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Boris Andreianov[*], Robert Eymard[†], Mustapha Ghilani[‡] and Nouzha Marhraoui[§¶]

Dedicated to Monique Madaune-Tort on the occassion of her 60th anniversary

Abstract

Starting from a two-phase flow model in porous media with the viscosity of the "mobile" phase going to infinity, the Generalized Richards Equation for the "viscous" phase:

$$\begin{cases} u_t - \operatorname{div}(k_w(u)\nabla p) \;=\; \overline{s} - \theta\, \underline{s} 1\!\!1_{[u=1]}, \\ k_a(u)\nabla(p + p_c(u)) = 0 \quad \text{a.e. in } \Omega \times (0,T) \end{cases}$$

was derived in the works [6] and [2] (see also [4]). We discuss intrinsic formulations (weak solutions, renormalized solutions) of this singular limit problem, using in particular the techniques developed by Plouvier-Debaigt, Gagneux et al. [13, 11, 12]. For the no-source case, we justify the equivalence of the Generalized Richards Equation and the classical Richards model.

Keywords: Flow in porous medium, two-phase flow model, Richards model, renormalized solutions.

1 Introduction

The widely accepted model for undersaturated water flow in porous medium, used in particular in the hydrogeology context, is the Richards equation (see [14], see also [5]). Let us present this model in a simplified mathematical setting. Assuming that gravity effects can be neglected and the porosity of the media does not vary, setting most of the physically meaningful constants equal to one, we can write the Richards model as follows:

$$\begin{cases} u_t - \operatorname{div}(k_w(u)\nabla p) = s_w, \\ u = {p_c}^{-1}(p_{atm} - p). \end{cases} \tag{1}$$

Here $(t,x) \in (0,T) \times \Omega$ with Ω a bounded Lipschitz domain of $\mathbb{R}^d$ ($d = 1, 2$ or 3); $u = u(t,x)$ is the water saturation function that takes values in $[0,1]$; $k_w(\cdot), p_c(\cdot) : [0,1] \mapsto \mathbb{R}$ are given nonlinear functions (permeability and capillar pressure, respectively) whose properties will be made precise later on; and $s_w = s_w(t,x)$ is a source term, for which a particular form will be fixed later. An important feature of the Richards model is that the reciprocal function of the capillar pressure (called the capacity function) is prolonged for all negative values by the value 1. Finally, $p = p(t,x)$ is the unknown liquid pressure function and p_{atm} is a reference (normalization) value for the pressure. Boundary conditions should be imposed; the simplest and most important ones are the homogeneous Neumann boundary conditions.

In the Richards equation, the conservation equation of the air phase is replaced by the assumption that the air pressure $p(x,t) + p_c(u(x,t))$ is equal to p_{atm} for (x,t) such that $1 - u(x,t) > 0$ (where $1 - u$

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is the air saturation), due to its high mobility (in practice, the ratio μ of air and water mobilities is of order 10^2). If this assumption is not done, one should consider the more precise two-phase flow model, that describes the conservation of both components (see, e.g., [9]):

$$\begin{cases} u_t - \operatorname{div}(k_w(u)\nabla p) = s_w, \\ (1-u)_t - \operatorname{div}(\mu k_a(u)\nabla(p + p_c(u))) = s_a. \end{cases} \tag{2}$$

Here, in addition to the previous notation, $\mu = const$ is the mobility coefficient, $s_a = s_a(t,x)$ is the corresponding source term, and $k_a(\cdot)$ is another nonlinearity (the mobile phase permeability). In this model, the pressure $p = p(t,x)$ should be normalized, e.g., by imposing a mean value zero in Ω for $p(t,\cdot)$, for $t \in (0,T)$.

It is a natural question to investigate the limit of the two-phase problem, as $\mu \to \infty$, and to compare it to the Richards model. This question was addressed in the work of Eymard, Ghilani and Marhrauoi [4] then pursued by Eymard, Henry and Hilhorst in [6] (from the theoretical perspective) and by the authors of the present note in [2] (mainly from the numerical analysis perspective).

In these works, it was shown that (under restrictions recalled in Section 2, and given some fixed initial condition $u_0 = u_0(x)$, source and sink terms, and the homogeneous Neumann boundary conditions) solutions (u_μ, p_μ) to the two-phase flow system (2) admit an accumulation point (u,p), as $\mu \to 0$. This accumulation point satisfies a formulation that we will call Generalized Richards Equation:

$$\begin{cases} u_t - \operatorname{div}(k_w(u)\nabla p) = s_w, \\ \quad \nabla(p + p_c(u)) = 0 \ \text{ a.e. on the set } [u < 1], \end{cases} \tag{3}$$

the source term s_w taking a particular form described in Section 2. Thus (3), obtained by a passage to the limit in a subsequence of solutions to (2), should be considered as a singular limit of the two-phase flow system.

One has to undertake the analysis of the Generalized Richards Equation (3), giving an intrinsic definition of solution (definition independent of a particular approximation procedure used to construct solutions), investigating existence and uniqueness of such solutions, and comparing Generalized Richards and classical Richards equations. In this note, we make first steps towards this goal.

In Section 2, we give a synthetic presentation of the theoretical results of [4, 6, 2]. In Section 3, we recall the weak formulation of (3) obtained in these works, and make precise the functional setting suitable for this definition. Then, following the works of Plouvier-Debaigt, Gagneux et al. [13, 11, 12], we give a notion of renormalized solution that allows to separate the singular set $[u = 1]$ from the remaining part of the physical domain. We show that weak solutions are renormalized ones; and, using an idea of [7], we prove an *incomplete contraction inequality* for renormalized solutions $u, \hat{u}$ with data $u_0, \hat{u}_0$, which takes the form

$$\|(u - \hat{u})^+\|_{L^\infty(0,T;L^1(\Omega))} \le \|(u_0 - \hat{u}_0)^+\|_{L^1(\Omega)} + \int_0^T\!\!\int_\Omega \overline{s}\,1\!\!1_{[u=\hat{u}=1]},$$

where $\overline{s}$ is the injection term for the model. The uniqueness follows directly only for the case the injection term is zero; and for this case, we show in Section 4 coincidence of Richards and Generalized Richards equations. In the final section, we give some conclusions and perspectives for a further study of the Generalized Richards Equation.

