MINIMA OF NON COERCIVE FUNCTIONALS

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Abstract. We study an integral non coercive functional defined on $H^1_0(\Omega)$, proving the existence of a minimum in $W^{1,1}_0(\Omega)$.

In this paper we study a class of integral functionals defined on $H^1_0(\Omega)$, but non coercive on the same space, so that the standard approach of the Calculus of Variations does not work. However, the functionals are coercive on $W^{1,1}_0(\Omega)$ and we will prove the existence of minima, despite the non reflexivity of $W^{1,1}_0(\Omega)$, which implies that, in general, the Direct Methods fail due to lack of compactness.

Let $J$ be the functional defined as

$$J(v) = \int_{\Omega} \frac{j(x, \nabla v)}{1 + b(x)|v|^2} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} f v, \quad v \in H^1_0(\Omega).$$

We assume that $\Omega$ is a bounded open set of $\mathbb{R}^N$, $N > 2$, that $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is such that $j(\cdot, \xi)$ is measurable on $\Omega$ for every $\xi$ in $\mathbb{R}^N$, $j(x, \cdot)$ is convex and belongs to $C^1(\mathbb{R}^N)$ for almost every $x$ in $\Omega$, and

$$\alpha |\xi|^2 \leq j(x, \xi) \leq \beta |\xi|^2,$$

(2)

$$|j_\xi(x, \xi)| \leq \gamma |\xi|,$$

for some positive $\alpha$, $\beta$ and $\gamma$, for almost every $x$ in $\Omega$, and for every $\xi$ in $\mathbb{R}^N$. We assume that $b$ is a measurable function on $\Omega$ such that

$$0 \leq b(x) \leq B,$$

(3)

for almost every $x$ in $\Omega$, where $B > 0$, while $f$ belongs to some Lebesgue space. For $k > 0$ and $s \in \mathbb{R}$, we define the truncation function as $T_k(s) = \max(-k, \min(s, k))$.

In [3] the minimization in $H^1_0(\Omega)$ of the functional

$$I(v) = \int_{\Omega} \frac{j(x, \nabla v)}{1 + |v|^2} - \int_{\Omega} f v, \quad 0 < \theta < 1, \quad f \in L^m(\Omega),$$

was studied. It was proved that $I(v)$ is coercive on the Sobolev space $W^{1,q}_0(\Omega)$, for some $q = q(\theta, m)$ in (1,2), and that $I(v)$ achieves its minimum on $W^{1,q}_0(\Omega)$. This approach does not work for $\theta > 1$ (see Remark 7 below). Here we will be able to overcome this difficulty thanks to the presence of the lower order term $\int_{\Omega} |v|^2$, which will yield the coercivity of $J$ on $W^{1,1}_0(\Omega)$; then we will prove the existence of minima in $W^{1,1}_0(\Omega)$, even if it is a non reflexive space.

Integral functionals like $J$ or $I$ are studied in [1], in the context of the Thomas-Fermi-von Weizsäcker theory.
We are going to prove the following result.

**Theorem 1.** Let $f \in L^2(\Omega)$. Then there exists $u$ in $W_0^{1,1}(\Omega) \cap L^2(\Omega)$ minimum of $J$, that is,

$$\int_\Omega j(x, \nabla u) \frac{1}{[1 + b(x)|u|^2]} + \frac{1}{2} \int_\Omega |u|^2 - \int_\Omega f u \leq \int_\Omega j(x, \nabla v) \frac{1}{[1 + b(x)|v|^2]} + \frac{1}{2} \int_\Omega |v|^2 - \int_\Omega f v,$$

for every $v \in H^1_0(\Omega)$. Moreover $T_k(u)$ belongs to $H^1_0(\Omega)$ for every $k > 0$.

In [2] we studied the following elliptic boundary problem:

$$\begin{cases}
-\text{div} \left( \frac{a(x) \nabla u}{1 + b(x)|u|^2} \right) + u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

under the same assumptions on $\Omega$, $b$ and $f$, with $0 < \alpha \leq a(x) \leq \beta$. It is easy to see that the Euler equation of $J$, with $j(x, \xi) = \frac{1}{2} a(x)|\xi|^2$, is not equation (5). Therefore Theorem 1 cannot be deduced from [2]. Nevertheless some technical steps of the two papers (for example, the a priori estimates) are similar.

