The non-perturbative analytical equation of state for the gluon matter: I

V Gogokhia, M Vasúth

The Non-Perturbative Analytical Equation of State for the Gluon Matter. I

V. Gogokhia and M. Vasúth

HAS, CRIP, RMKI, Depart. Theor. Phys., Budapest 114, P.O.B. 49, H-1525, Hungary

E-mail: gogohia@rmki.kfki.hu, vasuth@rmki.kfki.hu

Abstract. The effective potential approach for composite operators is generalized to non-zero temperatures in order to derive the equation of state for pure $SU(3)$ Yang-Mills fields. In the absence of external sources, this is nothing but the vacuum energy density. The key element of this derivation is the introduction of a temperature dependence into the expression for the bag constant. The non-perturbative analytical equation of state for gluon matter does not depend on the coupling constant, but instead introduces a dependence on the mass gap. This is responsible for the large-scale structure of the QCD ground state. The important thermodynamic quantities, such as the pressure, energy and entropy densities, etc., have been calculated. We show explicitly that the pressure may vary smoothly around $T_c = 266.5$ MeV, whereas all other thermodynamic quantities undergo drastic changes at this point. The proposed analytical approach makes it possible to control for the first time the thermodynamics of gluon matter at low temperatures, below T_c. We reproduce the properties of the so-called "fuzzy bag"-type models through the presence of the mass gap in our equation of state. An analytic calculation of the gluon condensate is obtained as a function of temperature.

PACS numbers: 11.10.Wx, 12.38.Mh, 12.38.Lg, 12.38.Aw

1. Introduction

The prediction of a possible existence of the quark-gluon plasma (QGP) created in the relativistic heavy ion collisions is one of the most interesting theoretical achievements of Quantum Chromodynamics (QCD) at non-zero temperatures and densities. A fairly full list of the relevant pioneering papers is given in Refs. [1, 2, 3]) and the present status of the investigations of the properties of QCD matter is described in Refs. [4, 5].

The equation of state (EoS) for the QGP has been derived analytically up to the order $g^6 \ln(1/g^2)$ by using the perturbation theory (PT) expansion for the evaluation of the corresponding thermodynamic potential term by term ([6, 7] and references therein). However, the most characteristic feature of the thermal PT expansion is its non-analytical dependence on the coupling constant g^2, which means that PT QCD is not applicable at finite temperatures. The problem is not the poor convergence of this series [6, 7, 8, 9] but rather the fact that a radius of convergence cannot even be defined; any next calculated term can be bigger than the previous one. This is an in-principle problem which cannot be overcome. From the strictly mathematical point of view, four-dimensional QCD at non-zero temperatures effectively becomes a three-dimensional theory. At the same time, three-dimensional QCD has more severe infrared singularities [10] and its coupling constant becomes dimensional. It is as a consequence of this that the dependence becomes non-analytical when using the dimensionless coupling constant g^2. One also needs to introduce three different scales, T, gT and g^2T, where T is the temperature, in order to try somehow to understand the dynamics of the QGP within the thermal PT QCD approach.

At present, the only practical method to investigate the problem is lattice QCD at finite temperature and baryon density, which has recently shown rapid progress ([4, 5, 11, 12, 13, 14, 15] and references therein). However, lattice QCD, being a very specific regularization scheme, is primary aimed at obtaining well-defined corresponding expressions in order to get realistic numbers for physical quantities. One may therefore get numbers and curves without understanding what the physics is behind them. Such understanding can only come from the dynamical theory, which is continuous QCD. For example, any description of the QGP has to be formulated within the framework of a dynamical theory. The need for an analytical EoS remains, but, of course, it should be essentially non-perturbative (NP), approaching the so-called Stefan-Boltzmann (SB) limit only at very hight temperatures. Thus the approaches of analytic NP QCD and lattice QCD to finite-temperature QCD do not exclude each other, but, on the contrary, should be complementary. This is especially true at low temperatures where the thermal QCD lattice calculations suffer from big uncertainties [4, 5, 11, 12, 13, 14, 15]. On the other hand, any analytic NP approach has to correctly reproduce thermal lattice QCD results at high temperatures (see papers cited above). There already exist interesting analytical approaches based on quasi-particle picture [16, 17, 18, 19, 20, 21, 22, 23, 24, 25] (and references therein) to analyze results of $SU(3)$ lattice QCD calculations for the QGP EoS.

The main purpose of this paper is to derive the NP analytical EoS for the gluon matter (GM), i.e., a system consisting purely of Yang-Mills (YM) fields without quark degrees of freedom. The formalism we use to generalize it to non-zero temperatures is the effective potential approach for composite operators [26]. In the absence of external sources it is nothing but the vacuum energy density (VED). The approach is NP from the very beginning, since it deals with the expansion of the corresponding skeleton vacuum loop diagrams. The key element is the extension of our initial work [27] to non-zero temperatures. This makes it possible to introduce the temperature-dependent bag constant (pressure) as a function of the mass gap. It is this which is responsible for the large-scale structure of the QCD ground state. The confining dynamics in the GM will therefore be nontrivially taken into account directly through the mass gap and via the temperature-dependent bag constant itself, but other NP effects will be also present. Let us note that the temperature-dependent bag constant within the thermal PT QCD has been introduced into the Gibbs equilibrium criteria for a phase transition [28] (see also Ref. [16]), and the effective potential approach has been already used in order to study the structure of QCD at very large baryon density for an arbitrary number of flavors [29].