2 The two-phase model and a singular limit formulation

The works [4, 6, 2], and also the present one, concern the particular situation where

– the gravity effects are neglected in (2);

– the initial condition $u_0 = u_0(x)$ is assumed to take values in some interval $[u_m, 1]$ with $u_m = const > 0$;
– the right-hand side terms in (2) are actually u-dependent, and take the particular form

$$s_w = f_\mu(c)\,\overline{s} - f_\mu(u)\,\underline{s}, \quad s_a = (1 - f_\mu(c))\,\overline{s} - (1 - f_\mu(u))\,\underline{s}. \tag{4}$$

In (4), $c = c(t,x) \geq u_m$ is the given saturation in water of the injected fluid, $\overline{s} = \overline{s}(t,x)$ and $\underline{s} = \underline{s}(t,x)$ are the intensities of sources and sinks, respectively, and f_μ determines the fractional flow of the water phase, given by

$$f_\mu(s) = \frac{k_w(s)}{M_\mu(s)} \;\text{ with }\; M_\mu(s) = k_w(s) + \mu k_a(s). \tag{5}$$

For the two-phase model, one requires that

$$\overline{s} \geq 0,\; \underline{s} \geq 0,\; \overline{s},\, \underline{s} \in L^2((0,T)\times\Omega), \text{ and } \int_\Omega (\,\overline{s}(x) - \,\underline{s}(x))dx = 0 \tag{6}$$

(the latter constraint is needed to ensure the total mass conservation for the two phases). The interpretation of (4)–(5) is clear: both sources and sinks operate on the mixture of the two phases, so that the quantities of water and air that actually enter or exit the medium depend on the water saturation ($c(t,x)$ at sources, $u(t,x)$ at sinks), on the permeabilities $k_w(\cdot), k_a(\cdot)$ of the phases, and on the mobility ratio μ.

The set of the above assumptions, together with realistic assumptions on the profiles $p_c(\cdot), k_w(\cdot), k_a(\cdot)$ (see (8) below) leads to the

$$\text{uniform in } \mu \text{ bound } u_\mu(t,x) \geq u_m \text{ on solutions of the two-phase flow system (2).} \tag{7}$$

This is important in order to avoid the degeneracy of the problem that happens at zero saturation. Indeed, typical nonlinearities $p_c(\cdot), k_w(\cdot), k_a(\cdot)$ satisfy

$$\begin{aligned}
&k_w(\cdot), k_a(\cdot) \text{ are continuous functions on } [0,1],\\
&k_w(\cdot) \text{ is non-decreasing, } k_w(0) = 0,\ k_w(1) = 1 \text{ and } k_w(u_m) > 0,\\
&k_a(\cdot) \text{ is non-increasing with } k_a(1) = 0,\ k_a(0) = 1 \text{ and } k_a(s) > 0 \text{ for all } s \in [0,1),\\
&p_c(\cdot) \text{ is continuous strictly decreasing function on } (0,1], \text{ normalized by } p_c(1) = 0,
\end{aligned} \tag{8}$$

thus the lower bound by $u_\mu \geq u_m = const > 0$ ensures that

$$k_w(u_\mu) \geq const > 0. \tag{9}$$

Further uniform estimates obtained in [4, 6, 2] are the following:

$$\int_0^T \int_\Omega k_w(u_\mu)|\nabla p_\mu|^2 \leq const, \tag{10}$$

$$\int_0^T \int_\Omega \mu k_a(u_\mu)|\nabla(p_\mu + p_c(u_\mu))|^2 \leq const, \tag{11}$$

moreover,

$$\int_0^T \int_\Omega |\nabla \zeta_a(u_\mu)|^2 \leq const \tag{12}$$

where we use an auxiliary continuous strictly increasing on $[u_m, 1]$ nonlinearity $\zeta_a(\cdot)$ defined, e.g., by

$$\zeta_a(s) := \int_s^1 \sqrt{k_a(\sigma)}\, dp_c(\sigma). \tag{13}$$

Actually, in absence of general uniqueness result for (2), in [6, 2] one constructs solutions (u_μ, p_μ) (using either the parabolic regularization by vanishing viscosity, or a specially designed finite volume numerical scheme) satisfying the estimates (9)–(12). Equipped with these estimates, using in addition the evolution equation on $(u_\mu)_t$ contained in system (2) along with time compactness arguments developed by Alt and Luckhaus [1], we can extract a (not labelled) sequence of values of μ going to infinity such that

$$\begin{aligned} &\zeta_a(u_\mu) \to Z \ \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ and a.e. on } (0,T)\times\Omega, \\ &p_\mu \to p \ \text{ weakly in } L^2(0,T;H^1(\Omega)), \quad \text{ as } \mu\to\infty. \end{aligned} \tag{14}$$

