots, 0, \xi_{i+1}, \ldots, \xi_0 \rangle], [\theta_0, \xi_0, \Delta] \). 

With a) and because of \( \text{Dom}(\mathcal{S}') = \text{Dom}(\mathcal{S}) + 1 \), we then have \( \text{Dom}(\mathcal{S}^*) = \text{Dom}(\mathcal{S}) + k + 1 \).

With b) and because of \( \mathcal{S}^* \uparrow \text{Dom}(\mathcal{S}) = \mathcal{S} \), we also have \( \mathcal{S}^* \uparrow \text{Dom}(\mathcal{S}) = \mathcal{S} \).

With c) and because of \( \text{AVAP}(\mathcal{S}') \subseteq \text{AVAP}(\mathcal{S}) \), we have that \( \text{AVAP}(\mathcal{S}^*) \subseteq \text{AVAP}(\mathcal{S}) \). Thus we have that \( \mathcal{S}^* \in \text{RCS} \setminus \{0\} \) and that clauses (i) to (iii) hold for \( \mathcal{S}^* \).

With d) and \( \zeta_i = \xi_{i+1} \) and \( \theta_i = \theta_i \), we also have

For all \( i < k-1: \ C(\mathcal{S}^* \uparrow \text{Dom}(\mathcal{S}) + i + 1) = \langle \land_{\xi_{i+2}} \ldots \land_{\xi_0} [\langle 0_i, \ldots, 0, \xi_0, \ldots, \xi_i \rangle], [\theta_0, \xi_0, \Delta] \rangle \).

With \( \text{Dom}(\mathcal{S}') = \text{Dom}(\mathcal{S}) + 1 \) we thus have

f) For all \( i < k-1: \ C(\mathcal{S}^* \uparrow \text{Dom}(\mathcal{S}) + i + 1 + 1) = \langle \land_{\xi_{i+3}} \ldots \land_{\xi_0} [\langle 0_i, \ldots, 0, \xi_0, \ldots, \xi_i \rangle], [\theta_0, \xi_0, \Delta] \rangle \).

Thus we have

g) For all \( i \) with \( 0 < i < k \): \( C(\mathcal{S}^* \uparrow \text{Dom}(\mathcal{S}) + i + 1) = \langle \land_{\xi_{i+1}} \ldots \land_{\xi_0} [\langle 0_i, \ldots, 0, \zeta_0, \ldots, \xi_i \rangle], [\theta_0, \zeta_0, \Delta] \rangle \).

We also have

h) For all \( i \) with \( 0 < i < k+1 \): \( [\langle 0_i, \ldots, 0_i \rangle, \xi_0, \ldots, [\theta_0, \zeta_0, \Delta]] = [\langle 0_i, \ldots, 0_i \rangle, \xi_0, \ldots, [\theta_0, \zeta_0, \Delta]] \).

h) can be shown by induction on \( i \). First, we have, with Theorem 1-28-(ii), that \( [\langle 0_i, \xi_0, \ldots, [\theta_0, \xi_0, \Delta]] = [\langle 0_i, \xi_0, \ldots, [\theta_0, \xi_0, \Delta]] \).

Now, suppose for \( i \) it holds that if \( 0 < i < k+1 \), then \( [\langle 0_i, \ldots, 0_i \rangle, \xi_0, \ldots, [\theta_0, \xi_0, \Delta]] = [\langle 0_i, \ldots, 0_i \rangle, \xi_0, \ldots, [\theta_0, \xi_0, \Delta]] \).

Suppose \( 0 < i+1 < k+1 \). Then we have \( i = 0 \) or \( 0 < i \). For \( i = 0 \), the statement follows in the same way as the induction basis. Now, suppose \( 0 < i \). With Theorem 1-28-(ii), we first have \( [\langle 0_i, \ldots, 0_i \rangle, \xi_0, \ldots, [\theta_0, \xi_0, \Delta]] = [\langle 0_i, \xi_0, \ldots, [\theta_0, \xi_0, \Delta]] \).

With the I.H., it then holds that \( [\langle 0_i, \xi_0, \ldots, [\theta_0, \xi_0, \Delta]] = [\langle 0_i, \xi_0, \ldots, [\theta_0, \xi_0, \Delta]] \).

Again with Theorem 1-28-(ii), we then have \( [\langle 0_i, \xi_0, \ldots, [\theta_0, \xi_0, \Delta]] \).
= [(\theta_0, \ldots, \theta_{i+1}), (\xi_0, \ldots, \xi_{i+1}), \Delta] and hence [(\theta_1, \ldots, \theta_{i+1}), (\xi_1, \ldots, \xi_{i+1}), [\theta_0, \xi_0, \Delta]] = [(\theta_0, \ldots, \theta_{i+1}), (\xi_0, \ldots, \xi_{i+1}), \Delta]. Therefore we have h).

With Dom(\delta') = Dom(\delta)+1 and C(\delta*\{Dom(\delta')\}) = C(\delta') = \langle \wedge \xi_1 \ldots \wedge \xi_n [\theta_0, \xi_0, \Delta]\rangle, we have C(\delta*\{Dom(\delta')+0+1\}) = \langle \wedge \xi_1 \ldots \wedge \xi_n [\theta_0, \xi_0, \Delta]\rangle. With g) and h), we thus get that clause (iv) holds:

For all \(i < k\): C(\delta*\{Dom(\delta)+i+1\}) = \langle \wedge \xi_{i+1} \ldots \wedge \xi_n [\theta_0, \ldots, \theta_i, \xi_{i+1}, \Delta]\rangle.

Last, it holds, with e), h) and \(\theta_i = \theta_{i+1}\) and \(\xi = \xi_{i+1}\) that

\[
C(\delta*) = [(\theta_0, \ldots, \theta_{i+1}), (\xi_0, \ldots, \xi_{i+1}), [\theta_0, \xi_0, \Delta]]
\]

= [(\theta_0, \ldots, \theta_{i}), (\xi_0, \ldots, \xi_{i}), [\theta_0, \xi_0, \Delta]]

= [(\theta_0, \ldots, \theta_{i}), (\xi_0, \ldots, \xi_{i}), \Delta].

Thus clause (v) holds as well, and hence the theorem holds for \(k+1\). ■

**Theorem 4.13. Induction basis for Theorem 4-14**

If \(\delta, \delta' \in \text{RCS}\{\emptyset\}\) and AVAS(\delta') = 0, then there is an \(\delta* \in \text{RCS}\{\emptyset\}\) such that

(i) \(C(\delta), C(\delta') \in \text{AVP}(\delta*)\)

(ii) \(\text{AVAP}(\delta*) \subseteq \text{AVAP}(\delta)\).

**Proof:** Suppose \(\delta, \delta' \in \text{RCS}\{\emptyset\}\) and suppose AVAS(\delta') = 0. If C(\delta) = C(\delta'), we can choose \(\delta\) as well as \(\delta'\) for \(\delta*\). Now, suppose C(\delta) \(\neq \) C(\delta'). With PAR \cap STSEQ(\delta) \cap STSEQ(\delta') = 0 and PAR \cap STSEQ(\delta) \cap STSEQ(\delta') \neq 0, we can then distinguish two cases.

**First case:** Suppose PAR \(\cap\) STSEQ(\delta) \(\cap\) STSEQ(\delta') = 0. There is an \(\alpha \in \text{CONST}(\text{STSEQ}(\delta) \cup \text{STSEQ}(\delta'))\). With Theorem 4-4, there is then an \(\delta* \in \text{RCS}\{\emptyset\}\) such that AVP(\delta) \(\cup\) AVP(\delta') \(\subseteq\) AVP(\delta*) and AVAP(\delta*) = AVAP(\delta) \(\cup\) \{\(\neg\alpha = \alpha^3\}\} \(\cup\) AVAP(\delta'). With Theorem 2-82, we have C(\delta) \(\in\) AVP(\delta) and C(\delta') \(\in\) AVP(\delta') and thus we have C(\delta), C(\delta') \(\in\) AVP(\delta*). With Theorem 4-7, there is then an \(\delta* \in \text{RCS}\{\emptyset\}\) such that AVAP(\delta*) \(\subseteq\) AVAP(\delta*)\{\(\neg\alpha = \alpha^3\}\} = (AVAP(\delta) \(\cup\) \{\(\neg\alpha = \alpha^3\}\) \(\cup\) AVAP(\delta')\}\{\(\neg\alpha = \alpha^3\}\} \(\subseteq\) AVAP(\delta) \(\cup\) AVAP(\delta') and C(\delta), C(\delta') \(\in\) AVP(\delta*), with which \(\delta*\) is the desired RCS-element.

**Second case:** Now, suppose PAR \(\cap\) STSEQ(\delta) \(\cap\) STSEQ(\delta') \(\neq\) 0. Then there occur \(k\) pairwise different parameters in \(\delta'\) for a \(k \in \mathbb{N}\{\emptyset\}\). Now, let \(\{\beta_0, \ldots, \beta_{k+1}\} = \text{PAR} \cap\)
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STSEQ(\(\xi\)'), where for all \(i, j < k\) with \(i \neq j\) it holds that \(\beta_i \neq \beta_j\). There are \(\beta_{*b}, \ldots, \beta_{*k-1} \in \text{PAR}'(\text{STSEQ}(\xi) \cup \text{STSEQ}(\xi'))\), where for all \(i, j < k\) it holds that if \(i \neq j\), then \(\beta_{*i} \neq \beta_{*j}\). Also, there are \(\xi_{0}, \ldots, \xi_{k-1} \in \text{VAR}'(\text{STSEQ}(\xi) \cup \text{STSEQ}(\xi'))\), where for all \(i, j < k\): If \(i \neq j\), then \(\xi_i \neq \xi_j\).

With Theorem 2-77 and AVAS(\(\xi\)) = \(\emptyset\), we also have AVAP(\(\xi\)) = \(\emptyset\). With Theorem 1-16, there is a \(\Delta \in \text{FORM}\), where FV(\(\Delta\)) \(\subset \{\xi_{0}, \ldots, \xi_{k-1}\} \cup \text{FV}(C(\xi')) = \{\xi_{0}, \ldots, \xi_{k-1}\}\) and ST(\(\Delta\)) \(\cap \{\beta_{0}, \ldots, \beta_{k-1}\} = \emptyset\), such that C(\(\xi\)) = \([\beta_{0}, \ldots, \beta_{k-1}], \langle \xi_{0}, \ldots, \xi_{k-1}\rangle, \Delta\) [1]. With Theorem 4-11, it then follows that there is \(\xi_{1} \in \text{RCS}\\{\emptyset\}\) such that \(\text{PAR} \cap \text{STSEQ}(\xi_{1}) = \text{PAR} \cap \text{STSEQ}(\xi)\), AVAP(\(\xi_{1}\)) \(\subset \text{AVAP}(\xi') = \emptyset\) and thus also AVAS(\(\xi_{1}\)) = \(\emptyset\) and C(\(\xi_{1}\)) = \(\{\beta_{0}, \ldots, \beta_{k-1}, \langle \xi_{0}, \ldots, \xi_{k-1}\rangle, \Delta\}\). With C(\(\xi\)) = \([\beta_{0}, \ldots, \beta_{k-1}, \langle \xi_{0}, \ldots, \xi_{k-1}\rangle, \Delta\) [1]), it follows that \(\text{PAR} \cap \text{ST}(\Delta) \subset \text{PAR} \cap \text{STSEQ}(\xi) = \{\beta_{0}, \ldots, \beta_{k-1}\} = \emptyset\) and thus, with ST(\(\Delta\)) \(\cap \{\beta_{0}, \ldots, \beta_{k-1}\} = 0\), it follows that \(\text{PAR} \cap \text{ST}(\Delta) = \text{PAR} \cap \text{ST}(\{\beta_{0}, \ldots, \beta_{k-1}\}) = \text{PAR} \cap \text{ST}(C(\xi_{1})) = \emptyset\).

We also have, with Theorem 4-10, that \(\xi_{2} = \{[\beta_{*b}, \ldots, \beta_{*k-1}], \langle \beta_{0}, \ldots, \beta_{k-1}, \xi_{1}\rangle\} \in \text{RCS}\) and Dom(AVS(\(\xi_{2}\))) = Dom(AVS(\(\xi_{1}\))) = \emptyset and hence also AVAP(\(\xi_{2}\)) = \emptyset. Moreover, we have PAR \(\cap \text{STSEQ}(\xi) \cap \text{STSEQ}(\xi_{2}) \subset \text{PAR} \cap \text{STSEQ}(\xi) \cap \{\beta_{*b}, \ldots, \beta_{*k-1}\} = \emptyset\). Furthermore, we have, because of PAR \(\cap \text{ST}(C(\xi_{1})) = \emptyset\), that C(\(\xi_{2}\)) = \([\beta_{*b}, \ldots, \beta_{*k-1}, \langle \beta_{0}, \ldots, \beta_{k-1}\rangle, C(\xi_{1})] = C(\xi_{1}) = \{\beta_{*b}, \ldots, \beta_{*k-1}, \langle \beta_{0}, \ldots, \beta_{k-1}\rangle, C(\xi_{1})\} \cap \text{STSEQ}(\xi_{2}) = \emptyset\).

There is an \(\alpha \in \text{CONST}(\text{ST}(\xi) \cup \text{ST}(\xi_{2}))\). With Theorem 4-4, there is then, because of PAR \(\cap \text{STSEQ}(\xi) \cap \text{STSEQ}(\xi_{2}) = \emptyset\), an \(\xi_{3} \in \text{RCS}\\{\emptyset\}\) such that:

a) \(\text{Dom}(\xi_{3}) = \text{Dom}(\xi_{2}) + 1 + \text{Dom}(\xi_{2})\),

b) \(\xi_{3} \cup \text{Dom}(\xi_{2}) = \xi_{2}\),

c) \(\xi_{3} \cap \text{Dom}(\xi_{2}) = \langle \text{Suppose } \alpha = \alpha'\rangle\),

d) \(\text{For all } i \in \text{Dom}(\xi_{2}) \text{ it holds that } \xi_{3} = \xi_{2} \cup \text{Dom}(\xi_{2}) + 1 + i\),

e) \(\text{Dom}(\text{AVS}(\xi_{3})) = \text{Dom}(\text{AVS}(\xi_{2})) \cup \{\text{Dom}(\xi_{2})\} \cup \{\text{Dom}(\xi_{2}) + 1 + l \mid l \in \text{Dom}(\text{AVS}(\xi_{3}))\})

f) \(\text{AVP}(\xi_{3}) = \text{AVP}(\xi_{2}) \cup \{\langle \alpha = \alpha'\rangle \}, \) \(\text{AVAP}(\xi_{3}) = \text{AVAP}(\xi_{2}) \cup \{\langle \alpha = \alpha'\rangle \}\), and

g) \(\text{AVAP}(\xi_{3}) = \text{AVAP}(\xi_{2}) \cup \{\langle \alpha = \alpha'\rangle \} \cup \text{AVAP}(\xi_{3}) = \text{AVAP}(\xi_{2}) \cup \{\langle \alpha = \alpha'\rangle \} \).
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h) \( \text{Dom}(\xi_4) = \text{Dom}(\xi_3) + k \)

i) \( \xi_4 \upharpoonright \text{Dom}(\xi_3) = \xi_3 \)

j) \( \text{AVAP}(\xi_4) \subseteq \text{AVAP}(\xi_3) = \text{AVAP}(\xi) \cup \{ \gamma \alpha = \alpha \gamma \} \)

k) For all \( i < k \): \( C(\xi_4 \upharpoonright \text{Dom}(\xi_3) + i + 1) = \mathbb{R}^{\xi_i + 1} \ldots \mathbb{R}^{\xi_k - 1} \left\langle \beta_0, \ldots, \beta_i, \xi_0, \ldots, \xi_i, \Delta \right\rangle \)

l) \( C(\xi_4) = \left\langle \beta_0, \ldots, \beta_{k-1}, \xi_0, \ldots, \xi_{k-1}, \Delta \right\rangle \)

Then we have \( C(\xi) = \left\langle \beta_0, \ldots, \beta_{k-1}, \xi_0, \ldots, \xi_{k-1}, \Delta \right\rangle = C(\xi_4) \in AVP(\xi_4) \). We also have: \( \xi_4 \upharpoonright \text{Dom}(\xi) = \xi_3 \upharpoonright \text{Dom}(\xi_3) = \xi_3 \).

Theorem 4-14. \( \text{CdE-}, \text{CI-}, \text{BI-}, \text{BE-} \) and \( \text{IE-preparation theorem} \)

If \( \xi, \xi' \in \text{RCS} \setminus \{ \emptyset \} \), then there is an \( \xi^* \in \text{RCS} \setminus \{ \emptyset \} \) such that

(i) \( C(\xi), C(\xi') \in \text{AVP}(\xi^*) \) and

(ii) \( \text{AVAP}(\xi^*) \subseteq \text{AVAP}(\xi) \cup \text{AVAP}(\xi') \).

Proof: Proof by induction on \( |\text{AVAS}(\xi')| \). For \( |\text{AVAS}(\xi')| = 0 \) the statement holds with Theorem 4-13. Now, suppose the statement holds for \( n \) and suppose \( \xi, \xi' \in \text{RCS} \setminus \{ \emptyset \} \) and \( |\text{AVAS}(\xi')| = n + 1 \). With Theorem 3-18, we then have \( \xi^* \upharpoonright \text{Dom}(\xi^*) ) \rightarrow C(\xi^*) \subseteq \text{CdIF}(\xi^*) \subseteq \text{RCS} \setminus \{ \emptyset \} \). With Theorem 3-19-(iv) and (v), we have \( |\text{AVAS}(\xi^*)| = n \) and, with Theorem 3-19-(ix), we have \( \text{AVAP}(\xi^*) \subseteq \text{AVAP}(\xi') \). With the I.H., it then holds that there is an \( \xi^2 \in \text{RCS} \setminus \{ \emptyset \} \) such that...
a) $C(\alpha_0), C(\alpha_1) \in AVP(\alpha_2)$ and
b) $AVAP(\alpha_3) \subseteq AVAP(\alpha) \cup AVAP(\alpha_1) \subseteq AVAP(\alpha) \cup AVAP(\alpha')$.

Now, let the following sentence sequences be defined, where $\alpha \in \text{CONST \ STSEQ}(\alpha_2)$:

\[
\begin{align*}
\alpha_3 &= \alpha_2 \cup \{\text{Dom}(\alpha_2), \text{Suppose } P(\max(\text{Dom}(AVS(\alpha)))) \}\} \\
\alpha_4 &= \alpha_3 \cup \{\text{Dom}(\alpha_3), \text{Therefore } \alpha = \alpha' \}\} \\
\alpha_5 &= \alpha_4 \cup \{\text{Dom}(\alpha_4), \text{Therefore } C(\alpha') \}\}.
\end{align*}
\]

With Theorem 1-12, we have $C(\alpha_3) \notin ISENT$ and thus $\alpha_3 \notin \text{CdIF}(\alpha_2) \cup \text{NIF}(\alpha_2) \cup \text{PEF}(\alpha_2)$. With Theorem 1-10 and Theorem 1-11, we have that $C(\alpha_4)$ is neither a negation nor a conditional and thus we have $\alpha_4 \notin \text{CdIF}(\alpha_3) \cup \text{NIF}(\alpha_3) \cup \text{PEF}(\alpha_3)$. If $P(\max(\text{Dom}(AVS(\alpha)))) = \alpha = \alpha'$, then we would have $\alpha \in \text{ST}(P(\max(\text{Dom}(AVS(\alpha)))) \subseteq \text{ST}(C(\alpha)) \subseteq \text{STSF}(AVP(\alpha)) \subseteq \text{STSEQ}(\alpha_2)$ and thus a contradiction. Therefore $\alpha_4 \notin \text{CdIF}(\alpha_3) \cup \text{NIF}(\alpha_3) \cup \text{PEF}(\alpha_3)$. If $\alpha_5 \in \text{CdIF}(\alpha_4) \cup \text{NIF}(\alpha_4) \cup \text{PEF}(\alpha_4)$, then we would have $\alpha \in \text{ST}(P(\max(\text{Dom}(AVS(\alpha)))) \cup \text{ST}(C(\alpha)) \subseteq \text{STSEQ}(\alpha_2)$ and thus again a contradiction. Therefore $\alpha_5 \notin \text{CdIF}(\alpha_4) \cup \text{NIF}(\alpha_4) \cup \text{PEF}(\alpha_4)$.

On the other hand, we have that $\alpha_3 \in \text{AF}(\alpha_2)$ and thus $\alpha_3 \in \text{RCS}$ and, with Theorem 3-15-(vi), $C(\alpha), C(\alpha_1), P(\max(\text{Dom}(AVS(\alpha)))) \in AVP(\alpha_2) \cup \{P(\max(\text{Dom}(AVS(\alpha))))\} = AVP(\alpha_3)$ and, with Theorem 3-15-(viii), $AVAP(\alpha_3) = AVAP(\alpha_2) \cup \{P(\max(\text{Dom}(AVS(\alpha))))\} \subseteq AVAP(\alpha) \cup AVAP(\alpha')$. Next, we have $\alpha_4 \in \text{IIF}(\alpha_2)$ and thus $\alpha_4 \in \text{RCS}$ and, with Theorem 3-25, $AVS(\alpha_4) = AVS(\alpha_3) \cup \{\text{Dom}(\alpha_3), \text{Therefore } \alpha = \alpha' \}$). Thus we have $AVAP(\alpha_4) = AVAP(\alpha_3) \subseteq AVAP(\alpha) \cup AVAP(\alpha')$ and $C(\alpha), C(\alpha_1), P(\max(\text{Dom}(AVS(\alpha)))) \in AVP(\alpha_3) \subseteq AVP(\alpha_4)$. Because of $C(\alpha_1) = P(\max(\text{Dom}(AVS(\alpha)))) \rightarrow C(\alpha')$, we have $\alpha_5 \in \text{CdEF}(\alpha_4) \subseteq \text{RCS} \setminus \{\emptyset\}$. With Theorem 3-25, we have $AVS(\alpha_5) = AVS(\alpha_4) \cup \{\text{Dom}(\alpha_4), \text{Therefore } C(\alpha') \}$). Thus we have $AVAP(\alpha_5) = AVAP(\alpha_4) \subseteq AVAP(\alpha) \cup AVAP(\alpha')$ and $C(\alpha) \in AVP(\alpha_4) \subseteq AVP(\alpha_5)$ and, with Theorem 2-82, $C(\alpha_5) = C(\alpha_3) \in AVP(\alpha_5)$ and $\alpha_5 \in \text{RCS} \setminus \{\emptyset\}$. $\alpha_5$ is thus the desired RCS-element. ■
4.2 Properties of the Deductive Consequence Relation

Now, we will establish some usual theorems about the deductive consequence relation. In particular, we will show that the deductive consequence relation is reflexive (Theorem 4-15), monotone (Theorem 4-16), closed under the introduction and elimination of logical operators (Theorem 4-18) and transitive (Theorem 4-19).

**Theorem 4-15. Extended reflexivity (AR)**
If $X \subseteq CFORM$ and $A \in X$, then $X \vdash A$.

*Proof:* Suppose $X \subseteq CFORM$ and $A \in X$. Then we have $A \in CFORM$ and, according to Definition 3-1, we have that $\{(0, \text{"Suppose } A\})\} \in AF(\emptyset) \subseteq RCS\setminus \{\emptyset\}$ and we have $C(\{(0, \text{"Suppose } A\})\}) = A$ and $AVP(\{(0, \text{"Suppose } A\})\}) = \{A\} \subseteq X$. With Theorem 3-12, we thus have $X \vdash A$. ■

**Theorem 4-16. Monotony**
If $X \vdash B$ and $X \subseteq Y \subseteq CFORM$, then $Y \vdash B$.

*Proof:* Suppose $X \vdash B$ and $X \subseteq Y \subseteq CFORM$. With Theorem 3-12, there is then an $\delta \in RCS\setminus \{\emptyset\}$ such that $AVP(\delta) \subseteq X$ and $C(\delta) = B$. Then we have $AVP(\delta) \subseteq Y$ and thus $Y \vdash B$. ■

**Theorem 4-17. Principium non contradictionis**
If $X \cup \{\Gamma\} \subseteq CFORM$, then $X \vdash \neg(\Gamma \land \neg \Gamma)$.

*Proof:* Suppose $X \cup \{\Gamma\} \subseteq CFORM$. Now, let $\delta$ be the following sentence sequence:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Suppose $\Gamma \land \neg \Gamma$</td>
</tr>
<tr>
<td>1</td>
<td>Therefore $\Gamma$</td>
</tr>
<tr>
<td>2</td>
<td>Therefore $\neg \Gamma$</td>
</tr>
<tr>
<td>3</td>
<td>Therefore $\neg(\Gamma \land \neg \Gamma)$</td>
</tr>
</tbody>
</table>

According to Definition 3-1, we have $\delta|1 \in AF(\emptyset) \subseteq RCS\setminus \{\emptyset\}$ and, with Theorem 3-15, we have $AVS(\delta|1) = \{(0, \text{"Suppose } \Gamma \land \neg \Gamma\})\} = \delta|1$ and $AVP(\delta|1) = \{\Gamma \land \neg \Gamma\}$ and $AVAS(\delta|1) = \{(0, \text{"Suppose } \Gamma \land \neg \Gamma\})\} \cupd AVP(\delta|1) = \{\Gamma \land \neg \Gamma\}$. According to Definition 3-5, we then have $\delta|2 \in CEF(\delta|1) \subseteq RCS\setminus \{\emptyset\}$. Since, with Theorem 1-8, $\Gamma$...
\( \land \lnot \Gamma' \not\in SF(\Gamma) \), we have, with Definition 3-2, Definition 3-10 and Definition 3-15, that \( \delta \vdash 2 \not\in CdIF(\delta \vdash 1) \cup NIF(\delta \vdash 1) \cup PEF(\delta \vdash 1) \). With Theorem 3-25, it then follows that AVS(\( \delta \vdash 2 \)) = AVS(\( \delta \vdash 1 \)) \cup \{1, \lnot \therefore \Gamma' \}\( = \delta \vdash 2 \). We also have with Theorem 3-27-(ii) and -(iii) that AVAS(\( \delta \vdash 2 \)) = AVAS(\( \delta \vdash 1 \)) and thus AVAP(\( \delta \vdash 2 \)) = AVAP(\( \delta \vdash 1 \)) = \{\lnot \Gamma' \land \lnot \Gamma'\}.

With Definition 3-5, we then have \( \delta \vdash 3 \in CEF(\delta \vdash 2) \subseteq RCS\{\emptyset\} \). Since, with Theorem 1-8, \( \Gamma' \land \lnot \Gamma' \not\in SF(\lnot \Gamma' \land \Gamma') \) and \( \Gamma \not\lnot \Gamma' \land \Gamma' \), we have, with Definition 3-2, Definition 3-10 and Definition 3-15 that \( \delta \vdash 3 \not\in CdIF(\delta \vdash 2) \cup NIF(\delta \vdash 2) \cup PEF(\delta \vdash 2) \). With Theorem 3-25, it then follows that AVS(\( \delta \vdash 3 \)) = AVS(\( \delta \vdash 2 \)) \cup \{1, \lnot \therefore \lnot \Gamma' \}\( = \delta \vdash 3 \) and, with Theorem 3-27-(ii) and -(iii), that AVAS(\( \delta \vdash 3 \)) = AVAS(\( \delta \vdash 2 \)) and thus that AVAP(\( \delta \vdash 3 \)) = AVAP(\( \delta \vdash 2 \)) = \{\lnot \Gamma' \land \lnot \Gamma'\}. Then we have \( 0 = \max(\text{Dom}(AVAS(\delta \vdash 3))) \) and \( 1, 2 \in \text{Dom}(AVAS(\delta \vdash 3))) \) and P(\( \delta \vdash 3_1 \)) = \( \Gamma' \) and P(\( \delta \vdash 3_2 \)) = \( \lnot \Gamma' \). According to Definition 3-10, we thus have \( \delta \in NIF(\delta \vdash 3) \). According to Theorem 3-20, we have AVAS(\( \delta \)) = AVAS(\( \delta \vdash 3 \)), \{0, \lnot \Gamma' \land \lnot \Gamma' \}\( = \emptyset \) and thus also AVAP(\( \delta \)) = \( \emptyset \). Hence we have \( \delta \in RCS\{\emptyset\} \) and AVAP(\( \delta \)) = \( \emptyset \) and C(\( \delta \)) = \( \lnot (\Gamma' \land \lnot \Gamma') \). With Theorem 3-12, we then have \( 0 \vdash \lnot (\Gamma' \land \lnot \Gamma') \) and thus it holds, with Theorem 4-16, that \( \overline{\overline{X}} \vdash \lnot (\Gamma' \land \lnot \Gamma') \).■

**Theorem 4-18. Closure under introduction and elimination**

If A, B, \( \Gamma \in \text{CFORM}, \theta_0, \theta_1 \in \text{CTERM}, \xi \in \text{VAR} \) and \( \Delta \in \text{FORM} \), where FV(\( \Delta \)) \subseteq \{\xi\}, then:

1. If \( \overline{\overline{X}} \vdash B \) and \( A \in X \), then \( X \backslash \{A\} \vdash \lnot A \rightarrow B \), (Cdl)
2. If \( \overline{\overline{X}} \vdash A \land Y \vdash \lnot A \rightarrow B \), then \( X \cup Y \vdash B \), (Cdl)
3. If \( \overline{\overline{X}} \vdash A \land Y \vdash B \), then \( X \cup Y \vdash \lnot A \land B \), (Cl)
4. If \( \overline{\overline{X}} \vdash A \rightarrow B \) or \( \overline{\overline{X}} \vdash B \rightarrow A \), then \( A \vdash A \land B \), (CE)
5. If \( \overline{\overline{X}} \vdash A \rightarrow B \) and \( Y \vdash B \rightarrow A \), then \( X \cup Y \vdash \lnot A \leftrightarrow B \), (BI)
6. If \( X \vdash B \) and \( A \in X \) and \( \overline{\overline{Y}} \vdash \lnot A \land B \in Y \), then \( (X \backslash \{A\}) \cup (Y \backslash \{B\}) \vdash A \leftrightarrow B \), (BI*)
7. If \( \overline{\overline{X}} \vdash A \land Y \vdash \lnot A \leftrightarrow B \) or \( Y \vdash \lnot B \leftrightarrow A \), then \( X \cup Y \vdash \lnot A \leftrightarrow B \), (BE)
8. If \( \overline{\overline{X}} \vdash A \lor X \vdash B \), then \( X \vdash A \lor B \), (DI)
9. If \( \overline{\overline{X}} \vdash A \lor B \) and \( Y \vdash A \rightarrow \Gamma \) and \( Z \vdash \lnot B \rightarrow \Gamma \), then \( X \cup Y \cup Z \vdash \Gamma \), (DE)
10. If \( \overline{\overline{X}} \vdash \lnot A \lor B \) and \( Y \vdash \lnot \Gamma \) and \( A \in Y \) and \( \overline{\overline{Z}} \vdash \Gamma \) and \( B \in Z \), then \( X \cup \overline{\overline{Y}} \backslash \{A\} \cup \overline{\overline{Z}} \backslash \{B\} \vdash \Gamma \), (DE*)
11. If \( \overline{\overline{X}} \vdash \Gamma \land \lnot \Gamma \) and \( A \in X \) and \( Y \vdash \lnot A \rightarrow \Gamma \), then \( (X \cup Y) \backslash \{A\} \vdash \lnot \lnot A \rightarrow \Gamma \), (NI)
12. If \( \overline{\overline{X}} \vdash \lnot \lnot \Gamma \), then \( X \vdash \Gamma \), (NE)
13. If \( \overline{\overline{X}} \vdash [\beta, \xi, \Delta] \) and \( \beta \not\in \text{STSF}(X \cup \{\Delta\}) \), then \( X \vdash \lnot \Lambda \xi \Delta \), (UI)
Hence we have \( \emptyset \in \emptyset \).