The present paper is organized as follows. In section 2 the effective potential approach for composite operators is discussed in general terms. An explicit expression is obtained for the VED for pure YM fields. In section 3 the expression for the gluon pressure is derived taking into account the expression for the bag constant at zero temperature as a function of the mass gap [27]. Taken together this makes it possible to derive a formula for the pressure that is suitable for the generalization to non-zero temperatures. This consists of the two independent parts, describing the NP and PT contributions to the gluon pressure. In section 4 the generalization to non-zero temperatures is performed using the imaginary-time formalism. All the analytic results for the gluon pressure as a function of temperature are collected together in section 5. In section 6, expressions are given for the main thermodynamic quantities, such as the entropy and energy densities, the heat capacity, etc., as functions of the pressure. In section 7 we discuss all the numerical results for the NP part of the gluon pressure. Section 8 concludes our discussion. Some explicit expressions for the summation of the thermal logarithms over the Matsubara frequencies are present in appendix A, while in appendix B a scale-setting scheme of our calculations is formulated.

2. The Vacuum Energy Density

The quantum part of the VED is determined by the effective potential approach for composite operators [26]. It is given in the form of the skeleton loops expansion, containing all the types of the QCD full propagators and vertices (for its pictorial representation, see Ref. [27]). Each vacuum skeleton loop is itself a sum of an infinite number of the corresponding PT vacuum loops (i.e., containing the point-like vertices and free propagators). Thus the effective potential approach makes it possible to calculate the VED from first principles, i.e., using only fundamental constituents of QCD: gluons and quarks and their interactions or, more precisely, their propagators and vertices of their interactions. The number of the vacuum skeleton loops is equal to the power of the Planck constant, $\hbar$, so that a series expansion for the effective potential is nothing other than the semiclassical WKB loop expansion, which, in general, is an asymptotic series. It has been widely used in quantum field theory [30]. In QCD the instantons have been discovered by using just this method [31] (and references therein).

It is instructive to begin with some general expressions at zero temperature in this formalism. The gluon part of the VED to leading order (the so-called log-loop level $\sim \hbar$, whose infinite series is shown in Fig. 1) is analytically given by the effective potential for composite operators [26] as follows:

$$V(D) = \frac{i}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} Sp\{\ln(D_0^{-1}D) - (D_0^{-1}D) + 1\}, \tag{1}$$

where $D(q)$ is the full gluon propagator and $D_0(q)$ is its free counterpart. The traces over space-time and color group indices are assumed. It is clear from this that the effective potential is normalized to the free PT vacuum to be zero, i.e., $V(D_0) = 0$. We note that the YM bag constant has been calculated to this order in Ref. [27].

The two-point Green's function, describing the full gluon propagator, is

$$D_{\mu\nu}(q) = -i \left\{ T_{\mu\nu}(q) d(-q^2;\xi) + \xi L_{\mu\nu}(q) \right\} \frac{1}{q^2}, \tag{2}$$

where ξ is the gauge-fixing parameter and

$$T_{\mu\nu}(q) = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} = g_{\mu\nu} - L_{\mu\nu}(q). \tag{3}$$

Its free counterpart $D_0 \equiv D^0_{\mu\nu}(q)$ is obtaining by replacing the full gluon propagator's Lorentz structure $d(-q^2;\xi)$ in Eq. (2) by unity, i.e.,

$$D^0_{\mu\nu}(q) = -i \left\{ T_{\mu\nu}(q) + \xi L_{\mu\nu}(q) \right\} \frac{1}{q^2}. \tag{4}$$

In order to evaluate the effective potential (1) with Eq. (2), we use the well-known identity [32]

$$Sp \ln(D_0^{-1}D) = 8 \times 4 \ln det(D_0^{-1}D) = 32 \ln[(3/4)d(-q^2;\xi) + (1/4)]. \tag{5}$$

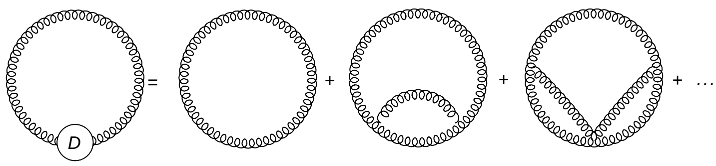

Figure 1. Infinite series for the gluon part of the VED to log-loop level.

Going over to four-dimensional Euclidean space in Eq. (1), one obtains ($\epsilon_g = V(D)$)

$$\epsilon_g = -16 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left[\ln[1 + 3d(q^2;\xi)] - \frac{3}{4} d(q^2;\xi) + a \right], \tag{6}$$

where the constant $a = (3/4) - 2\ln 2$ and an integration from zero to infinity over q^2 is assumed. The VED ϵ_g derived in Eq. (6) is already a colorless (color-singlet) quantity, since it has been summed over the color indices. It also does not depend explicitly on the unphysical (longitudinal) part of the full gluon propagator due to the product $(D_0^{-1}D)$ which, in turn, comes from the above-mentioned normalization of the free PT vacuum to zero. Thus it is worth emphasizing that only the transversal ("physical") degrees of freedom only of the gauge bosons contribute to this equation. As a consequence, in the effective potential approach to leading order there is no need for the ghost degrees of freedom from the very beginning in order to cancel the longitudinal ("unphysical") component of the full gluon propagator. This role is played by the normalization condition. However, this does not work for the higher order vacuum skeleton loops. In this case and beyond the PT at any gauge (i.e., in the general case) the cancelation of unphysical gluon modes should proceed with the help of ghosts [27].

An overall numerical factor 1/2 has been introduced into Eq. (1) in order to make the gluon degrees of freedom equal to $32/2 = 16 = 8 \times 2$, where 8 color of gluons times 2 helicity (transversal) degrees of freedom (see Eqs. (5) and (6)).

In what follows the Lorentz structure $d(q^2) \equiv d(q^2;\xi)$ will be called the full effective charge ("running") or the gluon propagator invariant function (its form factor), for convenience. For the generalization of Eq. (6) to non-zero temperatures, the most important thing is to introduce correctly the above-mentioned bag constant.