Because $\zeta_a(\cdot)$ is a homeomorphism of $[u_m,1]$ on $\zeta_a([u_m,1])$, we deduce that $u_\mu \to u := \zeta_a^{-1}(Z)$ a.e. on $(0,T)\times\Omega$. Moreover, the so created accumulation point (u,p) satisfies "$k_a(u)\big(\nabla(p+p_c(u))\big)=0$" (which can be expressed as $\nabla p = \nabla\zeta_a(u)/\sqrt{k_a(u)}$ a.e. on $[u<1]$) because we have

$$\int_0^T\!\!\int_\Omega \left|\sqrt{k_a(u)}\nabla p - \nabla\zeta_a(u))\right|^2 = 0, \tag{15}$$

due to the lower semicontinuity of the L^2 norm for the weak convergence. Thanks to properties (8), the fractional flow function $f_\mu(\cdot)$ in (5) tends to $1\!\!1_{\{1\}}(\cdot)$ pointwise on $[u_m,1]$ as $\mu\to\infty$. Thus we can only pass to the weak-* L^∞ limit in the sink term, which leads to the following source term at the limit:

$$\overline{s}\,1\!\!1_{[c=1]} - \theta\,\underline{s}\,1\!\!1_{[u=1]} \ \text{ for some measurable } [0,1]\text{-valued function } \theta=\theta(t,x).$$

This eventually leads to the following weak formulation for the accumulation point (u,p) of (u_μ,p_μ), derived by passage to the limit in (a subsequence of) weak formulations of (2):

$$\left.\begin{aligned} &u_t - \operatorname{div}\big(\mathbf{F}[u,p]\big) = \overline{s}\,1\!\!1_{[c=1]} - \theta\,\underline{s}\,1\!\!1_{[u=1]}, \\ &\mathbf{F}[u,p]\cdot\mathbf{n}\,|_{(0,T)\times\partial\Omega} = 0 \ \text{ and } \ u\,|_{t=0} = u_0 \end{aligned}\right. \quad \text{in } \mathcal{D}'([0,T)\times\overline{\Omega}) \tag{16}$$

with the flux

$$\mathbf{F}[u,p] := k_w(u)\nabla p \equiv \frac{k_w(u)}{\sqrt{k_a(u)}}\nabla\zeta_a(u)\,1\!\!1_{[u<1]} + \nabla p\,1\!\!1_{[u=1]}; \tag{17}$$

in addition,

$$p\,,\,\zeta_a(u) \in L^2(0,T;H^1(\Omega)) \text{ and } \int_\Omega p(t,\cdot) = 0 \text{ on } (0,T). \tag{18}$$

From (17), (18) and (6) we see in particular that

$$u_t \in (L^2(0,T;H^1(\Omega))' \ \text{ and the equality in (16) holds in } (L^2(0,T;H^1(\Omega))', \tag{19}$$

in the "variational" sense first introduced by Alt and Luckhaus in [1]. This means in particular that the initial condition holds in the following precise sense:

$$\begin{aligned} &\int_0^T < u_t\,,\,\xi >_{(H^1)',H^1} = -\int_0^T\!\!\int_\Omega u\,\xi_t - \int_\Omega u_0\xi(0,\cdot) \\ &\text{for all } \xi\in L^2(0,T;H^1(\Omega)) \text{ such that } \xi_t\in L^1((0,T)\times\Omega) \text{ and } \xi(T,\cdot)=0. \end{aligned} \tag{20}$$

Then it is possible to test (16) with functions of the kind $(H\circ\zeta_a)(u)$, with Lipschitz continuous $H(\cdot)$. Indeed, (20) leads to the celebrated Mignot-Bamberger/Alt-Luckhaus chain rule ("integration-by-parts") argument, see [1, 8, 10, 3]:

$$\begin{aligned} \int_0^T < u_t\,,\,H(\zeta_a(u)) >_{(H^1)',H^1} &= \int_\Omega \big(B_{H,a}(u)(T,\cdot) - B_{H,a}(u_0)\big) \\ &\quad \text{with } B_{H,a} : s \mapsto -\int_s^1 H(\zeta_a(\sigma))\,d\sigma. \end{aligned} \tag{21}$$

Notice that, actually, formulation (16) concerns the *triple* (u,p,θ). In the sequel, we will focus on description of the saturation u, while the functions p and θ will play the role of auxiliary quantities that describe the balance of fluxes and sources on the set $[u=1]$ only.

3 Weak and renormalized formulations of the Generalized Richards Equation

Let first provide an *intrinsic* weak formulation of the Generalized Richards Equation. Indeed, (16)–(18) "keeps memory" of the limiting process by which solutions were obtained, in particular, $\zeta_a(\cdot)$ keeps memory of the nonlinearity $k_a(\cdot)$. While in principle, it is possible that a singular limit of two-phase flow equations depend on the full set of data and non-linearities also for the "carefully neglected" air phase, we easily see that $k_a(\cdot)$ can be eliminated from the formulation (16)–(18). Indeed, we have

Lemma 3.1 *If (u,p,θ) verify the weak formulation (16)–(18), then*

$$\zeta_w(u) \in L^2(0,T;H^1(\Omega)), \quad \text{where } \zeta_w(z) := \int_z^1 \sqrt{k_w(\sigma)}\, dp_c(\sigma). \tag{22}$$

Moreover, if $p_c(\cdot)$ is bounded on $(0,1]$, then $\|\nabla\zeta_w(u)\|_{L^2((0,T)\times\Omega)}$ is estimated uniformly with respect to the lower bound u_m on the initial datum.

Notice that due to the specific structure of the source and sink terms, we have the lower bound (7) on the saturation, inherited at the limit $\mu \to \infty$.