First, we have

\[
\text{Proof: Suppose } A, B, \Gamma \in \text{CFORM}, \theta_0, \theta_1 \in \text{CTERM}, \xi \in \text{VAR} \text{ and } \Delta \in \text{FORM}, \text{ where } FV(\Delta) \subseteq \{\xi\}. \text{ First, we will deal with case (i), in which the set of premises is reduced. Then we will treat the cases (ii), (iii), (v), (vii) and (xvii), in which two premise sets are joined. In the cases (iv), (viii), (xii), (xiii), (xiv) and (xv), the premise set does not change. The remaining special cases will be dealt with in the order (vi), (ix), (x), (xi), (xvi), (xvii).}

\text{Ad (ii) (CdI): Suppose } X \vdash B \text{ and } A \in X. \text{ According to Theorem 3-12, there is then an } \emptyset \in \text{RCS}\{\emptyset\} \text{ such that } C(\emptyset) = B \text{ and } \text{AVAP}(\emptyset) \subseteq X. \text{ With Theorem 4-2, there is then an } \emptyset \in \text{RCS}\{\emptyset\} \text{ such that } \text{AVAP}(\emptyset) \subseteq \text{AVAP}(\emptyset) \text{ and } C(\emptyset) = C(\emptyset) \text{ and for all } i \in \text{Dom}^{2}(\text{AVAS}(\emptyset)):\text{ If } P(\emptyset) = A, \text{ then } i = \max(\text{Dom}(\text{AVAS}(\emptyset))). \text{ With Theorem 2-82, we then have } B = C(\emptyset) \subseteq \text{AVP}(\emptyset). \text{ With } A \in \text{AVP}(\emptyset) \text{ and } A \notin \text{AVP}(\emptyset), \text{ we can now distinguish two cases.}

\text{First case: Suppose } A \in \text{AVP}(\emptyset). \text{ Then we have } \text{AVAS}(\emptyset) \neq \emptyset \text{ and it holds for all } i \in \text{Dom}^{2}(\text{AVAS}(\emptyset)): P(\emptyset) = A \text{ iff } i = \max(\text{Dom}(\text{AVAS}(\emptyset))). \text{ With Theorem 3-18, we then have } \emptyset = \emptyset \in \emptyset \in \{\emptyset\} \in \{\emptyset\} \in \{\emptyset\} \in \{\emptyset\} \subseteq \text{RCS}\{\emptyset\} \subseteq \text{RCS}\{\emptyset\}. \text{ With Theorem 3-22, it then holds that } \text{AVP}(\emptyset) \subseteq \text{AVAP}(\emptyset) \subseteq \text{AVP}(\emptyset) \subseteq \text{AVP}(\emptyset) \subseteq X \setminus \{\emptyset\}. \text{ Hence we have } \emptyset \in \text{RCS}\{\emptyset\}, C(\emptyset) = A \rightarrow B^\ast \text{ and } \text{AVAP}(\emptyset) \subseteq X \setminus \{A\} \text{ and thus, with Theorem 3-12, } \emptyset \vdash A \rightarrow B^\ast.

\text{Second case: Now, suppose } A \notin \text{AVP}(\emptyset). \text{ Then we can extend } \emptyset \text{ as follows to an } \emptyset \in \text{SEQ with } \emptyset \in \text{Dom}(\emptyset) = \emptyset:\n
\begin{align*}
\emptyset^1 & = \emptyset^1 \\
\emptyset^2 & = \emptyset^1 \\
\emptyset^3 & = \emptyset^2 \\
\emptyset^4 & = \emptyset^3 \\
\end{align*}

First, we have \( \emptyset^4 \in \text{ASENT. With Theorem 1-8, Theorem 1-10 and Theorem 1-11, we have } C(\emptyset^1) \neq C(\emptyset^2) \text{ und } C(\emptyset^2) \neq C(\emptyset^3). \text{ We also have that } C(\emptyset^3) \text{ is neither a condi-}
tional nor a negation. We further have with Theorem 1-8 that \( C(\delta^3) = B \neq \neg A \rightarrow (A \land B) \) and that \( P(\delta^3_{\text{Dom}(\delta^3)}) = A \neq \neg(C(\delta^3_{\text{Dom}(\delta^3)}) \land \neg P(\delta^3_{\text{Dom}(\delta^3)}). \) With Theorem 2-42, Definition 2-11, Definition 2-12 and Definition 2-13, we then have that it holds for all \( k \) with \( 1 \leq k \leq 3 \) that there is no closed segment \( \mathfrak{A} \) in \( \delta^k \) such that \( \min(\text{Dom}(\mathfrak{A})) = \text{Dom}(\delta^3). \) With Theorem 2-47, we thus have for all \( k \) with \( 1 \leq k \leq 3 \) that there is no closed segment \( \mathfrak{A} \) in \( \delta^k \) such that \( \min(\text{Dom}(\mathfrak{A})) \leq \text{Dom}(\delta^3) \leq \max(\text{Dom}(\mathfrak{A})). \) Thus we also get that it holds for all \( k \) with \( 1 \leq k \leq 3 \) that \( \text{Dom}(\delta^3) = \max(\text{Dom}(\text{AVAS}(\delta^3))). \) With Theorem 3-19-(i), Theorem 3-20-(i), Theorem 3-21-(i) and Theorem 2-61, we then have for all \( k \) with \( 2 \leq k \leq 3 \) that \( \delta^k \not\in \text{CdIF}(\delta^{k-1}) \cup \text{NIF}(\delta^{k-1}) \cup \text{PEF}(\delta^{k-1}). \)

On the other hand, we first have, according to Definition 3-1, \( \delta^1 \in \text{AF}(\delta^1) \subseteq \text{RCS}\setminus\{\emptyset\} \) and, with Theorem 3-15, \( \text{AVS}(\delta^1) = \text{AVS}(\delta^1) \cup \{(\text{Dom}(\delta^1), "\text{Suppose } A")\} \) and \( (\text{Dom}(\delta^1), "\text{Suppose } A") \in \text{AVAS}(\delta^1) \cup \{(\text{Dom}(\delta^1), "\text{Suppose } A")\} = \text{AVS}(\delta^1) \) and \( B \in \text{AVP}(\delta^1) \subseteq \text{AVP}(\delta^1) \) and \( A \in \text{AVP}(\delta^1) \). Therefore we have second, according to Definition 3-4, \( \delta^2 \in \text{CEF}(\delta^1) \subseteq \text{RCS}\setminus\{\emptyset\} \) and, with Theorem 3-25, \( \text{AVS}(\delta^2) = \text{AVS}(\delta^1) \cup \{(\text{Dom}(\delta^2), "\text{Therefore } A \land B")\}. \) Thus we have \( (\text{Dom}(\delta^1), "\text{Suppose } A") \in \text{AVAS}(\delta^1) \) and \( A \land B \in \text{AVP}(\delta^1) \). Therefore we have third, according to Definition 3-5, \( \delta^3 \in \text{CEF}(\delta^2) \subseteq \text{RCS}\setminus\{\emptyset\} \) and, with Theorem 3-25, \( \text{AVS}(\delta^3) = \text{AVS}(\delta^2) \cup \{(\text{Dom}(\delta^3), "\text{Therefore } B")\}. \) Thus we have \( \text{Dom}(\delta^1) \in \text{Dom}(\delta^3) \) and \( \text{P}(\delta^3_{\text{Dom}(\delta^3)}) = A \) and \( (\text{Dom}(\delta^1), "\text{Suppose } A") \in \text{AVAS}(\delta^3) = \text{AVS}(\delta^3) \) and \( \text{P}(\delta^3_{\text{Dom}(\delta^3)}) = A \) and there is no \( l \) such that \( \text{Dom}(\delta^3) < l \leq \text{Dom}(\delta^3-1) \) and \( (l, \delta^3 l) \in \text{AVAS}(\delta^3) \). According to Definition 3-2, we thus have \( \delta^4 \in \text{CdIF}(\delta^3) \subseteq \text{RCS}\setminus\{\emptyset\} \) and, with Theorem 3-19-(iv) and -(v), \( \text{AVAS}(\delta^4) = \text{AVS}(\delta^3) \cup \{(\max(\text{Dom}(\text{AVAS}(\delta^3))), \delta^4_{\text{max}(\text{Dom}(\text{AVAS}(\delta^3)))}\) = \( \text{AVAS}(\delta^3) \cup \{(\text{Dom}(\delta^3), "\text{Suppose } A")\} = \text{AVS}(\delta^3) \cup \{(\text{Dom}(\delta^3), "\text{Suppose } A")\} = \text{AVAS}(\delta^3) \cup \{(\text{Dom}(\delta^3), "\text{Suppose } A")\} \subseteq \text{AVAS}(\delta^3). \) With Theorem 2-75, we then have \( \text{AVAP}(\delta^3) \subseteq \text{AVAP}(\delta^4) \) and, because of \( A \not\in \text{AVAP}(\delta^3) \) and \( \text{AVAP}(\delta^4) \subseteq \text{AVAP}(\delta) \subseteq X, \) we then also have \( \text{AVAP}(\delta^4) \subseteq \text{AVAP}(\delta) \setminus \{A\} \subseteq X \setminus \{A\}. \) Since \( C(\delta^4) = \neg A \rightarrow B, \) it holds, with Theorem 3-12, that \( X \setminus \{A\} \vdash \neg A \rightarrow B. \)

Ad (ii) (CdE), (iii) (Cl), (v) (Bl), (vii) (BE), (xviii) (IE): We prove (ii) exemplarily, clauses (iii), (v), (vii) and (xviii) are shown analogously. Suppose for (ii) that \( X \vdash A \) and \( Y \vdash \neg A \rightarrow B. \) According to Theorem 3-12, there are then \( \delta, \delta^1 \in \text{RCS}\setminus\{\emptyset\} \) such that \( \text{AVAP}(\delta) \subseteq X \) and \( C(\delta) = A \) and \( \text{AVAP}(\delta^1) \subseteq Y \) and \( C(\delta^1) = A \rightarrow B. \) With Theorem
4-14, there is then an STSF({}) such that A, \( \neg A \rightarrow B \) \( \in \) AVP(\( \delta_5 \)) and AVAP(\( \delta_5 \)) \( \subseteq \) AVAP(\( \delta_j \)) \( \cup \) AVAP(\( \delta_j' \)) \( \subseteq \) X \( \cup \) Y. According to Definition 3-3, we then have \( \delta_j^+ = \delta_j^* \cap \{0, \neg \text{Therefore } B\} \) \( \in \) CdEF(\( \delta_j^* \)) \( \subseteq \) RCS\( \setminus \{0\} \) and, with Theorem 3-27-(v), we have AVAP(\( \delta_j' \)) \( \subseteq \) AVAP(\( \delta_j^* \)) \( \subseteq \) X \( \cup \) Y and we have C(\( \delta_j^+ \)) = B. It then holds, with Theorem 3-12, that X \( \cup \) Y \( \vdash \) B.

**Ad (iv) (CE), (viii) (DI), (xii) (NE), (xiii) (UI), (xiv) (UE), (xv) (PI):** We prove (iv) exemplarily, clauses (viii), (xii), (xiii), (xiv) and (xv) are shown analogously. Suppose for (iv) that X \( \vdash \) \( \neg A \land B \) or X \( \vdash \) \( \neg B \land A \). Now, suppose X \( \vdash \) \( \neg A \land B \). According to Theorem 3-12, there is then an \( \delta_j \) \( \in \) RCS\( \setminus \{0\} \) such that AVAP(\( \delta_j \)) \( \subseteq \) X and C(\( \delta_j \)) = \( \neg A \land B \). With Theorem 2-82, we have \( \neg A \land B \) \( \in \) AVP(\( \delta_j \)) and thus, according to Definition 3-5, \( \delta_j' = \delta_j^\cap \{0, \neg \text{Therefore } A\} \) \( \in \) CEF(\( \delta_j \)) \( \subseteq \) RCS\( \setminus \{0\} \) and, with Theorem 3-27-(v), we have AVAP(\( \delta_j' \)) \( \subseteq \) AVAP(\( \delta_j \)) \( \subseteq \) X and we have C(\( \delta_j' \)) = A. With Theorem 3-12, we then have X \( \vdash \) A. In the case that X \( \vdash \) \( \neg B \land A \), one shows analogously that X \( \vdash \) A holds as well.

**Ad (vi) (BI*):** Suppose X \( \vdash \) B and A \( \in \) X and Y \( \vdash \) A and B \( \in \) Y. With (i), we then have X\( \setminus \{A\} \vdash \) \( \neg A \rightarrow B \) and Y\( \setminus \{B\} \vdash \) \( \neg B \rightarrow A \). With (v), it then holds that (X\( \setminus \{A\} \cup (Y\setminus \{B\}) \vdash \) \( \neg A \leftrightarrow B \).

**Ad (ix) (DE):** Suppose X \( \vdash \) \( A \lor B \) and Y \( \vdash \) \( A \rightarrow \Gamma \) and Z \( \vdash \) \( B \rightarrow \Gamma \). By double application of (iii), we then get X \( \cup \) Y \( \cup \) Z \( \vdash \) \( (A \lor B) \land ((A \rightarrow \Gamma) \land (B \rightarrow \Gamma)) \). With Theorem 3-12, there is then an \( \delta_j \) \( \in \) RCS\( \setminus \{0\} \) such that AVAP(\( \delta_j \)) \( \subseteq \) X \( \cup \) Y \( \cup \) Z and C(\( \delta_j \)) = \( (A \lor B) \land ((A \rightarrow \Gamma) \land (B \rightarrow \Gamma)) \). There is an \( a \in \text{CONST}\setminus\text{STSEQ}(\delta_j) \). Thus we can extend \( \delta_j \) as follows to an \( \delta_6 \) \( \in \) SEQ with \( \delta_6 \cup \text{Dom}(\delta_j) = \delta_j^\ast \):

\[
\begin{align*}
\delta_1^1 & = \delta_j \cup \{(\text{Dom}(\delta_j), \neg \text{Therefore } a = a)\} \\
\delta_2^1 & = \delta_1 \cup \{(\text{Dom}(\delta_1), \text{Therefore } A \lor B)\} \\
\delta_3^1 & = \delta_2 \cup \{(\text{Dom}(\delta_2), \text{Therefore } (A \rightarrow \Gamma) \land (B \rightarrow \Gamma))\} \\
\delta_4^1 & = \delta_3 \cup \{(\text{Dom}(\delta_3), \text{Therefore } A \rightarrow \Gamma)\} \\
\delta_5^1 & = \delta_4 \cup \{(\text{Dom}(\delta_4), \text{Therefore } B \rightarrow \Gamma)\} \\
\delta_6^1 & = \delta_5 \cup \{(\text{Dom}(\delta_5), \text{Therefore } \Gamma)\}.
\end{align*}
\]

First, we have \( \delta_6^{\text{Dom}(\delta_j)} \) \( \in \) ASENT. With \( a \in \text{CONST}\setminus\text{STSEQ}(\delta_j) \), we also have \( a \notin \text{STSF}(\{A, B, \Gamma\}) \) and thus we have for all \( k \) with 1 \( \leq \) k \( \leq \) 6: If \( i \in \text{Dom}(\delta_j^k) \), then: \( a \in \)
ST(δ_k) iff i = Dom(δ). Furthermore, it holds for all k with 1 ≤ k ≤ 6 that Dom(δ) ∈ Dom(AS(δ^k)). With Theorem 4-3, we thus have for all k with 1 ≤ k ≤ 6: There is no closed segment ∃l in δ^k such that min(Dom(∃l)) ≤ Dom(δ) ≤ max(Dom(∃l)). Thus we also get that for all k with 1 ≤ k ≤ 6 it holds that Dom(δ) = max(Dom(AVAS(δ^k))). With Theorem 3-19-(i), Theorem 3-20-(i), Theorem 3-21-(i) and Theorem 2-61, we then have that for all k with 2 ≤ k ≤ 6 it holds that δ_k ∈ CIR(δ^k-1) ∪ NIF(δ^k-1) ∪ PEF(δ^k-1).

On the other hand, we have first, according to Definition 3-1, δ^1 ∈ AF(δ) ⊆ RCS\{∅} and, with Theorem 3-15, AVS(δ^1) = AVS(δ) ∪ {(Dom(δ), "Suppose α = α′"); and AVAS(δ^1) = AVAS(δ) ∪ {(Dom(δ), "Suppose α = a′") and γ(A ∨ B) ∧ ((A → Γ) ∧ (B → Γ)) ∈ AVP(δ) ⊆ AVP(δ^1). Therefore we have second, according to Definition 3-5, δ^2 ∈ CEF(δ^1) ⊆ RCS\{∅} and, with Theorem 3-25, AVS(δ^2) = AVS(δ^1) ∪ {(Dom(δ^1), "Therefore A ∨ B′"). Thus we have AVAS(δ^2) = AVAS(δ^1), γ(A ∨ B) ∧ ((A → Γ) ∧ (B → Γ)) ∈ AVP(δ) ⊆ AVP(δ^2) and γ(A ∨ B′) ∈ AVP(δ^2). Therefore we have third, according to Definition 3-5, δ^3 ∈ CEF(δ^2) ⊆ RCS\{∅} and, with Theorem 3-25, AVS(δ^3) = AVS(δ^2) ∪ {(Dom(δ^2), "Therefore (A → Γ) ∧ (B → Γ)′")}. Thus we have AVAS(δ^3) = AVAS(δ^2), "A ∨ B′ ∈ AVP(δ^2) ⊆ AVP(δ^3) and γ(A → Γ) ∧ (B → Γ) ∈ AVP(δ^3). Therefore we have fourth, according to Definition 3-5, δ^4 ∈ CEF(δ^3) ⊆ RCS\{∅} and, with Theorem 3-25, AVS(δ^4) = AVS(δ^3) ∪ {(Dom(δ^3), "Therefore A → Γ′"). Thus we have AVAS(δ^4) = AVAS(δ^3), "A ∨ B′, γ(A → Γ) ∧ (B → Γ) ∈ AVP(δ^3) ⊆ AVP(δ^4) and γ(A → Γ) ∈ AVP(δ^4). Therefore we have fifth, according to Definition 3-5, δ^5 ∈ CEF(δ^4) ⊆ RCS\{∅} and, with Theorem 3-25, AVS(δ^5) = AVS(δ^4) ∪ {(Dom(δ^4), "Therefore B → Γ′"). Thus we have AVAS(δ^5) = AVAS(δ^4), "A ∨ B′, γ(A → Γ) ∈ AVP(δ^4) ⊆ AVP(δ^5) and γ(B → Γ′) ∈ AVP(δ^5). Finally, we have sixth, according to Definition 3-9, δ^6 ∈ DEF(δ^5) ⊆ RCS\{∅} and, with Theorem 3-25, AVS(δ^6) = AVS(δ^5) ∪ {(Dom(δ^5), "Therefore Γ′"). Thus we have AVAS(δ^6) = AVAS(δ^5), "A ∨ B′, γ(A → Γ) ∈ AVP(δ^5) and γ(G → Γ′) ∈ AVP(δ^6). With Theorem 4-7, there is then an δ^7 ∈ RCS\{∅} such that AVAP(δ^7) ⊆ AVAP(δ^6)\{"α = a′"} = AVAP(δ) ∪ {"α = a′")\{"α = a′"} = AVAP(δ)\{"α = a′"} ⊆ X ∪ Y ∪ Z\{"α = a′"} ⊆ X ∪ Y ∪ Z and C(δ^7) = Γ. With Theorem 3-12, we then have X ∪ Y ∪ Z ⊢ Γ.
**Ad (xi) (DE*)**: Suppose \( X \vdash \Gamma \land B^{\gamma} \) and \( Y \vdash \Gamma \) and \( A \in Y \) and \( Z \vdash \Gamma \) and \( B \in Z \). Then it holds with (i): \( Y \setminus \{ A \} \vdash \Gamma \land B^{\gamma} \) and \( Z \setminus \{ B \} \vdash \Gamma \land B^{\gamma} \). Then it holds with (ix): \( X \cup (Y \setminus \{ A \}) \cup (Z \setminus \{ B \}) \vdash \Gamma \).

**Ad (xi) (NI)**: Suppose \( X \vdash \Gamma \land B^{\gamma} \) and \( A \in X \cup Y \). If \( A = \Gamma^{\gamma} \land \neg \Delta^{\gamma} \) for a \( \Delta \in \text{CFORM} \), then it holds, with Theorem 4-17, that \( (X \cup Y) \setminus \{ A \} \vdash \neg((\Gamma^{\gamma} \land \neg \Delta^{\gamma})^{\gamma} = \neg \Delta^{\gamma} \). Now, suppose \( A \neq \Gamma^{\gamma} \land \neg \Delta^{\gamma} \) for all \( \Delta \). With (iii), it holds that \( X \cup Y \vdash \Gamma \land \neg \Gamma^{\gamma} \). Also, we have, again with Theorem 4-17, \( X \cup Y \vdash \neg((\Gamma^{\gamma} \land \neg \Gamma)^{\gamma} \land \neg((\Gamma^{\gamma} \land \neg \Gamma)))^{\gamma} \). Thus there is, with Theorem 3-12, an \( \delta \in \text{RCS}\{\emptyset\} \) such that \( \text{AVAP}(\delta) \subseteq (X \cup Y) \setminus \{ A \} \) and \( C(\delta) = (\Gamma^{\gamma} \land \neg \Gamma) \land \neg((\Gamma^{\gamma} \land \neg \Gamma))^{\gamma} \). Then we can extend \( \delta \) as follows to an \( \delta^{5} \in \text{SEQ} \) with \( \delta^{5} \upharpoonright \text{Dom}(\delta) = \delta_{1}^{\gamma} \):

\[
\begin{align*}
\delta_{1}^{\gamma} &= \delta_{1} \cup \{ \text{Dom}(\delta), \text{Suppose } A^{\gamma} \} \\
\delta_{2}^{\gamma} &= \delta_{1}^{\gamma} \cup \{ \text{Dom}(\delta^{2}), \text{Therefore } (\Gamma^{\gamma} \land \neg \Gamma) \land \neg((\Gamma^{\gamma} \land \neg \Gamma))^{\gamma} \} \\
\delta_{3}^{\gamma} &= \delta_{2}^{\gamma} \cup \{ \text{Dom}(\delta^{3}), \text{Therefore } \Gamma^{\gamma} \land \neg \Gamma^{\gamma} \} \\
\delta_{4}^{\gamma} &= \delta_{3}^{\gamma} \cup \{ \text{Dom}(\delta^{4}), \text{Therefore } \neg((\Gamma^{\gamma} \land \neg \Gamma))^{\gamma} \} \\
\delta_{5}^{\gamma} &= \delta_{4}^{\gamma} \cup \{ \text{Dom}(\delta^{5}), \text{Therefore } \neg A^{\gamma} \}.
\end{align*}
\]

First, we have \( \delta_{5}^{\gamma} \upharpoonright \text{Dom}(\delta) \in \text{ASENT} \). By hypothesis, we have \( C(\delta_{1}^{\gamma}) = A \neq C(\delta_{2}^{\gamma}) \). With Theorem 1-8, Theorem 1-10 and Theorem 1-11 we have \( C(\delta_{2}^{\gamma}) \neq C(\delta_{3}^{\gamma}) \) and \( C(\delta_{3}^{\gamma}) \neq C(\delta_{4}^{\gamma}) \). We also have that \( C(\delta_{2}^{\gamma}) \) and \( C(\delta_{3}^{\gamma}) \) are neither conditionals nor negations and that \( C(\delta_{4}^{\gamma}) \) is not a conditional and by hypothesis \( C(\delta_{5}^{\gamma}) = \neg((\Gamma^{\gamma} \land \neg \Gamma)^{\gamma} \neq \neg A^{\gamma} \). With Theorem 2-42, Definition 2-11, Definition 2-12 and Definition 2-13, we then have that it holds for all \( k \) with \( 1 \leq k \leq 4 \) that there is no closed segment \( \mathfrak{A} \) in \( \delta_{k}^{\gamma} \) such that \( \text{min}(\text{Dom}(\mathfrak{A})) = \text{Dom}(\delta) \). With Theorem 2-47, we then have that there is no closed segment \( \mathfrak{A} \) in \( \delta_{k}^{\gamma} \) such that \( \text{min}(\text{Dom}(\mathfrak{A})) \leq \text{Dom}(\delta) \leq \text{max}(\text{Dom}(\mathfrak{A})) \).

Thus we also get that it holds for all \( k \) with \( 1 \leq k \leq 4 \) that \( \text{Dom}(\delta) = \text{max}(\text{Dom}(\text{AVAS}(\delta^{5}))) \). With Theorem 3-19-(i), Theorem 3-20-(i), Theorem 3-21-(i) and Theorem 2-61, we thus have for all \( k \) with \( 2 \leq k \leq 4 \) that \( \delta_{k}^{\gamma} \notin \text{CdIF}(\delta_{k+1}^{\gamma}) \cup \text{NIF}(\delta_{k+1}^{\gamma}) \cup \text{PEF}(\delta_{k+1}^{\gamma}) \).

On the other hand, we have *first*, according to Definition 3-1, \( \delta_{1}^{\gamma} \in \text{AF}(\delta) \subseteq \text{RCS}\{\emptyset\} \) and, with Theorem 3-15, \( \text{AVS}(\delta_{1}^{\gamma}) = \text{AVS}(\delta) \cup \{ \text{Dom}(\delta), \text{Suppose } A^{\gamma} \} \) and \( \text{AVAS}(\delta_{1}^{\gamma}) = \text{AVAS}(\delta) \cup \{ \text{Dom}(\delta), \text{Suppose } A^{\gamma} \}, A \to ((\Gamma^{\gamma} \land \neg \Gamma) \land \neg((\Gamma^{\gamma} \land \neg \Gamma))^{\gamma} \in
AVP(Δ) ⊆ AVP(Δ′) and A ∈ AVP(Δ′). Then we have second, according to Definition 3-3, \(Δ^5\) ∈ CdEF(Δ′) ⊆ RCS\(\{0\}\) and, with Theorem 3-25, AVS(Δ′) = AVS(Δ) \cup \{(Dom(Δ)), \]) Therefore (Γ ∧ −Γ) ∧ −(Γ ∧ −Γ)\). Thus we have AVAS(Δ′) = AVAS(Δ′) and \(Γ ∧ −Γ\) ∈ AVP(Δ′). Therefore we have third, according to Definition 3-5, \(Δ^5\) ∈ CEFS(Δ′) ⊆ RCS\(\{0\}\) and, with Theorem 3-25, AVS(Δ′) = AVS(Δ′) \cup \{(Dom(Δ), \]) \Therefore Γ ∧ −Γ\}\). Thus we have AVAS(Δ′) = AVAS(Δ′), \(Γ ∧ −Γ\) ∈ AVP(Δ′) and \(Γ ∧ −Γ\) ∈ AVP(Δ′). Then we have fourth, according to Definition 3-5, \(Δ^4\) ∈ CEFS(Δ′) ⊆ RCS\(\{0\}\) and, with Theorem 3-25, AVS(Δ′) = AVS(Δ′) \cup \{(Dom(Δ), \]) \Therefore −(Γ ∧ −Γ)\}\). Thus we have AVAS(Δ′) = AVAS(Δ′) and (Dom(Δ), \]) \Therefore Γ ∧ −Γ\}\), (Dom(Δ), \]) \Therefore −(Γ ∧ −Γ)\}\) ∈ AVS(Δ′) and (Dom(Δ), \]) \Suppose A\)\) ∈ AVAS(Δ′) = AVAS(Δ′).

Thus we have Dom(Δ), Dom(Δ′) ∈ Dom(Δ′), where Dom(Δ) ≤ Dom(Δ′), \(P(Δ^4_{Dom(Δ)}) = A\) and (Dom(Δ), \(Δ^5_{Dom(Δ′)}) ∈ AVAS(Δ′), \(P(Δ^4_{Dom(Δ′)}) = Γ ∧ −Γ\) and \(P(Δ^4_{Dom(Δ′)}) = Γ ∧ −Γ\). Finally we thus have fifth, according to Definition 3-10, \(Δ^5\) ∈ NIF(Δ′) ⊆ RCS\(\{0\}\) and, with Theorem 3-20-(iv) and - (v), AVAS(Δ′) = AVAS(Δ′) \{(max(Dom(AVAS(Δ′))))\}, \(Δ^5_{max(Dom(AVAS(Δ′))}) = AVAS(Δ′)\} \{(Dom(Δ), \]) \Suppose A\)\} = AVAS(Δ′) \{(Dom(Δ), \]) \Suppose A\)\} = (AVAS(Δ′) \cup \{(Dom(Δ), \]) \Suppose A\)\} \{(Dom(Δ), \]) \Suppose A\)\} = AVAS(Δ′) \{(Dom(Δ), \]) \Suppose A\)\} ⊆ AVAS(Δ). With Theorem 2-75, we then have AVAP(Δ′) ⊆ AVAP(Δ) ⊆ (X ∪ Y)\{A\}. Since C(Δ′) = −A, it holds, with Theorem 3-12, that \((X ∪ Y)\{A\} ⊢ −A\).

Ad (xvi) (PE): Suppose \(X ⊢ −\forall ϵA\) and \(Y ⊢ Γ\) and \([β, ϵ, Δ] ∈ Y\) and \(β \notin STSF((Y\{[β, ϵ, Δ]\}) \cup \{Δ, Γ\})\). Then it holds, with (i), that \(Y\{[β, ϵ, Δ]\} ⊢ −[β, ϵ, Δ] → Γ\). We also have with \(Γ ∈ CFORM\): \([β, ϵ, Δ] → Γ\). Thus we have \([β, ϵ, Δ] → Γ\) and \(Γ \rightarrow Γ\) and thus we have \(Y\{[β, ϵ, Δ]\} ⊢ β, ϵ, Δ → Γ\). With \(β \notin STSF(Δ, Γ)\), we have \(β \notin ST(Δ → Γ)\). With \(Γ ∈ CFORM\) and \(FV(Δ) \subseteq \{ξ\}\), we also have \(FV(Δ → Γ) \subseteq \{ξ\}\). Since by hypothesis also \(β \notin ST(Δ → Γ)\), it then follows, with (xv), that \(Y\{[β, ϵ, Δ]\} ⊢ −[β, ϵ, Δ] → Γ\). With (iii), we then have \(X ∪ (Y\{[β, ϵ, Δ]\}) ⊢ −[β, ϵ, Δ] → Γ\) and \(ΔA\).
According to Theorem 3-12, there is thus an \( \mathcal{F}_1 \) \( \in \text{RCS}\{\emptyset\} \) such that \( \text{AVAP}(\mathcal{F}_1) \subseteq X \cup (Y \setminus \{β, ξ, Δ\}) \) and \( \text{C}(\mathcal{F}_1) = \{\forall ξ(Δ → Γ) \land \forall ζ(Δ → Γ)\} \). With Theorem 4-5, there is then an \( \mathcal{F}_* \in \text{RCS}\{\emptyset\} \) such that \( \text{AVAP}(\mathcal{F}_*) = \text{AVAP}(\mathcal{F}_1) \subseteq X \cup (Y \setminus \{β, ξ, Δ\}) \) and \( \{\forall ξ(Δ → Γ)\} \). Furthermore we have, with Theorem 2-42, Definition 2-11, Definition 2-12 and Definition 2-13, that it holds for all \( \mathcal{F}_* \) that \( \text{AVS}(\mathcal{F}_*) \) is not a conditional and that \( \text{C}(\mathcal{F}_*) \) is not a negation. In addition we have \( \text{C}(\mathcal{F}_*) = \{\forall ξ(Δ → Γ)\} \). With Theorem 1-8, we also have \( \text{C}(\mathcal{F}_*) \neq C(\mathcal{F}_*) \). Furthermore we have, with Theorem 1-10 and Theorem 1-11, that \( \text{C}(\mathcal{F}_*) \) is a conditional and that \( \text{C}(\mathcal{F}_*) \) is not a negation. In addition we have \( \text{C}(\mathcal{F}_*) = \{\forall ξ(Δ → Γ)\} \). With Theorem 2-42, Definition 2-11, Definition 2-12 and Definition 2-13, it then holds for all \( k \) with \( 1 ≤ k ≤ 4 \) that there is no closed segment \( \mathcal{S}_k \) in \( \mathcal{F}_n \) such that \( \min(\text{Dom}(\mathcal{S}_k)) = \text{Dom}(\mathcal{F}_*) \).
AVAS(δ^5) and \(\forall \xi (\Delta \to \Gamma)^\gamma\); \([\beta^*, \xi, \Delta] \in \text{AVP}(\delta^1) \cup \text{AVP}(\delta^2)\) and (Dom(\delta^*)-1, \(\delta^5_{\text{max}(\text{Dom}(\text{AVP}(\delta^4)))}) \in \text{AVS}(\delta^3).\) Therefore we have third, according to Definition 3-13, \(\delta^3 \in \text{UEF}(\delta^3) \subset \text{RCS}\{\emptyset\}\) and, with Theorem 3-25, \(\text{AVS}(\delta^3) = \text{AVS}(\delta^2) \cup \{(\text{Dom}(\delta^2), \text{Therefore } [\beta^*, \xi, \Delta] \to \Gamma^\gamma)\}.\) Thus we have (Dom(\(\delta^5\)), "Suppose \([\beta^*, \xi, \Delta]^\gamma\) \(\in \text{AVAS}(\delta^3)\) and (Dom(\(\delta^*\))-1, \(\delta^5_{\text{Dom}(\delta^*+1)} \in \text{AVS}(\delta^3)\) and \([\beta^*, \xi, \Delta] \in \text{AVP}(\delta^2) \subset \text{AVP}(\delta^3)\) and \([\beta^*, \xi, \Delta] \to \Gamma^\gamma \in \text{AVP}(\delta^3)\). Therefore we have fourth, according to Definition 3-3, \(\delta^5 \in \text{CdEF}(\delta^3) \subset \text{RCS}\{\emptyset\}\) and, with Theorem 3-25, \(\text{AVS}(\delta^4) = \text{AVS}(\delta^3) \cup \{(\text{Dom}(\delta^3), \text{Therefore } \Gamma^\gamma)\}.\) Thus we have (Dom(\(\delta^5\)), "Suppose \([\beta^*, \xi, \Delta]^\gamma\) \(\in \text{AVAS}(\delta^3)\) = \(\text{AVAS}(\delta^4)\) and (Dom(\(\delta^*\))-1, \(\delta^5_{\text{Dom}(\delta^*+1)}\), (Dom(\(\delta^*\))+3, \("\text{Therefore } \Gamma^\gamma\) \(\in \text{AVS}(\delta^4)\).\)

Altogether we thus have \(\beta^* \in \text{PAR}, \xi \in \text{VAR}, \Delta \in \text{FORM}, \text{FV}(\Delta) \subset \{\xi\}, \Gamma \in \text{CFORM}\)
\(\text{Dom}(\delta^*)-1 \in \text{Dom}(\delta^3), \text{P}(\delta^4_{\text{Dom}(\delta^*+1)}) = \forall \xi \Delta^\gamma\) and (Dom(\(\delta^*\))-1, \(\delta^5_{\text{Dom}(\delta^*+1)} \in \text{AVS}(\delta^3).\) \(\text{P}(\delta^4_{\text{Dom}(\delta^*+1)}) = [\beta^*, \xi, \Delta]\) and (Dom(\(\delta^*\)), \(\delta^5_{\text{Dom}(\delta^*+1)} \in \text{AVS}(\delta^4)), \text{P}(\delta^4_{\text{Dom}(\delta^*+1)}) = \Gamma, \beta^* \notin \text{STSF}(\{\Delta, \Gamma\})\) and there is no \(j \leq \text{Dom}(\delta^*)-1\) such that \(\beta^* \in \text{ST}(\delta^4)\) and there is no \(m\) such that \(\text{Dom}(\delta^*) < m \leq \text{Dom}(\delta^4)\) and (m, \(\delta^4_m \in \text{AVS}(\delta^4).\) Finally we thus have, according to Definition 3-15, \(\delta^5 \in \text{PEF}(\delta^4) \subset \text{RCS}\{\emptyset\}\) and, with Theorem 3-21-(iv) and -(v), \(\text{AVAS}(\delta^5) = \text{AVAS}(\delta^4) \cup \{(\text{max}(\text{Dom}(\text{AVS}(\delta^4))), \text{AVAS}(\delta^4) \cup \{(\text{Dom}(\delta^*), \text{Therefore } [\beta^*, \xi, \Delta]^\gamma)\} = \text{AVAS}(\delta^4) \cup \{(\text{Dom}(\delta^*), \text{Therefore } [\beta^*, \xi, \Delta]^\gamma)\} = \text{AVAS}(\delta^4) \cup \{(\text{Dom}(\delta^*), \text{Therefore } \Gamma^\gamma)\} = \text{AVS}(\delta^5) \cup \{(\text{Dom}(\delta^*), \text{Therefore } \Gamma^\gamma)\} = \text{AVS}(\delta^5) \cup \{(\text{Dom}(\delta^*), \text{Therefore } \Gamma^\gamma)\} \subset X \cup (Y \setminus \{[\beta, \xi, \Delta]\}).\) Since \(C(\delta^5) = \Gamma,\) it thus holds, with Theorem 3-12, that \(X \cup (Y \setminus \{[\beta, \xi, \Delta]\}) \vdash \Gamma.\)

Ad (xxvii) (II): Suppose \(X \subset \text{CFORM}\). According to Definition 3-16, we then have \(\{(0, \text{Therefore } \theta = \theta^\gamma)\} \in \text{IE}(\emptyset) \subset \text{RCS}\{\emptyset\}\) and we have \(\text{AVAS}(\{(0, \text{Therefore } \theta_0 = \theta_0^\gamma)\}) = \emptyset\) and hence, according to Definition 2-31, \(\text{AVAP}(\{(0, \text{Therefore } \theta_0 = \theta_0^\gamma)\}) = \emptyset\) and we have \(C(\{(0, \text{Therefore } \theta_0 = \theta_0^\gamma)\}) = \theta_0 = \theta_0^\gamma\) and thus, according to Theorem 3-12, \(\emptyset \vdash \theta_0 = \theta_0^\gamma.\) With Theorem 4-16, we hence have \(X \vdash \theta_0 = \theta_0^\gamma.\)
Theorem 4-19. Transitivity
If \( X \vdash M Y \) and \( Y \vdash B \), then \( X \vdash B \).