3. The gluon pressure at zero temperature

The vacuum of QCD is a very complicated confining medium and its dynamical and topological complexity means that its structure can be organized at various levels: classical and quantum (see, for example Refs. [33, 34, 35] and references therein). It is mainly NP by origin, character and magnitude, since the corresponding fine structure constant is large. However, the virtual gluon field configurations and excitations of the PT magnitude due to asymptotic freedom (AF) [36] are also present there.

One of the main dynamical characteristics of the true QCD ground state is the bag constant B. Its name has come from the famous bag models for hadrons [37, 38], but its present understanding (and thus modern definition) has nothing to do with the hadron properties. It is defined as the difference between the PT and the NP VEDs [39, 40, 41, 42]. We can symbolically write $B = VED^{PT} - VED$, where the VED is NP but "contaminated" by the PT contributions (i.e., this is a full VED like a full gluon propagator). Rewriting this as $B = VED^{PT} - VED = VED^{PT} - [VED - VED^{PT} + VED^{PT}] = VED^{PT} - [VED^{TNP} + VED^{PT}] = -VED^{TNP} > 0$, since the VED is always negative. The bag constant is nothing but the truly NP

(TNP) VED, apart from the sign, by definition, and thus is free of the PT contributions ("contaminations"). In order to consider it also as a physical characteristic of the true QCD ground state, the bag constant correctly calculated should satisfy some other requirements such as colorlessness and dependence on the physical degrees of freedom, finiteness, gauge-independence, no imaginary part (stable vacuum), etc.

In our previous work [27], we have already derived an expression for the bag constant, which satisfies the above conditions:

$$B_{YM} = 16 \int^{q^2_{eff}} \frac{\mathrm{d}^4 q}{(2\pi)^4} \left[\ln[1 + 3d^{TNP}(q^2)] - \frac{3}{4} d^{TNP}(q^2) \right], \tag{7}$$

where symbolically shown q^2_{eff} is the effective scale squared, separating the soft momenta from the hard ones in the integration over q^2. This is a general expression for any TNP effective charge in order to calculate the bag constant from first principles. It is defined as the special function of the TNP effective charge integrated out over the NP region (soft momenta region, $0 \leq q^2 \leq q^2_{eff}$).

Adding the bag constant to the both sides of Eq. (6), and introducing the gluon pressure $P_g = \epsilon_g + B_{YM}$, one obtains

$$P_g = B_{YM} - 16 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left[\ln[1 + 3d(q^2)] - \frac{3}{4} d(q^2) + a \right]. \tag{8}$$

The next step is to establish the relation between the full effective charge $d(q^2)$ and its TNP counterpart $d^{TNP}(q^2)$. In our previous work [43] it has been proven that the full effective charge depends explicitly and regularly on the scale Δ^2 (the so-called mass gap) responsible for the NP dynamics in QCD, i.e., $d(q^2) \equiv (q^2; \Delta^2)$. The above-mentioned symbolic subtraction at the fundamental gluon level can then be defined as [27, 43]: $d^{TNP}(q^2; \Delta^2) = d(q^2; \Delta^2) - d(q^2; \Delta^2 = 0) = d(q^2; \Delta^2) - d^{PT}(q^2)$. In this way the separation between the TNP effective charge and its PT counterpart becomes exact, but not unique. On how to make this separation exact and unique at the same time see subsection below. Evidently, both the TNP part and its PT counterpart are valid throughout the whole energy/momentum range, i.e, they are not asymptotics. Let us also emphasize the principle difference between $d(q^2)$ and $d^{TNP}(q^2)$. The former is the NP quantity "contaminated" by PT contributions, while the latter one, being also NP, is free of them.

However, this is not the whole story yet. Since the PT part $d^{PT}(q^2)$ contains the free gluon Lorentz structure $d^0(q^2) = 1$, it should be extracted explicitly as follows, where we have dropped the explicit dependence on Δ^2:

$$d(q^2) = d^{TNP}(q^2) + d^{PT}(q^2) = d^{TNP}(q^2) + 1 + d^{AF}(q^2), \tag{9}$$

and $d^{AF}(q^2)$ describes the part responsible for AF in the PT gluon effective charge. This procedure is necessary in order to maintain the normalization of the free PT vacuum to zero. In this connection, let us stress that the second equality in this relation does not imply any violation of AF in the full gluon effective charge $d(q^2)$, provided by the

PT effective charge $d^{PT}(q^2)$ in the first equality of the same relation. Extracting $d^0 = 1$ explicitly, we thereby subtract the divergent contribution associated with the constant a in Eq. (6), and thus the above-mentioned normalization condition will be automatically satisfied. We now replace all the effective charges as follows: $d^{TNP}(q^2) \equiv \alpha_s^{TNP}(q^2)$ and $d^{AF}(q^2) \equiv \alpha^{AF}(q^2)$. The explicit expression for the effective charge responsible for AF, that's $\alpha^{AF}(q^2)$, will be given in part II of our work, since its explicit expression is not used here.