Remark 3.1 *Under the assumptions we have taken, we have $1 \geq k_w(u) \geq k_w(u_m) > 0$, so that the functions such as $\nabla\zeta_w(u)$, $\nabla p_c(u)$, $\nabla g_w(u)$ with $g_w(s) = \int_s^1 k_w(\sigma)\, dp_c(\sigma)$ have the same integrability properties. To simplify the calculations, and also because of the last claim of Lemma 3.1, we prefer to work with $\zeta_w(u)$.*

Proof: Formally, we would like to take $\xi = -p_c(u)$ for the test function in (16). In order to do so, we use regularization. For $\delta > 0$, for $s \in [u_m, 1]$ consider the auxiliary nonlinearity

$$G_\delta(s) := \int_s^1 \sqrt{\frac{k_a(\sigma)}{k_a(\sigma)+\delta}}\, dp_c(\sigma) \equiv -\int_s^1 \frac{1}{\sqrt{k_a(\sigma)+\delta}}\, d\zeta_a(\sigma) \equiv H_\delta(\zeta_a(s));$$

here $H_\delta(\cdot)$ is a Lipschitz continuous function that we do not make explicit. Because $p_c(\cdot)$ is bounded on $[u_m,1]$, we have $\|G_\delta\|_\infty \leq const$ uniformly in δ; also the primitives $B_{H_\delta,a}$ defined as in (21) are uniformly bounded. In view of the "variational" reformulation explained in Section 2, we can use $G_\delta(u) \equiv H(\zeta_a(u))$ as a test function in (16) (it lies in $L^2(0,T;H^1(\Omega))$, due to (18)), and using the chain rule (21) and the aforementioned L^∞ bounds, we find that

$$I_\delta := \int_0^T\!\!\int_\Omega \frac{k_w(u)}{\sqrt{k_a(u)}} \frac{1}{\sqrt{k_a(\sigma)+\delta}} |\nabla\zeta_a(u)|^2 \leq \int_\Omega \big(B_{H_\delta,a}(u_0) - B_{H_\delta,a}(u)(T,\cdot)\big) + const(\|\,\overline{s}\|_1 + \|\,\underline{s}\|_1)$$

(notice that we have $\nabla\zeta_a(u) = 0$ a.e. on the set $[u=1] \equiv [\zeta_a(u)=0]$, by a well-known property of level sets of functions in Sobolev spaces). Due to the boundedness of $B_{H_\delta,a}$, the right-hand side above is bounded uniformly in δ. Now, due to the definition of $\zeta_a(\cdot)$, we find that the map

$$s \mapsto \zeta_{w,\delta}(s) := \int_s^1 \sqrt{k_w(\sigma)\sqrt{\frac{k_a(\sigma)}{k_a(\sigma)+\delta}}}\, dp_c(\sigma) \equiv -\int_s^1 \sqrt{\frac{k_w(\sigma)}{\sqrt{k_a(\sigma)(k_a(\sigma)+\delta)}}}\, d\zeta_a(\sigma)$$

is well defined and bounded on $[u_m,1]$. Then $I_\delta = \int_0^T\!\int_\Omega |\nabla\zeta_{w,\delta}(u)|^2$, thus we have a uniform $L^2(0,T;H^1(\Omega))$ estimate for the functions $\zeta_{w,\delta}(u)$. By weak compactness of bounded sets in $L^2(0,T;H^1(\Omega))$ and because $\zeta_{w,\delta}(\cdot)$ converges to $\zeta_w(\cdot)$ on $[u_m,1]$ as $\delta \to 0$, we deduce that also the limit $\zeta_w(u)$ of $\zeta_{w,\delta}(u)$ belongs to $L^2(0,T;H^1(\Omega))$.

At this point, we are also allowed to test (16) with $-p_c(u)$, since it lies in $L^2(0,T;H^1(\Omega))$ (recall that $1 \geq k(u) \geq k(u_m) > 0$). Proceeding as for the above estimates, we get the $L^2((0,T)\times\Omega)$ bound on $\nabla\zeta_w(u)$ that only depends on $\|p_c\|_\infty$ and $\|\,\overline{s}\|_1, \|\,\underline{s}\|_1$. This concludes the proof of the lemma. □

It should be noticed that since $k_w(u) \geq k_w(u_m) > 0$, the L^2 integrability of $\nabla\zeta_w(u)$ implies the one of $\zeta_a(u)$. Thus we can give an equivalent weak formulation of the one-phase flow model (16)–(18):

Definition 3.1 *Assume u_0 takes values in $[u_m, 1]$, the nonlinearities $k_w(\cdot)$ and $p_c(\cdot)$ satisfy the assumptions listed in (8); consider $\zeta_w(\cdot)$ defined in (22). Consider $\underline{s}$ and $\overline{s}$ (the latter one now replaces $\overline{s}\,1\!\!1_{[c=1]}$) two nonnegative functions in $L^2((0,T)\times\Omega)$.*

A triple (u,p,θ) such that $\zeta_w(u)$, $p \in L^2(0,T;H^1(\Omega))$, $u_t \in (L^2(0,T;H^1(\Omega)))'$, and θ is a $[0,1]$-valued measurable function on $(0,T)\times\Omega$ is a weak solution of the Generalized Richards Equation with initial condition u_0 and the homogeneous Neumann boundary condition if, firstly, $\nabla(p+p_c(u))1\!\!1_{[u=1]} = 0$ a.e. on $(0,T)\times\Omega$, secondly, (20) holds, and thirdly,

$$-\int_0^T\!\!\int_\Omega u\,\xi_t - \int_\Omega u_0\xi(0,\cdot) + \int_0^T\!\!\int_\Omega \big(\sqrt{k_w(u)}\nabla\zeta_w(u) + \nabla p\,1\!\!1_{[u=1]}\big)\cdot\nabla\xi = \int_0^T\!\!\int_\Omega \big(\overline{s} - \theta\,\underline{s}\,1\!\!1_{[u=1]}\big)\xi \tag{23}$$

for all $\xi \in L^2(0,T;H^1(\Omega))$ such that $\xi_t \in L^1((0,T)\times\Omega)$ and $\xi(T,\cdot) = 0$.