We will prove Theorem 1 by approximation. Therefore, we begin with the case of bounded data.

**Lemma 2.** If $g$ belongs to $L^\infty(\Omega)$, then there exists a minimum $w$ belonging to $H^1_0(\Omega) \cap L^\infty(\Omega)$ of the functional

$$v \in H^1_0(\Omega) \mapsto \int_\Omega j(x, \nabla v) \frac{1}{[1 + b(x)|v|^2]} + \frac{1}{2} \int_\Omega |v|^2 - \int_\Omega g v.$$

**Proof.** Since the functional is not coercive on $H^1_0(\Omega)$, we cannot directly apply the standard techniques of the Calculus of Variations. Therefore, we begin by approximating it. Let $M > 0$, and let $J_M$ be the functional defined as

$$J_M(v) = \int_\Omega j(x, \nabla v) \frac{1}{[1 + b(x)|T_M(v)|]} + \frac{1}{2} \int_\Omega |T_M(v)|^2 - \int_\Omega g v, \quad v \in H^1_0(\Omega).$$

Since $J_M$ is both weakly lower semicontinuous (due to the convexity of $j$ and to De Giorgi’s theorem, see [4]) and coercive on $H^1_0(\Omega)$, for every $M > 0$ there exists a minimum $w_M$ of $J_M$ on $H^1_0(\Omega)$. Let $A = \|g\|_{L^\infty(\Omega)}$, let $M > A$, and consider the inequality $J_M(w_M) \leq J_M(T_A(w_M))$, which holds true since $w_M$ is a minimum of $J_M$. We have

$$\int_\Omega j(x, \nabla w_M) \frac{1}{[1 + b(x)|T_M(w_M)|]} + \frac{1}{2} \int_\Omega |w_M|^2 - \int_\Omega g w_M \leq \int_\Omega j(x, \nabla T_A(w_M)) \frac{1}{[1 + b(x)|T_M(T_A(w_M))]|} + \frac{1}{2} \int_\Omega |T_A(w_M)|^2 - \int_\Omega g T_A(w_M) \leq \int_\Omega j(x, \nabla w_M) \frac{1}{[1 + b(x)|T_M(w_M)|]} + \frac{1}{2} \int_\Omega |T_A(w_M)|^2 - \int_\Omega g T_A(w_M),$$
where, in the last passage, we have used that $T_M(T_A(w_M)) = T_M(w_M)$ on the set \{|w_M| \leq A\}, and that $j(x,0) = 0$. Simplifying equal terms, we thus get

$$\int_{\{|w_M| > M\}} \frac{j(x, \nabla w_M)}{[1 + b(x)] T_M(w_M)]^2} + \frac{1}{2} \int_{\Omega} \{[w_M]^2 - [T_A(w_M)]^2\} \leq \int_{\Omega} g \{w_M - T_A(w_M)\}.$$ 

Dropping the first term, which is nonnegative, we obtain

$$\frac{1}{2} \int_{\Omega} \{w_M - T_A(w_M)\} [w_M + T_A(w_M)] \leq \int_{\Omega} g \{w_M - T_A(w_M)\},$$

which can be rewritten as

$$\frac{1}{2} \int_{\Omega} \{w_M - T_A(w_M)\} [w_M + T_A(w_M) - 2g] \leq 0.$$ 

We then have, since $w_M = T_A(w_M)$ on the set \{|w_M| \leq A\},

$$\frac{1}{2} \int_{\{w_M > A\}} \{w_M - A\} [w_M + A - 2g] + \frac{1}{2} \int_{\{w_M < A\}} \{w_M + A\} [w_M - A - 2g] \leq 0.$$ 