Substituting the decomposition (9) into Eq. (8), lengthy algebra leads to

$$P_g = P_{NP} + P_{PT} = B_{YM} + P_{YM} + P_{PT}, \tag{10}$$

i.e., $P_{NP} = B_{YM} + P_{YM}$. In Eq. (10) B_{YM} is given in Eq. (7), while

$$P_{YM} = -16 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left[\ln[1 + \frac{3}{4}\alpha_s^{TNP}(q^2)] - \frac{3}{4}\alpha_s^{TNP}(q^2) \right], \tag{11}$$

and

$$P_{PT} = -16 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left[\ln[1 + \frac{3\alpha^{AF}(q^2)}{4 + 3\alpha_s^{TNP}(q^2)}] - \frac{3}{4}\alpha^{AF}(q^2) \right]. \tag{12}$$

P_{YM} in Eq. (11) depends exclusively on the TNP effective charge. Together with the bag constant (7) it forms the NP part of the gluon pressure (10). P_{PT} in Eq. (12) contains the contribution which is mainly determined by the AF part of the PT effective charge, though the dependence on the TNP effective charge is also present (it is logarithmically suppressed in comparison to the pure AF term). If the interaction is switched off, i.e., putting formally $\alpha^{AF}(q^2) = \alpha_s^{TNP}(q^2) = 0$, then $P_g = 0$ in accordance with the initial normalization of the free PT vacuum to zero. Evidently, just the right-hand-side of Eq. (10) is to be generalized to non-zero temperature, on account of the explicit expressions (7), (11) and (12).

3.1. Confining effective charge

The only problem remaining is the explicit expression for the TNP effective charge. Evidently, it has to be consistent with our initial work [27], where the bag constant was evaluated at zero temperature, i.e.,

$$\alpha_s^{TNP}(q^2) \to \alpha_s^{INP}(q^2) = \frac{\Delta^2}{q^2}, \tag{13}$$

where the superscript "INP" stands for the intrinsically NP effective charge. Here $\Delta^2 \equiv \Delta^2_{JW}$ is the Jaffe-Witten (JW) mass gap, mentioned above, which is responsible for the large-scale structure of the QCD vacuum, and thus for its INP dynamics [44]. Let us note that how the mass gap appears in QCD has been shown in our work [43].

A few additional remarks are in order. In Ref. [45] it has been shown that the TNP part of the full gluon propagator as a function of the mass gap also contains a term that is regular at origin. It is for this reason that it is not uniquely separated from the PT gluon propagator, which effective charge that is always regular at origin. We distinguish between the INP and the PT effective charges not only by the presence of the mass gap, but by the character of the IR singularities as well [45, 46]. So only after the replacement (13) the INP effective charge is uniquely and exactly separated from its PT counterpart, and the obtained expression for the bag constant (7) becomes now *free of all the types of the PT contributions ("contaminations").*

In Ref. [46] we have shown that the so-called INP gluon propagator is, by construction, purely transversal in a gauge invariant way. It exactly converges to the gluon propagator, whose effective charge is Eq. (13), after the renormalization programme of the regularized mass gap is performed. This result has been obtained in the most general way, i.e., without making any truncations/approximations/assumptions or choosing a special gauge. *Hence the expression (13) is not an ansatz, and thus it is mathematically well justified.* However, it should also be physically well justified. The problem is that the gluon equation of motion is highly nonlinear, so the number of independent exact solutions is not fixed *a priori* (this number may be even bigger, depending on the different truncations/approximations/assumptions and the concrete gauge choice made). They should be considered on equal footing from the very beginning. In our previous work [45], at least the two different general types of exact solutions for the full gluon propagator have been found as a function of the regularized mass gap: the first is smooth at small gluon momentum, allowing for the gluons to acquire an effective gluon mass (the so-called massive solution). The second is singular in the $q^2 \to 0$ limit, so that the gluons always remain massless (the so-called nonlinear iteration solution). Only this solution has an affective charge (13) after the renormalization is completed. For the detailed arguments which makes its choice justified from the physical point of view as well (i.e., emphasizing its confining nature) see our paper [27]. In addition, let us note that the above-mentioned massive solution is not confining.

4. Generalization to non-zero temperatures

In the imaginary-time formalism [7, 47], all the four-dimensional integrals can be easily generalized to non-zero temperatures T according to the prescription (note that there is already Euclidean signature)

$$\int \frac{dq_0}{(2\pi)} \to T \sum_{n=-\infty}^{+\infty}, \qquad q^2 = \mathbf{q}^2 + q_0^2 = \mathbf{q}^2 + \omega_n^2 = \omega^2 + \omega_n^2, \ \omega_n = 2n\pi T, \tag{14}$$

i.e., each integral over q_0 of a loop momentum is to be replaced by the sum over the Matsubara frequencies labeled by n, which obviously assumes the replacement $q_0 \to \omega_n = 2n\pi T$ for bosons (gluons). In frequency-momentum space the INP effective charges (13) becomes

$$\alpha_s^{INP}(q^2) = \alpha_s^{INP}(\mathbf{q}^2, \omega_n^2) = \alpha_s^{TNP}(\omega^2, \omega_n^2) = \frac{\Delta^2}{\omega^2 + \omega_n^2}, \tag{15}$$

and it is also convenient to introduce the following notations:

$$T^{-1} = \beta, \quad \omega = \sqrt{\mathbf{q}^2}, \tag{16}$$

$\alpha^{AF}(q^2) = \alpha^{AF}(\mathbf{q}^2, \omega_n^2) = \alpha^{AF}(\omega^2, \omega_n^2)$, where, evidently, in all the expressions here and below $\mathbf{q}^2$ is the square of the three-dimensional loop momentum, in complete agreement with the relations (14).