Proof: First we show by induction on \(|Y|\) that the statement holds for all finite \( Y \): Suppose the statement holds for all \( k < |Y| \in \mathbb{N} \). Suppose \(|Y| = 0\). Now, suppose \( X \vdash M Y \) and \( Y \vdash B \). Then we have \( Y = \emptyset \subseteq X \subseteq \text{CFORM} \). With Theorem 4-16 follows \( X \vdash B \).

Now, suppose \( 0 < |Y| \) and suppose \( X \vdash M Y \) and \( Y \vdash B \). According to Definition 3-25, we then have \( X \cup Y \subseteq \text{CFORM} \) and for all \( \Delta \in Y \): \( X \vdash \Delta \). Now, suppose \( Y \vdash B \). Since \(|Y| \neq 0\), we have that there is an \( A \in Y \). With Theorem 4-18-(i), we then have \( Y \setminus \{A\} \vdash \neg A \rightarrow B \). Then we have \(|Y \setminus \{A\}| < |Y|\). By the I.H., we thus have \( X \vdash \neg A \rightarrow \) and, since \( A \in Y \), we also have \( X \vdash A \). With Theorem 4-18-(ii), we thus have \( X \vdash B \).

As the statement holds for finite \( Y \), it also holds in general: Suppose \( X \vdash M Y \) and \( Y \vdash B \). According to Definition 3-25, we have \( X \cup Y \subseteq \text{CFORM} \) and for all \( \Delta \in Y \): \( X \vdash \Delta \). Now, suppose \( Y \vdash B \). With Theorem 3-12, there is then an \( \bar{f} \in \text{RCS} \setminus \{\emptyset\} \) such that \( \text{AVAP}(\bar{f}) \subseteq Y \) and \( C(\bar{f}) = B \). According to Theorem 3-9, \( \text{AVAP}(\bar{f}) \) is finite and \( \text{AVAP}(\bar{f}) \subseteq \text{CFORM} \). According to Theorem 3-12, we have that \( \text{AVAP}(\bar{f}) \vdash B \). We also have with \( \text{AVAP}(\bar{f}) \subseteq Y \) that it holds for all \( \Gamma \in \text{AVAP}(\bar{f}) \) that \( X \vdash \Gamma \) and thus that \( X \vdash M \text{AVAP}(\bar{f}) \). Thus it then follows that \( X \vdash B \). ■

Theorem 4-20. Cut
If \( X \cup \{B\} \vdash A \) and \( Y \vdash B \), then \( X \cup Y \vdash A \).

Proof: Suppose \( X \cup \{B\} \vdash A \) and \( Y \vdash B \). With Theorem 4-18-(i), we then have \( X \setminus \{B\} \vdash \neg B \rightarrow A \) and thus with Theorem 4-16 that \( X \vdash \neg B \rightarrow A \). With Theorem 4-18-(ii), it thus holds that \( X \cup Y \vdash A \). ■

Theorem 4-21. Deduction theorem and its inverse
\( X \cup \{A\} \vdash B \) iff \( X \vdash \neg A \rightarrow B \).

Proof: First, suppose \( X \cup \{A\} \vdash B \). Then it holds, with Theorem 4-18-(i), that \( X \setminus \{A\} \vdash \neg A \rightarrow B \) and thus, with Theorem 4-16, that \( X \vdash \neg A \rightarrow B \). Now, suppose \( X \vdash \neg A \rightarrow
B^\bot. According to Definition 3-21 and Theorem 3-9, we then have \( \forall A \rightarrow B^\bot \in \text{CFORM} \) and thus also \( A \in \text{CFORM} \). With Theorem 4-15, we then have \( \{ A \} \vdash A \) and hence, with Theorem 4-18-(ii), \( X \cup \{ A \} \vdash B \). ■

**Theorem 4-22. Inconsistence and derivability**

\( X \vdash A \) iff \( X \cup \{ \neg A^\bot \} \) is inconsistent.

**Proof: (L-R):** First, suppose \( X \vdash A \). With Definition 3-21 and Theorem 3-9, we then have \( X \subseteq \text{CFORM} \) and \( A \in \text{CFORM} \). Then we have \( \neg A^\bot \in \text{CFORM} \) and it thus holds, with Theorem 4-16, that \( X \cup \{ \neg A^\bot \} \vdash A \), and, with Theorem 4-15, it holds that \( X \cup \{ \neg A^\bot \} \vdash \neg A^\bot \). According to Definition 3-24, we then have that \( X \cup \{ \neg A^\bot \} \) is inconsistent.

\((R-L):\) Now, suppose \( X \cup \{ \neg A^\bot \} \) is inconsistent. According to Definition 3-24, we then have \( X \cup \{ \neg A^\bot \} \subseteq \text{CFORM} \) and that there is a \( \Gamma \in \text{CFORM} \) such that \( X \cup \{ \neg A^\bot \} \vdash \neg \Gamma \). With Theorem 4-18-(xi), it then holds that \( X \setminus \{ \neg A^\bot \} \vdash \neg \Gamma \) and thus, with Theorem 4-16, that \( X \vdash \neg \neg \Gamma \). From this we get, with Theorem 4-18-(xii), that \( X \vdash A \). ■

**Theorem 4-23.** A set of propositions is inconsistent if and only if all propositions can be derived from it

\( X \) is inconsistent iff for all \( \Gamma \in \text{CFORM} \): \( X \vdash \Gamma \).

**Proof: (L-R):** First, suppose \( X \) is inconsistent. According to Definition 3-24, we then have \( X \subseteq \text{CFORM} \) and that there is an \( A \in \text{CFORM} \) such that \( X \vdash A \) and \( X \vdash \neg A^\bot \). Now, suppose \( \Gamma \in \text{CFORM} \). Then we have \( \neg \Gamma \in \text{CFORM} \). With Theorem 4-16, it then holds that \( X \cup \{ \neg \Gamma \} \vdash A \) and \( X \cup \{ \neg \Gamma \} \vdash \neg A^\bot \). Thus we have that \( X \cup \{ \neg \Gamma \} \) is inconsistent. According to Theorem 4-22, we then have \( X \vdash \Gamma \).

\((R-L):\) Now, suppose for all \( \Gamma \in \text{CFORM} \) it holds that \( X \vdash \Gamma \). There is a \( \Delta \in \text{CFORM} \). With \( \Delta \in \text{CFORM} \), we also have \( \neg \Delta^\bot \in \text{CFORM} \). Then we have \( X \vdash \Delta \) and \( X \vdash \neg \Delta^\bot \). With Definition 3-21, we then have \( X \subseteq \text{CFORM} \). According to Definition 3-24, we hence have that \( X \) is inconsistent. ■
**Theorem 4.24. Generalisation theorem**

If \( \xi \in \text{VAR}, \Delta \in \text{FORM} \), where \( \text{FV}(\Delta) \subseteq \{\xi\} \), \( \alpha \in \text{CONST} \) and \( X \vdash [\alpha, \xi, \Delta] \), where \( \alpha \notin \text{STSF}(X \cup \{\Delta\}) \), then \( X \vdash \Gamma \wedge \xi \Delta \).

**Proof:** Suppose \( \xi \in \text{VAR}, \Delta \in \text{FORM} \), where \( \text{FV}(\Delta) \subseteq \{\xi\} \), \( \alpha \in \text{CONST} \) and \( X \vdash [\alpha, \xi, \Delta] \), where \( \alpha \notin \text{STSF}(X \cup \{\Delta\}) \). According to Theorem 3-12, there is then an \( \mathfrak{S} \in \text{RCS}\backslash\{\emptyset\} \) such that \( \text{AVAP}(\mathfrak{S}) \subseteq X \) and \( C(\mathfrak{S}) = [\alpha, \xi, \Delta] \). There is a \( \beta \in \text{PAR}\backslash\text{STSEQ}(\mathfrak{S}) \).

With Theorem 4-9, there is then an \( \mathfrak{S}^* \in \text{RCS}\backslash\{\emptyset\} \) such that:

\[
\begin{align*}
&\text{a)} \quad \alpha \notin \text{STSEQ}(\mathfrak{S}^*), \\
&\text{b)} \quad \text{AVAP}(\mathfrak{S}) = \{[\alpha, \beta, B] \mid B \in \text{AVAP}(\mathfrak{S}^*)\}, \text{and} \\
&\text{c)} \quad C(\mathfrak{S}) = [\alpha, \beta, C(\mathfrak{S}^*)].
\end{align*}
\]

Since it holds for all \( \Gamma \in \text{AVAP}(\mathfrak{S}) \) that \( \alpha \notin \text{ST}(\Gamma) \), it holds with b) for all \( B \in \text{AVAP}(\mathfrak{S}^*) \) that \( \beta \notin \text{ST}(B) \) and thus that \( \beta \notin \text{STSF}(\text{AVAP}(\mathfrak{S}^*)) \). For if \( \beta \in \text{ST}(\Gamma) \) for a \( \Gamma \in \text{AVAP}(\mathfrak{S}^*) \), then we would have \( \alpha \in \text{ST}([\alpha, \beta, \Gamma]) \) and, with b), we would have \( [\alpha, \beta, \Gamma] \in \text{AVAP}(\mathfrak{S}) \subseteq X \). Thus we would have that \( \alpha \in \text{STSF}(X) \), which contradicts the hypothesis. With b), we thus have \( \text{AVAP}(\mathfrak{S}) = \{[\alpha, \beta, B] \mid B \in \text{AVAP}(\mathfrak{S}^*)\} = \{B \mid B \in \text{AVAP}(\mathfrak{S}^*)\} = \text{AVAP}(\mathfrak{S}^*) \).

With c), it holds that \( [\alpha, \xi, \Delta] = C(\mathfrak{S}) = [\alpha, \beta, C(\mathfrak{S}^*)] \). According to the initial assumption and with a), we have \( \alpha \notin \text{ST}(\Delta) \cup \text{ST}(C(\mathfrak{S}^*)) \). With Theorem 1-23, we thus have \( C(\mathfrak{S}^*) = [\beta, \xi, \Delta] \). Then we have \( \beta \notin \text{ST}(\Delta) \), because otherwise we would have, with \( [\alpha, \xi, \Delta] = C(\mathfrak{S}) \), that \( \beta \in \text{ST}(C(\mathfrak{S})) \subseteq \text{STSEQ}(\mathfrak{S}) \), which contradicts the choice of \( \beta \). With Definition 3-12, we thus have \( \mathfrak{S}^* \cup \{(\text{Dom}(\mathfrak{S}^*), \Gamma \wedge \xi \Delta)\} \in \text{UIF}(\mathfrak{S}^*) \subseteq \text{RCS}\backslash\{\emptyset\} \). With Theorem 3-26-(v), it then holds that \( \text{AVAP}(\mathfrak{S}^*) \cup \{(\text{Dom}(\mathfrak{S}^*), \Gamma \wedge \xi \Delta)\} \subseteq \text{AVAP}(\mathfrak{S}^*) = \text{AVAP}(\mathfrak{S}) \subseteq X \). With Theorem 3-12, we hence have \( X \vdash \Gamma \wedge \xi \Delta \). ■

**Theorem 4-25. Multiple IE**

If \( k \in \mathbb{N}\backslash\{0\}, \{\theta_0, \ldots, \theta_{k-1}\} \subseteq \text{CTERM}, \{\xi_0, \ldots, \xi_{k-1}\} \subseteq \text{VAR}, \) where for all \( i, j \in k \) with \( i \neq j \) also \( \xi_i \neq \xi_j \), \( \Delta \in \text{FORM} \), where \( \text{FV}(\Delta) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \), and \( X \vdash [\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1}\rangle, \Delta \) and for all \( i < k: X \vdash \theta_i = \theta_i^* \), then \( X \vdash [\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1}\rangle, \Delta \).

**Proof:** By induction on \( k \). For \( k = 1 \), the statement follows with Theorem 4-18-(xviii).

Now, suppose the statement holds for \( k \) and suppose \( \{\theta_0, \ldots, \theta_k\}, \{\theta_0^*, \ldots, \theta_k^*\} \subseteq \text{SORT, VAR} \subseteq \text{FORM} \), and for all \( i < k: X \vdash \theta_i = \theta_i^* \). Let \( k' = k-1 \). Then \( X \vdash [\theta_0, \ldots, \theta_{k-1}], \langle \xi_0, \ldots, \xi_{k-1}\rangle, \Delta \). With Theorem 4-24, it holds that \( X \vdash \Gamma \wedge \xi \Delta \). Therefore, \( X \vdash [\theta_0, \ldots, \theta_k], \langle \xi_0, \ldots, \xi_k\rangle, \Delta \).
CTERM, \{ξ_0, ..., ξ_k\} ⊆ VAR, where for all \(i, j < k+1\) with \(i \neq j\) also \(ξ_i \neq ξ_j\), \(Δ \in\) FORM, where \(FV(Δ) \subseteq \{ξ_0, ..., ξ_k\}\), and \(X \vdash [θ_0, ..., θ_b], \langle ξ_0, ..., ξ_b\rangle, Δ\) and for all \(i < k+1\): \(X \vdash \overset{\text{r}}{θ_i} = θ_i'\).

With Theorem 1-28-(ii), we then have that \([θ_0, ..., θ_b], \langle ξ_0, ..., ξ_b\rangle, Δ\) = \([θ_0, ξ_k, [θ_1, ..., θ_{k-1}], \langle ξ_1, ..., ξ_{k-1}\rangle, Δ]\) and thus \(X \vdash [θ_k, ξ_k, [θ_1, ..., θ_{k-1}], \langle ξ_1, ..., ξ_{k-1}\rangle, Δ]\), where, with \(FV(Δ) \subseteq \{ξ_0, ..., ξ_k\}\), it holds that \(FV([θ_1, ..., θ_{k-1}], \langle ξ_1, ..., ξ_{k-1}\rangle, Δ]) \subseteq \{ξ_k\}\). With \(X \vdash \overset{\text{r}}{θ_k} = θ_k'\) and Theorem 4-18-(xviii), we then have \(X \vdash [θ_k', ξ_k, [θ_0, ..., θ_{k-1}], ξ_0, ..., ξ_{k-1}, Δ]\) and thus, again with Theorem 1-28-(ii), that \(X \vdash [θ_k', ..., θ_{b-1}, ξ_k', [θ_0, ..., θ_{k-1}], ξ_0, ..., ξ_{k-1}, Δ]\). With Theorem 1-29-(ii), we have \([θ_0, ..., θ_{k-1}, θ_b', ξ_0, ..., ξ_{k-1}, Δ]\) and thus, again with Theorem 1-29-(ii), that \(X \vdash [θ_0', ..., θ_{b-1}', ξ_0, ..., ξ_{k-1}, Δ]\). ■
5 Model-theory

In this chapter we will develop a classical model-theoretic consequence concept for the language L. First, we will define the concepts we need, in particular model-theoretic satisfaction and based on it the model-theoretic consequence relation, and prove some basic theorems about them (5.1). Subsequently, we will prove some theorems on the closure of the model-theoretic consequence relation (5.2). Consequently, in ch. 6, we can then prove the correctness and completeness of the Speech Act Calculus relative to the model-theoretic consequence concept developed in ch. 5.1.

5.1 Satisfaction Relation and Model-theoretic Consequence

The development of the model-theoretic consequence concept proceeds in the standard way. First, we will define interpretation functions, models and parameter assignments. This suffices to assign each closed term a denotation (Definition 5-6), where the usual definition is mirrored in Theorem 5-2. Subsequently, we can determine under which conditions a model and a parameter assignment satisfy a formula (Definition 5-8). The usual definition is here mirrored by Theorem 5-4. Then, we will prove a coincidence and a substitution lemma (Theorem 5-5 and Theorem 5-6) as well as some other theorems that are needed for the further account. Finally, we will introduce further usual concepts, among them the model-theoretic consequence (Definition 5-10), which is used in the formulation of correctness and completeness.

Definition 5-1. Interpretation function

$I$ is an interpretation function for $D$ iff

- $D$ is a set and $I$ is a function with $\text{Dom}(I) = \text{CONST} \cup \text{FUNC} \cup \text{PRED}$ and
  - (i) For all $\alpha \in \text{CONST}$: $I(\alpha) \in D$,
  - (ii) For all $\phi \in \text{FUNC}$: If $\phi$ is $r$-ary, then $I(\phi)$ is an $r$-ary function over $D$,
  - (iii) For all $\Phi \in \text{PRED}$: If $\Phi$ is $r$-ary, then $I(\Phi) \subseteq rD$, and
  - (iv) $I(\text{=}^r) = \{ \langle a, a \rangle \mid a \in D \}$.

---

**Definition 5-2. Model**

\( M \) is a model iff there is \( D, I \) such that \( I \) is an interpretation function for \( D \) and \( M = (D, I) \).

**Note:** The non-emptiness of \( D \) is ensured by \( \text{CONST} \neq \emptyset \) and clause (i) of Definition 5-1. In contrast to the usual procedure, we will not use variable assignments, but parameter assignments. So, parameters, in keeping with their role in the calculus, fulfill tasks in the model-theory that are often given to free variables. Accordingly, quantificational formulas (e.g. \( \forall \xi \Delta \)) are not evaluated for \( \Delta \), but for a suitable parameter instantiation (e.g. \( [\beta, \xi, \Delta] \)) (cf. Definition 5-7 and Theorem 5-4).

**Definition 5-3. Parameter assignment**

\( b \) is a parameter assignment for \( D \) iff \( b \) is a function with \( \text{Dom}(b) = \text{PAR} \) and \( \text{Ran}(b) \subseteq D \).

**Definition 5-4. Assignment variant**

\( b' \) is in \( \beta \) an assignment variant of \( b \) for \( D \) iff \( b' \) and \( b \) are parameter assignments for \( D \) and \( \beta \in \text{PAR} \) and \( b' \setminus \{ (\beta, b'(\beta)) \} \subseteq b \).

**Definition 5-5. Term denotation functions for models and parameter assignments**

\( F \) is a term denotation function for \( D, I, b \) iff

\[(D, I) \text{ is a model and } b \text{ is a parameter assignment for } D \text{ and } F \text{ is a function on } \text{CTERM} \text{ and:} \]

(i) If \( \alpha \in \text{CONST} \), then \( F(\alpha) = I(\alpha) \),
(ii) If \( \beta \in \text{PAR} \), then \( F(\beta) = b(\beta) \), and
(iii) If \( \phi \in \text{FUNC}, \phi \text{ r-ary, and } \theta_0, \ldots, \theta_{r-1} \in \text{CTERM} \), then \( F(\phi(\theta_0, \ldots, \theta_{r-1})) = I(\phi)(F(\theta_0), \ldots, F(\theta_{r-1})) \).
Theorem 5-1. For every model \((D, I)\) and parameter assignment \(b\) for \(D\) there is exactly one term denotation function

If \((D, I)\) is a model and \(b\) is a parameter assignment for \(D\), then there is exactly one \(F\) such that \(F\) is a term denotation function for \(D, I, b\).

Proof: Suppose \((D, I)\) is a model and \(b\) is a parameter assignment for \(D\). With the theorems on unique readability (Theorem 1-10 and Theorem 1-11) there is then exactly one function \(F\) on CTERM such that clauses (i) to (iii) of Definition 5-5 are satisfied for \(F\) and thus, according to Definition 5-5, exactly one term denotation function for \(D, I, b\). ■

Definition 5-6. Term denotation operation (TD)

\[
\text{TD}(\theta, D, I, b) = a
\]

iff

(i) There is a term denotation function \(F\) for \(D, I, b\) and \(\theta \in \text{CTERM}\) and \(a = F(\theta)\)

or

(ii) There is no term denotation function for \(D, I, b\) or \(\theta \notin \text{CTERM}\) and \(a = \emptyset\).

The following theorem mirrors the usual definition of term denotations for models and parameter assignments:

Theorem 5-2. Term denotations for models and parameter assignments

If \((D, I)\) is a model and \(b\) is a parameter assignment for \(D\), then:

(i) If \(\alpha \in \text{CONST}\), then \(\text{TD}(\alpha, D, I, b) = I(\alpha)\),

(ii) If \(\beta \in \text{PAR}\), then \(\text{TD}(\beta, D, I, b) = b(\beta)\), and

(iii) If \(\varphi \in \text{FUNC}\), where \(\varphi\) \(r\)-ary ist, and \(\theta_0, \ldots, \theta_{r-1} \in \text{CTERM}\), then \(\text{TD}(\varphi(\theta_0, \ldots, \theta_{r-1}), D, I, b) = I(\varphi)(\text{TD}(\theta_0, D, I, b), \ldots, \text{TD}(\theta_{r-1}, D, I, b))\).

Proof: Suppose \((D, I)\) is a model and \(b\) is a parameter assignment for \(D\). With Theorem 5-1, there is then exactly one term denotation function \(F\) for \(D, I, b\). According to Definition 5-6, we then have for all \(\theta \in \text{CTERM}\): \(\text{TD}(\theta, D, I, b) = F(\theta)\). From this, the statement then follows with Definition 5-5. ■
**Definition 5-7.** Satisfaction functions for models and parameter assignments

*F* is a satisfaction function for \( D, I \)

iff

\((D, I)\) is a model, \( F \) is a function on \( \text{CFORM} \times \{ b \mid b \text{ is a parameter assignment for } D \} \), \( \text{Ran}(F) = \{0, 1\} \) and for all parameter assignments \( b \) for \( D \):

(i) If \( \Phi \in \text{PRED}, \Phi r\text{-ary, and } \theta_0, \ldots, \theta_{r-1} \in \text{CTERM} \) then:

\[ F(\overset{\cdots}{\Phi(\theta_0, \ldots, \theta_{r-1})}, b) = 1 \text{ iff } (\text{TD}(\theta_0, D, I, b), \ldots, \text{TD}(\theta_{r-1}, D, I, b)) \in I(\Phi), \]

(ii) If \( A \in \text{CFORM} \), then:

\[ F(\overset{\cdots}{\neg A}, b) = 1 \text{ iff } F(A, b) = 0, \]

(iii) If \( A, B \in \text{CFORM} \), then:

\[ F(\overset{\cdots}{A \land B}, b) = 1 \text{ iff } F(A, b) = 1 \text{ and } F(B, b) = 1, \]

(iv) If \( A, B \in \text{CFORM} \), then:

\[ F(\overset{\cdots}{A \lor B}, b) = 1 \text{ iff } F(A, b) = 1 \text{ or } F(B, b) = 1, \]

(v) If \( A, B \in \text{CFORM} \), then:

\[ F(\overset{\cdots}{A \rightarrow B}, b) = 1 \text{ iff } F(A, b) = 0 \text{ or } F(B, b) = 1, \]

(vi) If \( A, B \in \text{CFORM} \), then:

\[ F(\overset{\cdots}{A \leftrightarrow B}, b) = 1 \text{ iff } F(A, b) = F(B, b), \]

(vii) If \( \xi \in \text{VAR}, \Delta \in \text{FORM} \) and \( \text{FV}(\Delta) \subseteq \{\xi\} \), then:

\[ F(\overset{\cdots}{\xi \Delta}, b) = 1 \]

iff

there is \( \beta \in \text{PAR}\setminus\text{ST}(\Delta) \) such that for all \( b' \) that are in \( \beta \) assignment variants of \( b \) for \( D \):

\[ F([\beta, \xi, \Delta], b') = 1, \]

and

(viii) If \( \xi \in \text{VAR}, \Delta \in \text{FORM} \) and \( \text{FV}(\Delta) \subseteq \{\xi\} \), then:

\[ F(\overset{\cdots}{\forall \xi \Delta}, b) = 1 \]

iff

there is \( \beta \in \text{PAR}\setminus\text{ST}(\Delta) \) and \( b' \) that is in \( \beta \) an assignment variant of \( b \) for \( D \) such that

\[ F([\beta, \xi, \Delta], b') = 1. \]

**Theorem 5-3.** For every model \((D, I)\) there is exactly one satisfaction function

If \((D, I)\) is a model, then there is exactly one satisfaction function for \( D, I \).

**Proof:** Suppose \((D, I)\) is a model. With the theorems on unique readability (Theorem 1-10 and Theorem 1-11), there is then exactly one function \( F \) on \( \text{CFORM} \times \{ b \mid b \text{ is a parameter assignment for } D \} \) such that clauses (i) to (viii) of Definition 5-7 are satisfied for \( F \). Hence there is exactly one satisfaction function for \( D, I \). ■

**Definition 5-8.** 4-ary model-theoretic satisfaction predicate \( (\ldots, \ldots, \ldots, \bowtie \ldots) \)

\( D, I, b \models \Gamma \)

iff

\( \Gamma \in \text{CFORM}, b \) is a parameter assignment for \( D \) and there is a satisfaction function \( F \) for \( D, I \) such that \( F(\Gamma, b) = 1. \)
The following theorem mirrors the usual definition of model-theoretic consequence in the grammatical framework chosen here. In this, we use the contradictory predicate for '..,...,..', i.e. '..,...,..\neq..', in the usual way.

**Theorem 5-4. Usual satisfaction concept**

If \((D, I)\) is a model, \(b\) is a parameter assignment for \(D, A, B \in \text{CFORM}, \xi \in \text{VAR}, \Phi \in \text{PRED}, \Phi \ r\text{-ary}, \theta_0, \ldots, \theta_{r-1} \in \text{CTERM}, \Delta \in \text{FORM}\), where \(\text{FV}(\Delta) \subseteq \{\xi\}\), then:

(i) \(D, I, b \models (\Phi(\theta_0, \ldots, \theta_{r-1}))\) iff \((\text{TD}(\theta_0, D, I, b), \ldots, \text{TD}(\theta_{r-1}, D, I, b)) \in I(\Phi)\),

(ii) \(D, I, b \models (\neg\Phi)\) iff \(D, I, b \not\models \Phi\),

(iii) \(D, I, b \models (\Phi \land \Psi)\) iff \(D, I, b \models \Phi\) and \(D, I, b \models \Psi\),

(iv) \(D, I, b \models (\Phi \lor \Psi)\) iff \(D, I, b \models \Phi\) or \(D, I, b \models \Psi\),

(v) \(D, I, b \models (\Phi \rightarrow \Psi)\) iff \(D, I, b \models \Psi\) or \(D, I, b \not\models \Phi\),

(vi) \(D, I, b \models (\Phi \leftrightarrow \Psi)\) iff \(D, I, b \models \Phi \iff \Psi\),

(vii) \(D, I, b \models (\forall \xi \Delta)\) iff there is a \(\beta \in \text{PAR}\setminus\text{ST}(\Delta)\) such that for all \(b'\) that are in \(\beta\) assignment variants of \(b\) for \(D\), \(D, I, b' \models [\beta, \xi, \Delta]\), and

(viii) \(D, I, b \models (\exists \xi \Delta)\) iff there is a \(\beta \in \text{PAR}\setminus\text{ST}(\Delta)\) and a \(b'\) that is in \(\beta\) an assignment variant of \(b\) for \(D\) such that \(D, I, b' \models [\beta, \xi, \Delta]\).

**Proof:** Let \((D, I)\) be a model, \(b\) a parameter assignment for \(D, A, B \in \text{CFORM}, \xi \in \text{VAR}, \Phi \in \text{PRED}, \Phi \ r\text{-ary}, \theta_0, \ldots, \theta_{r-1} \in \text{CTERM}, \Delta \in \text{FORM}\), where \(\text{FV}(\Delta) \subseteq \{\xi\}\). With Theorem 5-3, there is then exactly one satisfaction function \(F\) for \(D, I\). With Definition 5-8, it then follows that for all \(\Gamma \in \text{CFORM}: D, I, b \models \Gamma\) iff \(F(\Gamma, b) = 1\) and \(D, I, b \not\models \Gamma\) iff \(F(\Gamma, b) = 0\). From this, the statement then follows with Definition 5-7. ■

**Theorem 5-5. Coincidence lemma**

If \((D, I)\) and \((D, I')\) are models and \(b, b'\) are parameter assignments for \(D\), then:

(i) For all \(\theta \in \text{CTERM}: \text{If } I \models \text{SE}(\theta) = I' \models \text{SE}(\theta) \text{ and } b \models \text{ST}(\theta) = b' \models \text{ST}(\theta), \text{ then } \text{TD}(\theta, D, I, b) = \text{TD}(\theta, D, I', b'),\)

(ii) For all \(\Gamma \in \text{CFORM}: \text{If } I \models \text{SE}(\Gamma) = I' \models \text{SE}(\Gamma) \text{ and } b \models \text{ST}(\Gamma) = b' \models \text{ST}(\Gamma), \text{ then } D, I, b \models \Gamma \iff D, I', b' \models \Gamma.\)

**Proof:** Ad (i): Let \((D, I)\) and \((D, I')\) be models and \(b, b'\) parameter assignments for \(D\). The proof is carried out by induction on the complexity of \(\theta \in \text{TERM}. First, suppose \(\theta \in \text{ATERM} \cap \text{CTERM}\) and suppose \(I \models \text{SE}(\theta) = I' \models \text{SE}(\theta)\) and \(b \models \text{ST}(\theta) = b' \models \text{ST}(\theta)\). Then we
have $\theta \in \text{CONST} \cup \text{PAR}$. Now, suppose $\theta \in \text{CONST}$. Then it holds with $\{\theta\} = \text{SE}(\theta) \cap \text{CONST}$ and $I|\text{SE}(\theta) = I'|\text{SE}(\theta)$ and Theorem 5-2-(i) that $\text{TD}(\theta, D, I, b) = I(\theta) = I'(\theta) = \text{TD}(\theta, D, I', b')$. Now, suppose $\theta \in \text{PAR}$. Then it holds with $\{\theta\} = \text{ST}(\theta) \cap \text{PAR}$ and $b|\text{ST}(\theta) = b'|\text{ST}(\theta)$ and Theorem 5-2-(ii) that $\text{TD}(\theta, D, I, b) = b(\theta) = b'(\theta) = \text{TD}(\theta, D, I', b')$.