Since $|g| \leq A$, we have $A - 2g \geq -A$, and $-A - 2g < A$, so that

$$0 \leq \frac{1}{2} \int_{\{w_M > A\}} \{w_M - A\}^2 + \frac{1}{2} \int_{\{w_M < A\}} \{w_M + A\}^2 \leq 0,$$

which then implies that $\text{meas}(\{|w_M| \geq A\}) = 0$, and so $|w_M| \leq A$ almost everywhere in $\Omega$. Recalling the definition of $A$, we thus have

$$\|w_M\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}. \tag{6}$$ 

Since $M > \|g\|_{L^\infty(\Omega)}$, we thus have $T_M(w_M) = w_M$. Starting now from $J_M(w_M) \leq J_M(0) = 0$ we obtain, by (6),

$$\int_{\Omega} \frac{j(x, \nabla w_M)}{[1 + b(x)] |w_M|^2} + \frac{1}{2} \int_{\Omega} |w_M|^2 \leq \int_{\Omega} g w_M \leq \text{meas}(\Omega) \|g\|_{L^\infty(\Omega)}^2,$$

which then implies, by (1) and (3), and dropping the nonnegative second term,

$$\frac{\alpha}{[1 + B \|g\|_{L^\infty(\Omega)}]^2} \int_{\Omega} |\nabla w_M|^2 \leq \text{meas}(\Omega) \|g\|_{L^\infty(\Omega)}^2.$$ 

Thus, \{w_M\} is bounded in $H^1_0(\Omega) \cap L^\infty(\Omega)$, and so, up to subsequences, it converges to some function $w$ in $H^1_0(\Omega) \cap L^\infty(\Omega)$ weakly in $H^1_0(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in $\Omega$. We prove now that

$$\int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)] |w|^2} \leq \liminf_{M \to +\infty} \int_{\Omega} \frac{j(x, \nabla w_M)}{[1 + b(x)] |w_M|^2}. \tag{7}$$
Indeed, since \( j \) is convex, we have
\[
\int_{\Omega} \frac{j(x, \nabla w_M)}{[1 + b(x)|w_M|^2]} \geq \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)|w_M|^2]} - \int_{\Omega} \frac{j(\xi, \nabla w)}{[1 + b(x)|w_M|^2]} \cdot \nabla [w_M - w].
\]

Using assumption (1), the fact that \( w \) belongs to \( H^1_0(\Omega) \), the almost everywhere convergence of \( w_M \) to \( w \) and Lebesgue’s theorem, we have
\[
\lim_{M \to +\infty} \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)|w_M|^2]} = \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)|w|^2]}.
\]

Using (8) and (9), we have that (7) holds true. On the other hand, since \( \nabla w_M \) tends to \( \nabla w \) weakly in the same space, we thus have that
\[
\lim_{M \to +\infty} \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)|w_M|^2]} \cdot \nabla [w_M - w] = 0.
\]

Using (8) and (9), we have that (7) holds true. On the other hand, using (1) and Lebesgue’s theorem again, it is easy to see that
\[
\lim_{M \to +\infty} \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|T_M(v)|]^2} = \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|v|^2]} \quad \forall v \in H^1_0(\Omega).
\]

Thus, starting from \( J_M(w_M) \leq J_M(v) \), we can pass to the limit as \( M \) tends to infinity (using also the strong convergence of \( w_M \) to \( w \) in \( L^2(\Omega) \)), to have that \( w \) is a minimum.

As stated before, we prove Theorem 1 by approximation. More in detail, if \( f_n = T_n(f) \) then Lemma 2 with \( g = f_n \) implies that there exists a minimum \( u_n \) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \) of the functional
\[
J_n(v) = \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|v|^2]} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} f_n v, \quad v \in H^1_0(\Omega).
\]

In the following lemma we prove some uniform estimates on \( u_n \).

**Lemma 3.** Let \( u_n \) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \) be a minimum of \( J_n \). Then
\[
\int_{\Omega} \frac{|
abla u_n|^2}{[1 + b(x)|u_n|^2]} \leq \frac{1}{2\alpha} \int_{\Omega} |f|^2;
\]
\[
\int_{\Omega} |\nabla T_k(u_n)|^2 \leq \frac{(1 + B k)^2}{2\alpha} \int_{\Omega} |f|^2;
\]
\[
\int_{\Omega} |u_n|^2 \leq 4 \int_{\Omega} |f|^2.
\]
\[
\int_{\Omega} |\nabla u_n| \leq \left[ \frac{1}{2\alpha} \int_{\Omega} |f|^2 \right]^{\frac{1}{2}} \left( \text{meas}(\Omega)^{\frac{1}{2}} + 2B \left[ \int_{\Omega} |f|^2 \right]^{\frac{1}{2}} \right);
\]

(14) \[
\int_{\Omega} |G_k(u_n)|^2 \leq 4 \int_{\{u_n \geq k\}} |f|^2,
\]
where \( G_k(s) = s - T_k(s) \) for \( k \geq 0 \) and \( s \) in \( \mathbb{R} \).