Introducing the temperature dependence into the right-hand-side of the relation (10), we finally obtain

$$P_g(T) = P_{NP}(T) + P_{PT}(T) = B_{YM}(T) + P_{YM}(T) + P_{PT}(T). \tag{17}$$

4.1. Derivation of $B_{YM}(T)$

It is convenient to begin with Eq. (7) for the bag constant. In frequency-momentum space the temperature-dependent YM bag pressure becomes

$$B_{YM}(T) = 16 \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, T \sum_{n=-\infty}^{+\infty} \left[\ln[1 + 3\alpha_s^{TNP}(\mathbf{q}^2, \omega_n^2)] - \frac{3}{4}\alpha_s^{TNP}(\mathbf{q}^2, \omega_n^2) \right]. \tag{18}$$

After the substitution of the expression from Eq. (15), one obtains

$$B_{YM}(T) = 16 \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, T \sum_{n=-\infty}^{+\infty} \left[\ln \left(\frac{\omega'^2 + \omega_n^2}{\omega^2 + \omega_n^2} \right) - \frac{3}{4} \frac{\Delta^2}{\omega^2 + \omega_n^2} \right], \tag{19}$$

where we introduced the notations:

$$\omega' = \sqrt{\mathbf{q}^2 + 3\Delta^2} = \sqrt{\omega^2 + m'^2_{eff}}, \quad m'_{eff} = \sqrt{3}\Delta. \tag{20}$$

One of the attractive features of the confining effective charge (15) is that it allows for an exact summation over the Matsubara frequencies (see appendix A). Substituting all our results of the summation into Eq. (19), dropping the β-independent terms [7], and performing almost trivial integration over angular variables, one gets

$$B_{YM}(T) = -\frac{8}{\pi^2} \int_0^{\omega_{eff}} \mathrm{d}\omega\, \omega^2 \left[\frac{3}{4}\Delta^2 \frac{1}{\omega} \frac{1}{e^{\beta\omega} - 1} - 2\beta^{-1} \ln \left(\frac{1 - e^{-\beta\omega'}}{1 - e^{-\beta\omega}} \right) \right]. \tag{21}$$

The ω_{eff}, which is the three-dimensional analog of q_{eff} in Eq. (7), is discussed in appendix B, where numerical values are given.

It is convenient to present the integral (21) as a sum of several terms

$$B_{YM}(T) = -\frac{6}{\pi^2}\Delta^2 B_{YM}^{(1)}(T) - \frac{16}{\pi^2} T \left[B_{YM}^{(2)}(T) - B_{YM}^{(3)}(T) \right], \tag{22}$$

where the explicit expressions of all the terms are given as

$$\begin{aligned}
B_{YM}^{(1)}(T) &= \int_0^{\omega_{eff}} \mathrm{d}\omega \frac{\omega}{e^{\beta\omega} - 1}, \\
B_{YM}^{(2)}(T) &= \int_0^{\omega_{eff}} \mathrm{d}\omega\, \omega^2 \ln \left(1 - e^{-\beta\omega} \right), \\
B_{YM}^{(3)}(T) &= \int_0^{\omega_{eff}} \mathrm{d}\omega\, \omega^2 \ln \left(1 - e^{-\beta\omega'} \right).
\end{aligned} \tag{23}$$

Here and below $N = (e^{\beta\omega} - 1)^{-1}$ is the Bose-Einstein distribution (the so-called gluon mean number [7]).

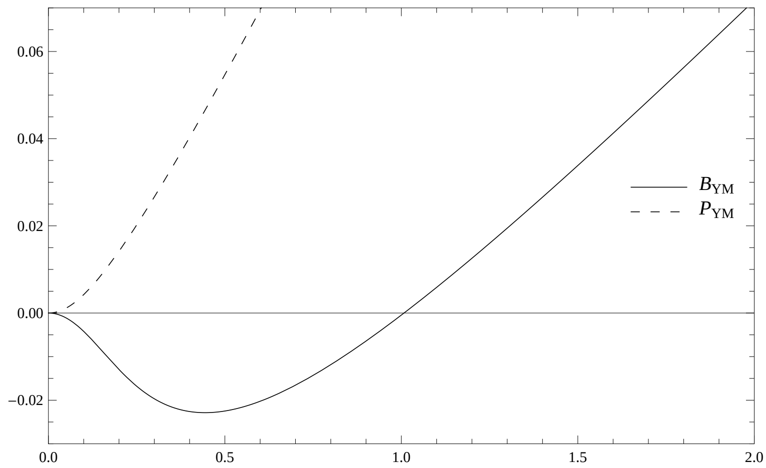

Figure 2. The bag constant (22) and the NP YM part (27) in GeV^4 units as functions of temperature T in GeV units. The bag constant at $T = 1$ GeV is zero, so up to this value it is responsible for the NP vacuum contributions to the pressure (17) and hence (29).

4.2. Derivation of $P_{YM}(T)$

In frequency-momentum space Eq. (11) becomes

$$P_{YM}(T) = -16 \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, T \sum_{n=-\infty}^{+\infty} \left[\ln[1 + \frac{3}{4}\alpha_s^{TNP}(\mathbf{q}^2, \omega_n^2)] - \frac{3}{4}\alpha_s^{TNP}(\mathbf{q}^2, \omega_n^2) \right], \quad (24)$$

thus determining the temperature-dependent YM part of the NP pressure of Eq. (17). After substituting of the expression (15), one obtains

$$P_{YM}(T) = -16 \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, T \sum_{n=-\infty}^{+\infty} \left[\ln\left(\frac{\bar{\omega}^2 + \omega_n^2}{\omega^2 + \omega_n^2}\right) - \frac{3}{4}\frac{\Delta^2}{\omega^2 + \omega_n^2} \right], \quad (25)$$

where we introduced the following notations:

$$\bar{\omega} = \sqrt{\mathbf{q}^2 + \frac{3}{4}\Delta^2} = \sqrt{\omega^2 + \bar{m}_{eff}^2}, \quad \bar{m}_{eff} = \frac{\sqrt{3}}{2}\Delta = \frac{1}{2}m'_{eff}. \quad (26)$$