Now, suppose the statement holds for $\theta_0, \ldots, \theta_{r-1} \in \text{TERM}$ and suppose $\varphi \in \text{FUNC}, \varphi$ $r$-ary, and suppose $\{\varphi(\theta_0, \ldots, \theta_{r-1})\} \subseteq \text{TERM} \cap \text{CTERM}$ and suppose $I|\text{SE}(\{\varphi(\theta_0, \ldots, \theta_{r-1})\}) = I'|\text{SE}(\{\varphi(\theta_0, \ldots, \theta_{r-1})\})$ and $b|\text{ST}(\{\varphi(\theta_0, \ldots, \theta_{r-1})\}) = b'|\text{ST}(\{\varphi(\theta_0, \ldots, \theta_{r-1})\})$.

With $\text{FV}(\{\varphi(\theta_0, \ldots, \theta_{r-1})\}) = \bigcup \{\text{FV}(\theta_i) \mid i < r\}$, it then holds for all $\theta_i$ with $i < r$ that $\theta_i \in \text{CTERM}$. We also have, with $\bigcup \{\text{SE}(\theta_i) \mid i < r\} \subseteq \text{SE}(\{\varphi(\theta_0, \ldots, \theta_{r-1})\})$ and $\bigcup \{\text{ST}(\theta_i) \mid i < r\} \subseteq \text{ST}(\{\varphi(\theta_0, \ldots, \theta_{r-1})\})$, for all $i < r$: $I'|\text{SE}(\theta_i) = I'|\text{SE}(\theta_i)$ and $b'|\text{ST}(\theta_i) = b'|\text{ST}(\theta_i)$. With the I.H., it thus holds for all $i < r$ that $\text{TD}(\theta_i, D, I) = \text{TD}(\theta_i, D, I', b')$. With $\varphi \in \text{SE}(\{\varphi(\theta_0, \ldots, \theta_{r-1})\}) \cap \text{FUNC}$, we have by hypothesis that $I(\varphi) = I'(\varphi)$. Thus it holds that

$$\begin{align*}
\text{TD}(\{\varphi(\theta_0, \ldots, \theta_{r-1})\}, D, I, b) &= I(\varphi)(\text{TD}(\theta_0, D, I, b), \ldots, \text{TD}(\theta_{r-1}, D, I, b)) \\
&= I(\varphi)(\text{TD}(\theta_0, D, I', b'), \ldots, \text{TD}(\theta_{r-1}, D, I', b'))) \\
&= I'|\text{SE}(\theta_0, \ldots, \theta_{r-1}), D, I', b').
\end{align*}$$

\textit{Ad (ii):} The proof is carried out by induction on the degree of a formula. For this, suppose the theorem holds for all $A \in \text{FORM}$ with $\text{FDEG}(A) < k$. Now, let $(D, I), (D, I')$ be models, $b, b'$ parameter assignments for $D$ and suppose $\Gamma \in \text{CFORM}$ and suppose $I'|\text{SE}(\Gamma) = I'|\text{SE}(\Gamma)$ and $b'|\text{ST}(\Gamma) = b'|\text{ST}(\Gamma)$ and suppose $\text{FDEG}(\Gamma) = k$.

Suppose $\text{FDEG}(\Gamma) = 0$. Then we have $\Gamma \in \text{AFORM}$. Then there are $\theta_0, \ldots, \theta_{r-1} \in \text{TERM}$ and $\Phi \in \text{PRED}, \Phi$ $r$-ary, such that $\Gamma = \{\Phi(\theta_0, \ldots, \theta_{r-1})\}$. Then it holds, with $\text{FV}(\{\Phi(\theta_0, \ldots, \theta_{r-1})\}) = \bigcup \{\text{FV}(\theta_i) \mid i < r\}$, $\bigcup \{\text{SE}(\theta_i) \mid i < r\} \subseteq \text{SE}(\{\Phi(\theta_0, \ldots, \theta_{r-1})\})$ and $\bigcup \{\text{ST}(\theta_i) \mid i < r\} \subseteq \text{ST}(\{\Phi(\theta_0, \ldots, \theta_{r-1})\})$ and with $\Gamma \in \text{CTERM}$, $I'|\text{SE}(\theta_i) = I'|\text{SE}(\theta_i)$ and $b'|\text{ST}(\theta_i) = b'|\text{ST}(\theta_i)$. With (i), we thus have for all $i < r$: $\text{TD}(\theta_i, D, I, b) = \text{TD}(\theta_i, D, I', b')$. With $\Phi \in \text{SE}(\{\Phi(\theta_0, \ldots, \theta_{r-1})\}) \cap \text{PRED}$, we have by hypothesis $I(\Phi) = I'(\Phi)$. With Theorem 5-4-(i), it thus holds that
\[ D, I, b \models \Gamma \]

iff
\[ D, I, b \models \forall \Phi(\theta_0, \ldots, \theta_{r-1}) \]

iff
\[ \langle \text{TD}(\theta_0, D, I, b), \ldots, \text{TD}(\theta_{r-1}, D, I, b) \rangle \in I(\Phi) \]

iff
\[ \langle \text{TD}(\theta_0, D', I', b'), \ldots, \text{TD}(\theta_{r-1}, D', I', b') \rangle \in I'(\Phi) \]

iff
\[ D, I', b' \models \forall \Phi(\theta_0, \ldots, \theta_{r-1}) \]

iff
\[ D, I', b' \models \Gamma. \]

Now, suppose \( \text{FDEG}(\Gamma) \neq 0 \). Then we have \( \Gamma \in \text{CONFORM} \cup \text{QFORM} \). We can distinguish seven cases. First: Suppose \( \Gamma = \neg \Phi \). Then we have \( \text{FDEG}(\Phi) < \text{FDEG}(\Gamma) \). According to the assumption for \( \Gamma \), we then have that \( \Phi \in \text{CFORM}, I \vdash \text{SE}(\Phi) = I' \vdash \text{SE}(\Phi) \) and \( b \vdash \text{ST}(\Phi) = b' \vdash \text{ST}(\Phi) \). With Theorem 5-4-(ii) and the I.H., we thus have

\[ D, I, b \models \Gamma \]

iff
\[ D, I, b \models \neg \Phi \]

iff
\[ D, I, b \not\models \Phi \]

iff
\[ D, I', b' \not\models \Phi \]

iff
\[ D, I', b' \models \neg \Phi \]

iff
\[ D, I', b' \models \Gamma. \]

Second: Suppose \( \Gamma = \Phi \). Then we have \( \text{FDEG}(\Phi) < \text{FDEG}(\Gamma) \) and \( \text{FDEG}(\Phi) < \text{FDEG}(\Gamma) \). According to assumption for \( \Gamma \), we then have \( \Phi, \psi \in \text{CTERM}, I \vdash \text{SE}(\Phi \cup \psi) = I' \vdash \text{SE}(\Phi \cup \psi) \) and \( b \vdash \text{ST}(\Phi \cup \psi) = b' \vdash \text{ST}(\Phi \cup \psi) \). With Theorem 5-4-(iii) and the I.H., it then holds that

\[ D, I, b \models \Gamma \]

iff
\[ D, I, b \models \Phi \]

iff
\[ D, I, b \models \Phi \text{ and } D, I, b \models \psi \]

iff
\[ D, I', b' \models \Phi \text{ and } D, I', b' \models \psi \]

iff
\[ D, I', b' \models \Gamma. \]
The third to fifth cases are treated analogously.

Sixth: Suppose $\Gamma = \forall \zeta \Delta$. According to the assumption for $\Gamma$, we then have $\text{FV}(\Delta) \subseteq \{\zeta\}$, $I \models \text{SE}(\Delta) = I^I \models \text{SE}(\Delta)$ and $b^I \models \text{ST}(\Delta) = b^I \models \text{ST}(\Delta)$. Now, suppose $D, I, b \models \forall \zeta \Delta$. With Theorem 5-4-(vii), there is then a $\beta \in \text{PAR}\setminus \text{ST}(\Delta)$ such that for all $b'$ that are in $\beta$ assignment variants of $b$ for $D$ it holds that $D, I, b' \models [\beta, \zeta, \Delta]$. Now, suppose $b_1'$ is in $\beta$ an assignment variant of $b'$ for $D$. Now, let $b_1' = (b^I \setminus \{(\beta, b' (\beta))\}) \cup \{(\beta, b_1' (\beta))\}$. Then $b_1'$ is in $\beta$ an assignment variant of $b'$ for $D$ and thus it holds that $D, I, b_1 \models [\beta, \zeta, \Delta]$. Since $\beta \notin \text{ST}(\Delta)$ and $b^I \models \text{ST}(\Delta) = b^I \models \text{ST}(\Delta)$, we have for all $\beta' \in \text{ST}(\Delta) \cap \text{PAR}$ that $b_1(\beta') = b(\beta') = b'(\beta') = b_1'(\beta')$. Since also $b_1(\beta) = b_1'(\beta)$ and $\text{ST}([\beta, \zeta, \Delta]) \subseteq \text{ST}(\Delta) \cup \{\beta\}$, we thus have that $b_1 \models \text{ST}([\beta, \zeta, \Delta]) = b_1' \models \text{ST}([\beta, \zeta, \Delta])$. Also, we have $I \models \text{SE}([\beta, \zeta, \Delta]) = I \models \text{SE}([\beta, \zeta, \Delta]) \cap (\text{CONST} \cup \text{FUNC} \cup \text{PRED}) = I \models \text{SE}([\beta, \zeta, \Delta]) \cup (\text{CONST} \cup \text{FUNC} \cup \text{PRED}) = I \models \text{SE}([\beta, \zeta, \Delta])$.

Moreover, we have $[\beta, \zeta, \Delta] \in \text{CFORM}$ and, with Theorem 1-13, we have $\text{FDEG}([\beta, \zeta, \Delta]) < \text{FDEG}(\Gamma)$. According to the I.H., we thus have that with $D, I, b_1 \models [\beta, \zeta, \Delta]$ it also holds that $D, I', b_1 \models [\beta, \zeta, \Delta]$. Therefore we have for all $b_1''$ that are in $\beta$ assignment variants of $b'$ for $D$: $D, I', b_1'' \models [\beta, \zeta, \Delta]$ and hence, according to Theorem 5-4-(vii), $D, I', b' \models \forall \zeta \Delta$. The right-left-direction is shown analogously.

Seventh: Suppose $\Gamma = \forall \zeta \Delta$. According to the assumption for $\Gamma$, we then have $\text{FV}(\Delta) \subseteq \{\zeta\}$, $I \models \text{SE}(\Delta) = I \models \text{SE}(\Delta)$ and $b \models \text{ST}(\Delta) = b \models \text{ST}(\Delta)$. Now, suppose $D, I, b \models \forall \zeta \Delta$. With Theorem 5-4-(viii), there is then $\beta \in \text{PAR} \setminus \text{ST}(\Delta)$ and $b_1$ that is in $\beta$ assignment variant of $b$ for $D$ such that $D, I, b_1 \models [\beta, \zeta, \Delta]$. Now, let $b_1' = (b^I \setminus \{(\beta, b'(\beta))\}) \cup \{(\beta, b_1(\beta))\}$. Then $b_1'$ is in $\beta$ an assignment variant of $b'$ for $D$. Since $\beta \notin \text{ST}(\Delta)$ and $b \models \text{ST}(\Delta) = b \models \text{ST}(\Delta)$, it then holds for all $\beta' \in \text{ST}(\Delta) \cap \text{PAR}$ that $b_1(\beta') = b(\beta') = b'(\beta') = b_1'(\beta')$. Since also $b_1(\beta) = b_1'(\beta)$ and $\text{ST}([\beta, \zeta, \Delta]) \subseteq \text{ST}(\Delta) \cup \{\beta\}$, we thus have that $b_1 \models \text{ST}([\beta, \zeta, \Delta]$. The right-left-direction is shown analogously.
Using the coincidence lemma, we can now prove the substitution lemma:

**Theorem 5-6. Substitution lemma**

If \((D, I), (D, I')\) are models, \(b, b'\) are parameter assignments for \(D, \xi \in \text{VAR}, \theta, \theta' \in \text{CTERM}\) and \(\text{TD}(\theta, D, I, b) = \text{TD}(\theta', D, I', b')\) then:

(i) For all \(\theta' \in \text{TERM}\) with \(\text{FV}(\theta') \subseteq \{\xi\}\), \(I|\text{SE}(\theta') = I'|\text{SE}(\theta')\) and \(b|\text{ST}(\theta') = b'|\text{ST}(\theta)\) it holds that \(\text{TD}([\theta', \xi, \theta'], D, I, b) = \text{TD}([\theta', \xi, \theta'], D, I', b')\), and

(ii) For all \(\Delta \in \text{FORM}\) with \(\text{FV}(\Delta) \subseteq \{\xi\}\), \(I|\text{SE}(\Delta) = I'|\text{SE}(\Delta)\) and \(b|\text{ST}(\Delta) = b'|\text{ST}(\Delta)\) it holds that \(D, I, b \models [\theta, \xi, \Delta]\) if \(D, I', b' \models [\theta', \xi, \Delta]\).

**Proof:** Ad (i): Let \((D, I), (D, I')\) be models, \(b, b'\) parameter assignments for \(D, \xi \in \text{VAR}, \theta, \theta' \in \text{CTERM}\) and \(\text{TD}(\theta, D, I, b) = \text{TD}(\theta', D, I', b')\). The proof is carried out by induction on the complexity of \(\theta' \in \text{TERM}\). First, suppose \(\theta' \in \text{ATERM}\), where \(\text{FV}(\theta') \subseteq \{\xi\}\). \(I|\text{SE}(\theta') = I'|\text{SE}(\theta')\) and \(b|\text{ST}(\theta') = b'|\text{ST}(\theta)\). Then we have \(\theta^+ \in \text{CONST} \cup \text{PAR} \cup \text{VAR}\). Now, suppose \(\theta' \in \text{CONST}\). Then we have \([\theta, \xi, \theta^+] = \theta^+ = [\theta', \xi, \theta^+]\) and thus it holds, with \(\text{SE}(\theta') = \{\theta^+\}\), \(I|\text{SE}(\theta') = I'|\text{SE}(\theta')\) and Theorem 5-2-(i), that \(\text{TD}([\theta, \xi, \theta^+], D, I, b) = \text{TD}(\theta^+, D, I, b)\). Now, suppose \(\theta^+ \in \text{PAR}\). Then we have \([\theta, \xi, \theta^+] = \theta^+ = [\theta', \xi, \theta^+]\) and thus it holds, with \(\text{ST}(\theta') = \{\theta^+\}\), \(b|\text{ST}(\theta') = b'|\text{ST}(\theta)\) and Theorem 5-2-(ii), that \(\text{TD}([\theta, \xi, \theta^+], D, I, b) = \text{TD}(\theta^+, D, I, b)\). Now, suppose \(\theta^+ \in \text{VAR}\). Then we have \(\theta^+ = \xi\). Then we have \([\theta, \xi, \theta^+] = \theta\) and \([\theta', \xi, \theta^+] = \theta'\). By hypothesis, we thus have \(\text{TD}([\theta, \xi, \theta^+], D, I, b) = \text{TD}(\theta, D, I, b) = \text{TD}(\theta', D, I, b)\).
Now, suppose the statement holds for \( \theta^0, \ldots, \theta^{r-1} \in \text{TERM} \) and suppose \( \phi \in \text{FUNC}, \phi \) r-ary, and suppose \( \theta^r = \tau(\theta^0, \ldots, \theta^{r-1}) \in \text{FTERM} \), where \( \text{FV}(\tau(\theta^0, \ldots, \theta^{r-1})) \subseteq \{ \xi \} \), \( I \text{SE}(\tau(\theta^0, \ldots, \theta^{r-1})) = I \text{SE}(\tau(\theta^0, \ldots, \theta^{r-1})) \) and \( b \text{ST}(\tau(\theta^0, \ldots, \theta^{r-1})) = b \text{ST}(\tau(\theta^0, \ldots, \theta^{r-1})) \). Then it holds, with \( \text{FV}(\tau(\theta^0, \ldots, \theta^{r-1})) = \cup \{ \text{FV}(\theta^i) \mid i < r \} \), \( \cup \{ \text{SE}(\theta^i) \mid i < r \} \subseteq \text{SE}(\tau(\theta^0, \ldots, \theta^{r-1})) \) and \( \cup \{ \text{ST}(\theta^i) \mid i < r \} \subseteq \text{ST}(\tau(\theta^0, \ldots, \theta^{r-1})) \), for all \( i < r \) that \( \text{FV}(\theta^i) \subseteq \{ \xi \} \), \( I \text{SE}(\theta^i) = I \text{SE}(\theta^i) \) and \( b \text{ST}(\theta^i) = b \text{ST}(\theta^i) \).

With the I.H., it thus holds for all \( i < r \) that \( \text{TD}([\theta, \xi, \theta^i], D, I, b) = \text{TD}([\theta', \xi, \theta^i], D, I, b') \). With \( \phi \in \text{SE}(\tau(\theta^0, \ldots, \theta^{r-1})) \cap \text{FUNC} \), we have, by hypothesis, also \( I(\phi) = I(\phi) \).

With Theorem 5.2-2(ii), we hence have

\[
\text{TD}([\theta, \xi, \tau(\theta^0, \ldots, \theta^{r-1})], D, I, b) =
\text{TD}(\tau([\theta, \xi, \theta^0], \ldots, [\theta, \xi, \theta^{r-1}]), D, I, b)
= I(\phi)(\text{TD}([\theta, \xi, \theta^0], D, I, b), \ldots, \text{TD}([\theta, \xi, \theta^{r-1}], D, I, b)))
= I(\phi)(\text{TD}([\theta', \xi', \theta^0], D, I', b'), \ldots, \text{TD}([\theta', \xi', \theta^{r-1}], D, I', b'))
= \text{TD}(\tau([\theta', \xi', \theta^0], \ldots, [\theta', \xi', \theta^{r-1}]), D, I', b')
= \text{TD}([\theta', \xi', \tau(\theta^0, \ldots, \theta^{r-1})], D, I', b').
\]

Ad (ii): The proof is carried out by induction on the degree of a formula. For this, suppose the theorem holds for all \( A \in \text{FORM} \) with \( \text{FDEG}(A) < k \). Let now \( (D, I), (D, I') \) be models, \( b, b' \) parameter assignments for \( D, \xi \in \text{VAR}, 0, \theta' \in \text{FTERM} \) and \( \text{TD}(\theta, D, I, b) = \text{TD}(\theta', D, I', b') \) and suppose \( \Delta \in \text{FORM} \), where \( \text{FV}(\Delta) \subseteq \{ \xi \} \), \( I(\Delta) = I(\Delta) \) and \( b \text{ST}(\Delta) = b' \text{ST}(\Delta) \), and suppose \( \text{FDEG}(\Delta) = k \). Suppose \( \text{FDEG}(\Delta) = 0 \). Then we have \( \Delta \in \text{AFORM} \). Then there are \( \theta^0, \ldots, \theta^{r-1} \in \text{TERM} \) and \( \Phi \in \text{PRED} \), where \( \Phi \) is r-ary, such that \( \Delta = \tau(\Phi(\theta^0, \ldots, \theta^{r-1})) \). With \( \text{FV}(\tau(\Phi(\theta^0, \ldots, \theta^{r-1}))) = \cup \{ \text{FV}(\Phi(\theta^i)) \mid i < r \}, \cup \{ \text{SE}(\Phi(\theta^i)) \mid i < r \} \subseteq \text{SE}(\tau(\Phi(\theta^0, \ldots, \theta^{r-1}))) \) and \( \cup \{ \text{ST}(\Phi(\theta^i)) \mid i < r \} \subseteq \text{ST}(\tau(\Phi(\theta^0, \ldots, \theta^{r-1}))) \) and the assumption for \( \Delta \), it then holds for all \( i < r \) that \( \text{FV}(\Phi(\theta^i)) \subseteq \{ \xi \} \), \( I(\Phi(\theta^i)) = I(\Phi(\theta^i)) \) and \( b \text{ST}(\Phi(\theta^i)) = b' \text{ST}(\Phi(\theta^i)). \) With (i), we thus have for all \( i < r \) that \( \text{TD}([\theta, \xi, \Phi(\theta^i)], D, I, b) = \text{TD}([\theta', \xi, \Phi(\theta^i)], D, I', b') \). With \( \Phi \in \text{SE}(\tau(\Phi(\theta^0, \ldots, \theta^{r-1}))) \cap \text{PRED} \), we have, by hypothesis, that \( I(\Phi) = I(\Phi) \). With Theorem 5.4-1(i), we hence have
5.1 Satisfaction Relation and Model-theoretic Consequence

\( D, I, b \models [\theta, \xi, \Delta] \)

iff

\( D, I, b \models [\theta, \xi, \Phi(\theta_0^+, \ldots, \theta_{r-1}^+)] \)

iff

\( D, I, b \models [\theta, \xi, \Phi([\theta_0^+, \ldots, \theta_{r-1}^+])] \)

iff

\( \langle \text{TD}([\theta, \xi, \theta_0^+], D, I, b), \ldots, \text{TD}([\theta, \xi, \theta_{r-1}^+], D, I, b) \rangle \in I(\Phi) \)

iff

\( \langle \text{TD}([\theta', \xi, \theta_0^+], D, I', b'), \ldots, \text{TD}([\theta', \xi, \theta_{r-1}^+], D, I', b') \rangle \in I'(\Phi) \)

iff

\( D, I', b' \models [\theta', \xi, \Phi(\theta_0^+, \ldots, \theta_{r-1}^+)] \)

iff

\( D, I', b' \models [\theta', \xi, \Delta] \).

Now, suppose \( \text{FDEG}(\Delta) \neq 0 \). Then we have \( \Delta \in \text{CONFORM} \cup \text{QFORM} \). We can distinguish seven cases. First: Suppose \( \Delta = \neg A \). Then we have \( \text{FDEG}(A) < \text{FDEG}(\Delta) \). According to the assumption for \( \Delta \), we also have \( \text{FV}(A) \subseteq \{\xi\} \), \( I|\text{SE}(A) = I'|\text{SE}(A) \) and \( b|\text{ST}(A) = b'|\text{ST}(A) \). With the I.H. and Theorem 5-4-(ii), it then follows that

\( D, I, b \models [\theta, \xi, \Delta] \)

iff

\( D, I, b \models [\theta, \xi, \neg A] \)

iff

\( D, I, b \models \neg [\theta, \xi, A] \)

iff

\( D, I, b \models [\theta, \xi, A] \)

iff

\( D, I', b' \models \neg [\theta, \xi, A] \)

iff

\( D, I', b' \models [\theta, \xi, \neg A] \)

iff

\( D, I', b' \models [\theta, \xi, \Delta] \).

Second: Suppose \( \Delta = A \land B \). Therefore \( \text{FDEG}(A) < \text{FDEG}(\Delta) \) and \( \text{FDEG}(B) < \text{FDEG}(\Delta) \). According to the assumption for \( \Delta \), we also have \( \text{FV}(A) \lor \text{FV}(B) \subseteq \{\xi\} \), \( I|\text{SE}(A \lor SE(B)) = I'|\text{SE}(A \lor SE(B)) \) and \( b|\text{ST}(A \lor ST(B)) = b'|\text{ST}(A \lor ST(B)) \). With the I.H. and Theorem 5-4-(iii), it then follows that
The third to fifth cases are treated analogously.

Sixth: Suppose $\Delta = \lnot\zeta A$. According to the assumption for $\Delta$, we then have $FV(A) \subseteq \{\xi, \zeta\}$, $I|SE(A) = I'|SE(A)$ and $b|ST(A) = b'|ST(A)$. Suppose $\zeta = \xi$. Then we have $[0, \zeta, \Delta] = [0, \xi, \zeta A\zeta] = [0, \xi, \zeta A\zeta] = [0, \zeta, \Delta]$ and hence $[0, \zeta, \Delta] = \Delta = [0, \zeta, \Delta]$. Also, we have $FV(\Delta) = \emptyset$ and hence $\Delta \subseteq CFORM$. Since, by hypothesis, $I|SE(\Delta) = I'|SE(\Delta)$ and $b|ST(\Delta) = b'|ST(\Delta)$ we thus have, with Theorem 5-5-(ii), that $D, I, b \models [0, \zeta, \Delta]$ if $D, I, b \models \Delta$ if $D, I, b' \models \Delta$ if $D, I, b' \models [0, \zeta, \Delta]$. Now, suppose $\zeta \neq \xi$. Then we have $[0, \zeta, \Delta] = \lnot\zeta[0, \zeta, \Delta]$ and $[0, \xi, \Delta] = \lnot\zeta[0, \xi, \Delta]$. With $\zeta \neq \xi$ and $\zeta, \xi \not\in ST(\theta^\vee)$ for all $\theta^\vee \in CTERM$ and Theorem 1-25-(ii), we also have for all $\beta^\vee \in \text{PAR}: [\beta^\vee, \zeta, [0, \xi, \Delta]] = [0, \xi, [\beta^\vee, \zeta, A]]$ and $[\beta^\vee, \xi, [0, \zeta, \Delta]] = [0, \xi, [\beta^\vee, \zeta, A]]$.

Now, suppose $D, I, b \models \lnot\zeta[0, \zeta, A]$. With Theorem 5-4-(vii), there is then a $\beta^\vee \in \text{PAR}\setminus ST([0, \zeta, A])$ such that for all $b^\vee$ that are in $\beta^\vee$ assignment variants of $b$ for $D$ it holds that $D, I, b^\vee \models [\beta^\vee, \zeta, [0, \xi, A]]$. Now, let $\beta^\vee \in \text{PAR}\setminus ST([0, \zeta, A]) \cup ST(\theta) \cup ST(\theta^\vee)$. Then suppose $b_1 \models \beta^\vee \zeta[0, \zeta, A]$ and $[\beta^\vee, \zeta, [0, \zeta, A]]$. Now, let $b_2 = (b \setminus \{\beta^\vee, b(\beta^\vee)\}) \cup \{\beta^\vee, b_1(\beta^\vee)\}$. Then $b_2$ is in $\beta^\vee$ an assignment variant of $b$ for $D$ and $b_2(\beta^\vee) = b_1(\beta^\vee)$. Now, let $b_2 = (b \setminus \{\beta^\vee, b(\beta^\vee)\}) \cup \{\beta^\vee, b_1(\beta^\vee)\}$. Then $b_2$ is in $\beta^\vee$ an assignment variant of $b$ for $D$ and thus we have $D, I, b_2 \models [\beta^\vee, \zeta, [0, \zeta, A]]$. Also, we have $\text{TD}(\beta^\vee, D, I, b_2) = b_2(\beta^\vee) = b_1(\beta^\vee) = \text{TD}(\beta^\vee, D, I, b_1)$. Also, we have, according to the assumption for $\beta^\vee$ and $\beta^\vee$, that $\beta^\vee, \beta^\vee \not\in ST([0, \zeta, A])$ and thus $b_2|ST([0, \zeta, A]) = \emptyset$.
\( b \models \text{ST}([\theta, \xi, A]) = b \models \text{ST}([\theta, \xi, A]) \). Also, we trivially have that \( I \models \text{SE}([\theta, \xi, A]) = I \models \text{SE}([\theta, 
abla \xi, A]) \). Further, we have \( \text{FV}([\theta, \xi, A]) \subseteq \{\xi\} \) and, with Theorem 1-13, we have \( \text{FDEG}([\theta, \xi, A]) = \text{FDEG}(A) < \text{FDEG}(\Delta) \). By the I.H., we thus have, because of \( D, I, b \models [\beta^+, \xi, [\theta, \xi, A]] = [\theta, \xi, [\beta^+, \xi, A]] \), that also \( D, I, b_1 \models [\beta^+, \xi, [\theta, \xi, A]] = [\theta, \xi, [\beta^+, \xi, A]] \).

With \( \beta'' \not\in \text{ST}(\theta) \), we have that \( b_1 \models \text{ST}(\theta) = b_1 \models \text{ST}(\theta') \) and, with \( \beta'' \not\in \text{ST}(\theta') \), we have that \( b_1 \models [\beta'' \models \text{ST}(\theta') = b_1 \models \text{ST}(\theta') \), and, because we trivially have \( I \models \text{SE}(\theta) = I \models \text{SE}(\theta) \) and \( I \models \text{SE}(\theta') = I \models \text{SE}(\theta') \), we thus have, according to Theorem 5-5-(i), that \( \text{TD}(\theta, D, I, b_1) = \text{TD}(\theta, D, I, b) \) and \( \text{TD}(\theta', D, I', b_1') = \text{TD}(\theta', D, I', b') \). By our intial hypothesis, we thus have \( \text{TD}(\theta, D, I, b_1) = \text{TD}(\theta', D, I', b_1') \). With \( b \models \text{ST}(A) = b \models [\beta'' \models \text{ST}(A), b_1[\beta'' \models [\beta^+, \xi, A]] = b_1[\beta^+ \models [\beta^+, \xi, A]] \) and \( \text{ST}([\beta'', \xi, A]) \subseteq \text{ST}(A) \cup \{\beta''\} \), we also have \( b \models \text{ST}([\beta'', \xi, A]) = b \models \text{ST}([\beta'', \xi, A]) \). We also have: \( I \models \text{SE}(\beta'', \xi, A) \cap (\text{CONST} \cup \text{FUNC} \cup \text{PRED}) = I \models \text{SE}(\beta'', \xi, A) \cap (\text{CONST} \cup \text{FUNC} \cup \text{PRED}) = I \models \text{SE}(\beta'', \xi, A) \cap (\text{CONST} \cup \text{FUNC} \cup \text{PRED}) = I \models \text{SE}(\beta'', \xi, A) \) and hence \( I \models \text{SE}(\beta'', \xi, A) = I \models \text{SE}(\beta'', \xi, A) \). Further, we have \( \text{FV}([\beta'', \xi, A]) \subseteq \{\xi\} \) and, with Theorem 1-13, we have \( \text{FDEG}([\beta'', \xi, A]) < \text{FDEG}(\Delta) \). By the I.H. it thus holds, because of \( D, I, b_1 \models [\theta, \xi, [\beta'', \xi, A]] \), that also \( D, I', b_1' \models [\theta', \xi, [\beta'', \xi, A]] = [\beta'', \xi, [\theta', \xi, A]] \). Therefore we have for all \( b'' \) that are in \( \beta'' \) assignment variants of \( b' \) for \( D \) that \( D, I', b'' \models [\beta'', \xi, [\theta', \xi, A]] \) and hence we have, according to Theorem 5-4-(vii), that \( D, I, b' \models [\lnot \lor \xi[\theta', \xi, A]] \). The right-left-direction is shown analogously.