Definition 3.1 is indeed intrinsic, in the sense that it does not make appeal to the data of the neglected mobile phase. Notice that $\nabla p_c(u)$ in the above definition makes sense because $p_c(u)$ is a Lipschitz function of $\zeta_w(u)$, being understood that $k_w(u) \geq k_w(u_m) > 0$ (cf. Remark 3.1).

The results of [6] and of [2] provide existence for the weak formulation of Definition 3.1. As a matter of fact, uniqueness of a weak solution triple (u,p,θ) should not be true in general. Clearly, the main difficulty in treating Generalized Richards Equation lies in treatment of the set $[u=1]$, called "saturated region", and on the interplay that could take place between the pressure p and the "sink efficiency ratio" θ in the saturated region; we expect that, whenever $\underline{s}1\!\!1_{[u=1]}$ is not zero, infinitely many solutions (u,p,θ) corresponding to the same saturation u may co-exist. These different solutions may correspond to different convergent approximations by the two-phase flow equations (2).

Remark 3.2 *To give an idea of the possible interplay between p and θ in the saturated region, let us assume that u is regular enough, namely, upon a choice of the representative of u, $u \in W^{1,1}((0,T);L^1(\Omega))$ and for all $t \in (0,T)$, the set $\Omega_t := \{x \in \Omega \,|\, u(t,x) = 1\}$ is a set with Lipschitz boundary which depends nicely on t. Such assumptions allow to separate the unsaturated zone $\mathcal{S}^c := \{(x,t) \,|\, t \in (0,T),\ x \in \Omega\setminus\Omega_t\}$ where u is governed by a parabolic equation, and the saturated region $\mathcal{S} := [u=1]$ where p is governed by an elliptic equation with θ-dependent right-hand side.*

In this case, it is easily seen from Definition 3.1 that p is uniquely defined by u in $\mathcal{S}^c$. Consequently, the weak normal trace $\gamma_{ext}(k_w(u)\nabla p\cdot\mathbf{n}_t)$ of $(k_w(u)\nabla p)(\cdot,t)$ on $\partial\Omega_t$ is uniquely defined ($\mathbf{n}_t$ being the outer unit normal vector to $\partial\Omega_t$). These data are transmitted to the elliptic equation on p in the zone $\mathcal{S}$. Namely, due to the regularity of u assumed above, the couple (p,θ) solves the equation

$$\begin{cases} -k_w(1)\Delta p(\cdot,t) = \overline{s}(\cdot,t) - \theta(\cdot,t)\,\underline{s}(\cdot,t) & \text{in the interior of } \Omega_t, \\ k_w(1)\nabla p(t,\cdot)\cdot\mathbf{n}_t = \gamma_{ext}(k_w(u)\nabla p\cdot\mathbf{n}_t) & \text{on } \partial\Omega_t. \end{cases} \tag{24}$$

Then the only constraint on θ is the compatibility condition

$$\int_{\partial\Omega_t}\gamma_{ext}(k_w(u)\nabla p\cdot\mathbf{n}_t)(t,\cdot) + \int_{\Omega_t}\overline{s}(\cdot,t) = \int_{\Omega_t}\theta(\cdot,t)\,\underline{s}(\cdot,t); \tag{25}$$

whenever $\theta(t,\cdot)$ verifies (25), there exists a unique $p(t,\cdot)$ solving (24).

Therefore we now focus on the information about the saturation u contained in the weak formulation of Definition 3.1. Following the idea of the papers [13, 11, 12], let us carry out a renormalization of the formulation (23), cutting off the saturated region $[u=1]$.

In the sequel, for the sake of simplicity let us take the assumptions (that are realistic):

$$\begin{aligned}&p_c \text{ is absolutely continuous on } [u_m,1], \text{ i.e., } {p_c}'(\cdot) \text{ is an } L^1([u_m,1]) \text{ function;}\\ &\text{moreover, } \inf_{\sigma\in[u_m,1-\alpha]} {p_c}'(\sigma)<0 \text{ for every } \alpha>0.\end{aligned} \tag{26}$$

Consider the sequence of Lipschitz continuous, piecewise affine truncation functions $(T_n(\cdot))_{n\in\mathbb{N}}$ on $[0,1]$:

$$T_n|_{[0,1-\frac{1}{n}]}\equiv 1,\;\; T_n|_{[1-\frac{1}{2n},1]}\equiv 0,\;\text{ and }\; T_n'|_{[1-\frac{1}{n},1-\frac{1}{2n}]}\equiv 2n.$$

Then the following properties are obvious:

$$b_n(s):=\int_0^s T_n(\sigma)\,d\sigma \;\text{ tends to the identity function;} \tag{27}$$

$$c_n(s)=\int_0^s (1-T_n(\sigma))\,d\sigma \;\text{ tends to the zero function.} \tag{28}$$

We define in addition the auxiliary nonlinearities

$$\varphi_n(s)=\int_s^1 k_w(\sigma)T_n(\sigma)\,dp_c(\sigma)\equiv-\int_s^1\sqrt{k_w(\sigma)}T_n(\sigma)\,d\zeta_w(\sigma)\quad\text{and}\quad \psi_n(s)=\int_s^1\sqrt{k_w(\sigma)T_n'(\sigma)p_c'(\sigma)}\,d\sigma.$$

We are now in a position to define renormalized solutions of the Generalized Richards Equation.