**Proof.** The minimality of \( u_n \) implies that \( J_n(u_n) \leq J_n(0) \), that is,

(15) \[
\int_{\Omega} \frac{j(x, \nabla u_n)}{1 + b(x)|u_n|^2} + \frac{1}{2} \int_{\Omega} u_n^2 \leq \int_{\Omega} f_n u_n.
\]

Using (1) on the left hand side, and Young’s inequality on the right hand side gives

\[
\alpha \int_{\Omega} \frac{|\nabla u_n|^2}{1 + b(x)|u_n|^2} + \frac{1}{2} \int_{\Omega} u_n^2 \leq \frac{1}{2} \int_{\Omega} u_n^2 + \frac{1}{2} \int_{\Omega} f_n^2,
\]

which then implies (10). Let now \( k \geq 0 \). The above estimate, and (3), give

\[
\frac{1}{(1 + Bk)^2} \int_{\Omega} |\nabla T_k(u_n)|^2 \leq \int_{\{u_n \leq k\}} \frac{|\nabla u_n|^2}{1 + b(x)|u_n|^2} \leq \frac{1}{2\alpha} \int_{\Omega} |f|^2,
\]

and therefore (11) is proved. On the other hand, dropping the first positive term in (15) and using Hölder’s inequality on the right hand side, we have

\[
\frac{1}{2} \int_{\Omega} |u_n|^2 \leq \int_{\Omega} |f_n u_n| \leq \left[ \int_{\Omega} |f_n|^2 \right]^{\frac{1}{2}} \left[ \int_{\Omega} |u_n|^2 \right]^{\frac{1}{2}},
\]

that is, (12) holds. Hölder’s inequality, assumption (3), and estimates (10) and (12) give (13):

(16) \[
\int_{\Omega} |\nabla u_n| \leq \left[ \int_{\Omega} \frac{|\nabla u_n|^2}{[1 + b(x)|u_n|^2]} \right]^{\frac{1}{2}} \left[ \int_{\Omega} [1 + b(x)|u_n|^2]^{\frac{1}{2}} \right]^{\frac{1}{2}} \\
\leq \left[ \frac{1}{2\alpha} \int_{\Omega} |f|^2 \right]^{\frac{1}{2}} \left( \text{meas}(\Omega)^{\frac{1}{2}} + 2B \left[ \int_{\Omega} |f|^2 \right]^{\frac{1}{2}} \right).
\]

We are left with estimate (14). Since \( J_n(u_n) \leq J_n(T_k(u_n)) \) we have

\[
\frac{1}{2} \int_{\Omega} \frac{j(x, \nabla u_n)}{1 + b(x)|u_n|^2} + \frac{1}{2} \int_{\Omega} |u_n|^2 - \int_{\Omega} f_n u_n
\]

\[
\leq \frac{1}{2} \int_{\Omega} \frac{j(x, \nabla T_k(u_n))}{1 + b(x)|T_k(u_n)|^2} + \frac{1}{2} \int_{\Omega} |T_k(u_n)|^2 - \int_{\Omega} f_n T_k(u_n).
\]

Recalling the definition of \( G_k(s) \), and using that \( |s|^2 - |T_k(s)|^2 \geq |G_k(s)|^2 \), the last inequality implies

\[
\frac{1}{2} \int_{\Omega} \frac{j(x, \nabla G_k(u_n))}{1 + b(x)|u_n|^2} + \frac{1}{2} \int_{\Omega} |G_k(u_n)|^2 \leq \int_{\Omega} f_n G_k(u_n).
\]
Dropping the first term of the left hand side and using Hölder’s inequality on the right one, we obtain
\[ \frac{1}{2} \int_\Omega |G_k(u_n)|^2 \leq \left[ \int_{\{ |u_n| \geq k \}} |f|^2 \right]^\frac{1}{2} \left[ \int_\Omega |G_k(u_n)|^2 \right]^\frac{1}{2}, \]
that is, (14) holds.