**Seventh:** Suppose \( \Delta = [\lor \xi A^\top] \). According to the assumption for \( \Delta \), we then have \( \text{FV}(A) \subseteq \{\xi, \xi\} \), \( I \models \text{SE}(A) = I \models \text{SE}(A) \) and \( b \models \text{ST}(A) = b \models \text{ST}(A) \). Suppose \( \xi = \xi \). Then we have \([0, 
abla \xi A^\top] = \lor \xi A^\top = [\theta', \xi, [\lor \xi A^\top] = [\theta', \xi, \Delta] \) and hence \([\theta, 
abla \xi, \Delta] = \Delta = [\theta', \xi, \Delta] \). Also, we have \( \text{FV}(\Delta) = 0 \) and hence \( \Delta \in \text{CFORM} \). Since by hypothesis \( I \models \text{SE}(\Delta) = I \models \text{SE}(\Delta) \) and \( b \models \text{ST}(\Delta) = b \models \text{ST}(\Delta) \), we thus have, with Theorem 5-5-(ii) that \( D, I, b \models [\theta, 
abla \xi, \Delta] \) iff \( D, I, b \models \Delta \) iff \( D, I', b' \models \Delta \) iff \( D, I', b' \models [\theta', \xi, \Delta] \). Now, suppose \( \xi \neq \xi \). Then we have \([0, \xi, \Delta] = [\lor \xi[0, \xi, A^\top] \) and \([0', \xi, \Delta] = [\lor \xi[0', \xi, A^\top] \). With \( \xi \neq \xi \) and \( \xi \neq \xi \) \( \not\in \text{ST}(\theta') \) for all \( \theta' \in \text{CFORM} \) and Theorem 1-25-(ii), it holds for all \( \beta^+ \in \text{PAR} \) that \( [\beta^+, \xi, \Delta = [\theta', \xi, [\beta^+, \xi, A]] = [\theta', \xi, [\beta^+, \xi, A]] \).
Now, suppose $D, I, b \models \forall \varphi[\theta, \zeta, A]$. With Theorem 5-4-(viii), there is then $\beta^+ \in \text{PAR}\setminus\text{ST}([\theta, \zeta, A])$ and $b_1$, that is in $\beta^+$ an assignment variant of $b$ for $D$ such that $D, I, b_1 \models [\beta^+, \zeta, [\theta, \zeta, A]]$. Now, let $\beta^\theta \in \text{PAR}\setminus(\text{ST}([\theta, \zeta, A]) \cup \text{ST}(\theta) \cup \text{ST}(\theta'))$. Now, let $b_1' = (b \setminus \{(\beta^\theta, b'(\beta^\theta))\}) \cup \{(\beta^\theta, b_1(\beta^\theta))\}$. Then $b_1'$ is in $\beta^\theta$ an assignment variant of $b'$ for $D$ and $b_1'(\beta^\theta) = b_1(\beta^\theta)$. Now, let $b_2 = (b \setminus \{(\beta^\theta, b'(\beta^\theta))\}) \cup \{(\beta^\theta, b_1'(\beta^\theta))\}$. Then $b_2$ is in $\beta^\theta$ an assignment variant of $b$ for $D$ and $\text{TD}(\beta^\theta, D, I, b_2) = b_2(\beta^\theta) = b_1'(\beta^\theta) = b_1(\beta^\theta) = \text{TD}(\beta^\theta, D, I, b_1)$. According to the assumption for $\beta^+$ and $\beta^\theta$, we also have that $\beta^+, \beta^\theta \not\in \text{ST}([\theta, \zeta, A])$ and thus that $b_2|\text{ST}([\theta, \zeta, A]) = b|\text{ST}([\theta, \zeta, A]) = b_1|\text{ST}([\theta, \zeta, A])$. We trivially have $I|\text{SE}([\theta, \zeta, A]) = I|\text{SE}([\theta, \zeta, A])$. Also, we have $I|\text{FV}([\theta, \zeta, A]) \subseteq \{\zeta\}$ and, with Theorem 1-13, we have $\text{FDEG}([\theta, \zeta, A]) = \text{FDEG}(A) < \text{FDEG}(\Delta)$. By the I.H., it thus holds, because of $D, I, b_1 \models [\beta^+, \zeta, [\theta, \zeta, A]]$, that $D, I, b_2 \models [\beta^+, \zeta, [\theta, \zeta, A]] = [\theta, \zeta, [\beta^\theta, \zeta, A]]$.

With $\beta^\theta \not\in \text{ST}(\theta)$ and $\beta^\theta \not\in \text{ST}(\theta')$, we have $b_2|\text{ST}(\theta) = b|\text{ST}(\theta)$ and $b_1'|\text{ST}(\theta') = b'|\text{ST}(\theta')$ and hence, according to Theorem 5-5-(i), we have $\text{TD}(\theta, D, I, b_2) = \text{TD}(\theta, D, I, b')$. By our initial hypothesis, we thus have $\text{TD}(\theta, D, I, b_2) = \text{TD}(\theta', D, I', b')$. With $b|\text{ST}(A) = b_1'|\text{ST}(A)$, $b_2(\beta^\theta) = b_1'(\beta^\theta)$ and $\text{ST}([\beta^\theta, \zeta, A]) \subseteq \text{ST}(A) \cup \{\beta^\theta\}$, we also have $b_2'|\text{ST}([\beta^\theta, \zeta, A]) = b_1'|\text{ST}([\beta^\theta, \zeta, A])$ and it holds that $I|\text{SE}([\beta^\theta, \zeta, A]) = I|(\text{SE}([\beta^\theta, \zeta, A]) \cap \text{CONST} \cup \text{FUNC} \cup \text{PRED})) = I|(\text{SE}(A) \cap \text{CONST} \cup \text{FUNC} \cup \text{PRED})) = I|\text{SE}(A) = I|\text{SE}(A) = I|\text{SE}(A) = I|(\text{SE}(A) \cap \text{CONST} \cup \text{FUNC} \cup \text{PRED})) = I|(\text{SE}(\{\beta^\theta, \zeta, A\}) \cap \text{CONST} \cup \text{FUNC} \cup \text{PRED})) = I|\text{SE}(\{\beta^\theta, \zeta, A\}) \cap \text{CONST} \cup \text{FUNC} \cup \text{PRED}) = I|(\text{SE}(\{\beta^\theta, \zeta, A\}) \cap \text{CONST} \cup \text{FUNC} \cup \text{PRED})) = I|\text{SE}(\{\beta^\theta, \zeta, A\})$ and hence it holds that $I|\text{SE}([\beta^\theta, \zeta, A]) = I|\text{SE}([\beta^\theta, \zeta, A])$. Further we have $\text{FV}([\beta^\theta, \zeta, A]) \subseteq \{\zeta\}$ and, with Theorem 1-13, we have $\text{FDEG}([\beta^\theta, \zeta, A]) < \text{FDEG}(\Delta)$. By the I.H., it thus holds, because of $D, I, b_2 \models [\theta, \zeta, [\beta^\theta, \zeta, A]]$, that $D, I, b_1' \models [\theta, \zeta, [\beta^\theta, \zeta, A]] = [\beta^\theta, \zeta, [\theta, \zeta, A]]$ and hence, according to Theorem 5-4-(viii), that $D, I, b' \models \forall \varphi[\theta, \zeta, A]$.

The right-left-direction is shown analogously. ■

Now we will prove some consequences of the substitution lemma in order to facilitate some later proofs.
Theorem 5-7. Coreferentiality

If \((D, I)\) is a model, \(b\) is a parameter assignment for \(D\), \(\xi \in \text{VAR}\), \(\theta, \theta' \in \text{TERM}\) and \(\text{TD}(\theta, D, I, b) = \text{TD}(\theta', D, I, b)\), then:

(i) For all \(\theta' \in \text{TERM}\) with \(\text{FV}(\theta') \subseteq \{\xi\}\) it holds that \(\text{TD}([\theta, \xi, \theta'], D, I, b) = \text{TD}([\theta', \xi, \theta'], D, I, b)\), and

(ii) For all \(\Delta \in \text{FORM}\) with \(\text{FV}(\Delta) \subseteq \{\xi\}\) it holds that \(D, I, b \models [\theta, \xi, \Delta]\) iff \(D, I, b \models [\theta', \xi, \Delta]\).

Proof: Suppose \((D, I)\) is a model, \(b\) is a parameter assignment for \(D\), \(\xi \in \text{VAR}\), \(\theta, \theta' \in \text{TERM}\) and \(\text{TD}(\theta, D, I, b) = \text{TD}(\theta', D, I, b)\). Then we trivially have for all \(\mu \in \text{TERM} \cup \text{FORM}\): \(I|\text{SE}(\mu) = I|\text{SE}(\mu)\) and \(b|\text{ST}(\mu) = b|\text{ST}(\mu)\) and thus the statement follows with Theorem 5-6. ■

Theorem 5-8. Invariance of the satisfaction of quantificational formulas with respect to the choice of parameters

If \((D, I)\) is a model, \(b\) is a parameter assignment for \(D\), \(\xi \in \text{VAR}\), \(\Delta \in \text{FORM}\), with \(\text{FV}(\Delta) \subseteq \{\xi\}\) and \(\beta \in \text{PAR} \setminus \text{ST}(\Delta)\), then:

(i) \(D, I, b \models \forall \xi \Delta\) iff for all \(b'\) that are in \(\beta\) assignment variants of \(b\) for \(D\) it holds that \(D, I, b' \models [\beta, \xi, \Delta]\), and

(ii) \(D, I, b \models \forall \xi \Delta\) iff there is a \(b'\) that is in \(\beta\) assignment variant of \(b\) for \(D\) such that \(D, I, b' \models [\beta, \xi, \Delta]\).

Proof: Suppose \((D, I)\) is a model, \(b\) is a parameter assignment for \(D\), \(\xi \in \text{VAR}\), \(\Delta \in \text{FORM}\) with \(\text{FV}(\Delta) \subseteq \{\xi\}\) and \(\beta \in \text{PAR} \setminus \text{ST}(\Delta)\). Ad (i): The right-left-direction follows directly with Theorem 5-4-(vii). Now, for the left-right-direction, suppose \(D, I, b \models \forall \xi \Delta\). Then there is a \(\beta^* \in \text{PAR} \setminus \text{ST}(\Delta)\) such that for all \(b^*\) that are in \(\beta^*\) assignment variants of \(b\) for \(D\) it holds that \(D, I, b^* \models [\beta^*, \xi, \Delta]\). Now, suppose \(b'\) is in \(\beta\) an assignment variant of \(b\) for \(D\). Now, let \(b^* = (b \setminus \{\beta^*, b(\beta^*)\}) \cup \{\beta^*, b(\beta^*)\}\). Then \(b^*\) is in \(\beta^*\) an assignment variant of \(b\) for \(D\) and hence we have \(D, I, b^* \models [\beta^*, \xi, \Delta]\). We also have \(\text{TD}(\beta^*, D, I, b^*) = b^*(\beta^*) = b'(\beta) = \text{TD}(\beta, D, I, b')\). With \(\beta, \beta^* \not\in \text{ST}(\Delta)\), we further have \(b^*|\text{ST}(\Delta) = b'|\text{ST}(\Delta) = b'|\text{ST}(\Delta)\). With Theorem 5-6-(ii), we hence have \(D, I, b' \models [\beta, \xi, \Delta]\).

Ad (ii): The right-left-direction follows directly with Theorem 5-4-(viii). Now, for the left-right-direction, suppose \(D, I, b \models \forall \xi \Delta\). Then there is a \(\beta^* \in \text{PAR} \setminus \text{ST}(\Delta)\) and \(b^*\) that
is in $\beta^*$ an assignment variant of $b$ for $D$ such that $D, I, b^* \models [\beta^*, \xi, \Delta]$. Now, let $b' = (b \setminus \{(\beta, b(\beta))\}) \cup \{(\beta, b^*(\beta^*))\}$. Then $b'$ is in $\beta$ an assignment variant of $b$ for $D$ and we have $TD(\beta^*, D, I, b^*) = b^*(\beta^*) = b'(\beta) = TD(\beta, D, I, b')$. With $\beta, \beta^* \not\in ST(\Delta)$ we have again $b^* \models ST(\Delta) = b' \models ST(\Delta)$. With Theorem 5-6-(ii), we hence have $D, I, b' \models [\beta, \xi, \Delta]$. ■

**Theorem 5-9. Simple substitution lemma for parameter assignments**

If $(D, I)$ is a model, $b$ is a parameter assignment for $D$, $\xi \in \text{VAR}$, $\beta \in \text{PAR}$ and $\theta \in \text{CTERM}$, then:

(i) If $b'$ is in $\beta$ an assignment variant of $b$ for $D$ and $b'(\beta) = TD(\theta, D, I, b)$, then for all $\theta^+ \in \text{TERM}$ with $\text{FV}(\theta^+) \subseteq \{\xi\}$ and $\beta \not\in ST(\theta^+)$: $TD([\theta, \xi, \theta^+], D, I, b) = TD([\beta, \xi, \theta^+], D, I, b')$, and

(ii) If $b'$ is in $\beta$ an assignment variant of $b$ for $D$ and $b'(\beta) = TD(\theta, D, I, b)$, then for all $\Delta \in \text{FORM}$ with $\text{FV}(\Delta) \subseteq \{\xi\}$ and $\beta \not\in ST(\Delta)$: $D, I, b \models [\theta, \xi, \Delta]$ iff $D, I, b' \models [\beta, \xi, \Delta]$.

**Proof:** Suppose $(D, I)$ is a model, $b$ is a parameter assignment for $D$, $\xi \in \text{VAR}$, $\beta \in \text{PAR}$ and $\theta \in \text{CTERM}$. Now, suppose $b'$ is in $\beta$ an assignment variant of $b$ for $D$, where $b'(\beta) = TD(\theta, D, I, b)$. Now, suppose $\mu \in \text{TERM} \cup \text{FORM}$ with $\text{FV}(\mu) \subseteq \{\xi\}$ and $\beta \not\in ST(\mu)$. Then we trivially have $I|SE(\mu) = I|SE(\mu)$. With $\beta \not\in ST(\mu)$, we also have $b|ST(\mu) = b'|ST(\mu)$. By hypothesis, we also have $TD(\beta, D, I, b') = b'(\beta) = TD(\theta, D, I, b)$.

According to Theorem 5-6-(i), we then have for all $\theta^+ \in \text{TERM}$ with $\text{FV}(\theta^+) \subseteq \{\xi\}$ and $\beta \not\in ST(\theta^+)$: $TD([\theta, \xi, \theta^+], D, I, b) = TD([\beta, \xi, \theta^+], D, I, b')$, and, with Theorem 5-6-(ii), we have for all $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi\}$ and $\beta \not\in ST(\Delta)$: $D, I, b \models [\theta, \xi, \Delta]$ iff $D, I, b' \models [\beta, \xi, \Delta]$. ■

**Definition 5-9. 4-ary model-theoretic satisfaction for sets**

$D, I, b \models X$

iff

$(D, I)$ is a model, $b$ is a parameter assignment for $D$, $X \subseteq \text{FORM}$ and for all $\Delta \in X$: $D, I, b \models \Delta$. 

Definition 5-10. Model-theoretic consequence
\[ X \models \Gamma \]
iff
\[ X \cup \{ \Gamma \} \subseteq \text{CFORM} \text{ and for all } D, I, b: \text{ if } D, I, b \models X, \text{ then } D, I, b \models \Gamma. \]

Definition 5-11. Validity
\[ \models \Gamma \text{ iff } \emptyset \models \Gamma. \]

Definition 5-12. Satisfiability
\[ \Gamma \text{ is satisfiable} \]
iff
\[ \Gamma \in \text{CFORM} \text{ and there is } D, I, b \text{ such that } D, I, b \models \Gamma. \]

In Definition 5-8 to Definition 5-12 we introduced some of the usual model-theoretic concepts. With the next Definition, we will now add a 3-ary satisfaction concept for propositions that aims especially at parameter-free propositions. Subsequently, we will introduce concepts for sets of propositions that are analogous to the concepts we introduced for closed formulas in Definition 5-10 to Definition 5-13, in the same way as we did with Definition 5-9 for the satisfaction concept for closed formulas defined in Definition 5-8.

Definition 5-13. 3-ary model-theoretic satisfaction
\[ D, I \models \Gamma \]
iff
\[ (D, I) \text{ is a model and for all } b \text{ that are parameter assignments for } D \text{ it holds that } D, I, b \models \Gamma. \]

Definition 5-14. 3-ary model-theoretic satisfaction for sets
\[ D, I \models X \]
iff
\[ (D, I) \text{ is a model, } X \subseteq \text{CFORM} \text{ and for all } \Delta \in X \text{ it holds that } D, I \models \Delta. \]

Definition 5-15. Model-theoretic consequence for sets
\[ X \models Y \]
iff
\[ X \cup Y \subseteq \text{CFORM} \text{ and for all } \Delta \in Y \text{ it holds that } X \models \Delta. \]
Definition 5-16. Validity for sets
\[ M \models X \iff X \subseteq \text{CFORM and for all } \Delta \in X \text{ it holds that } \models \Delta. \]

Definition 5-17. Satisfiability for sets
\[ X \text{ is satisfiable}_M \iff X \subseteq \text{CFORM and there is } D, I, b \text{ such that } D, I, b \models X. \]

In the following the context will always indicate if we deal with propositions or with sets of propositions. Therefore, we will suppress the index 'M' when using concepts defined in Definition 5-9 and Definition 5-14 to Definition 5-17. Now, we will define the closure of a set of propositions under the model-theoretic consequence relation. The remaining part of this section contains only some simple supporting theorems.

Definition 5-18. The closure of a set of propositions under model-theoretic consequence
\[ X^\models = \{ \Delta \mid \Delta \in \text{CFORM and } X \models \Delta \}. \]

Theorem 5-10. Satisfaction carries over to subsets
If \( D, I, b \models X \), then it holds for all \( Y \subseteq X \) that \( D, I, b \models Y \).

Proof: Follows directly from Definition 5-9. ■

Theorem 5-11. Satisfiability carries over to subsets
If \( X \) is satisfiable, then it holds for all \( Y \subseteq X \) that \( Y \) is satisfiable.

Proof: Follows directly from Definition 5-17 and Theorem 5-10. ■

Theorem 5-12. Consequence relation and satisfiability
If \( X \cup \{ \Gamma \} \subseteq \text{CFORM} \), then: \( X \models \Gamma \iff X \cup \{ \lnot \Gamma \} \) is not satisfiable.

Proof: Suppose \( X \cup \{ \Gamma \} \subseteq \text{CFORM} \). Suppose \( X \models \Gamma \). Then we have for all \( D, I, b \): If \( D, I, b \models X \), then \( D, I, b \models \Gamma \). Suppose for contradiction that \( X \cup \{ \lnot \Gamma \} \) is satisfiable. Then there would be \( D, I, b \) such that \( D, I, b \models X \cup \{ \lnot \Gamma \} \). With Definition 5-9 and Theorem 5-4-(ii), it then follows that \( D, I, b \not\models \Gamma \). On the other hand, we would have,
with Theorem 5-10, that $D, I, b \models X$ and thus, by hypothesis, that $D, I, b \models \Gamma$. Contradiction!

Now, suppose $X \cup \{\neg \Gamma\}$ is not satisfiable. Then there is no $D, I, b$ such that $D, I, b \models X \cup \{\neg \Gamma\}$. With Definition 5-9 there is then no $D, I, b$ such that $D, I, b \models X$ and $D, I, b \models \neg \Gamma$. Now, suppose $D, I, b \models X$. Then $(D, I)$ is a model and $b$ is a parameter assignment for $D$ and $D, I, b \not\models \neg \Gamma$. According to Theorem 5-4-(ii), we then have $D, I, b \not\models \Gamma$. Therefore we have for all $D, I, b$: If $D, I, b \models X$, then $D, I, b \models \Gamma$. Hence we have $X \models \Gamma$. ■
5.2 Closure of the Model-theoretic Consequence Relation

The following section leads to correctness. For each rule of the Speech Act Calculus (cf. ch. 3.1) (or for each extension operation (cf. ch. 3.2)), we will therefore prove a model-theoretic theorem that corresponds to the respective closure clause in ch. 4.2, i.e. to Theorem 4-15 (AR) or to one of the clauses of Theorem 4-18. First, however, we will prove the monotony of the model-theoretic consequence relation (cf. Theorem 4-16).

**Theorem 5-13. Model-theoretic monotony**

If $X' \subseteq X \subseteq \text{CFORM}$ and $X' \models \Gamma$, then $X \models \Gamma$.

*Proof: Suppose $X' \subseteq X \subseteq \text{CFORM}$ and $X' \models \Gamma$. Then we have for all $D, I, b$: If $D, I, b \models X'$, then $D, I, b \models \Gamma$. Now, suppose $D, I, b \models X$. Then it holds, with $X' \subseteq X$ and Theorem 5-10, that $D, I, b \models X'$. By hypothesis, it thus holds that $D, I, b \models \Gamma$. Therefore we have for all $D, I, b$: If $D, I, b \models X$, then $D, I, b \models \Gamma$. Therefore $X \models \Gamma$. ■

**Theorem 5-14. Model-theoretic counterpart of AR**

If $X \subseteq \text{CFORM}$ and $A \in X$, then $X \models A$.

*Proof: Suppose $X \subseteq \text{CFORM}$ and $A \in X$. According to Definition 5-9, we then have for all $D, I, b$: If $D, I, b \models X$, then $D, I, b \models A$ and thus we have $X \models A$. ■

**Theorem 5-15. Model-theoretic counterpart of CdI**

If $X \models \neg B$ and $A \in X$, then $X \setminus \{A\} \models \neg \neg A \rightarrow B$.

*Proof: Suppose $X \models \neg B$ and $A \in X$. Now, suppose $D, I, b \models X \setminus \{A\}$. Then $(D, I)$ is a model and $b$ is a parameter assignment for $D$ and for all $\Delta \in X \setminus \{A\}$ it holds that $D, I, b \models \Delta$. Then we have either $D, I, b \models A$ or $D, I, b \not\models A$. In the first case, it holds that $D, I, b \models \Delta$ for all $\Delta \in X$, and hence we have $D, I, b \models X$. By hypothesis, it then follows that also $D, I, b \models \neg B$. With Theorem 5-4-(v), it then follows that $D, I, b \models \neg \neg A \rightarrow B$. The same holds if $D, I, b \not\models A$. Therefore we have for all $D, I, b$ that if $D, I, b \models X \setminus \{A\}$, then $D, I, b \models \neg \neg A \rightarrow B$. Therefore $X \setminus \{A\} \models \neg \neg A \rightarrow B$. ■
Theorem 5-16. Model-theoretic counterpart of CdE
If \( X \models \left\langle A \rightarrow B \right\rangle \) and \( Y \models A \), then \( X \cup Y \models B \).

Proof: Suppose \( X \models \left\langle A \rightarrow B \right\rangle \) and \( Y \models A \). Suppose \( D, I, b \models X \cup Y \). Then \( (D, I) \) is a model and \( b \) is a parameter assignment for \( D \) and, with Theorem 5-10, we have \( D, I, b \models X \) and \( D, I, b \models Y \). By hypothesis, it then follows that \( D, I, b \models A \) and \( D, I, b \models \left\langle A \rightarrow B \right\rangle \). With \( D, I, b \models \left\langle A \rightarrow B \right\rangle \) and Theorem 5-4-(v), we then have \( D, I, b \not\models A \) or \( D, I, b \models B \). With \( D, I, b \models A \), we thus have \( D, I, b \models B \). Therefore we have for all \( D, I, b \), that if \( D, I, b \models X \cup Y \), then also \( D, I, b \models B \). Therefore \( X \cup Y \models B \). 

Theorem 5-17. Model-theoretic counterpart of CI
If \( X \models A \) and \( Y \models B \), then \( X \cup Y \models \left\langle A \land B \right\rangle \).

Proof: Suppose \( X \models A \) and \( Y \models B \). Suppose \( D, I, b \models X \cup Y \). Then \( (D, I) \) is a model and \( b \) is a parameter assignment for \( D \) and, with Theorem 5-10, we have \( D, I, b \models X \) and \( D, I, b \models Y \). By hypothesis, it then follows that also \( D, I, b \models A \) and \( D, I, b \models B \). With Theorem 5-4-(iii), it then follows that \( D, I, b \models \left\langle A \land B \right\rangle \). Therefore we have for all \( D, I, b \) that if \( D, I, b \models X \cup Y \), then also \( D, I, b \models \left\langle A \land B \right\rangle \). Therefore \( X \cup Y \models \left\langle A \land B \right\rangle \).

Theorem 5-18. Model-theoretic counterpart of CE
If \( X \models \left\langle A \land B \right\rangle \), then \( X \models A \) and \( X \models B \).

Proof: Suppose \( X \models \left\langle A \land B \right\rangle \). Suppose \( D, I, b \models X \). Then \( (D, I) \) is a model and \( b \) is a parameter assignment for \( D \) and by hypothesis we have \( D, I, b \models \left\langle A \land B \right\rangle \). With Theorem 5-4-(iii), it then follows that \( D, I, b \models A \) and \( D, I, b \models B \). Therefore we have for all \( D, I, b \) that if \( D, I, b \models X \), then also \( D, I, b \models A \) and \( D, I, b \models B \). Therefore \( X \models A \) and \( X \models B \).
Theorem 5-19. Model-theoretic counterpart of BI
If \( X \models A \rightarrow B \) and \( Y \models B \rightarrow A \), then \( X \cup Y \models A \leftrightarrow B \).

Proof: Suppose \( X \models A \rightarrow B \) and \( Y \models B \rightarrow A \). Suppose \( D, I, b \models X \cup Y \). Then \( (D, I) \) is a model and \( b \) is a parameter assignment for \( D \) and, with Theorem 5-10, we have \( D, I, b \models X \) and \( D, I, b \models Y \). By hypothesis, it then follows that \( D, I, b \models A \rightarrow B \) and \( D, I, b \models B \rightarrow A \). With Theorem 5-4-(v), it then follows that \( (i) \) \( D, I, b \not\models A \) or \( D, I, b \models B \) and \( (ii) \) that \( D, I, b \not\models B \) or \( D, I, b \models A \). Suppose (the first case of \( (i) \)) \( D, I, b \not\models A \). With (ii), it then holds that \( D, I, b \models B \). Suppose (the second case of \( (i) \)) \( D, I, b \not\models B \). With (ii), it then holds that \( D, I, b \models A \). Therefore we have \( D, I, b \models A \) and \( D, I, b \not\models B \) or \( D, I, b \not\models A \) and \( D, I, b \not\models B \). With Theorem 5-4-(vi), it then follows that \( D, I, b \models A \leftrightarrow B \). Therefore we have for all \( D, I, b \) that if \( D, I, b \models X \cup Y \), then also \( D, I, b \models A \leftrightarrow B \). Therefore \( X \cup Y \models A \leftrightarrow B \). ■

We include a variant of Theorem 5-19 as a corollary. Here it is not required that some conditionals have to be model-theoretic consequences of some sets of propositions.

Theorem 5-20. Model-theoretic counterpart of BI*
If \( X \models B \) and \( A \in X \) and \( Y \models A \) and \( B \in Y \), then \( (X \setminus \{A\}) \cup (Y \setminus \{B\}) \models A \leftrightarrow B \).

Proof: Suppose \( X \models B \) and \( A \in X \) and \( Y \models A \) and \( B \in Y \). According to Theorem 5-15, we then have \( X \setminus \{A\} \models A \rightarrow B \) and \( Y \setminus \{B\} \models B \rightarrow A \). With Theorem 5-19, it then follows that \( (X \setminus \{A\}) \cup (Y \setminus \{B\}) \models A \leftrightarrow B \). ■

Theorem 5-21. Model-theoretic counterpart of BE
If \( X \models A \leftrightarrow B \) or \( X \models B \leftrightarrow A \) and \( Y \models A \), then \( X \cup Y \models B \).

Proof: Suppose \( X \models A \leftrightarrow B \) or \( X \models B \leftrightarrow A \) and \( Y \models A \). Now, suppose \( D, I, b \models X \cup Y \). Then \( (D, I) \) is a model and \( b \) is a parameter assignment for \( D \) and, with Theorem 5-10, we have \( D, I, b \models X \) and \( D, I, b \models Y \). By hypothesis, it then follows that \( D, I, b \models A \). Now, suppose \( X \models A \leftrightarrow B \). Then we have \( D, I, b \models A \leftrightarrow B \). With Theorem 5-4-(vi), it then follows that \( D, I, b \models A \) and \( D, I, b \not\models B \) or \( D, I, b \not\models A \) and \( D, I, b \not\models B \).
B. Now, suppose $X \models \neg B \leftrightarrow A \neg$. Then we have $D, I, b \models \neg B \leftrightarrow A \neg$. With Theorem 5-4-(vi), it then follows again that $D, I, b \models A$ and $D, I, b \models B$ or $D, I, b \not\models A$ and $D, I, b \not\models B$. However, since $D, I, b \models A$, it cannot be the case that $D, I, b \not\models A$ and $D, I, b \not\models B$. Thus we have $D, I, b \models A$ and $D, I, b \models B$. Therefore we have for all $D, I, b$ that if $D, I, b \models X \cup Y$, then also $D, I, b \models B$. Therefore $X \cup Y \models B$. 

**Theorem 5-22. Model-theoretic counterpart of DI**

If $X \models A$ or $X \models B$, then $X \models \neg \neg A \lor \neg B$.

*Proof:* Suppose $X \models A$ or $X \models B$. Suppose $D, I, b \models X$. Then $(D, I)$ is a model and $b$ is a parameter assignment for $D$. By hypothesis, we also have $D, I, b \models A$ or $D, I, b \models B$.

With Theorem 5-4-(iv), we have in both cases $D, I, b \models \neg \neg A \lor \neg B$. Therefore we have for all $D, I, b$ that if $D, I, b \models X$, then also $D, I, b \models \neg \neg A \lor \neg B$. Therefore $X \models \neg \neg A \lor \neg B$. 

**Theorem 5-23. Model-theoretic counterpart of DE**

If $X \models \neg \neg A \lor \neg B$ and $Y \models \neg A \rightarrow \Gamma$ and $Z \models \neg B \rightarrow \Gamma$, then $X \cup Y \cup Z \models \Gamma$.

*Proof:* Suppose $X \models \neg \neg A \lor \neg B$ and $Y \models \neg A \rightarrow \Gamma$ and $Z \models \neg B \rightarrow \Gamma$. Suppose $D, I, b \models X \cup Y \cup Z$. Then $(D, I)$ is a model and $b$ is a parameter assignment for $D$ and, with Theorem 5-10, we have $D, I, b \models X$ and $D, I, b \models Y$ and $D, I, b \models Z$. By hypothesis, it then follows that $D, I, b \models \neg \neg A \lor \neg B$ and $D, I, b \models \neg A \rightarrow \Gamma$ and $D, I, b \models \neg B \rightarrow \Gamma$.

With Theorem 5-4-(iv) and -(v), we then have: (i) $D, I, b \models A$ or $D, I, b \models B$ and (ii) $D, I, b \not\models A$ or $D, I, b \models \Gamma$ and (iii) $D, I, b \not\models B$ or $D, I, b \models \Gamma$. Suppose (the first case of (i)) $D, I, b \models A$. With (ii), we then have $D, I, b \models \Gamma$. Suppose (the second case of (i)) $D, I, b \models B$. With (iii), we then have $D, I, b \models \Gamma$. Thus we have in both cases $D, I, b \models \Gamma$. Therefore we have for all $D, I, b$ that if $D, I, b \models X \cup Y \cup Z$, then also $D, I, b \models \Gamma$. Therefore $X \cup Y \cup Z \models \Gamma$.

We include a variant of Theorem 5-23 as a corollary. Here it is not required that some conditionals have to be model-theoretic consequences of some sets of propositions.
Theorem 5-24. Model-theoretic counterpart of $DE^*$
If $X \vDash \neg \Psi \land A \land B$ and $Y \vDash \Gamma$ and $A \in Y$ and $Z \vDash \Gamma$ and $B \in Z$, then $X \cup (Y \setminus \{A\}) \cup (Z \setminus \{B\}) \vDash \Gamma$.

Proof: Suppose $X \vDash \neg \Psi$ and $Y \vDash \Gamma$ and $A \in Y$ and $Z \vDash \Gamma$ and $B \in Z$. According to Theorem 5-15, we then have $Y \setminus \{A\} \vDash \neg \Psi$ and $Z \setminus \{B\} \vDash \neg \Psi$. With Theorem 5-23, it then follows that $X \cup (Y \setminus \{A\}) \cup (Z \setminus \{B\}) \vDash \Gamma$. ■

Theorem 5-25. Model-theoretic counterpart of $NI$
If $X \vDash B$ and $Y \vDash \neg \Psi$ and $A \in X \cup Y$, then $(X \cup Y) \setminus \{A\} \vDash \neg \Psi$.

Proof: Suppose $X \vDash B$ and $Y \vDash \neg \Psi$ and $A \in X \cup Y$. Suppose $D, I, b \vDash (X \cup Y) \setminus \{A\}$. Then $(D, I)$ is a model and $b$ is a parameter assignment for $D$ such that for all $\Delta \in (X \cup Y) \setminus \{A\}$ it holds that $D, I, b \vDash \Delta$. Suppose for contradiction that $D, I, b \vDash A$. Then we would have for all $\Delta \in X$ and for all $\Delta \in Y$: $D, I, b \vDash \Delta$ and thus $D, I, b \vDash X$ and $D, I, b \vDash Y$. By hypothesis, it would then follow that $D, I, b \vDash B$ and $D, I, b \vDash \neg \Psi$. With Theorem 5-4-(ii), it would then follow that $D, I, b \vDash B$ and $D, I, b \not\models \neg \Psi$. Sed certe hoc esse non potest. Therefore $D, I, b \not\models A$ and thus $D, I, b \not\models \neg \Psi$. Therefore we have for all $D, I, b$ that if $D, I, b \vDash (X \cup Y) \setminus \{A\}$, then also $D, I, b \vDash \neg \Psi$. Therefore $(X \cup Y) \setminus \{A\} \vDash \neg \Psi$. ■

Theorem 5-26. Model-theoretic counterpart of $NE$
If $X \vDash \neg \neg \Psi$, then $X \models A$.

Proof: Suppose $X \vDash \neg \neg \Psi$. Suppose $D, I, b \models X$. Then $(D, I)$ is a model and $b$ is a parameter assignment for $D$ and, by hypothesis, we also have $D, I, b \models \neg \neg \Psi$. With Theorem 5-4-(ii), it then follows that $D, I, b \not\models \neg \Psi$. Applying Theorem 5-4-(ii) again yields $D, I, b \models A$. Therefore we have for all $D, I, b$: If $D, I, b \models X$, then $D, I, b \models A$. Therefore $X \models A$. ■
Theorem 5-27. Model-theoretic counterpart of UI
If $\beta \in \text{PAR}, \xi \in \text{VAR}, A \in \text{FORM}$, where $\text{FV}(A) \subseteq \{\xi\}$, and $X \models [\beta, \xi, A]$ and $\beta \not\in \text{STSF}(X \cup \{A\})$, then $X \models \text{Ú} \xi A$.