Definition 3.2 *Take the assumptions of Definition 3.1.*

A couple (u,θ) such that $\varphi_n(u),\psi_n(u)\in L^2(0,T;H^1(\Omega))$, $b_n(u)_t\in(L^2(0,T;H^1(\Omega))'+L^1((0,T)\times\Omega)$, and θ is a $[0,1]$-valued measurable function on $(0,T)\times\Omega$ is a renormalized solution of the Generalized Richards Equation with initial condition u_0 and the homogeneous Neumann boundary condition if

- *for every $n\in\mathbb{N}$, the* renormalized formulation on $[u<1]$*:*

$$\begin{cases} b_n(u)_t-\Delta\varphi_n(u)-|\nabla\psi_n(u)|^2=\overline{s}\,T_n(u)\\ \varphi_n(u)\cdot\mathbf{n}|_{(0,T)\times\partial\Omega}=0 \;\text{ and }\; b_n(u)|_{t=0}=b_n(u_0)\end{cases}\quad \text{in } L^2(0,T;(H^1(\Omega))')+L^1((0,T)\times\Omega) \tag{29}$$

  holds[1] *in the "variational" sense of [1] (the term $\int_0^T\int_\Omega b_n(u)_t\xi$ is given sense using the initial condition and the integration-by-parts formula analogous to* (20)*).*

- *there exists $p\in L^2(0,T;H^1(\Omega))$ such that the following constraint holds at the limit $n\to\infty$*

$$\begin{aligned}&\lim_{n\to\infty}\int_0^T\int_\Omega|\nabla\psi_n(u)|^2\,\xi\;=\iint_{[u=1]}\big[(\,\overline{s}-\theta\,\underline{s})\,\xi-\nabla p\cdot\nabla\xi\big]\\ &\text{for all } \xi\in L^2(0,T;H^1(\Omega))\cap L^\infty((0,T)\times\Omega)\;.\end{aligned} \tag{30}$$

Remark 3.3 *The constraint* (30) *can be localized in time, upon taking $\xi 1\!\!1_{[0,t_0]}(t)$. It implies, in particular, the following constraint from which p is eliminated:*

$$\text{for all } t_0\in(0,T)\qquad \lim_{n\to\infty}\int_0^{t_0}\int_\Omega|\nabla\psi_n(u)|^2=\int_0^{t_0}\int_{[u=1]}(\,\overline{s}-\theta\,\underline{s}). \tag{31}$$

Let us stress that the components p and θ are eliminated from the renormalized formulation (29), because "it only sees" the set $[u<1]$ and we have $\theta=0$, $\nabla p=-\nabla p_c(u)=\sqrt{k_w(u)}\nabla\zeta_w(u)$ on the set $[u<1]$. As it is usual in the context of renormalized solutions, some information concerning the quantities p and θ on the singular set (here, this is the saturated region $[u=1]$) and also the behaviour of u near the boundary of this region is retained by means of the limit (30).

[1] As in [13, 11, 12], one can replace T_n by more general nonlinearities, and re-define b_n, φ_n, ψ_n accordingly.

Remark 3.4 *In the renormalized formulation* (29),(31)*, the* L^2 *integrability assumption of the source and sink terms* $\overline{s}$, $\underline{s}$ *can be replaced by the* L^1 *integrability provided the* $H^1-(H^1)'$ *duality is replaced by* $X-X'$ *duality with* $X=L^\infty\cap H^1$*, and the test functions* ξ *in* (29) *are chosen accordingly; notice that the chain rule* (21) *extends to this setting ([3]).*

Now, we will show

Proposition 3.1 *A weak solution of the Generalized Richards Equation is also its renormalized solution.*

Proof: Since u takes values in $[u_m,1]$, $\varphi_n(u)=H(\zeta_w(u))$ for some Lipschitz continuous function $H(\cdot)$ that we need not make precise. Due to assumptions (26) and the definition of $T_n(\cdot)$, writing

$$T_n(s)\equiv\int_s^1 T_n'(\sigma)\,d\sigma\equiv-\int_{[s,1]\cap[s,1-\frac{1}{2n}]}T_n'(\sigma)\frac{1}{\sqrt{k_w(\sigma)p_c{}'(\sigma)}}\,d\zeta_w(\sigma)$$

we see that also $T_n(u)$ is the composition of $\zeta_w(u)$ by a Lipschitz continuous function. Analogous property holds for $\psi_n(u)$. Therefore for all $n\in\mathbb{N}$, $T_n(u)$, $\varphi_n(u)$ and $\psi_n(u)$ belong to $L^2(0,T;H^1(\Omega))$ whenever u satisfies (22). Moreover, $T_n(\cdot)$ are bounded. It is therefore possible to take $T_n(u)\xi$ (say, with $\xi\in\mathcal{D}([0,T]\times\Omega)$) and then $(1-T_n(u))\xi$ as test functions in the weak formulation (23). Using the generalized chain rule analogous to (21) for the term $\int_0^T < u_t\,,\,T_n(u)\,\xi >_{(H^1)',H^1}$, we find

$$\begin{aligned}\int_\Omega\big(b_n(u(T,\cdot))\xi(T,\cdot)-b_n(u_0)\xi(0,\cdot)\big)-\int_0^T\int_\Omega b_n(t)\cdot\xi_t\\+\int_0^T\int_\Omega\big(\nabla\varphi_n(u)\cdot\nabla\xi-|\nabla\psi_n(u)|^2\,\xi\big)=\int_0^T\int_\Omega\overline{s}\,T_n(u)\,\xi.\end{aligned}\tag{32}$$

This means in particular that in the sense of distributions, $b_n(u)_t=\Delta\varphi_n(u)+|\nabla\psi_n(u)|^2+\overline{s}\,T_n(u)$, of which the right-hand side belongs to the space $L^2(0,T;(H^1(\Omega))')+L^1((0,T)\times\Omega)$. Then, using the generalization of [3] of the chain rule argument, we can interpret the first line of (32) as

$$\int_0^T < b_n(u)_t\,,\,\xi >_{(H^1)'+L^1,H^1\cap L^\infty}$$

(being understood that $(H^1)'+L^1\subset(H^1\cap L^\infty)')$ and take $\xi\in L^2(0,T;H^1(\Omega))\cap L^\infty((0,T)\times\Omega)$ with $\xi_t\in L^1((0,T)\times\Omega)$. We end up with (29), which is the renormalized formulation on $[u<1]$ of the Generalized Richards Equation.