\[ \text{Lemma 4.} \] Let \( u_n \) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \) be a minimum of \( J_n \). Then there exists a subsequence, still denoted by \( \{ u_n \} \), and a function \( u \) in \( W^{1,1}_0(\Omega) \cap L^2(\Omega) \), with \( T_k(u) \) in \( H^1_0(\Omega) \) for every \( k > 0 \), such that \( u_n \) converges to \( u \) almost everywhere in \( \Omega \), strongly in \( L^2(\Omega) \) and weakly in \( W^{1,1}_0(\Omega) \), and \( T_k(u_n) \) converges to \( T_k(u) \) weakly in \( H^1_0(\Omega) \). Moreover,
\[ \lim_{n \to +\infty} \frac{\nabla u_n}{1 + b(x)|u_n|} = \frac{\nabla u}{1 + b(x)|u|} \quad \text{weakly in} \quad (L^2(\Omega))^N. \]

\[ \text{Proof.} \] By (13), the sequence \( u_n \) is bounded in \( W^{1,1}_0(\Omega) \). Therefore, it is relatively compact in \( L^1(\Omega) \). Hence, up to subsequences still denoted by \( u_n \), there exists \( u \) in \( L^1(\Omega) \) such that \( u_n \) almost everywhere converges to \( u \). From Fatou’s lemma applied to (12) we then deduce that \( u \) belongs to \( L^2(\Omega) \).

We are going to prove that \( u_n \) strongly converges to \( u \) in \( L^2(\Omega) \). Let \( E \) be a measurable subset of \( \Omega \); then by (14) we have
\[ \int_E |u_n|^2 \leq 2 \int_E |T_k(u_n)|^2 + 2 \int_E |G_k(u_n)|^2 \leq 2k^2 \text{meas}(E) + 2 \int_\Omega |G_k(u_n)|^2 \leq 2k^2 \text{meas}(E) + 8 \int_{\{ |u_n| \geq k \}} |f|^2. \]
Since \( u_n \) is bounded in \( L^2(\Omega) \) by (12), we can choose \( k \) large enough so that the second integral is small, uniformly with respect to \( n \); once \( k \) is chosen, we can choose the measure of \( E \) small enough such that the first term is small. Thus, the sequence \( \{ u^2_n \} \) is equiintegrable and so, by Vitali’s theorem, \( u_n \) strongly converges to \( u \) in \( L^2(\Omega) \).

Now we to prove that \( u_n \) weakly converges to \( u \) in \( W^{1,1}_0(\Omega) \). Let \( E \) be a measurable subset of \( \Omega \). By Hölder’s inequality, assumption (3), and (10), one has, for \( i \in \{1, \ldots, N\} \),
\[ \int_E \left| \frac{\partial u_n}{\partial x_i} \right| \leq \int_E |\nabla u_n| \leq \left[ \int_E \frac{|\nabla u_n|^2}{1 + b(x)|u_n|^2} \right]^\frac{1}{2} \left[ \int_\Omega [1 + b(x)|u_n|]^2 \right]^\frac{1}{2} \leq \left[ \frac{1}{2^\alpha} \int_\Omega |f|^2 \right]^\frac{1}{2} \left[ \int_\Omega [1 + B|u_n|]^2 \right]^\frac{1}{2}. \]
Since the sequence \( \{ u_n \} \) is compact in \( L^2(\Omega) \), this estimate implies that the sequence \( \{ \frac{\partial u_n}{\partial x_i} \} \) is equiintegrable. Thus, by Dunford-Pettis
theorem, and up to subsequences, there exists \( Y_i \in L^1(\Omega) \) such that \( \frac{\partial u_n}{\partial x_i} \) weakly converges to \( Y_i \) in \( L^1(\Omega) \). Since \( \frac{\partial u_n}{\partial x_i} \) is the distributional partial derivative of \( u_n \), we have, for every \( n \in \mathbb{N} \),

\[
\int_{\Omega} \frac{\partial u_n}{\partial x_i} \varphi = -\int_{\Omega} u_n \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega).
\]

We now pass to the limit in the above identities, using that \( \partial_i u_n \) weakly converges to \( Y_i \) in \( L^1(\Omega) \), and that \( u_n \) strongly converges to \( u \) in \( L^2(\Omega) \): we obtain

\[
\int_{\Omega} Y_i \varphi = -\int_{\Omega} u \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega).
\]

This implies that \( Y_i = \frac{\partial u}{\partial x_i} \), and this result is true for every \( i \). Since \( Y_i \) belongs to \( L^1(\Omega) \) for every \( i \), \( u \) belongs to \( W^{1,1}_0(\Omega) \), as desired.