Proof: Suppose $\beta \in \text{PAR}, \xi \in \text{VAR}, A \in \text{FORM}$, where $\text{FV}(A) \subseteq \{\xi\}$, and $X \models [\beta, \xi, A]$ and $\beta \not\in \text{STSF}(X \cup \{A\})$. Suppose $D, I, b \models X$. Then $(D, I)$ is a model and $b$ is a parameter assignment for $D$. Suppose $b'$ in $\beta$ an assignment variant of $b$ for $D$. Suppose $\Delta \in X$. Therefore $D, I, b \models \Delta$. We have, by hypothesis, $\beta \not\in \text{ST}(\Delta)$. Therefore we have $b|\text{ST}(\Delta) = b'|\text{ST}(\Delta)$. According to Theorem 5-5-(ii) it then follows that also $D, I, b' \models \Delta$. Therefore $D, I, b' \models \Delta$ for all $\Delta \in X$ and hence $D, I, b' \models X$. With $X \models [\beta, \xi, A]$, we then have also $D, I, b' \models [\beta, \xi, A]$. Therefore we have for all $b'$ that are in $\beta$ an assignment variant of $b$ for $D$: $D, I, b' \models [\beta, \xi, A]$. With Theorem 5-4-(vii) follows $D, I, b \models \text{Ú} \xi A$. Therefore we have for all $D, I, b$: If $D, I, b \models X$, then also $D, I, b \models \text{Ú} \xi A$. Therefore $X \models \text{Ú} \xi A$. ■

Theorem 5-28. Model-theoretic counterpart of UE
If $\theta \in \text{CTERM}, \xi \in \text{VAR}, A \in \text{FORM}$, where $\text{FV}(A) \subseteq \{\xi\}$, and $X \models \text{Ú} \xi A$, then $X \models [\theta, \xi, A]$.

Proof: Suppose $\theta \in \text{CTERM}, \xi \in \text{VAR}, A \in \text{FORM}$, where $\text{FV}(A) \subseteq \{\xi\}$, and $X \models \text{Ú} \xi A$. Suppose $D, I, b \models X$. Then $(D, I)$ is a model and $b$ is a parameter assignment for $D$ and, by hypothesis, $D, I, b \models \text{Ú} \xi A$. According to Theorem 5-4-(vii) there is then a $\beta \in \text{PAR}\setminus\text{ST}(A)$ such that for all $b'$ that are in $\beta$ an assignment variant of $b$ for $D$ it holds that $D, I, b' \models [\beta, \xi, A]$. Suppose $b^* = (b\setminus\{(\beta, b(\beta))\}) \cup \{(\beta, \text{TD}(\theta, D, I, b))\}$. Obviously $b^*$ is in $\beta$ an assignment variant of $b$ for $D$. Therefore $D, I, b^* \models [\beta, \xi, A]$. With $b^*(\beta) = \text{TD}(\theta, D, I, b)$ and $\beta \not\in \text{ST}(A)$ it follows then with Theorem 5-9-(ii) that $D, I, b \models [\theta, \xi, A]$. Therefore we have for all $D, I, b$: If $D, I, b \models X$, then $D, I, b \models [\theta, \xi, A]$. Therefore $X \models [\theta, \xi, A]$. ■
Theorem 5-29. Model-theoretic counterpart of PI
If \( \theta \in \text{CTERM}, \xi \in \text{VAR}, A \in \text{FORM} \), where \( \text{FV}(A) \subseteq \{\xi\} \), and \( X \models \emptyset, \xi, A \), then \( X \models \forall \xi A^\ast \).

Proof: Suppose \( \theta \in \text{CTERM}, \xi \in \text{VAR}, A \in \text{FORM} \), where \( \text{FV}(A) \subseteq \{\xi\} \), and \( X \models \emptyset, \xi, A \). Suppose \( D, I, b \models X \). Then \( (D, I) \) is a model and \( b \) is a parameter assignment for \( D \) and, by hypothesis, we have \( D, I, b \models \emptyset, \xi, A \). Now, let \( \beta \in \text{PAR}\setminus\text{ST}(A) \) and let \( b^* = (b \setminus \{\beta, b(\beta)\}) \cup \{\beta, \text{TD}(\emptyset, D, I, b)\} \). Then \( b^* \) is in \( \beta \) an assignment variant of \( b \) for \( D \). With \( b^*(\beta) = \text{TD}(\emptyset, D, I, b), \beta \not\in \text{ST}(A) \) and Theorem 5-9-(ii), it then follows that \( D, I, b^* \models [\beta, \xi, A] \). With Theorem 5-4-(viii), it then follows that \( D, I, b \models \forall \xi A^\ast \). Therefore we have for all \( D, I, b \): If \( D, I, b \models X \), then \( D, I, b \models \forall \xi A^\ast \). Therefore \( X \models \forall \xi A^\ast \). ■

Theorem 5-30. Model-theoretic counterpart of PE
If \( \beta \in \text{PAR}, \xi \in \text{VAR}, A \in \text{FORM} \), where \( \text{FV}(A) \subseteq \{\xi\} \), and \( X \models \forall \xi A^\ast \) and \( Y \models B \) and \( \{\beta, \xi, A\} \subseteq Y \) and \( \beta \not\in \text{STSF}(\Psi\setminus\{\beta, \xi, A\}) \cup \{A, B\} \), then \( X \cup \{Y \setminus \{\beta, \xi, A\}\} \models B \).

Proof: Suppose \( \beta \in \text{PAR}, \xi \in \text{VAR}, A \in \text{FORM} \), where \( \text{FV}(A) \subseteq \{\xi\} \), \( X \models \forall \xi A^\ast \), \( Y \models B \), \( \{\beta, \xi, A\} \subseteq Y \) and \( \beta \not\in \text{STSF}(\Psi\setminus\{\beta, \xi, A\}) \cup \{A, B\} \). Suppose \( D, I, b \models X \cup \{Y \setminus \{\beta, \xi, A\}\} \). Then \( (D, I) \) is a model and \( b \) is a parameter assignment for \( D \) and, with Theorem 5-10, we have \( D, I, b \models X \) and \( D, I, b \models Y \setminus \{\beta, \xi, A\} \). By hypothesis, it then follows that \( D, I, b \models \forall \xi A^\ast \). Since \( \beta \not\in \text{ST}(A) \), there is then, according to Theorem 5-8-(ii), a \( b' \) that is in \( \beta \) an assignment variant of \( b \) for \( D \) such that \( D, I, b' \models [\beta, \xi, A] \). Now, suppose \( \Delta' \in Y \). Then we have \( \Delta' \in Y \setminus \{\beta, \xi, A\} \) or \( \Delta' = [\beta, \xi, A] \). In the first case, we have \( D, I, b \models \Delta' \). Since \( \beta \not\in \text{ST}(A') \), we have \( b|\text{ST}(A') = b'|\text{ST}(A') \). By Theorem 5-5-(ii), it then follows that \( D, I, b' \models \Delta' \). For the second case, we already have \( D, I, b' \models [\beta, \xi, A] \). Therefore \( D, I, b' \models \Delta' \) for all \( \Delta' \in Y \) and hence \( D, I, b' \models Y \). By hypothesis, it then follows that \( D, I, b' \models B \). Since \( \beta \not\in \text{ST}(B) \), we have \( b|\text{ST}(B) = b'|\text{ST}(B) \). With Theorem 5-5-(ii), it then follows that \( D, I, b \models B \). Therefore we have for all \( D, I, b \): If \( D, I, b \models X \cup \{Y \setminus \{\beta, \xi, A\}\} \), then \( D, I, b \models B \). Therefore \( X \cup \{Y \setminus \{\beta, \xi, A\}\} \models B \). ■
Theorem 5-31. Model-theoretic counterpart of $\mathcal{II}$
For all $X \subseteq \text{CFORM}$ and $\theta \in \text{CTERM}$: $X \models \theta = \theta$.

Proof: Suppose $X \subseteq \text{CFORM}$ and $\theta \in \text{CTERM}$. Suppose $(D, I)$ is a model and $b$ is a parameter assignment for $D$. With $\langle \text{TD}(\theta, D, I, b), \text{TD}(\theta, D, I, b) \rangle \in \{\langle a, a \rangle \mid a \in D\}$, we have $\langle \text{TD}(\theta, D, I, b), \text{TD}(\theta, D, I, b) \rangle \in I(\models \equiv)$. According to Theorem 5-4-(i), it then follows that $D, I, b \models \theta = \theta$. Therefore we have for all $D, I, b$: If $D, I, b \models X$, then $D, I, b \models \theta = \theta$. Therefore $X \models \theta = \theta$. ■

Theorem 5-32. Model-theoretic counterpart of $\mathcal{IE}$
If $\theta_0, \theta_1 \in \text{CTERM}, \xi \in \text{VAR}, \Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi\}$, and $X \models \theta_0 = \theta_1$ and $Y \models [\theta_0, \xi, \Delta]$, then $X \cup Y \models [\theta_1, \xi, \Delta]$.

Proof: Suppose $\theta_0, \theta_1 \in \text{CTERM}, \xi \in \text{VAR}, \Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi\}$, and $X \models \theta_0 = \theta_1$ and $Y \models [\theta_0, \xi, \Delta]$. Now, suppose $D, I, b \models X \cup Y$. Then $(D, I)$ is a model and $b$ is a parameter assignment for $D$ and, with Theorem 5-10, we have $D, I, b \models X$ and $D, I, b \models Y$. By hypothesis, it then follows that $D, I, b \models \theta_0 = \theta_1$ and $D, I, b \models [\theta_0, \xi, \Delta]$. By Theorem 5-4-(i), we then have that $\langle \text{TD}(\theta_0, D, I, b), \text{TD}(\theta_1, D, I, b) \rangle \in I(\models \equiv) = \{\langle a, a \rangle \mid a \in D\}$. Thus we have $\text{TD}(\theta_0, D, I, b) = \text{TD}(\theta_1, D, I, b)$. According to Theorem 5-7-(ii), it then follows, with $D, I, b \models [\theta_0, \xi, \Delta]$, that also $D, I, b \models [\theta_1, \xi, \Delta]$. Therefore we have for all $D, I, b$: If $D, I, b \models X \cup Y$, then $D, I, b \models [\theta_1, \xi, \Delta]$. Therefore $X \cup Y \models [\theta_1, \xi, \Delta]$. ■
6 Correctness and Completeness of the Speech Act Calculus

After having established the Speech Act Calculus and a model-theory, we now have to show that the respective consequence relations are equivalent. As usual, this adequacy proof contains two parts: First the proof of the correctness of the Speech Act Calculus relative to the model-theory. Informally: Everything that is derivable also follows model-theoretically (6.1). Second the proof of the completeness of the Speech Act Calculus relative to the model-theory. Informally: Everything that follows model-theoretically is also derivable (6.2).

Note that our talk of the correctness and completeness of the Speech Act Calculus follows the usual custom. On the other hand, one could also read the two results obversely, i.e. so that we show in ch. 6.1 that the model-theoretic consequence relation is complete relative to the calculus. In ch. 6.2 we would then accordingly show that the model-theoretic consequence relation is correct relative to the calculus. We do not follow this alternative way of interpreting the results in order to avoid confusion. However, even if we speak of correctness and completeness in the usual way, we do not want to insinuate that the model-theoretic consequence relation is in some way superior to the deductive consequence relation established by the calculus or that calculi have to be justified by reference to model-theoretic concepts of consequence and not the other way round. The adequacy result just says that Speech Act Calculus and classical first-order model-theory are associated with equivalent consequence relations.

6.1 Correctness of the Speech Act Calculus

The following section consists mainly of one single proof, namely the proof of Theorem 6-1, which says that in each derivation \( \mathcal{F} \) the conclusion is a model-theoretic consequence of AVAP(\( \mathcal{F} \)). The proof is carried out by induction on the length of a derivation. Using the I.H., we will show that for all 17 possible extensions of \( \mathcal{F} \mid \text{Dom}(\mathcal{F})^{-1} \) to \( \mathcal{F} \) it holds that AVAP(\( \mathcal{F} \)) \( \vdash C(\mathcal{F}) \). In doing this, we will first deal with the more interesting cases, i.e. those cases in which the set of available assumptions is reduced or augmented by the extension of \( \mathcal{F} \mid \text{Dom}(\mathcal{F})^{-1} \) to \( \mathcal{F} \). These four cases are AF, CdIF, NIF and PEF (or AR, CdI, NI and PE). For the remaining 13 cases, we can then exclude that the last step in
the derivation under consideration belongs to one of the first four cases. The correctness of the Speech Act Calculus relative to the model-theory is then established at the end of the section in Theorem 6-2.

**Theorem 6-1. Main correctness proof**

If $\mathcal{H} \in \text{RCS} \{\emptyset\}$, then $\text{AVAP}(\mathcal{H}) \models C(\mathcal{H})$.

**Proof:** Proof by induction on $|\mathcal{H}|$. For this, suppose the theorem holds for all $l < |\mathcal{H}|$ and suppose $\mathcal{H} \in \text{SEQ}$ and for all $j < \text{Dom}(\mathcal{H})$: $\mathcal{H}_{j} \cup j \in \text{RCE}(\mathcal{H}_{j})$. Also, with Theorem 3-8, it holds for all $j \in \text{Dom}(\mathcal{H})$ that $\mathcal{H}_{j} \cup j \in \text{RCS} \{\emptyset\}$. With this and the I.H., we have for all $0 < j < \text{Dom}(\mathcal{H})$:

$\text{AVAP}(\mathcal{H}_{j}) \models C(\mathcal{H}_{j})$.

According to Theorem 3-6 and Definition 3-18, we have $\text{C}(\mathcal{H}_{j}) \iff \text{P}(\mathcal{H}_{j}) \models C(\mathcal{H}_{j})$.

We further have that $\mathcal{H}_{j} \in \text{AF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{CIF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{NIF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{PEF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{CDIf}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{NIF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{PEF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{IF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1}$.

We further have that $\mathcal{H}_{j} \in \text{AF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{CIF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{NIF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{PEF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{CDIf}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{NIF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{PEF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \cup \text{IF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1}$.

**AF:** Suppose $\mathcal{H}_{j} \in \text{AF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1}$. According to Theorem 3-15-(viii), we then have $C(\mathcal{H}_{j}) \in \text{AVAP}(\mathcal{H}_{j})$. Theorem 5-14 then yields $\text{AVAP}(\mathcal{H}_{j}) \models C(\mathcal{H}_{j})$.

**CDIF:** Suppose $\mathcal{H}_{j} \in \text{CDIF}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1}$. According to Theorem 3-19-(x), we then have $C(\mathcal{H}_{j}) = \{P(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1}) \subseteq \text{C}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1}\}$. We have $\text{AVAP}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \models C(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1}$. With Theorem 3-19-(ix), we have $\text{AVAP}(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1} \models C(\mathcal{H}_{j}) \text{ Dom}(\mathcal{H}_{j})^{-1}$. With Theorem 5-15, it then follows that $\text{AVAP}(\mathcal{H}) \text{ Dom}(\mathcal{H})^{-1} \models C(\mathcal{H}) \text{ Dom}(\mathcal{H})^{-1}$.
\[ \rightarrow C(\delta) \upharpoonright \text{Dom}(\delta)-1 \]}. Theorem 5-13 then yields \( \text{AVAP}(\delta) \models \Gamma \left( C(\delta) \upharpoonright \text{Dom}(\delta)-1 \right) \) and thus \( \text{AVAP}(\delta) \models C(\delta) \).

\((\text{NIF}): \) Suppose \( \delta \in \text{NIF}(\delta) \upharpoonright \text{Dom}(\delta)-1 \). According to Theorem 3-20-(x), we then have \( C(\delta) = \Gamma \left( \neg \left( \text{AVAP}(\delta) \upharpoonright \text{Dom}(\delta)-1 \right) \right) \). With Theorem 3-20-(i) and Theorem 2-92, there is \( \Gamma \in \text{CFORM} \) and \( j \in \text{Dom}(\delta)-1 \) such that \( \text{max}(\text{Dom}((\text{AVAS}(\delta) \upharpoonright \text{Dom}(\delta)-1))) \leq j \) and either \( \text{P}(\delta_j) = \Gamma \) and \( \text{P}(\delta_{\text{Dom}(\delta)-2}) = \Gamma \) or \( \text{P}(\delta_j) = \Gamma \) and \( \text{P}(\delta_{\text{Dom}(\delta)-2}) = \Gamma \) and \((j, \delta_j) \in \text{AVS}(\delta) \upharpoonright \text{Dom}(\delta)-1 \). Thus we have either \( \Gamma = C(\delta)[j+1] \) and \( \neg \Gamma = C(\delta) \upharpoonright \text{Dom}(\delta)-1 \) or \( \neg \Gamma = C(\delta)[j+1] \) and \( \Gamma = C(\delta) \upharpoonright \text{Dom}(\delta)-1 \). First suppose \( \Gamma = C(\delta)[j+1] \) and \( \neg \Gamma = C(\delta) \upharpoonright \text{Dom}(\delta)-1 \). Then we have \( \text{AVAP}(\delta)[j+1] \models \Gamma \) and \( \text{AVAP}(\delta) \upharpoonright \text{Dom}(\delta)-1 \) \( \models \neg \Gamma \).

Also, we have that \( \Gamma \) is available in \( \delta[j] \upharpoonright \text{Dom}(\delta)-1 \) at \( j \) and thus, according to Theorem 3-29-(iv), \( \text{AVAP}(\delta)[j+1] \subset \text{AVAP}(\delta) \upharpoonright \text{Dom}(\delta)-1 \). With Theorem 5-13, we thus also have \( \text{AVAP}(\delta) \upharpoonright \text{Dom}(\delta)-1 \) \( \models \Gamma \). Second suppose \( \neg \Gamma = C(\delta)[j+1] \) and \( \Gamma = C(\delta) \upharpoonright \text{Dom}(\delta)-1 \). Then we have \( \text{AVAP}(\delta)[j+1] \models \neg \Gamma \) and \( \text{AVAP}(\delta) \upharpoonright \text{Dom}(\delta)-1 \) \( \models \Gamma \).

Also, \( \neg \Gamma \) is then available in \( \delta[j] \upharpoonright \text{Dom}(\delta)-1 \) at \( j \) and hence we have, again with Theorem 3-29-(iv), that \( \text{AVAP}(\delta)[j+1] \subset \text{AVAP}(\delta) \upharpoonright \text{Dom}(\delta)-1 \) and thus, with Theorem 5-13, that \( \text{AVAP}(\delta) \upharpoonright \text{Dom}(\delta)-1 \) \( \models \neg \Gamma \). Thus we have in both cases that \( \text{AVAP}(\delta) \upharpoonright \text{Dom}(\delta)-1 \) \( \models \Gamma \) and \( \text{AVAP}(\delta) \upharpoonright \text{Dom}(\delta)-1 \) \( \models \neg \Gamma \). With Theorem 3-20-(ix), we have \( \text{AVAP}(\delta) \upharpoonright \text{Dom}(\delta)-1 = \text{AVAP}(\delta) \cup \{\text{P}(\delta_{\text{max}}(\text{Dom}(\text{AVAS}(\delta) \upharpoonright \text{Dom}(\delta)-1)))\} \). Thus we have \( \text{AVAP}(\delta) \cup \{\text{P}(\delta_{\text{max}}(\text{Dom}(\text{AVAS}(\delta) \upharpoonright \text{Dom}(\delta)-1)))\} \models \Gamma \) and \( \text{AVAP}(\delta) \cup \{\text{P}(\delta_{\text{max}}(\text{Dom}(\text{AVAS}(\delta) \upharpoonright \text{Dom}(\delta)-1)))\} \models \neg \Gamma \). With Theorem 5-25 (where \( X \) as well as \( Y \) are instantiated by \( \text{AVAP}(\delta) \cup \{\text{P}(\delta_{\text{max}}(\text{Dom}(\text{AVAS}(\delta) \upharpoonright \text{Dom}(\delta)-1)))\} \) and Theorem 5-13, it then follows that \( \text{AVAP}(\delta) \models \neg \Gamma \) and \( \text{AVAP}(\delta) \upharpoonright \text{Dom}(\delta)-1 \) \( \models \neg \Gamma \).
It holds that $\text{AVAP}(\bar{\mathfrak{S}}|\max(\text{Dom}(\text{AVAS}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1)))) \subseteq \text{AVAP}(\bar{\mathfrak{S}})$. According to Theorem 3-21-(iii), we first have $(\max(\text{Dom}(\text{AVAS}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1))))-1$, $\forall \xi \Delta^n \in \text{AVS}(\bar{\mathfrak{S}})$ because $(\max(\text{Dom}(\text{AVAS}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1))))-1$, $\bar{\mathfrak{S}}(\max(\text{Dom}(\text{AVAS}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1))))-1 < \max(\text{Dom}(\text{AVAS}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1))))$. Therefore $\forall \xi \Delta^n$ is available in $\bar{\mathfrak{S}}$ at $(\max(\text{Dom}(\text{AVAS}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1))))-1$. With Theorem 3-29-(ii), it then follows that AVAP($\bar{\mathfrak{S}}|\max(\text{Dom}(\text{AVAS}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1)))) \subseteq \text{AVAP}(\bar{\mathfrak{S}})$. With Theorem 5-13, we then have AVAP($\bar{\mathfrak{S}}) \vdash \forall \xi \Delta^n$.

We already have $\beta \notin \text{STSF}({\{\Delta, C(\bar{\mathfrak{S}})\}})$. Since there is no $\gamma < (\max(\text{Dom}(\text{AVAS}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1))))-1$ such that $\beta \in \text{ST}(\bar{\mathfrak{S}})$, there is no $\gamma \in \text{Dom}(\text{AVAS}(\bar{\mathfrak{S}}|\max(\text{Dom}(\text{AVAS}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1))))))$ such that $\beta \in \text{ST}(\bar{\mathfrak{S}}) = \text{ST}(\text{P}(\bar{\mathfrak{S}}))$ and $j \neq (\max(\text{Dom}(\text{AVAS}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1))))-1).$ With Theorem 3-21-(iv) and -(v), we therefore have that there is no $\gamma \in \text{Dom}(\text{AVAS}(\bar{\mathfrak{S}}))$ such that $\beta \in \text{ST}(\text{P}(\bar{\mathfrak{S}})).$ Thus we have $\beta \notin \text{STSF}(	ext{AVAP}(\bar{\mathfrak{S}}))$ and thus $\beta \notin \text{STSF}(	ext{AVAP}(\bar{\mathfrak{S}}) \cup \{\Delta, C(\bar{\mathfrak{S}})\})$ and finally $\beta \notin \text{STSF}((\text{AVAP}(\bar{\mathfrak{S}})|\{[\beta, \xi, \Delta]\}) \cup \{\Delta, C(\bar{\mathfrak{S}})\})$. According to Theorem 5-30 (where $X$ is instantiated by AVAP($\bar{\mathfrak{S}})$ and $Y$ is instantiated by AVAP($\bar{\mathfrak{S}}) \cup \{[\beta, \xi, \Delta]\}$), we hence have AVAP($\bar{\mathfrak{S}}) \vdash C(\bar{\mathfrak{S}})$.

Second case: Now, suppose $\bar{\mathfrak{S}} \notin \text{AF}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1) \cup \text{CdIF}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1) \cup \text{NIF}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1) \cup \text{PEF}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1)$. According to Theorem 3-28, we then have AVAP($\bar{\mathfrak{S}}) = \text{AVAP}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1)$. We can distinguish 13 subcases.

(CdEF, CIF, BIF, BEF, IEF): Suppose $\bar{\mathfrak{S}} \in \text{CdEF}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1)$. According to Definition 3-3, there is then $\Delta \in \text{CFORM}$ such that $\Delta, \bar{\mathfrak{S}} \Rightarrow \text{C(\bar{\mathfrak{S}})} \in \text{AVP}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1)$. Because of $\Delta, \bar{\mathfrak{S}} \Rightarrow \text{C(\bar{\mathfrak{S}})} \in \text{AVP}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1)$ there are $j, l \in \text{Dom}(\bar{\mathfrak{S}})-1$ such that $\Delta$ is available in $\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1$ at $j$ and $\bar{\mathfrak{S}} \Rightarrow \text{C(\bar{\mathfrak{S}})} \in \text{AVP}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1$ at $l$. Then we have $\text{C}(\bar{\mathfrak{S}})|j+1) = \Delta$ and $\text{C}(\bar{\mathfrak{S}})|l+1) = \bar{\Delta} \Rightarrow \text{C(\bar{\mathfrak{S}})}$. Then we have AVAP($\bar{\mathfrak{S}}) j+1) \vdash \Delta$ and AVAP($\bar{\mathfrak{S}}) l+1) \vdash \bar{\Delta} \Rightarrow \text{C(\bar{\mathfrak{S}})}$. With Theorem 3-29-(iv), it then follows that AVAP($\bar{\mathfrak{S}}) j+1) \subseteq \text{AVAP}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1)$ and AVAP($\bar{\mathfrak{S}}) l+1) \subseteq \text{AVAP}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1)$. Since AVAP($\bar{\mathfrak{S}}) = \text{AVAP}(\bar{\mathfrak{S}}|\text{Dom}(\bar{\mathfrak{S}})-1), we thus have AVAP($\bar{\mathfrak{S}}) j+1) \subseteq \text{AVAP}(\bar{\mathfrak{S}})$ and AVAP($\bar{\mathfrak{S}}) l+1) \subseteq \text{AVAP}(\bar{\mathfrak{S}})$ and thus, with Theorem 5-13, also AVAP($\bar{\mathfrak{S}}) \vdash \Delta$ and AVAP($\bar{\mathfrak{S}}) \vdash \bar{\Delta} \Rightarrow \text{C(\bar{\mathfrak{S}})}$. Theorem 5-16 then yields AVAP($\bar{\mathfrak{S}}) \vdash \text{C(\bar{\mathfrak{S}}).}$ Similarly one shows for CIF with Theorem 5-17, for BIF with Theorem 5-19, for BEF with Theorem 5-21 and for IEF with Theorem 5-32 that AVAP($\bar{\mathfrak{S}}) \vdash \text{C(\bar{\mathfrak{S}}).}$
(CEF, DIF): Suppose \( s \in \text{CEF}(\Delta|\text{Dom}(s)-1) \). According to Definition 3-5, there is then \( \Delta \in \text{CFORM} \) such that "\( \Delta \land C(s) \)" \( \in \text{AVP}(s|\text{Dom}(s)-1) \) or "\( C(s) \land \Delta \)" \( \in \text{AVP}(s|\text{Dom}(s)-1) \). Because of "\( \Delta \land C(s) \)" \( \in \text{AVP}(s|\text{Dom}(s)-1) \) or "\( C(s) \land \Delta \)" \( \in \text{AVP}(s|\text{Dom}(s)-1) \) there is \( j \in \text{Dom}(s)-1 \) such that "\( \Delta \land C(s) \)" or "\( C(s) \land \Delta \)" is available in \( s|\text{Dom}(s)-1 \) at \( j \). Then we have \( C(s)|j+1 = \Delta \land C(s) \) or \( C(s)|j+1 = C(s) \land \Delta \). Then we have \( \text{AVAP}(s|j+1) \models \Delta \land C(s) \) or \( \text{AVAP}(s|j+1) \models C(s) \land \Delta \). With Theorem 3-29-(iv), it follows that \( \text{AVAP}(s|j+1) \subseteq \text{AVAP}(s|\text{Dom}(s)-1) = \text{AVAP}(s) \).

With Theorem 5-13, we thus have \( \text{AVAP}(s) \models \Delta \land C(s) \) or \( \text{AVAP}(s) \models C(s) \land \Delta \). Theorem 5-18 yields in both cases \( \text{AVAP}(s) \models C(s) \). For DIF one shows similarly, with Theorem 5-22, that \( \text{AVAP}(s) \models C(s) \).

(DEF): Suppose \( s \in \text{DEF}(\Delta|\text{Dom}(s)-1) \). According to Definition 3-9, there are then \( B, \Delta \in \text{CFORM} \) such that "\( B \lor \Delta \)" \( \land C(s) \)" \( \in \text{AVP}(s|\text{Dom}(s)-1) \) or "\( C(s) \lor \Delta \)" \( \in \text{AVP}(s|\text{Dom}(s)-1) \). Then there are \( j, k, l \in \text{Dom}(s)-1 \) such that "\( B \lor \Delta \)" \( \land C(s) \)" is available in \( s|\text{Dom}(s)-1 \) at \( j \) and "\( B \lor \Delta \)" \( \land C(s) \)" is available in \( s|\text{Dom}(s)-1 \) at \( k \) and "\( \Delta \lor C(s) \)" \( \in \text{AVP}(s|\text{Dom}(s)-1) \) at \( l \). Then we have \( C(s)|j+1 = B \lor \Delta \) and \( C(s)|k+1 = B \lor \Delta \) and \( C(s)|l+1 = \Delta \lor C(s) \). Then it holds that \( \text{AVAP}(s|j+1) \models B \lor \Delta \) and \( \text{AVAP}(s|k+1) \models B \lor \Delta \) and \( \text{AVAP}(s|l+1) \models \Delta \lor C(s) \). With Theorem 3-29-(iv), it then follows that \( \text{AVAP}(s|j+1) \subseteq \text{AVAP}(s|\text{Dom}(s)-1) \) and \( \text{AVAP}(s|k+1) \subseteq \text{AVAP}(s|\text{Dom}(s)-1) \) and \( \text{AVAP}(s|l+1) \subseteq \text{AVAP}(s|\text{Dom}(s)-1) \) and thus \( \text{AVAP}(s|j+1) \subseteq \text{AVAP}(s) \) and \( \text{AVAP}(s|k+1) \subseteq \text{AVAP}(s) \) and \( \text{AVAP}(s|l+1) \subseteq \text{AVAP}(s) \). With Theorem 5-13, we thus have \( \text{AVAP}(s) \models B \lor \Delta \) and \( \text{AVAP}(s) \models B \lor \Delta \) and \( \text{AVAP}(s) \models \Delta \lor C(s) \). Theorem 5-23 then yields \( \text{AVAP}(s) \models C(s) \).

(NEF, UEF, PIF): Suppose \( s \in \text{NEF}(\Delta|\text{Dom}(s)-1) \). According to Definition 3-11, we then have "\( \neg \neg C(s) \)" \( \in \text{AVP}(s|\text{Dom}(s)-1) \). Then there is \( j \in \text{Dom}(s)-1 \) such that "\( \neg \neg C(s) \)" is available in \( s|\text{Dom}(s)-1 \) at \( j \). Then we have \( C(s)|j+1 = \neg \neg C(s) \). Then we have \( \text{AVAP}(s|j+1) \models \neg \neg C(s) \). With Theorem 3-29-(iv), it follows that \( \text{AVAP}(s|j+1) \subseteq \text{AVAP}(s|\text{Dom}(s)-1) = \text{AVAP}(s) \). With Theorem 5-13, we thus have \( \text{AVAP}(s) \models \neg \neg C(s) \). Theorem 5-26 then yields \( \text{AVAP}(s) \models C(s) \). Similarly, one shows for UEF with Theorem 5-28 and for PIF with Theorem 5-29 that in both cases \( \text{AVAP}(s) \models C(s) \).

(UIF): Suppose \( s \in \text{UIF}(\Delta|\text{Dom}(s)-1) \). According to Definition 3-12 there is then \( \beta \in \text{PAR}, \: \xi \in \text{VAR} \) and \( \Delta \in \text{FORM} \), where \( \text{FV}(\Delta) \subseteq \{\xi\} \), such that "\( [\beta, \xi, \Delta] \in \text{AVP}(s|\text{Dom}(s)-1) \) and \( \beta \not\in \text{STSF}(|\Delta| \cup \text{AVP}(s|\text{Dom}(s)-1)) \) and \( C(s) = \neg \neg \beta \).
Then there is \( j \in \text{Dom}(\delta) \) such that \([\beta, \xi, \Delta]\) is available in \( \delta|\text{Dom}(\delta) \) at \( j \). Then we have \( C(\delta|_{j+1}) = [\beta, \xi, \Delta] \). Then it holds that \( \text{AVAP}(\delta|_{j+1}) \models [\beta, \xi, \Delta] \). With Theorem 3-29-(iv), it follows that \( \text{AVAP}(\delta|_{j+1}) \subseteq \text{AVAP}(\delta|\text{Dom}(\delta)-1) = \text{AVAP}(\delta) \). With Theorem 5-13, we thus have \( \text{AVAP}(\delta) \models [\beta, \xi, \Delta] \). With \( \text{AVAP}(\delta|\text{Dom}(\delta)-1) = \text{AVAP}(\delta) \), it follows from \( \beta \notin \text{STSF}(\{\Delta\} \cup \text{AVAP}(\delta|\text{Dom}(\delta)-1)) \) that \( \beta \notin \text{STSF}(\{\Delta\} \cup \text{AVAP}(\delta)) \). Theorem 5-27 then yields \( \text{AVAP}(\delta) \models C(\delta) \).