Comparing Eqs. (19) and (25) one can write down the final result directly. For this purpose, in the system of Eqs. (22)-(23) one must change the overall sign, replace ω' by $\bar{\omega}$ and integrate from zero to infinity. Thus, one obtains

$$P_{YM}(T) = \frac{6}{\pi^2}\Delta^2 P_{YM}^{(1)}(T) + \frac{16}{\pi^2} T \left[P_{YM}^{(2)}(T) - P_{YM}^{(3)}(T) \right], \quad (27)$$

where

$$\begin{aligned} P_{YM}^{(1)}(T) &= \int_0^\infty \mathrm{d}\omega \frac{\omega}{e^{\beta\omega} - 1} = \frac{\pi^2}{6} T^2, \\ P_{YM}^{(2)}(T) &= \int_0^\infty \mathrm{d}\omega\ \omega^2 \ln\left(1 - e^{-\beta\omega}\right), \\ P_{YM}^{(3)}(T) &= \int_0^\infty \mathrm{d}\omega\ \omega^2 \ln\left(1 - e^{-\beta\bar{\omega}}\right). \end{aligned} \quad (28)$$

The bag pressure (22) and the NP YM part (27) in GeV^4 units as functions of temperature T are shown in Fig. 2. The scale-setting scheme of all our numerical calculations in this paper is presented in appendix B.

5. The gluon pressure at non-zero temperatures

Summing up all the expressions and integrals (22)-(23) and (27)-(28), the gluon pressure (17) becomes

$$P_g(T) = P_{NP}(T) + P_{PT}(T), \tag{29}$$

where

$$\begin{aligned} P_{NP}(T) &= B_{YM}(T) + P_{YM}(T) \\ &= \frac{6}{\pi^2}\Delta^2 P_1(T) + \frac{16}{\pi^2} T[P_2(T) + P_3(T) - P_4(T)], \end{aligned} \tag{30}$$

and

$$P_1(T) = \int_{\omega_{eff}}^{\infty} \mathrm{d}\omega \frac{\omega}{e^{\beta\omega} - 1}, \tag{31}$$

while

$$\begin{aligned} P_2(T) &= \int_{\omega_{eff}}^{\infty} \mathrm{d}\omega\ \omega^2 \ln\left(1 - e^{-\beta\omega}\right), \\ P_3(T) &= \int_0^{\omega_{eff}} \mathrm{d}\omega\ \omega^2 \ln\left(1 - e^{-\beta\omega'}\right), \\ P_4(T) &= \int_0^{\infty} \mathrm{d}\omega\ \omega^2 \ln\left(1 - e^{-\beta\bar{\omega}}\right). \end{aligned} \tag{32}$$

Let us recall once more that in all these integrals $\beta = T^{-1}$, ω_{eff} along with the mass gap Δ^2 is fixed (appendix B), while ω' and $\bar{\omega}$ are given by the relations (20) and (26), respectively. In the formal PT $\Delta^2 = 0$ limit it follows that $\bar{\omega} = \omega' = \omega$ and the combination $P_2(T) + P_3(T) - P_4(T)$ becomes identically zero. Thus the NP part (30) of the gluon pressure (29) in this limit vanishes.

In frequency-momentum space the PT part (12) of the gluon pressure (29) is

$$P_{PT}(T) = -\frac{8}{\pi^2}\int_0^{\infty} \mathrm{d}\omega\, \omega^2\, T \sum_{n=-\infty}^{+\infty} \left[\ln\left(1 + \frac{3\alpha^{AF}(\omega^2, \omega_n^2)}{4 + 3\alpha_s^{TNP}(\omega^2, \omega_n^2)}\right) - \frac{3}{4}\alpha^{AF}(\omega^2, \omega_n^2)\right], \tag{33}$$

where the trivial integration over angular variables has already been carried out. Unfortunately, one cannot perform analytically (i.e., exactly) the summation over the Matsubara frequencies in this integral, using the AF expression for $\alpha^{AF}(\omega^2, \omega_n^2)$ [36, 46]. At this stage the PT part (33) remains undetermined and will be numerically evaluated elsewhere. Note that, since it becomes zero when the interaction is switched formally off ($\alpha^{AF}(\omega^2, \omega_n^2) = 0$), both terms (30) and (33) and hence the gluon pressure (29) satisfy the initial normalization condition.

Confining dynamics in Eq. (29) is implemented via the bag pressure (22) and the mass gap itself. However, other NP effects are also present via $P_{YM}(T)$ and $P_{PT}(T)$, given in Eqs. (27) and (33), respectively, though in Eq. (33) they are logarithmically suppressed.

6. Main thermodynamic quantities

Together with the pressure $P(T)$, the main thermodynamic quantities are the entropy density $s(T)$ and the energy density $\epsilon(T)$. The general formulae which connect them are [7]

$$
\begin{aligned}
s(T) &= \frac{\partial P(T)}{\partial T}, \\
\epsilon(T) &= T\left(\frac{\partial P(T)}{\partial T}\right) - P(T) = Ts(T) - P(T)
\end{aligned}
\tag{34}
$$

for pure YM fields, i.e., when the chemical potential vanishes. Let us note that in quantum statistics the pressure $P(T)$ is up to a sign equal to the thermodynamic potential $\Omega(T)$, i.e., $P(T) = -\Omega(T) > 0$ [7].

Other thermodynamic quantities of interest are the heat capacity $c_V(T)$ and the velocity of sound squared $c_s^2(T)$, which are defined as follows:

$$
c_V(T) = \frac{\partial \epsilon(T)}{\partial T} = T\left(\frac{\partial s(T)}{\partial T}\right), \tag{35}
$$

and

$$
c_s^2(T) = \frac{\partial P(T)}{\partial \epsilon(T)} = \frac{s(T)}{c_V(T)}, \tag{36}
$$

i.e., they are defined through the second derivative of the pressure. The so-called conformity

$$
C(T) = \frac{P(T)}{\epsilon(T)} \tag{37}
$$

mimics the behavior of the speed of sound squared (36) but without involving such a differentiation.