Since by (11) it follows that the sequence \( \{T_k(u_n)\} \) is bounded in \( H^1_0(\Omega) \), and since \( u_n \) tends to \( u \) almost everywhere in \( \Omega \), then \( T_k(u_n) \) weakly converges to \( T_k(u) \) in \( H^1_0(\Omega) \), and \( T_k(u) \) belongs to \( H^1_0(\Omega) \) for every \( k \geq 0 \).

Finally, we prove (17). Let \( \Phi \) be a fixed function in \( (L^\infty(\Omega))^N \). Since \( u_n \) almost everywhere converges to \( u \) in \( \Omega \), we have

\[
\lim_{n \to +\infty} \frac{\Phi}{1 + b(x)|u_n|} = \frac{\Phi}{1 + b(x)|u|} \quad \text{almost everywhere in } \Omega.
\]

By Egorov’s theorem, the convergence is therefore quasi uniform; i.e., for every \( \delta > 0 \) there exists a subset \( E_\delta \) of \( \Omega \), with \( \text{meas}(E_\delta) < \delta \), such that

\[
\lim_{n \to +\infty} \frac{\Phi}{1 + b(x)|u_n|} = \frac{\Phi}{1 + b(x)|u|} \quad \text{uniformly in } \Omega \setminus E_\delta.
\]

We now have

\[
\left| \int_{\Omega} \frac{\nabla u_n}{1 + b(x)|u_n|} \cdot \Phi - \int_{\Omega} \frac{\nabla u}{1 + b(x)|u|} \cdot \Phi \right| \\
\leq \int_{\Omega \setminus E_\delta} \frac{\nabla u_n}{1 + b(x)|u_n|} \cdot \Phi - \int_{\Omega \setminus E_\delta} \nabla u \cdot \frac{\Phi}{1 + b(x)|u|} \\
+ \|\Phi\|_{L^\infty(\Omega)} \int_{E_\delta} ||\nabla u_n| + |\nabla u||.
\]

Using the equiintegrability of \( |\nabla u_n| \) proved above, and the fact that \( |\nabla u| \) belongs to \( L^1(\Omega) \), we can choose \( \delta \) such that the second term of the right hand side is arbitrarily small, uniformly with respect to \( n \), and then use (18) to choose \( n \) large enough so that the first term is arbitrarily small. Hence, we have proved that

\[
\lim_{n \to +\infty} \frac{\nabla u_n}{1 + b(x)|u_n|} = \frac{\nabla u}{1 + b(x)|u|} \quad \text{weakly in } (L^1(\Omega))^N.
\]

On the other hand, from (10) it follows that the sequence \( \frac{\nabla u_n}{1 + b(x)|u_n|} \) is bounded in \( (L^2(\Omega))^N \), so that it weakly converges to some function \( \sigma \).
in the same space. Since (19) holds, we have that $\sigma = \frac{\nabla u}{1 + b(x)u}$, and (17) is proved.

Remark 5. The fact that we need to prove (17) is one of the main differences with the paper [2].

Proof of Theorem 1. Let $u_n$ be as in Lemma 4. The minimality of $u_n$ implies that

$$\int_{\Omega} j(x, \nabla u_n) \leq \int_{\Omega} j(x, \nabla v) + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} f_n v$$

for every $v$ in $H_0^1(\Omega)$. The result will then follow by passing to the limit in the previous inequality. The right hand side of (20) is easy to handle since $f_n$ converges to $f$ in $L^2(\Omega)$. Let us study the limit of the left hand side of (20). The convexity of $j$ implies that

$$\int_{\Omega} \frac{j(x, \nabla u_n)}{1 + b(x)|u_n|^2} \geq \int_{\Omega} \frac{j(x, \nabla T_k(u))}{1 + b(x)|u|^2}$$

$$- \int_{\Omega} \frac{j_k(x, \nabla T_k(u))}{1 + b(x)|u_n|} \cdot \left( \frac{\nabla u_n}{1 + b(x)|u_n|} - \frac{\nabla T_k(u)}{1 + b(x)|u|} \right).$$