\((IIF): \) Suppose \( \delta \in \text{IIF}(\delta|\text{Dom}(\delta)-1) \). According to Definition 3-16 there is then \( \theta \in \text{CTERM} \) such that \( C(\delta) = \langle \theta = \emptyset \rangle \). Theorem 5-31 yields \( \text{AVAP}(\delta) \models C(\delta) \).  

**Theorem 6-2. Correctness of the Speech Act Calculus relative to the model-theory**

For all \( X, \Gamma \): If \( X \models \Gamma \), then \( X \models \Gamma \).

**Proof:** Suppose \( X \models \Gamma \). According to Theorem 3-12, we then have that \( X \subseteq \text{CFORM} \) and that there is \( \delta \in \text{RCS}\setminus\{\emptyset\} \) such that \( \Gamma = C(\delta) \) and \( \text{AVAP}(\delta) \subseteq X \). Theorem 6-1 then yields \( \text{AVAP}(\delta) \models \Gamma \). With Theorem 5-13 and \( \text{AVAP}(\delta) \subseteq X \), it follows that \( X \models \Gamma \).  

\( \blacksquare \)
6.2 Completeness of the Speech Act Calculus

In the following we will prove the completeness of the Speech Act Calculus relative to the model-theoretic consequence relation for L defined in Definition 5-10. To do this, we will show that consistent sets are satisfiable. Since CFORM, the set of closed L-formulas, is denumerably infinite, it suffices to show this for denumerably infinite sets. For this, we choose the method of constructing Hintikka sets and showing that Hintikka sets are satisfied by the respective canonical term structure. For this purpose, L has to be expanded to the language $L_H$, which results from L by adding denumerably infinitely many new individual constants to the vocabulary of L:

**Definition 6-1.** The vocabulary of $L_H$ (CONTEXP, PAR, VAR, FUNC, PRED, CON, QUANT, PERF, AUX)

The vocabulary of $L_H$ contains the following pairwise disjunct sets: the denumerably infinite set $\text{CONTEXP} = \text{CONST} \cup \text{CONTEXP}_{\text{NEW}}$, where $\text{CONTEXP}_{\text{NEW}} = \{c^*_i \mid i \in \mathbb{N}\}$ (and for all $i, j \in \mathbb{N}$ with $i \neq j$: $c^*_i \neq c^*_j$ and $c^*_i \in \{c^*_i \}$ and $\text{CONST} \cap \text{CONTEXP}_{\text{NEW}} = \emptyset$), and PAR, VAR, FUNC, PRED, CON, QUANT, PERF, AUX.

Note: In the remainder of this section we adopt the following notation: For all expressions $P$ that are defined by definition $D$ let $P_H$ be the expression defined for $L_H$ instead of $L$ and let $D_H$ be the corresponding definition and for all theorems $T$ let $T_H$ be the corresponding theorem for $L_H$. As for the relationship of $P$ and $P_H$, it holds that suitable restrictions of $P_H$ and $P_H(a)$ to $L$ lead back to $P$ and $P(a)$, respectively. For example, we have: (i) $\text{PEXP} = \text{PEXP}_H \cap \text{PEXP}$, $\text{TERM} = \text{TERM}_H \cap \text{PEXP}$, $\text{FORM} = \text{FORM}_H \cap \text{PEXP}$, $\text{SENT} = \text{SENT}_H \cap \text{PEXP}$, $\text{SEQ} = \text{SEQ}_H \cap \text{SEQ}$, $\text{RCS} = \text{RCS}_H \cap \text{SEQ}$. (ii) $\text{ST} = \text{ST}_H \cap \text{PEXP}$, $\text{STSEQ} = \text{STSEQ}_H \cap \text{SEQ}$, $\text{STSF} = \text{STSF}_H \cap \text{Pot(FORM)}$, $P = P_H \cap \text{SENT}$, $C = C_H \cap \text{SEQ}$, $\text{AVAP} = \text{AVAP}_H \cap \text{SEQ}$. (iii) If $\delta \in \text{SEQ}$, then $\text{RCE}(\delta) = \text{RCE}_H(\delta) \cap \text{SEQ}$.

Many of these relationships can be shown without much technical difficulties but require quite some tedious writing. Therefore, we will not reproduce the proofs here. Where the relationships are not immediately obvious or where there are particular complications in a proof, we will execute the proofs. For example, we will show that $\text{RCS} \subseteq \text{RCS}_H$ in

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Theorem 6-6. In Theorem 6-3-(i), we will show that models \( \text{mod}_{L_H} \) can be transformed into models by restricting the respective interpretation function \( \text{mod}_{I_H} \) on \( \text{PEXP} \) (or, more precisely: \( \text{CONST} \cup \text{FUNC} \cup \text{PRED} \)). For the substitution operation, the equivalence for L-arguments is trivial. To avoid a clutter of indices behind square brackets (cf. the proof of Theorem 6-10), we will therefore suppress the H-index for the substitution operator.

The following theorems first secure the connection between satisfiability in L and \( L_H \) (Theorem 6-3 to Theorem 6-5) and between consistency in L and \( L_H \) (Theorem 6-6 to Theorem 6-8). Then we will define Hintikka sets (Definition 6-2). Subsequently, we will show that all consistent sets of L-propositions have a Hintikka superset (Theorem 6-9) and that all Hintikka sets are satisfiable\( _H \) (Theorem 6-10). From this, we will then derive the completeness of the Speech Act Calculus (Theorem 6-11).

**Theorem 6-3.** Restrictions of \( L_H \)-models on L are \( L \)-models

(i) If \((D, I)\) is a model\( _{L_H} \), then \((D, I|(\text{CONST} \cup \text{FUNC} \cup \text{PRED}))\) is a model,

(ii) \( b \) is a parameter assignment\( _H \) for \( D \) iff \( b \) is a parameter assignment for \( D \), and

(iii) \( b' \) is in \( \beta \) an assignment variant\( _H \) of \( b \) for \( D \) iff \( b' \) is in \( \beta \) an assignment variant of \( b \) for \( D \).

**Proof:** Ad (i): Suppose \((D, I)\) is a model\( _{L_H} \). According to Definition 5-2\( _H \), \( I \) is then an interpretation function\( _H \) for \( D \). According to Definition 5-1\( _H \), we then have \( \text{Dom}(I) = \text{CONSTEXP} \cup \text{FUNC} \cup \text{PRED} \). With \( \text{CONST} \subseteq \text{CONSTEXP} \), we then have \( \text{Dom}(I|(\text{CONST} \cup \text{FUNC} \cup \text{PRED})) = \text{CONST} \cup \text{FUNC} \cup \text{PRED} \) and for all \( \mu \in \text{CONST} \cup \text{FUNC} \cup \text{PRED} \) it holds that \( I|(\text{CONST} \cup \text{FUNC} \cup \text{PRED})(\mu) = I(\mu) \). Thus it follows, with Definition 5-1\( _H \) and Definition 5-1, that \( I|(\text{CONST} \cup \text{FUNC} \cup \text{PRED}) \) is an interpretation function for \( D \) and thus that \((D, I|(\text{CONST} \cup \text{FUNC} \cup \text{PRED}))\) is a model.

Ad (ii): With Definition 5-3\( _H \) and Definition 5-3 it holds that

\[ b \text{ is a parameter assignment}_{L_H} \text{ for } D \]
iff
\[ b \text{ is a function with } \text{Dom}(b) = \text{PAR} \text{ such that for all } \beta \in \text{PAR}: b(\beta) \in D \]
iff
\[ b \text{ is a parameter assignment for } D \].

Ad (iii): With Definition 5-4\( _H \), (ii) and Definition 5-4 it holds that
6.2 Completeness of the Speech Act Calculus

Theorem 6-4. $L_{lf}$-models and their $L$-restrictions behave in the same way with regard to $L$-entities

If $(D, I)$ is a model$_{lf}$ and $b$ is a parameter assignment$_{lf}$ for $D$, then for all $\theta \in \mathit{CTERM}$, $\Gamma \in \mathit{CFORM}$ and $X \subseteq \mathit{CFORM}$:

(i) $\mathit{TD}_{lf}(\theta, D, I, b) = \mathit{TD}(\theta, D, I|(\mathit{CONST} \cup \mathit{FUNC} \cup \mathit{PRED}), b)$

(ii) $D, I, b \models_{lf} \Gamma$ iff $D, I|(\mathit{CONST} \cup \mathit{FUNC} \cup \mathit{PRED}), b \models \Gamma$, and

(iii) $D, I, b \models_{lf} X$ iff $D, I|(\mathit{CONST} \cup \mathit{FUNC} \cup \mathit{PRED}), b \models X$.

Proof: The proof for (i) and (ii) is analogous to the proof of the coincidence lemma (Theorem 5-5) by induction on the complexity of terms and formulas. Additionally, one has to use Theorem 6-3. (iii) then follows from (ii) and Definition 5-9$_{lf}$ and Definition 5-9.

Theorem 6-5. A set of $L$-propositions is $L_{lf}$-satisfiable if and only if it is $L$-satisfiable

If $X \subseteq \mathit{CFORM}$, then: $X$ is satisfiable$_{lf}$ iff $X$ is satisfiable.

Proof: Suppose $X \subseteq \mathit{CFORM}$. Now, suppose $X$ is satisfiable$_{lf}$. According to Definition 5-17$_{lf}$, there are then $D, I, b$ such that $D, I, b \models_{lf} X$. With Theorem 6-4, it then follows that $D, I|(\mathit{CONST} \cup \mathit{FUNC} \cup \mathit{PRED}), b \models X$ and thus we have that $X$ is satisfiable.

Now, suppose $X$ is satisfiable. Then there is $D^-, I^-, b^-$ such that $D^-, I^-, b^- \models X$. We have that there is an $a \in D$. Now, let $I^+ = I^- \cup (\mathit{CONSTNEW} \times \{a\})$. Then $(D, I^+)$ is a model$_{lf}$ and $b^-$ is a parameter assignment$_{lf}$ and $I^+|(\mathit{CONST} \cup \mathit{FUNC} \cup \mathit{PRED}) = I^+$. With Theorem 6-4, it then follows that $D^-, I^+, b^- \models_{lf} X$ and hence that $X$ is satisfiable$_{lf}$.
Theorem 6-6. L-sequences are RCS\(_H\)-elements if and only if they are RCS-elements

If \( \bar{s} \in \text{SEQ} \), then: \( \bar{s} \in \text{RCS}_H \) iff \( \bar{s} \in \text{RCS} \).

Proof: The proof is to be carried out by induction on \( \text{Dom}(\bar{s}) \). The induction basis is given with \( \emptyset \in \text{RCS}_H \cap \text{RCS} \) and one easily shows for \( \bar{s} \in \text{SEQ} \) with \( 0 < \text{Dom}(\bar{s}) \) that if the statement holds for \( \bar{s}|\text{Dom}(\bar{s})-1 \), it also holds for \( \bar{s} \). ■

Theorem 6-7. An L-proposition is LH-derivable from a set of L-propositions if and only if it is L-derivable from that set

If \( X \cup \{ \Gamma \} \subseteq \text{CFORM} \), then: \( X \vdash_H \Gamma \) iff \( X \vdash \Gamma \).

Proof: Suppose \( X \cup \{ \Gamma \} \subseteq \text{CFORM} \). Then the right-left-direction follows directly with Theorem 3-12, Theorem 6-6 and Theorem 3-12\(_H\). Now, for the left-right-direction, suppose \( X \vdash_H \Gamma \). According to Theorem 6-6, we then have for such \( \bar{s}^* \in \text{RCS}\_H\setminus\{\emptyset\} \), \( \text{AVAP}_H(\bar{s}^*) = \text{AVAP}_H(\bar{s}) \) and \( \text{CH}(\bar{s}^*) = \text{CH}(\bar{s}) \). Now, suppose \( |\text{CONSTNEW} \cap \text{STSEQ}_H(\bar{s})| = k \). According to Theorem 4-9\(_H\), there is then an \( \bar{s}^* \in \text{RCS}\_H\setminus\{\emptyset\} \) with \( \text{AVAP}_H(\bar{s}^*) \subseteq X \) and \( \text{K}_H(\bar{s}^*) = \text{K}_H(\bar{s}) \). Therefore we have \( \text{CH}(\bar{s}^*) = \text{CH}(\bar{s}) \). From this, we then get \( X \vdash \Gamma \).
If \( \text{Theorem 6-8.} \) According to the I.H., there is then an \( \mathcal{F}' \) such that \( \text{AVAP}(\mathcal{F}') = \text{AVAP}(\mathcal{F}) \) and \( \text{C}_H(\mathcal{F}') = \text{C}_H(\mathcal{F}) \) and \( \mathcal{F}' \in \text{SEQ} \cap \text{RCS}_H\{\emptyset\} \).

**Theorem 6-8.** A set of \( L \)-propositions is \( L \)-consistent if and only if it is \( L \)-consistent

If \( X \subseteq \text{CFORM} \), then: \( X \) is consistent iff \( X \) is consistent.

**Proof:** Suppose \( X \subseteq \text{CFORM} \) and suppose \( X \) is not consistent. With Theorem 4-23, it then holds for all \( \Delta \in \text{CFORM}_H \) that \( X \vdash \Delta \). Then we have \( X \vdash \text{c}_0 = \text{c}_0 \) and \( X \vdash \text{c}_0 = \text{c}_0 \). It holds that \( \text{c}_0 = \text{c}_0 \) and \( \text{c}_0 = \text{c}_0 \) \( \in \text{CFORM} \) and thus it follows with Theorem 6-7 that \( X \vdash \text{c}_0 = \text{c}_0 \) and \( X \vdash \text{c}_0 = \text{c}_0 \). Hence \( X \) is not consistent. Now, suppose \( X \) is not consistent. Then there is \( A \in \text{CFORM} \subseteq \text{CFORM}_H \) such that \( X \vdash A \) and \( X \vdash \text{¬} A \). With Theorem 6-7 we then also have \( X \vdash A \) and \( X \vdash \text{¬} A \) and thus that \( X \) is not consistent.

**Definition 6-2.** Hintikka set

\( X \) is a Hintikka set

\( X \subseteq \text{CFORM}_H \) and:

(i) If \( A \in \text{AFORM}_H \cap X \), then \( \text{¬} A \notin X \),

(ii) If \( A \in \text{CFORM}_H \) and \( \text{¬} A \in X \), then \( A \in X \),

(iii) If \( A, B \in \text{CFORM}_H \) and \( A \land B \in X \), then \( \{A, B\} \subseteq X \),

(iv) If \( A, B \in \text{CFORM}_H \) and \( \text{¬} (A \land B) \in X \), then \( \{\text{¬} A, \text{¬} B\} \cap X \neq \emptyset \),

(v) If \( A, B \in \text{CFORM}_H \) and \( A \lor B \in X \), then \( \{A, B\} \cap X \neq \emptyset \),

(vi) If \( A, B \in \text{CFORM}_H \) and \( \text{¬} (A \lor B) \in X \), then \( \{\text{¬} A, \text{¬} B\} \subseteq X \),

(vii) If \( A, B \in \text{CFORM}_H \) and \( A \rightarrow B \in X \), then \( \{\text{¬} A, B\} \cap X \neq \emptyset \),

(viii) If \( A, B \in \text{CFORM}_H \) and \( \text{¬} (A \rightarrow B) \in X \), then \( \{A, \text{¬} B\} \subseteq X \),

(ix) If \( A, B \in \text{CFORM}_H \) and \( A \leftrightarrow B \in X \), then \( \{A, B\} \subseteq X \) or \( \{\text{¬} A, \text{¬} B\} \subseteq X \),

(x) If \( A, B \in \text{CFORM}_H \) and \( \text{¬} (A \leftrightarrow B) \in X \), then \( \{A, \text{¬} B\} \subseteq X \) or \( \{\text{¬} A, B\} \subseteq X \),

(xi) If \( \xi \in \text{VAR}, \Delta \in \text{FORM}_H \), where \( \text{FV}_H(\Delta) \subseteq \{\xi\} \), and \( \text{¬} \Delta \in X \), then it holds for all \( \theta \in \text{CTERM}_H \) that \( [\theta, \xi, \Delta] \in X \),

(xii) If \( \xi \in \text{VAR}, \Delta \in \text{FORM}_H \), where \( \text{FV}_H(\Delta) \subseteq \{\xi\} \), and \( \text{¬} \Delta \in X \), then there is a \( \theta \in \text{CTERM}_H \) such that \( \text{¬} [\theta, \xi, \Delta] \in X \).

(xiii) If \( \xi \in \text{VAR}, \Delta \in \text{FORM}_H \), where \( \text{FV}_H(\Delta) \subseteq \{\xi\} \), and \( \text{¬} \Delta \in X \), then there is a \( \theta \in \text{CTERM}_H \) such that \( [\theta, \xi, \Delta] \in X \).

(xiv) If \( \xi \in \text{VAR}, \Delta \in \text{FORM}_H \), where \( \text{FV}_H(\Delta) \subseteq \{\xi\} \), and \( \text{¬} \Delta \in X \), then it holds for all \( \theta \in \text{CTERM}_H \) that \( \text{¬} [\theta, \xi, \Delta] \in X \).
(xv) If $\theta \in \mathit{CTERM}_H$, then $\langle \theta, \theta \rangle \in X$.

(xvi) If $\theta_0, \ldots, \theta_i \in \mathit{CTERM}_H$, $\theta_0', \ldots, \theta_{r-1}' \in \mathit{CTERM}_H$, for all $i < r$: $\langle \theta_i, \theta_i' \rangle \in X$ and $\phi \in \mathit{FUNC}$, $\phi$ $r$-ary, then $\langle \phi(\theta_0, \ldots, \theta_i), \phi(\theta_0', \ldots, \theta_{r-1}') \rangle \in X$, and

(xvii) If $\theta_0, \ldots, \theta_i \in \mathit{CTERM}_H$, $\theta_0', \ldots, \theta_{r-1}' \in \mathit{CTERM}_H$, for all $i < r$: $\langle \theta_i, \theta_i' \rangle \in X$ and $\Phi$ $r$-ary, and $\langle \Phi(\theta_0, \ldots, \theta_{r-1}) \rangle \in X$, then $\langle \Phi(\theta_0', \ldots, \theta_{r-1}') \rangle \in X$.

**Theorem 6-9.** Hintikka-supersets for consistent sets of $L$-propositions

If $X \subseteq \mathit{CFORM}$ and $X$ is consistent, then there is a $Y \subseteq \mathit{CFORM}_H$ such that

(i) $Y$ is a Hintikka set, and

(ii) $X \subseteq Y$.

**Proof:** Suppose $X \subseteq \mathit{CFORM}$ and $X$ is consistent. Now, let $g$ be a bijection between $\mathbb{N}$ and $\mathit{CFORM}_H$. Using $g$ and the (inverse of) the CANTOR pairing function $C$, we will now define an enumeration of the $\Gamma \in \mathit{CFORM}_H$ in which each proposition occurs denumerably infinitely many times as value. For this, let $F = \{(k, \Gamma) \mid$ There is $i, j \in \mathbb{N}, k = \frac{(i + j)(i + j + 1)}{2} + j$ and $\Gamma = g(j)\}$. Then $F$ is a function from $\mathbb{N}$ to $\mathit{CFORM}_H$. First, we have $\text{Dom}(F) \subseteq \mathbb{N}$. Now, suppose $k \in \mathbb{N}$. With the surjectivity of the CANTOR pairing function and $\text{Dom}(g) = \mathbb{N}$, it then holds that there are $i, j \in \mathbb{N}$ and $\Gamma \in \mathit{CFORM}_H$ such that $k = \frac{(i + j)(i + j + 1)}{2} + j$ and $\Gamma = g(j)$. Therefore we have also $\mathbb{N} \subseteq \text{Dom}(F)$ and hence $\text{Dom}(F) = \mathbb{N}$.

According to the definitions of $F$ and $g$, we have $\text{Ran}(F) \subseteq \mathit{CFORM}_H$. Now, suppose $(k, \Gamma), (k, \Gamma^*) \in F$. Then there are $i, j$ and $i', j'$ so that $\frac{(i + j)(i + j + 1)}{2} + j = k = \frac{(i' + j')(i' + j' + 1)}{2} + j'$ and $\Gamma = g(j)$ and $\Gamma^* = g(j')$. Because of the injectivity of the CANTOR pairing function, we then have $i = i'$ and $j = j'$ and thus $\Gamma = g(j) = g(j') = \Gamma^*$. Also, we have for all $l \in \mathbb{N}$ and all $\Gamma \in \mathit{CFORM}_H$: There is a $k > l$ such that $F(k) = \Gamma$. To see this, suppose $l \in \mathbb{N}$ and $\Gamma \in \mathit{CFORM}_H$. Then there is an $s \in \mathbb{N}$ such that $\Gamma = g(s)$. Then we have $l \leq \frac{(i + j)(i + j + 1)}{2} + s < \frac{(i + 1 + j)(i + 1 + j + 1)}{2} + s$ and $F\left(\frac{(i + 1 + j)(i + 1 + j + 1)}{2} + s\right) = g(s) = \Gamma$.

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16 For the CANTOR pairing function $C: \mathbb{N} \times \mathbb{N} \xrightarrow{\exists} \mathbb{N}$ with $C(i, j) = (i + j) \cdot (i + j + 1)/2 + j$ see, for example, DEISER, O.: Mengenlehre, p. 112–113.
Using $F$, we will now define a function $G$ on $\mathbb{N}$, with which we will generate the desired Hintikka-superset for $X$. For this, let $G(0) = X$. For all $k \in \mathbb{N}$ let $G(k+1)$ be as follows: If $F(k) \in G(k)$, then:

(i*) If $F(k) = \gamma \Phi(\theta_0, \ldots, \theta_{r-1})^\gamma$, then $G(k+1) = G(k) \cup \{\gamma \Phi(\theta_0, \ldots, \theta_{r-1})^\gamma\}$ For all $i < r$: $\gamma \theta_i = \theta_i^\gamma \in G(k)$, or $G(k) \cup \{\gamma \Phi(\theta_0, \ldots, \theta_{r-1})^\gamma\} = \theta_0$ and for all $i < s$: $\gamma \theta_i = \theta_i^\gamma \in G(k)$,

(ii*) If $F(k) = \gamma \neg \Phi(\theta_0, \ldots, \theta_{r-1})^\gamma$, then $G(k+1) = G(k)$,

(iii*) If $F(k) = \gamma \neg \neg \Phi$, then $G(k+1) = G(k) \cup \{\Phi\}$,

(iv*) If $F(k) = \gamma \neg \Phi \wedge \Phi$, then $G(k+1) = G(k) \cup \{\Phi, \neg \Phi\}$, or $G(k+1) = G(k) \cup \{\neg \neg \Phi\}$ otherwise,

(vii*) If $F(k) = \gamma \neg \neg \Phi \wedge \neg \Phi$, then $G(k+1) = G(k) \cup \{\neg \neg \Phi\}$, if $G(k) \cup \{\neg \Phi\}$ is consistent,

(vii*) If $F(k) = \gamma \neg \neg \Phi \wedge \neg \Phi$, then $G(k+1) = G(k) \cup \{\neg \neg \Phi\}$, if $G(k) \cup \{\neg \Phi\}$ is consistent,

(vi*) If $F(k) = \gamma \Phi \vee \neg \Phi$, then $G(k+1) = G(k) \cup \{\Phi\}$, if $G(k) \cup \{\neg \Phi\}$ is consistent,

(vi*) If $F(k) = \gamma \Phi \vee \neg \Phi$, then $G(k+1) = G(k) \cup \{\Phi\}$, if $G(k) \cup \{\neg \Phi\}$ is consistent,

(vi*) If $F(k) = \gamma \Phi \vee \neg \Phi$, then $G(k+1) = G(k) \cup \{\Phi\}$, if $G(k) \cup \{\neg \Phi\}$ is consistent,

(vi*) If $F(k) = \gamma \Phi \vee \neg \Phi$, then $G(k+1) = G(k) \cup \{\Phi\}$, if $G(k) \cup \{\neg \Phi\}$ is consistent,

(vi*) If $F(k) = \gamma \Phi \vee \neg \Phi$, then $G(k+1) = G(k) \cup \{\Phi\}$, if $G(k) \cup \{\neg \Phi\}$ is consistent,

(vi*) If $F(k) = \gamma \Phi \vee \neg \Phi$, then $G(k+1) = G(k) \cup \{\Phi\}$, if $G(k) \cup \{\neg \Phi\}$ is consistent,

(vi*) If $F(k) = \gamma \Phi \vee \neg \Phi$, then $G(k+1) = G(k) \cup \{\Phi\}$, if $G(k) \cup \{\neg \Phi\}$ is consistent,
If \( F(k) \not\in G(k) \), then: If \( F(k) = \gamma \theta = \theta^\gamma \) for a \( \theta \in \text{CTERM}_{\text{H}} \), then \( G(k+1) = G(k) \cup \{ \gamma \theta = \theta^\gamma \} \), \( G(k+1) = G(k) \) otherwise.

Note that \( G \) is well-defined, because no \( \alpha \in \text{CONSTNEW} \) is a subterm of a \( \Gamma \in X \subseteq \text{CFORM} \) and because for every \( k \in \mathbb{N} \) at most one element of \( \text{CONSTNEW} \) can be added to the subterms of elements of \( G(k) \) in the step from \( G(k) \) to \( G(k+1) \): For all \( k \in \mathbb{N} \) it holds that \( \text{CONSTNEW} \setminus \text{STSF}_{\text{H}}(G(k)) \) is denumerably infinite.

According to the construction of \( G \) it now holds that

a) \( X = G(0) \subseteq \cup \text{Ran}(G) \),

b) For all \( k \in \mathbb{N} \): \( G(k) \) is consistent\( \text{H} \),

c) If \( l \leq k \), then \( G(l) \subseteq G(k) \),

d) If \( Y \subseteq \cup \text{Ran}(G) \) and \( |Y| \in \mathbb{N} \), then there is a \( k \in \mathbb{N} \) such that \( Y \subseteq G(k) \),

e) \( \cup \text{Ran}(G) \) is consistent\( \text{H} \).

a) follows directly from the definition of \( G \). Now ad b): By hypothesis, \( G(0) = X \subseteq \text{CFORM} \) is consistent and thus, with Theorem 6-8, also consistent\( \text{H} \). Now, suppose for \( k \) it holds that \( G(k) \) is consistent\( \text{H} \). Suppose for contradiction that \( G(k+1) \) is inconsistent\( \text{H} \). Then we have not for all \( \Gamma \in G(k+1) \) that \( G(k) \vdash \Gamma \), because otherwise, we would have, with Theorem 4-19\( \text{H} \) that \( G(k) \) is also inconsistent\( \text{H} \). Thus it is not the case that \( G(k+1) \subseteq G(k) \cup \{ \gamma \theta = \theta^\gamma \} \) for a \( \theta \in \text{CTERM}_{\text{H}} \). Therefore we have \( F(k) \in G(k) \). For this case, the cases (i*) to (iv*), (vii*), (ix*) and (xii*) are excluded for the same reason (this is easily established with the \( L_{\text{H}} \)-versions of the theorems in ch. 4.2). Therefore we have \( F(k) \in G(k) \) and \( F(k) = \gamma \lnot (A \land B) \) or \( F(k) = \gamma A \lor B \) or \( F(k) = \gamma A \rightarrow B \) or \( F(k) = \gamma A \leftrightarrow B \) or \( F(k) = \gamma \lnot (A \leftrightarrow B) \) or \( F(k) = \gamma \lnot A \lor \gamma \lnot B \) or \( F(k) = \gamma \lnot A \rightarrow \gamma \lnot B \). Suppose \( F(k) = \gamma \lnot (A \land B) \), according to (v*), then we have \( G(k+1) = G(k) \cup \{ \gamma \lnot \delta \Delta \} \), if \( G(k) \cup \{ \gamma \lnot \Delta \} \) is consistent\( \text{H} \), \( G(k+1) = G(k) \cup \{ \gamma \lnot B \} \) otherwise. Then we have that \( G(k) \cup \{ \gamma \lnot \Delta \} \) is inconsistent\( \text{H} \) and \( G(k+1) = G(k) \cup \{ \gamma \lnot B \} \) is inconsistent\( \text{H} \). With Theorem 4-22\( \text{H} \), it then holds that \( G(k) \vdash_{\text{H}} A \) and \( G(k) \vdash_{\text{H}} B \) and hence that \( G(k) \vdash_{\text{H}} \gamma A \land B \). Thus we would have that \( G(k) \) is inconsistent\( \text{H} \). Contradiction! The other cases for connective formulas are shown analogously. Now, suppose \( F(k) = \gamma \lnot A \lor \gamma \lnot B \). According to (xiii*), we
then have \( G(k+1) = G(k) \cup \{ \neg \neg \alpha, \xi, \Delta \} \) for the \( \alpha \in \text{CONSTNEW} \) with the smallest index for which it holds that \( \alpha \notin \text{STSF}_H(G(k)) \). Then we would have that \( G(k) \cup \{ \neg \neg \alpha, \xi, \Delta \} \) is inconsistent. Then we would have \( G(k) \vdash \alpha, \xi, \Delta \). But then we would have, because of \( \alpha \notin \text{STSF}_H(G(k)) \) and \( \neg \neg \alpha, \xi, \Delta \in G(k) \), that \( \alpha \notin \text{STSF}_H(G(k) \cup \{ \Delta \}) \) and thus, with Theorem 4-24_H, that \( G(k) \vdash \neg \neg \alpha, \xi, \Delta \). Then \( G(k) \) would be inconsistent. Contradiction! The case \( F(k) = \neg \neg \alpha, \xi, \Delta \) is treated analogously. Hence we have b).

By induction on \( k \), one can easily show that c) holds by the definition of \( G \). Thus we have also d). To see this, suppose \( Y \subseteq \cup \text{Ran}(G) \) and \( |Y| \in \mathbb{N} \). Then we have for all \( \Gamma \in Y \): There is an \( l \in \mathbb{N} \) such that \( \Gamma \in G(l) \). Now, let \( k = \max\{|l| \mid \text{There is a } \Gamma \in Y \text{ such that } \Gamma \in G(l)\} \). Then it holds with c) for all \( \Gamma \in Y: \Gamma \in G(k) \).

Thus we have also e). To see this, suppose for contradiction that \( \cup \text{Ran}(G) \) is inconsistent. Then there would be a finite inconsistent subset \( Y \) of \( \cup \text{Ran}(G) \) and thus a \( k \in \mathbb{N} \) such that \( G(k) \) is inconsistent, which contradicts b).

Now, we can show that \( \cup \text{Ran}(G) \) is a Hintikka set. First we have, with e), that clause (i) of Definition 6-2 holds. Now, suppose \( \neg\neg A \in \cup \text{Ran}(G) \). Then there is an \( l \in \mathbb{N} \) such that \( \neg\neg A \in G(l) \). Then there is a \( k > l \) such that \( \neg\neg A = F(k) \). With c), we then have \( \neg\neg A \in G(k) \). According to (iii*), we then have \( A \in G(k+1) \) and thus \( A \in \cup \text{Ran}(G) \). Thus clause (ii) of Definition 6-2 holds. The other cases for connective formulas (clauses (iii) to (x) of Definition 6-2) and the two particular cases (clauses (xii) and (xiii) of Definition 6-2) are shown analogously.

Now, suppose \( \theta \in \text{CTERM}_H \). Then there is a \( k \in \mathbb{N} \) such that \( \neg\neg \theta = F(k) \). Then it holds: If \( \neg\neg \theta \notin G(k) \), then \( \neg\neg \theta \in G(k+1) \) and hence in both cases: \( \neg\neg \theta \in \cup \text{Ran}(G) \). Thus we have on the one hand, that clause (xv) of Definition 6-2 holds. On the other hand, we thus have that the two universal cases, clauses (xi) and (xiv) of Definition 6-2, hold. To see this, suppose \( \neg\neg \theta \in \cup \text{Ran}(G) \). Now, suppose \( \theta \in \text{CTERM}_H \). Then we have (as we have just shown) \( \neg\neg \theta \in G(l) \) for an \( l \in \mathbb{N} \) and we have \( \neg\neg \theta \in G(i) \) for an \( i \in \mathbb{N} \). Then there is a \( k > l, i \) such that \( \neg\neg \theta = F(k) \). With c), we then have \( \neg\neg \theta \in G(k) \). According to (xii*), we then have \( [0, \xi, \Delta] \in G(k+1) \) and thus \( [0, \xi, \Delta] \in \cup \text{Ran}(G) \). Thus clause (xi) of Definition 6-2 holds. Clause (xiv) is shown analogously.
Now, we still have to show the two IE-clauses, i.e. clauses (xvi) and (xvii), of Definition 6-2. First ad (xvi): Suppose \( \theta^*_0, \ldots, \theta^*_s, 1 \in \text{CTERM}_{H}\), \( \theta^*_0, \ldots, \theta^*_s, 1 \in \text{CTERM}_{H} \), for all \( i < s \): \( \theta^*_i = \theta^*_i \gamma \in \text{URan}(G) \) and \( \varphi \in \text{FUNC}, \varphi \) s-ary. As we have already shown, it holds that \( \varphi(\theta^*_0, \ldots, \theta^*_s, 1) = \varphi(\theta^*_0, \ldots, \theta^*_s, 1) \gamma \in \text{URan}(G) \). With d), there is thus an \( l \in \mathbb{N} \) such that for all \( i < s \): \( \theta^*_i = \theta^*_i \gamma \in G(l) \) and \( \varphi(\theta^*_0, \ldots, \theta^*_s, 1) = \varphi(\theta^*_0, \ldots, \theta^*_s, 1) \gamma \in G(l) \). Then there is a \( k > l \) such that the same holds for \( G(k) \) and \( F(k) = \varphi(\theta^*_0, \ldots, \theta^*_s, 1) = \varphi(\theta^*_0, \ldots, \theta^*_s, 1) \gamma \). With (i*), we then have \( \varphi(\theta^*_0, \ldots, \theta^*_s, 1) = \varphi(\theta^*_0, \ldots, \theta^*_s, 1) \gamma \). With (i*), we then have \( \varphi(\theta^*_0, \ldots, \theta^*_s, 1) \gamma \in G(k+1) \subset \text{URan}(G) \).