Further, replacing $T_n(u)$ by $(1-T_n(u))$ in the above arguments, we find the identities

$$\begin{aligned}\int_0^T\int_\Omega\Big(-c_n(u)\xi_t+k_w(u)(1-T_n(u))\nabla p\cdot\nabla\xi+|\nabla\psi_n(u)|^2\xi\Big)\\=\int_0^T\int_\Omega(\,\overline{s}-\theta\;\underline{s}\;1\!\!1_{[u=1]})(1-T_n(u))\xi+\int_\Omega c_n(u_0)\xi(\cdot,0).\end{aligned}\tag{33}$$

for $\xi\in\mathcal{D}([0,T)\times\overline{\Omega})$. Letting $n\to\infty$, we make vanish the terms containing $c_n(u)$ (see (28)); we also have $k_w(u)(1-T_n(u))\to k_w(1)1\!\!1_{[u=1]}=1\!\!1_{[u=1]}$, so that we find the equality of (30) for regular ξ; this equality can be extended by the density of $\mathcal{D}([0,T)\times\overline{\Omega})$ in $L^2(0,T;H^1(\Omega))$.

This concludes the proof. □

The renormalized formulation on $[u<1]$ has the advantage of being easily exploited. Indeed, the theory of degenerate *elliptic-parabolic* problems of the kind (29) is well established: we refer in particular to Alt and Luckhaus [1], Otto [10], Carrillo and Wittbold [3]. Thus u a weak solution in the sense of Definition 3.1 is also the unique solution of (29).

At this point, we use the idea of Igbida, Sbihi and Wittbold [7]: instead of working directly with the renormalized formulation, one takes advantage of the well-known properties of equation (29), and concludes by exploiting the constraint (30). In particular, the following L^1 contraction and order-preservation principle can be easily established following [10, 3] (the only difference with the known results is that here, we have the homogeneous Neumann boundary condition): for $u, \hat{u}$ associated with data $(u_0, \overline{s})$ and $(\hat{u}_0, \widehat{\overline{s}})$, respectively,

$$\text{for a.e. } t \in (0,T) \quad \int_\Omega (b_n(u) - b_n(\hat{u}))^+(t,\cdot) \le \int_\Omega (b(u_0) - b(\hat{u}_0))^+ \\ + \int_0^t \int_\Omega \operatorname{sgn}^+(b_n(u) - b_n(\hat{u})) \left(\overline{s}\, T_n(u) - \widehat{\overline{s}}\, T_n(\hat{u}) + |\nabla \psi_n(u)|^2 - |\nabla \psi_n(\hat{u})|^2 \right). \tag{34}$$

For the proof, either one can use the nonlinear semigroup theory, as in [3]; or, following the idea of [10], one uses the *doubling of variables in time*. Let us briefly recall the technique of Otto [10]. One writes $u = u(t,x)$ and $\hat{u} = \hat{u}(\tau, x)$. Taking $H_\alpha(\cdot)$ a Lipschitz approximation of $\operatorname{sgn}^+(\cdot)$, taking $\delta_\beta(\cdot)$ a smooth approximation of the Dirac mass, one uses $H_\alpha(\varphi_n(u(t,\cdot)) - \varphi_n(\hat{u}(\tau,\cdot)))\delta_\beta(t-\tau)$ as the test function in (29), written for both u and $\hat{u}$. The time evolution terms (in t and in τ) are treated using the generalization of the chain rule (21) (see [10, 3]). As $\alpha \to 0$, this technique makes appear the term $(b_n(u)(t,\cdot) - b_n(\hat{u})(\tau,\cdot))^+$ weighted by $\delta_\beta(t-\tau)$ and its derivatives; then, letting $\beta \to 0$, with the classical Kruzhkov techniques we identify t and τ and find (34).

Theorem 3.1 *Assume $u, \hat{u}$ are renormalized solutions of the Generalized Richards Equation corresponding to data $(u_0, \overline{s})$ and $(\hat{u}_0, \widehat{\overline{s}})$. Then we have the following* incomplete contraction inequality*:*

$$\text{for a.e. } t \in (0,T) \quad \int_\Omega (u - \hat{u})^+(t,\cdot) \le \int_\Omega (u_0 - \hat{u}_0)^+ + \int_0^t \int_\Omega sgn^+(u - \hat{u})(\overline{s} - \widehat{\overline{s}}) + \int_0^t \int_{[u=1=\hat{u}]} \overline{s}. \tag{35}$$

In particular, (35) *holds true for weak solutions of the Generalized Richards Equation corresponding to data $(u_0, \overline{s}, \underline{s})$ and $(\hat{u}_0, \widehat{\overline{s}}, \widehat{\underline{s}})$.*

Proof: We let $n \to \infty$ in (34). Using (27) and the fact that $T_n(u) \to 1\!\!1_{[u<1]}$ pointwise, we find

$$\int_\Omega (u - \hat{u})^+(t,\cdot) \le \int_\Omega (u_0 - \hat{u}_0)^+ \\ + \lim_{n\to\infty} \int_0^t \int_\Omega \operatorname{sgn}^+(b_n(u) - b_n(\hat{u})) \Big(\overline{s}\, T_n(u) - \widehat{\overline{s}}\, T_n(\hat{u}) + |\nabla \psi_n(u)|^2 - |\nabla \psi_n(\hat{u})|^2 \Big) \tag{36}$$

for a.e. $t \in (0,T)$. Let us treat the right-hand side term per term. The last one in non-positive, and we drop it. The last but one term in lower bounded by the left-hand side, and then also by the right-hand side of inequality (31). The integrands of the first two terms tend to $1\!\!1_{[\hat{u}<u<1]}\, \overline{s}$ and to $1\!\!1_{[\hat{u}<u]} \widehat{\overline{s}}$, respectively; we can use the dominated convergence theorem for these two terms. Eventually, we find