By (17), assumptions (1) and (2), and Lebesgue’s theorem, we have

$$\liminf_{n \to +\infty} \int_{\Omega} \frac{j(x, \nabla u_n)}{1 + b(x)|u_n|^2} \geq \int_{\Omega} \frac{j(x, \nabla T_k(u))}{1 + b(x)|u|^2}$$

$$- \int_{\Omega} \frac{j_k(x, \nabla T_k(u))}{1 + b(x)|u|} \cdot \frac{\nabla [u - T_k(u)]}{1 + b(x)|u|},$$

that is, since $j_k(x, \nabla T_k(u)) \cdot \nabla (u - T_k(u)) = 0$,

$$\int_{\Omega} \frac{j(x, \nabla T_k(u))}{1 + b(x)|u|^2} \leq \liminf_{n \to +\infty} \int_{\Omega} \frac{j(x, \nabla u_n)}{1 + b(x)|u_n|^2}.$$
so that \( u \) is a minimum of \( J \); its regularity has been proved in Lemma 4.

**Remark 6.** If we suppose that the coefficient \( b(x) \) satisfies the stronger assumption

\[
0 < A \leq b(x) \leq B , \quad \text{almost everywhere in } \Omega ,
\]

it is possible to prove that \( J(u) \leq J(w) \) not only for every \( w \) in \( H^1_0(\Omega) \), but also for the test functions \( w \) such that

\[
\begin{align*}
T_k(w) & \text{ belongs to } H^1_0(\Omega) \text{ for every } k > 0 , \\
\log(1 + A |w|) & \text{ belongs to } H^1_0(\Omega) , \\
w & \text{ belongs to } L^2(\Omega).
\end{align*}
\]

Indeed, if \( w \) is as in (22), we can use \( T_k(w) \) as test function in (4) and we have

\[
J(u) \leq J(T_k(w)) = \int_{\Omega} j(x, \nabla T_k(w)) + \frac{1}{2} \int_{\Omega} |T_k(w)|^2 - \int_{\Omega} f T_k(w).
\]

In the right hand side is possible to pass to the limit, as \( k \) tends to infinity, so that we have \( J(u) \leq J(w) \), for every test function \( w \) as in (22).

**Remark 7.** We explicitly point out the differences, concerning the coercivity, between the functionals studied in [3] and the functionals studied in this paper. Indeed, let \( 0 < \rho < \frac{N-2}{2} \), and consider the sequence of functions

\[
v_n = \exp \left[ T_n \left( \frac{1}{|x|^\rho} - 1 \right) \right] - 1 ,
\]

defined in \( \Omega = B_1(0) \). Then

\[
\log(1 + |v_n|) = T_n \left( \frac{1}{|x|^\rho} - 1 \right) ,
\]

is bounded in \( H^1_0(\Omega) \) (since the function \( v(x) = \frac{1}{|x|^\rho} - 1 \) belongs to \( H^1_0(\Omega) \) by the assumptions on \( \rho \)), but, by Levi’s theorem,

\[
\lim_{n \to +\infty} \int_{\Omega} |\nabla v_n| = \rho \int_{\Omega} \frac{\exp \left[ \frac{1}{|x|^\rho} - 1 \right]}{|x|^{\rho+1}} = +\infty .
\]

Hence, the functional

\[
v \in H^1_0(\Omega) \mapsto \int_{\Omega} \frac{|\nabla v|^2}{(1 + |v|)^2} = \int_{\Omega} |\nabla \log(1 + |v|)|^2 ,
\]

which is of the type studied in [3], is non coercive on \( W^{1,1}_0(\Omega) \). On the other hand, recalling (16), we have

\[
\int_{\Omega} |\nabla v| = \int_{\Omega} \frac{|\nabla v|}{1 + |v|} (1 + |v|) \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(1 + |v|)^2} + \frac{1}{2} \int_{\Omega} (1 + |v|)^2 .
\]
Thus, the functional

\[ v \in H^1_0(\Omega) \mapsto \int_{\Omega} |\nabla v|^2 + \frac{1}{2} \int_{\Omega} \left( \frac{1}{1 + |v|} \right)^2 |v|^2, \]

which is of the type studied here, is coercive on \( W^{1,1}_0(\Omega) \).

References


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