Now ad (xvii): Suppose \( \theta_0, \ldots, \theta_{r-1} \in \text{CTERM}_{H}, \theta'_0, \ldots, \theta'_{r-1} \in \text{CTERM}_{H} \), for all \( i < r \): \( \theta_i = \theta_i \gamma \in \text{URan}(G) \) and \( \Phi \in \text{PRED}, \Phi \) r-ary, and \( \Phi(\theta_0, \ldots, \theta_{r-1}) \gamma \in \text{URan}(G) \). With d), there is then an \( l \in \mathbb{N} \) such that for all \( i < r \): \( \theta_i = \theta_i \gamma \in G(l) \) and \( \Phi(\theta_0, \ldots, \theta_{r-1}) \gamma \in G(l) \). Then there is a \( k > l \) such that the same holds for \( G(k) \) and \( F(k) = \Phi(\theta_0, \ldots, \theta_{r-1}) \gamma \). With (i*), we then have \( \Phi(\theta'_0, \ldots, \theta'_{r-1}) \gamma \in G(k+1) \subset \text{URan}(G) \).

**Theorem 6-10.** Every Hintikka set is \( H \)-satisfiable

If \( X \) is a Hintikka set, then \( X \) is satisfiable.

**Proof:** Suppose \( X \) is a Hintikka set. Now, let \( A = \{ (\theta, 0') \mid (\theta, 0') \in \text{CTERM}_{H} \times \text{CTERM}_{H} \) and \( \gamma \theta = \theta \gamma \in X \} \).

Then it holds that \( A \) is an equivalence relation on \( \text{CTERM}_{H} \). Concerning reflexivity, we have, according to Definition 6-2-(xv), that \( \gamma \theta = \theta \gamma \in X \) and thus \( (\theta, \theta) \in A \). Now for symmetry, suppose \( (\theta, 0') \in A \). Then we have \( \gamma \theta = \theta \gamma \in X \) and, as we have just shown, \( \gamma \theta = \theta \gamma \in X \). Thus we have \( \gamma \theta = \theta \gamma \in X \) and \( \gamma \theta = \theta \gamma \in X \) and thus (with \( \theta \) for \( \theta_0, \theta_1, \) and \( \theta'_0 \) and \( \theta'_0 \) and \( \theta_0 = \theta \) for \( \Phi(\theta_0, \theta_1) \gamma \) and \( \theta = \theta \) for \( \Phi(\theta'_0, \theta'_1) \gamma \)), according to Definition 6-2-(xvii), also \( \theta = \theta \gamma \in X \). Therefore \( (\theta, \theta) \in A \). Now for transitivity, suppose \( (\theta, 0') \in A \) and \( (0', 0^*) \in A \). Then it holds: \( \gamma \theta = \theta \gamma \in X \) and \( \gamma \theta = \theta \gamma \in X \). Also, as we have shown, it holds that \( \gamma \theta = \theta \gamma \in X \). Thus it holds (with \( \theta \) for \( \theta_0, \theta'_0 \) and \( \theta' \) for \( \theta_1, \theta'_1 \) and \( \theta^* \) for \( \theta'_1 \) and \( \gamma \theta = \theta \gamma \) for \( \Phi(\theta_0, \theta_1) \gamma \) and \( \gamma \theta = \theta \gamma \) for \( \Phi(\theta'_0, \theta'_1) \gamma \)), according to Definition 6-2-(xvii), also that \( \gamma \theta = \theta \gamma \in X \) and thus that \( (\theta, 0^*) \in A \).
Now, for all \( \theta \in \text{CTERM}_H \) let \( [\theta]_A = \{ \theta' \mid (\theta, \theta') \in A \} \). Since \( A \) is an equivalence relation on \( \text{CTERM}_H \), it then follows that

a) For all \( \theta \in \text{CTERM}_H \): \( \theta \in [\theta]_A \).

b) For all \( \theta, \theta' \in \text{CTERM}_H \): \( [\theta]_A = [\theta']_A \) iff \( (\theta, \theta') \in A \) iff \( \theta = \theta'^* \in X \).

c) For all \( \theta, \theta' \in \text{CTERM}_A \): If \( [\theta]_A \cap [\theta']_A \neq \emptyset \), then \( [\theta]_A = [\theta']_A \).

The second equivalence in b) follows from the definition of \( A \).

Now, let \( D_x = \text{CTERM}_H/A = \{ [\theta]_A \mid \theta \in \text{CTERM}_H \} \). In addition, let \( I_x \) be a function with \( \text{Dom}(I_x) = \text{CONST} \cup \text{CONSTNEW} \cup \text{FUNC} \cup \text{PRED} \), where for all \( \alpha \in \text{CONST} \cup \text{CONSTNEW} \): \( I_x(\alpha) = [\alpha]_A \) and for all \( \varphi \in \text{FUNC} \): If \( \varphi \) \( r \)-ary, then \( I_x(\varphi) = \{([\theta_0]_A, \ldots, [\theta_r]_A), [\theta^*]_A) \mid ([\theta_0], \ldots, [\theta_r], \theta^*) \in \text{CTERM}_H \times \text{CTERM}_H \) and \( \varphi(\theta_0, \ldots, \theta_r) = \theta^* \in X \} \) and for all \( \Phi \in \text{PRED} \): If \( \Phi \) \( r \)-ary, then \( I_x(\Phi) = \{ ([\theta_0]_A, \ldots, [\theta_r]_A) \mid (\theta_0, \ldots, \theta_r) \in \text{CTERM}_H \) and \( \Phi(\theta_0, \ldots, \theta_r) = \theta^* \in X \} \). Lastly, let \( b_x \) be a function with \( \text{Dom}(b_x) = \text{PAR} \) and for all \( \beta \in \text{PAR} \): \( b_x(\beta) = [\beta]_A \).

According to Definition 5-1_H, \( I_x \) is then an interpretation function \( H \) for \( D_x \). First, it holds for all \( \alpha \in \text{CONST} \cup \text{CONSTNEW} \): \( I_x(\alpha) = [\alpha]_A \in D_x \). Now, suppose \( \varphi \in \text{FUNC} \), \( \varphi \) \( r \)-ary. Then we have \( I_x(\varphi) = \{ ([\theta_0]_A, \ldots, [\theta_r]_A), [\theta^*]_A) \mid ([\theta_0], \ldots, [\theta_r], \theta^*) \in \text{CTERM}_H \times \text{CTERM}_H \) and \( \varphi(\theta_0, \ldots, \theta_r) = \theta^* \in X \} \). Thus we have \( I_x(\varphi) \subseteq \text{CTERM}_H \times D_x \). Now, suppose \( \langle a_0, \ldots, a_{r-1} \rangle \in \text{CTERM}_H \) such that for all \( i < r \): \( a_i = [\theta_i]_A \). With Definition 6-2-(xv), we also have \( \varphi(\theta_0, \ldots, \theta_r) = \varphi(\theta_0, \ldots, \theta_r) \in X \) and thus \( ([\theta_0]_A, \ldots, [\theta_r]_A), [\varphi(\theta_0, \ldots, \theta_r)]_A) \in I_x(\varphi) \) and therefore \( \langle a_0, \ldots, a_{r-1} \rangle \in \text{Dom}(I_x(\varphi)) \). Now, suppose \( \langle a_0, \ldots, a_{r-1}, a^* \rangle \in I_x(\varphi) \) and \( \langle a_0, \ldots, a_{r-1}, a^* \rangle \in I_x(\varphi) \).

Then there are \( \theta_0, \ldots, \theta_{r-1} \) and \( \theta^* \) such that for all \( i < r \): \( a_i = [\theta_i]_A \) and \( a^* = [\theta^*]_A \) and \( ([\theta_0], \ldots, [\theta_r], [\varphi(\theta_0, \ldots, \theta_r)]_A) \in I_x(\varphi) \).

Then there are \( \theta_0, \ldots, \theta_{r-1} \) and \( \theta^* \) such that for all \( i < r \): \( \theta_i = [\theta_i]_A \) and \( \theta_i = [\theta^*]_A \) and \( ([\theta_0], \ldots, [\theta_r], [\varphi(\theta_0, \ldots, \theta_r)]_A) \in I_x(\varphi) \).

Thus it holds that for all \( i < r \): \( \theta_i \in A \) and thus \( \varphi(\theta_i, \theta_i) \in X \). According
Also, we can easily convince ourselves that $\theta$ is a $\in$ and then:

$$\Phi(\theta, \ldots, \theta_{r-1}) = \theta^* \in X \text{ and thus, according to Definition 6-2-(xv), we then also have } [\Phi(\theta, \ldots, \theta_{r-1})]_a = [\theta^*]_a \text{ and thus } a^* = [\theta^*]_a = \alpha^r. \text{ Altogether, we thus have that } I_a(\Phi) \text{ is an } r-\text{ary function over } D_a. \text{ Furthermore, we have for all } \Phi \in \text{PRED: If } \Phi \text{ is } r-\text{ary, then } I_a(\Phi) \subseteq \mathcal{T}_{D_a}. \text{ Lastly, we have } I_a(\ulcorner \varphi \urcorner) = \{ \langle a, a' \rangle \mid a \in D_a \}. \text{ To see this, suppose } \langle a, a' \rangle \in I_a(\ulcorner \varphi \urcorner). \text{ Then there are } \theta, \theta' \in \text{TERM}_{H\downarrow} \text{ such that } a = [\theta]_a \text{ and } a' = [\theta']_a \text{ and } \ulcorner \varphi \urcorner = \theta^* \in X. \text{ With b), we thus have } a = [\theta]_a = [\theta']_a = a'. \text{ Now, suppose } a \in D_a. \text{ Then there is a } \theta \in \text{TERM}_{H\downarrow} \text{ such that } a = [\theta]_a. \text{ According to Definition 6-2-(xv), we have } \ulcorner \varphi \urcorner = \theta^* \in X \text{ and thus } \langle a, a' \rangle \in I_a(\ulcorner \varphi \urcorner). \text{ According to Definition 5-2_{H\downarrow}, } (D_a, I_a) \text{ is hence a model}_{H\downarrow}. \text{ Also, we can easily convince ourselves that } b_\alpha \text{ is a parameter assignment}_{H\downarrow} \text{ for } D_a.

Moreover, it holds for all $\varphi \in \text{FUNC}$ that if $\varphi$ is $r$-ary and $\theta_0, \ldots, \theta_{r-1} \in \text{TERM}_{H\downarrow}$, then $I_a(\varphi)(\langle \theta_0 \rangle_a, \ldots, \langle \theta_{r-1} \rangle_a) = [\varphi(\theta_0, \ldots, \theta_{r-1})]_a$. To see this, suppose $\varphi \in \text{FUNC}, \varphi$ is $r$-ary and $\theta_0, \ldots, \theta_{r-1} \in \text{TERM}_{H\downarrow}$. With Definition 6-2-(xv), we have $\ulcorner \varphi(\theta_0, \ldots, \theta_{r-1}) \urcorner = \varphi(\theta_0, \ldots, \theta_{r-1}) \in X \text{ and thus } \langle \langle \theta_0 \rangle_a, \ldots, \langle \theta_{r-1} \rangle_a \rangle, \langle \varphi(\theta_0, \ldots, \theta_{r-1}) \rangle_a \rangle \in I_a(\varphi). \text{ Thus we have } I_a(\varphi)(\langle \theta_0 \rangle_a, \ldots, \langle \theta_{r-1} \rangle_a) = [\varphi(\theta_0, \ldots, \theta_{r-1})]_a.$

Now we will show that for all $\Phi \in \text{PRED}: \text{ If } \Phi \text{ is } r$-ary and $\theta_0, \ldots, \theta_{r-1} \in \text{TERM}_{H\downarrow}$, then: $\langle \theta_0 \rangle_a, \ldots, \langle \theta_{r-1} \rangle_a \rangle \in I_a(\Phi)$ iff $\ulcorner \Phi(\theta_0, \ldots, \theta_{r-1}) \urcorner \in X$. For this, suppose $\Phi \in \text{PRED}, \Phi$ is $r$-ary and $\theta_0, \ldots, \theta_{r-1} \in \text{TERM}_{H\downarrow}$. First, suppose $\langle \theta_0 \rangle_a, \ldots, \langle \theta_{r-1} \rangle_a \rangle \in I_a(\Phi). \text{ Then there are } \theta'_0, \ldots, \theta'_{r-1} \text{ such that for all } i < r: [\theta_i] = [\theta'_i]_a \text{ and } \langle \theta'_0, \ldots, \theta'_{r-1} \rangle \in \ulcorner \text{TERM}_{H\downarrow} \text{ and } \ulcorner \Phi(\theta'_0, \ldots, \theta'_{r-1}) \urcorner \in X. \text{ With the symmetry shown above, it then follows that for all } i < r: \ulcorner \theta_i \urcorner = \theta^* \in X. \text{ Also, we have } \ulcorner \Phi(\theta'_0, \ldots, \theta'_{r-1}) \urcorner \in X. \text{ Now, suppose } \ulcorner \Phi(\theta_0, \ldots, \theta_{r-1}) \urcorner \in X. \text{ Then it follows easily that } \langle \theta_0 \rangle_a, \ldots, \langle \theta_{r-1} \rangle_a \rangle \in I_a(\Phi). \text{ Moreover, it follows with Theorem 5-2_{H\downarrow} \text{ by induction on the complexity of } \theta \text{ that for all } \theta \in \text{TERM}_{H\downarrow}: \text{TD}(\theta, D_a, I_a, b_\alpha) = [\theta]_a. \text{ To see this, suppose } \alpha \in \text{CONST} \cup \text{CONST}_{NEW}. \text{ Then we have } \text{TD}(\alpha, D_a, I_a, b_\alpha) = I_a(\alpha) = [\alpha]_a. \text{ Suppose } \beta \in \text{PAR}. \text{ Then we have } \text{TD}(\beta, D_a, I_a, b_\alpha) = b_\alpha(\beta) = [\beta]_a. \text{ Now, suppose the statement holds for } \theta_0, \ldots, \theta_{r-1} \in \ldots$
6.2 Completeness of the Speech Act Calculus

CTERMH and suppose \( \varphi(\theta_0, \ldots, \theta_{r-1}) \in \text{FTERMH} \). Then we have TD\(_H(\varphi(\theta_0, \ldots, \theta_{r-1}),\ D_x, I_x, b_x) = I_x(\varphi(\langle \theta_0 \rangle, \ldots, \langle \theta_{r-1} \rangle)) = \langle \varphi(\theta_0, \ldots, \theta_{r-1}) \rangle \).

Furthermore, it follows that for all \( \Lambda \in \text{AFORMH} \): \( D_x, I_x, b_x \models H \Lambda \) iff \( \Lambda \in X \). To see this, suppose \( \Lambda \in \text{AFORMH} \). Then there are \( \Phi \in \text{PRED}, \Phi \) \( r \)-ary, and \( \theta_0, \ldots, \theta_{r-1} \in \text{CTERMH} \) such that \( \Lambda = \varphi(\theta_0, \ldots, \theta_{r-1}) \). Then it holds that \( D_x, I_x, b_x \models H \Lambda \) iff \( \langle \theta_0 \rangle, \ldots, \langle \theta_{r-1} \rangle \rangle \in I_x(\Phi) \) iff \( \langle \varphi(\theta_0, \ldots, \theta_{r-1}) \rangle \in X \) iff \( \Lambda \in X \).

Now we will show by induction on FDEG\(_H(\Gamma) \): If \( \Gamma \in X \), then \( D_x, I_x, b_x \models H \Gamma \) and if \( \neg \Gamma \in X \), then \( D_x, I_x, b_x \not\models H \Gamma \). From this follows immediately \( D_x, I_x, b_x \models H X \) and thus that \( X \) is satisfiable\(_H \).

Suppose the statement holds for all \( k < \text{FDEG}_H(\Gamma) \). Now, suppose \( \text{FDEG}_H(\Gamma) = 0 \). Then we have \( \Gamma \in \text{AFORMH} \). Now, suppose \( \Gamma \in X \). Then it holds that \( D_x, I_x, b_x \models H \Gamma \). Now, suppose \( \neg \Gamma \in X \). With Definition 6-2-(i), we then have \( \Gamma \not\in X \) and thus \( D_x, I_x, b_x \not\models H \Gamma \).

Now, suppose \( \text{FDEG}_H(\Gamma) > 0 \). Then we have \( \Gamma \in \text{CONFORMH} \cup \text{QFORMH} \). First, we will now show: If \( \Gamma \in X \), then \( D_x, I_x, b_x \models H \Gamma \). For this, suppose \( \Gamma \in X \). We can distinguish seven cases. First: Suppose \( \Gamma = \neg \text{B} \). Then we have \( \text{FDEG}_H(\text{B}) < \text{FDEG}_H(\Gamma) \) and thus, according to the I.H., \( D_x, I_x, b_x \not\models H \text{B} \) and hence \( D_x, I_x, b_x \models H \neg \text{B} = \Gamma \). Second: Suppose \( \Gamma = \text{A} \wedge \text{B} \). With Definition 6-2-(iii), it then holds that \( \text{A}, \text{B} \in X \). Since \( \text{FDEG}_H(\text{A}) < \text{FDEG}_H(\Gamma) \) and \( \text{FDEG}_H(\text{B}) < \text{FDEG}_H(\Gamma) \), we thus have, according to the
I.H., that $D_\alpha, I_\alpha, b_\alpha \models_{\text{H}} A$ and $D_\alpha, I_\alpha, b_\alpha \models_{\text{H}} B$ and thus $D_\alpha, I_\alpha, b_\alpha \models_{\text{H}} \langle A \land B \rangle = \Gamma$. The third to fifth case are treated analogously.

Sixth: Suppose $\Gamma = \langle \neg \land \bar{\xi} \bar{\Delta} \rangle$. With Definition 6-2-(xi), it then holds that $[\theta, \xi, \Delta] \in X$ for all $\theta \in \text{TERMF}$. Since, according to Theorem 1-13H, it holds for all $\theta \in \text{TERMF}$ that $FDEG_{\text{H}}([\theta, \xi, \Delta]) < FDEG_{\text{H}}(\Gamma)$, we thus have, according to the I.H., for all $\theta \in \text{TERMF}$:

$D_\alpha, I_\alpha, b_\alpha \models_{\text{H}} [\theta, \xi, \Delta]$. Now, let $\beta \in \text{PARST}_{\text{H}}(\Delta)$ and let $b'$ be in $\beta$ an assignment variant of $b_\alpha$ for $D_\alpha$. Then we have $b'(\beta) \in D_\alpha$ and hence there is a $\theta \in \text{TERMF}$ such that $b'(\beta) = [\theta]_\alpha$. Then we have $TD_{\text{H}}(\theta, D_\alpha, I_\alpha, b_\alpha) = [\theta]_\alpha$ and hence $b'(\beta) = TD_{\text{H}}(\theta, D_\alpha, I_\alpha, b_\alpha)$. Because of $D_\alpha, I_\alpha, b_\alpha \models_{\text{H}} [\theta, \xi, \Delta]$, it then follows, with Theorem 5-9H-(ii), that $D_\alpha, I_\alpha, b' \models_{\text{H}} [\beta, \xi, \Delta]$. Therefore we have for all $b'$ that are in $\beta$ assignment variants$_{\text{H}}$ of $b_\alpha$ for $D_\alpha$: $D_\alpha, I_\alpha, b' \models_{\text{H}} [\beta, \xi, \Delta]$. According to Theorem 5-8H-(i), we hence have $D_\alpha, I_\alpha, b_\alpha \models_{\text{H}} \langle \neg \land \bar{\xi} \bar{\Delta} \rangle = \Gamma$.

Seventh: Suppose $\Gamma = \langle \vee \bar{\xi} \bar{\Delta} \rangle$. With Definition 6-2-(xiii), there is then a $\theta \in \text{TERMF}$ such that $[\theta, \xi, \Delta] \in X$. According to Theorem 1-13H, we then have $FDEG_{\text{H}}([\theta, \xi, \Delta]) < FDEG_{\text{H}}(\Gamma)$. According to the I.H., we thus have $D_\alpha, I_\alpha, b_\alpha \models_{\text{H}} [\theta, \xi, \Delta]$. Now, let $\beta \notin \text{ST}_{\text{H}}(\Delta)$. Now, let $b' = (b \setminus \{ (\beta, b_\alpha(\beta)) \} \cup \{ (\beta, [\theta]_\alpha) \})$. Then $b'$ is in $\beta$ an assignment variant of $b_\alpha$ for $D_\alpha$ with $b'(\beta) = [\theta]_\alpha$. Also, we have $TD_{\text{H}}(\theta, D_\alpha, I_\alpha, b_\alpha) = [\theta]_\alpha$ and hence $b'(\beta) = TD_{\text{H}}(\theta, D_\alpha, I_\alpha, b_\alpha)$. Because of $D_\alpha, I_\alpha, b_\alpha \models_{\text{H}} [\theta, \xi, \Delta]$, it then follows, with Theorem 5-9H-(ii), that $D_\alpha, I_\alpha, b' \models_{\text{H}} [\beta, \xi, \Delta]$. Therefore there is a $b'$ that is in $\beta$ an assignment variant$_{\text{H}}$ of $b_\alpha$ for $D_\alpha$ such that $D_\alpha, I_\alpha, b' \models_{\text{H}} [\beta, \xi, \Delta]$. According to Theorem 5-8H-(ii), we hence have $D_\alpha, I_\alpha, b_\alpha \models_{\text{H}} \langle \vee \bar{\xi} \bar{\Delta} \rangle = \Gamma$.

Now, we will show that if $\neg \langle \bar{\Gamma} \rangle \in X$, then $D_\alpha, I_\alpha, b_\alpha \not\models_{\text{H}} \Gamma$. Suppose $\langle \neg \bar{\Gamma} \rangle \in X$. Remember that, by hypothesis, $0 < FDEG_{\text{H}}(\Gamma)$. Thus we can distinguish seven cases. First: Suppose $\Gamma = \langle \neg B \rangle$. With Definition 6-2-(ii), we then have $B \in X$. Since $FDEG_{\text{H}}(B) < FDEG_{\text{H}}(\Gamma)$, we then have, according to the I.H., that $D_\alpha, I_\alpha, b_\alpha \models_{\text{H}} B$. With Theorem 5-4H-(ii), we then have $D_\alpha, I_\alpha, b_\alpha \not\models_{\text{H}} \langle \neg B \rangle = \Gamma$. Second: Suppose $\Gamma = \langle A \land B \rangle$. With Definition 6-2-(iv), we then have $\langle \neg A \rangle \in X$ or $\langle \neg B \rangle \in X$. Since $FDEG_{\text{H}}(A) < FDEG_{\text{H}}(\Gamma)$ and $FDEG_{\text{H}}(B) < FDEG_{\text{H}}(\Gamma)$, we then have, according to the I.H., that $D_\alpha, I_\alpha,
For all \( \not\models_H A \) or \( D_x, I_x, b_x \not\models_H B \). With Theorem 5-4_H(iii), it follows that \( D_x, I_x, b_x \not\models_H \forall A \land B^\gamma = \Gamma \). The third to fifth case are treated analogously.

**Sixth:** Suppose \( \Gamma = \neg \land \xi \Delta \). With Definition 6-2-(xii), there is then a \( \theta \in \text{CTERM}_H \) such that \( \neg[\theta, \xi, \Delta] \in X \). According to Theorem 1-13_H, we have \( \text{FDEG}_H([\theta, \xi, \Delta]) < \text{FDEG}_H(\Gamma) \). According to the I.H., we thus have \( D_x, I_x, b_x \not\models_H [\theta, \xi, \Delta] \). Now, let \( \beta \not\in \text{ST}_H(\Delta) \). Let \( b' \) be in \( \beta \) the assignment variant_H of \( b_x \) for \( D_x \) with \( b'(\beta) = [\theta]_x \). Then we have \( \text{TD}_H(\theta, D_x, I_x, b_x) = [\theta]_x \) and hence \( b'(\beta) = \text{TD}_H(\theta, D_x, I_x, b_x) \). Because of \( D_x, I_x, b_x \not\models_H [\theta, \xi, \Delta] \), it then follows, with Theorem 5-9_H(ii), that \( D_x, I_x, b' \not\models_H [\beta, \xi, \Delta] \). Therefore there is a \( b' \) that is in \( \beta \) an assignment variant_H of \( b_x \) for \( D_x \) such that \( D_x, I_x, b' \not\models_H [\beta, \xi, \Delta] \). With Theorem 5-8_H(i), we hence have \( D_x, I_x, b_x \not\models_H \neg \forall \xi \Delta \).

**Seventh:** Suppose \( \Gamma = \neg \forall \xi \Delta \). With Definition 6-2-(xiv), it then holds for all \( \theta \in \text{CTERM}_H \) that \( \neg[\theta, \xi, \Delta] \in X \). According to Theorem 1-13_H, it holds for all \( \theta \in \text{CTERM}_H \) that \( \text{FDEG}_H([\theta, \xi, \Delta]) < \text{FDEG}_H(\Gamma) \). According to the I.H., it thus holds for all \( \theta \in \text{CTERM}_H \) that \( D_x, I_x, b_x \not\models_H [\theta, \xi, \Delta] \). Now, let \( \beta \not\in \text{ST}_H(\Delta) \) and suppose \( b' \) is in \( \beta \) an assignment variant_H of \( b_x \) for \( D_x \). Then we have \( b'(\beta) \in D_x \) and hence there is a \( \theta \in \text{CTERM}_H \) such that \( b'(\beta) = [\theta]_x \). Then we have \( \text{TD}_H(\theta, D_x, I_x, b_x) = [\theta]_x \) and hence \( b'(\beta) = \text{TD}_H(\theta, D_x, I_x, b_x) \). Because of \( D_x, I_x, b_x \not\models_H [\theta, \xi, \Delta] \), it then follows, with Theorem 5-9_H(ii), that \( D_x, I_x, b' \not\models_H [\beta, \xi, \Delta] \). Therefore we have for all \( b' \) that are in \( \beta \) assignment variants_H of \( b_x \) for \( D_x \) that \( D_x, I_x, b' \not\models_H [\beta, \xi, \Delta] \). With Theorem 5-8_H(ii), we hence have \( D_x, I_x, b_x \not\models_H \neg \forall \xi \Delta \).

Thus we have shown: If \( \Gamma \in X \), then \( D_x, I_x, b_x \models_H \Gamma \) and if \( \neg \Gamma \in X \), then \( D_x, I_x, b_x \not\models_H \Gamma \). According to Definition 5-17_H and Definition 5-9_H, it follows from the first part alone that \( X \) is satisfiable_H.

**Theorem 6-11.** Model-theoretic consequence implies deductive consequence

For all \( X, \Gamma \): If \( X \models \Gamma \), then \( X \vdash \Gamma \).

**Proof:** Suppose \( X \models \Gamma \). According to Definition 5-10, we then have \( X \cup \{\Gamma\} \subseteq \text{CFORM} \) and thus also \( X \cup \{\neg \Gamma\} \subseteq \text{CFORM} \). With Theorem 5-12, we have that \( X \cup \{\neg \Gamma\} \) is not satisfiable. Now, suppose for contradiction that \( X \cup \{\neg \Gamma\} \) is consistent. With
Theorem 6-9, there would then be a Hintikka set $Z$ such that $X \cup \{\neg\Gamma\} \subseteq Z$. With Theorem 6-10, $Z$ would be satisfiable. With Theorem 5-11, we would then have that $X \cup \{\neg\Gamma\}$ is satisfiable. But then we would have, with Theorem 6-5, that $X \cup \{\neg\Gamma\}$ is satisfiable. Contradiction! Therefore $X \cup \{\neg\Gamma\}$ is not consistent and thus inconsistent. With Theorem 4-22, it then follows that $X \vdash \Gamma$. ■

**Theorem 6-12. Compactness theorem**

(i) If $X \models \Gamma$, then there is a $Y \subseteq X$ such that $|Y| \in \mathbb{N}$ and $Y \models \Gamma$,

(ii) If $X \subseteq \text{CFORM}$, then: $X$ is satisfiable iff it holds for all $Y \subseteq X$ with $|Y| \in \mathbb{N}$ that $Y$ is satisfiable.

**Proof: Ad (i):** Suppose $X \models \Gamma$. With Theorem 6-11, it then follows that $X \models \Gamma$. According to Definition 3-21, there is therefore an $\tilde{\mathcal{H}}$ such that $\tilde{\mathcal{H}}$ is a derivation of $\Gamma$ from $\text{AVAP}(\tilde{\mathcal{H}})$ and $\text{AVAP}(\tilde{\mathcal{H}}) \subseteq X$. According to Theorem 3-9, we then have $|\text{AVAP}(\tilde{\mathcal{H}})| \in \mathbb{N}$. According to Definition 3-20, we also have $\tilde{\mathcal{H}} \in \text{RC\{\emptyset\}}$ and thus, with Theorem 6-1, also $\text{AVAP}(\tilde{\mathcal{H}}) \models \Gamma$. Hence we have (i).

**Ad (ii):** Suppose $X \subseteq \text{CFORM}$. The left-right-direction follows directly from Theorem 5-11. Now, for the right-left-direction suppose all $Y \subseteq X$ with $|Y| \in \mathbb{N}$ are satisfiable. Suppose for contradiction that $X$ is not satisfiable. With Definition 5-17, there would then be no $D, I, b$ such that $D, I, b \models X$. According to Definition 5-10, we would then have $X \models \neg(c_0 = c_0) \land \neg:\neg(c_0 = c_0)\neg$. With (i), there is then $Y \subseteq X$ such that $|Y| \in \mathbb{N}$ and $Y \models \neg(c_0 = c_0) \land \neg:\neg(c_0 = c_0)\neg$. Suppose for contradiction that there are $D, I, b$ such that $D, I, b \models Y$. According to Definition 5-9, $(D, I)$ would then be a model and $b$ would be a parameter assignment for $D$. According to Definition 5-10, we would also have $D, I, b \models \neg(c_0 = c_0) \land \neg:\neg(c_0 = c_0)\neg$. With Theorem 5-4-(ii) and -(iii), it would then hold that $D, I, b \models \neg c_0 = c_0$ and $D, I, b \not\models c_0 = c_0$. Contradiction! Thus $Y$ is not satisfiable though $|Y| \in \mathbb{N}$, which contradicts the assumption. Hence $X$ is satisfiable. ■
7 Retrospects and Prospects

We have developed a pragmatised natural deduction calculus for which it holds that: (i) Every sentence sequence $\mathcal{A}$ is not a derivation of a proposition from a set of propositions or there is exactly one proposition $\Gamma$ and one set of propositions $X$ such that $\mathcal{A}$ is a derivation of $\Gamma$ from $X$, where this can be determined for every sentence sequence without recourse to any meta-theoretical means of commentary. (ii) The classical first-order model-theoretic consequence relation is equivalent to the consequence relation for the calculus. We assumed a language $L$, where $L$ is an arbitrary but fixed language with certain properties: The development of the calculus and its meta-theory can therefore be applied to all suitable languages.

We believe that this calculus is suited to support the claim that usual practices of inference can be established or modelled solely by setting up systems of rules, where the implementation of these practices does not require any meta-theoretical support practices (like, for example, an additional practice of commenting). Confessionally: Inferring in a language consists in the performance of (rule-respecting) speech acts in this language and not in the performance of speech acts in this language and concomitant meta-theoretical speech acts. For short: Inferring in a language is performing speech acts in this language. These theses have to be substantiated philosophically.

Also, some further meta-theoretical work seems in order, e.g. extending the completeness result to non-denumerably infinite languages and a precise investigation of the relationships between the individual rules of the calculus. So, one could investigate in which sense the logical operators are interdefinable. Also, it seems worthwhile to examine how the approach we have taken can be extended so as to include speech-act rules for the speech acts of positing-as-axiom, defining, stating and adducing-as-reason and for the use of modal and description operators etc. Further, it has to be examined how derivations in the calculus can be simplified by introducing admissible rules. Last but not least, a propaedeutic version of the calculus is to be established, where such a version should also demonstrate that in order to establish the availability concepts and the rules of the calculus solely for application purposes, one does not require genuinely set-theoretical vocabulary.
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