The thermodynamic quantity of a special interest is the thermal expectation value of the trace of the energy momentum tensor. This trace anomaly relation measures the deviation of the difference

$$
\epsilon(T) - 3P(T) \tag{38}
$$

from zero at finite temperatures, in the high temperature limit it must vanish. As a consequence it is very sensitive to the NP contributions to the EoS. It also assists in the temperature dependence of the gluon condensate [48] (see Ref. [49]), namely

$$
< G^2 >_T = < G^2 >_0 - [\epsilon(T) - 3P(T)], \tag{39}
$$

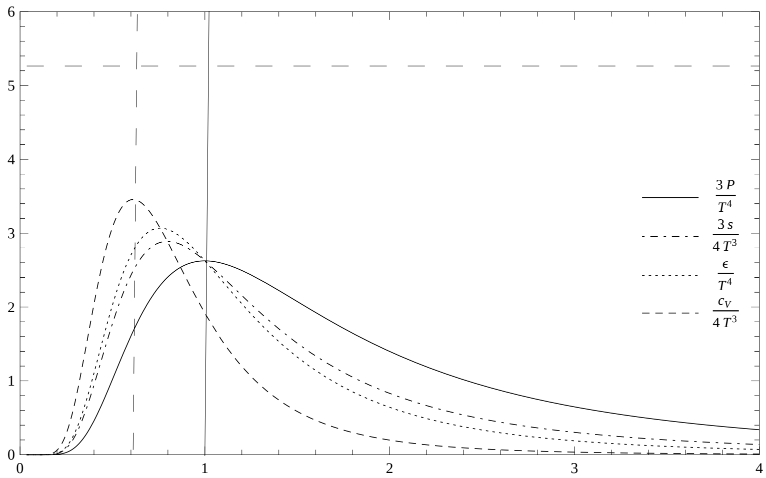

Figure 3. The NP pressure, the entropy and energy densities, the heat capacity as functions of T/T_c. The NP pressure has a maximum at $T_c = 266.5$ MeV. Here the horizontal dashed line is the general SB number (40), while the vertical dashed line at $0.6T_c$ separates the low-temperatures region from the transition region $(0.6 - 1)T_c$.

where $< G^2 >_0 \equiv < G^2 >_{T=0}$ denotes the gluon condensate at zero temperature, whose numerical value is discussed in appendix B.

As mentioned above, the main purpose of this paper is to investigate analytically and calculate the NP part (30) of Eq. (29), while the corresponding investigation of the PT part (33) will be the subject of subsequent work. This will make it possible to complete the investigation, derivation and final calculation of the full GM EoS.

6.1. SB limit

The high-temperature behavior of all the thermodynamic quantities is governed by the SB ideal gas limit, when the matter can be described in terms of non-interacting massless particles (gluons). In this limit these quantities satisfy special relations (see, for example Ref. [7])

$$\frac{3P_{SB}(T)}{T^4} = \frac{\epsilon_{SB}(T)}{T^4} = \frac{3s_{SB}(T)}{4T^3} = \frac{c_{V(SB)}(T)}{4T^3} = \frac{24}{45}\pi^2 \approx 5.26, \ T \to \infty, \tag{40}$$

and

$$C_{SB}(T) = c^2_{s(SB)}(T) = \frac{1}{3}, \quad T \to \infty, \tag{41}$$

on account of the previous relations and their definitions in Eqs. (36) and (37). The trace anomaly relation (38) also satisfies the SB limit, namely

$$\epsilon_{SB}(T) - 3P_{SB}(T) = 0, \quad T \to \infty, \tag{42}$$

as it comes out from the relations (40). In what follows its right-hand-side will be called the general SB number.

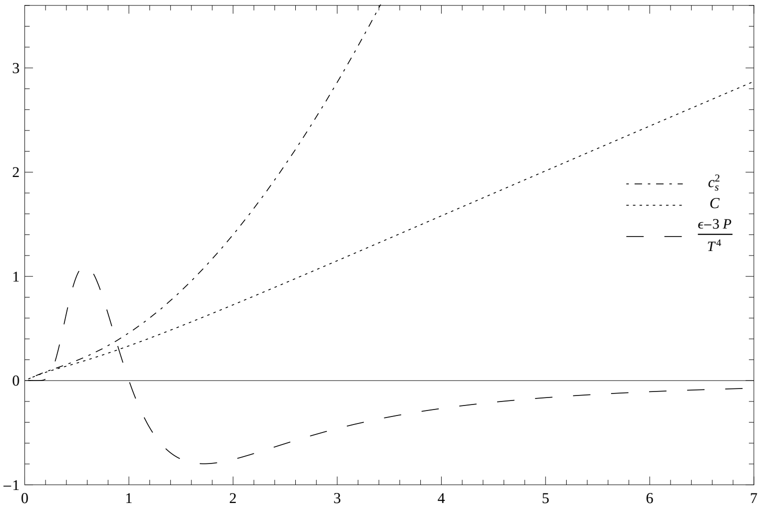

Figure 4. The NP velocity of sound, conformity and the trace anomaly relation as functions of T/T_c.