$$\int_\Omega (u - \hat{u})^+(t,\cdot) \le \int_\Omega (u_0 - \hat{u}_0)^+ + \int_0^t \int_{[\hat{u}<u<1]} \overline{s} - \int_0^t \int_{[\hat{u}<u<1]} \widehat{\overline{s}} + \int_0^t \int_{[u=1]} (\overline{s} - \theta\, \underline{s}) \\ \le \int_\Omega (u_0 - \hat{u}_0)^+ + \int_0^t \int_{[\hat{u}<u]} (\overline{s} - \widehat{\overline{s}}) - \int_0^t \int_{[\hat{u}<u=1]} \overline{s} + \int_0^t \int_{[u=1]} \overline{s}. \tag{37}$$

This inequality is precisely (35), which was to be proved.

The last claim of the Theorem follows readily. Indeed, from the existence results on Eymard, Henry, Hilhorst [6] and of the authors [2], taking into account the analysis of the beginning of this section we derive existence of a weak solution. Then by Proposition 3.1 we deduce uniqueness of a unique renormalized solution which is also the unique weak solution of the Generalized Richards Equation. □

Remark 3.5 *Clearly, the above technique for estimating $(u - \hat{u})^+$ does not look optimal: in particular, the passage from* (36) *to* (37) *includes several rather rough estimates. For instance, we have used only the weakest form* (31) *of the constraint* (30)*. For more subtle use of* (30) *(where the information on ∇p has to be eliminated anyway, because no sign could be given to this term), one can use it with test functions that are constant a.e. in the saturated region $[u = 1]$.*

Yet inequality (35) *remains the best of what we were able to prove with this kind of ideas.*

4 The no-source case: equivalence of Generalized and classical Richards equations

Although in general, the result of Theorem 3.1 does not yield uniqueness of a weak or a renormalized solution to the Generalized Richards Equation, it is immediate that under the *no-source assumption* $\overline{s} = 0$ the *incomplete contraction inequality* (35) becomes a true comparison property.

In this section, we show that not only this leads to well-posedness of the Generalized Richards Equation in the no-source case, but actually this model is equivalent to the classical Richards model without source.

Proposition 4.1 *Assume that* $\overline{s} = 0$ *a.e. on* $(0,T) \times \Omega$.[2]

Then for every initial datum u_0 *taking values in* $[u_m, 1]$ *there exists a unique* u *such that* (u, p, θ) *is a weak solution of the Generalized Richards Equation, and it is also its unique renormalized solution. Moreover, in this case we have* $\theta\, \underline{s} = 0$ *a.e. on* $(0,T) \times \Omega$*; and the saturation* u *of the solution coincides with the unique solution of the classical Richards equation* (1)*.*

Proof: The uniqueness claim (for u) in the case of data with $\overline{s} = 0$ is immediate from (35); existence was already established. In order to observe that we necessarily have $\theta\, \underline{s} = 0$ a.e. on $[u = 1]$, it is enough to use (31) which readily yields $\iint_{[u=1]} \theta\, \underline{s} \le 0$.

It remains to identify the saturation u with $\hat{u}$ the unique solution of the classical Richards equation (1). Indeed, let us recall that (1), if written in terms of the unknown function p, falls into the elliptic-parabolic framework of [1, 10, 3, 7]). Then, in absence of a source s_w, existence, uniqueness and comparison principle for variational solutions of (1) is well known, for arbitrary measurable data u_0 taking values in $[0, 1]$.

Now, let $(\hat{u}, \hat{p})$ be the solution of the classical Richards model with initial datum u_0 and the homogeneous Neumann boundary condition. Setting $\hat{\theta} = 0$, we readily see that the triple $(\hat{u}, \hat{p}, \hat{\theta})$ is a weak solution of the Generalized Richards Equation. Thus, by the uniqueness result above we do have $u = \hat{u}$. This ends the proof. □

Remark 4.1 *Making appeal to the Richards model, or to the comparison principle for solutions of the Generalized Richards Equation, we can extend the notion of weak solution to general data, for instance in the case where* $p_c(\cdot)$ *is bounded at zero. Indeed, in this case the last estimate on Lemma 3.1 allows to extend Definition 3.1 to general* $[0, 1]$*-valued data* u_0*; and existence can be established by truncating the data at levels* $u_m = \delta$ *going to zero, which leads to a monotone non-increasing sequence of saturations* u_δ *which clearly converges (see e.g. [7] and references therein for this technique of monotone approximations).*

Remark 4.2 *It should be noticed that, although we have not assumed that the sink (or water production) intensity* $\underline{s}$ *is zero, Proposition 4.1 shows that the effective production intensity* $\theta\, \underline{s}$ *remains zero if* $\overline{s} = 0$*.*

This feature of the model is somewhat disappointing, and it can be seen as an argument in favor of the complete two-phase model (2) *with large but finite air mobility* μ*. Compared to its singular limit, the two-phase model has this more realistic qualitative property: it allows for water production in saturated or close to saturation regions that correspond to the set* $[u :\ f_\mu(u)$ *is far enough from* $0]$*.*

5 Conclusions

We have presented a one-phase model originating as a singular limit of the two-phase flow, analyzed the notion of solution and investigated a uniqueness approach to this problem. It turns out that in the case without water injection in the media, there is well-posedness to the singular limit model, but also, the model actually coincides with the classical Richards one.

[2] recall that this corresponds to $\overline{s}1\!\!1_{[c=1]} = 0$, if we consider u as limit of the two-phase flow approximations with $c = c(t, x)$ the saturation in water of the injected fluid

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