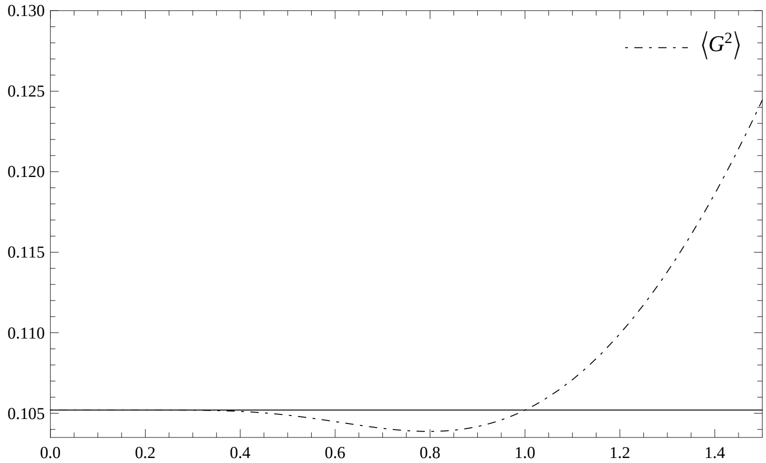

Figure 5. The NP gluon condensate (39) in GeV^4 units as a function of T/T_c. It shows little temperature dependence below T_c and grows rapidly above T_c. Here the solid line is its value at zero temperature $< G^2 >_0$ (see appendix B).

7. Numerical results and discussion

All our numerical results obtained for the thermodynamical quantities discussed in the previous section and calculated with the help of the NP pressure (30) are shown in Figs. 3, 4 and 5. The NP gluon pressure has a maximum at some finite ("characteristic") temperature, $T = T_c = 266.5$ MeV. This means that it may change continuously its regime in the near neighborhood of this point in order for its full counterpart to achieve the thermodynamic SB limit (40) at high temperatures. However, this is not possible for the other NP thermodynamic quantities, as is seen from the curves shown in Fig. 3 (none of the power-type fall off around T_c can be smoothly transformed into the almost constant behavior at high temperatures). In order to achieve the corresponding thermodynamic SB limits at high temperatures their full counterparts would have to undergo drastic changes around this point. As we already know from the thermodynamics of $SU(3)$ lattice QCD [49] (see Refs. [50, 51, 52]) the energy and entropy densities have discontinuities (jump or, equivalently, step discontinuities) at a temperature of about $T_c = 260 - 270$ MeV. Our characteristic temperature $T_c = 266.5$ MeV is surprisingly very close to this value. If we were not aware of the thermal lattice QCD results then we would be able to predict them. Since we are aware of them, these lattice results confirm our expectation of a sharp changes in the behavior of the entropy and energy densities in the region where the pressure is continuous.

One of the interesting features of our calculations is seen in Fig. 3. At the characteristic temperature T_c the NP pressure, the entropy and energy densities satisfy a SB-type relations, namely

$$\frac{3P_{NP}(T_c)}{T_c^4} = \frac{\epsilon_{NP}(T_c)}{T_c^4} = \frac{3s_{NP}(T_c)}{4T_c^3} = \frac{12}{45}\pi^2 \approx 2.63, \tag{43}$$

where, obviously, the right-hand-side is half of the general SB number (40). In turn this yields

$$C_{NP}(T_c) = \frac{1}{3}, \quad \epsilon_{NP}(T_c) - 3P_{NP}(T_c) = 0, \tag{44}$$

so that the NP trace anomaly relation approaches zero from below in the $T \to \infty$ ($\beta \to 0$) limit (see Fig. 4), i.e, satisfying the SB relation (42) in this limit as well.

Since the NP entropy and energy densities satisfy the SB-type relations (43), we expect for their full counterparts to have step discontinuities at T_c, as discussed above. The NP heat capacity and the velocity of sound squared do not satisfy the SB-type relations (43), as can be seen in Figs. 3 and 4. We expect, therefore, for the full heat capacity to have an essential discontinuity at T_c, since, in general, $[c_V(T_c)]^{-1} = 0$, and hence $c_s^2(T_c) = 0$ due to the relation (36). They are defined through derivatives of the entropy and energy densities in Eqs. (35)-(36), i.e., they involve the second derivatives of the pressure. That is a reason why these thermodynamic quantities are too sensitive to the dynamical structure of the GM in the near neighborhood of T_c.

Moreover, due to the SB-type relations (43)-(44), and on account of the SB relations (40), for example it follows that

$$\Big[P_{SB}(T) - 2P_{NP}(T)\Big]_{T=T_c} = 0, \ \Big\{\frac{\partial}{\partial T}\Big[P_{SB}(T) - 2P_{NP}(T)\Big]\Big\}_{T=T_c} = 0. \tag{45}$$

Combining these relations again with the SB-type relations (43)-(44) it is possible to derive some other exact relations between different thermodynamic quantities at T_c. Their importance will be explicitly shown in part II of our investigation, where they will be extensively exploited.

From Fig. 3 it clearly follows that the temperature range for the NP thermodynamic quantities can be divided into the three different intervals:

(i). The low-temperatures interval up to its upper bound of about $0.6T_c$. It is defined by the first maximum of the heat capacity which appears in this region before temperature reaches T_c. This region is obviously dominated by the NP contributions (30) to the gluon pressure (29). The behavior of all the NP thermodynamic quantities are shown in Figs. 5, 6 and 7. We do not expect any serious changes in the behavior of their full counterparts in this region. However, whatever changes may occur they will be under our control, since the NP part (30) is exactly calculated. The important observation is that all the NP thermodynamic quantities which have been properly scaled go down exponentially as temperature approaches zero. This is a general feature of the behavior of all the full thermodynamic quantities far below T_c; otherwise their zero temperature limit would not be realized. The almost constant behavior of the NP velocity of sound squared and conformity seen in Fig. 7 is due to the fact that they are the ratios of the corresponding thermodynamic quantities, see Eqs. (36)-(37). It seems that for the first time it is possible to predict the behavior of all the important thermodynamic quantities in this interval. We shall refer to the GM in this region as a "confining phase" (there are no lattice data for this region at all).

(ii). The transition interval $(0.6-1)T_c$. In this region all the thermodynamic quantities should undergo sharp changes in their regimes, so that their full counterparts will be able to achieve their corresponding SB limits at high temperatures. The full GM pressure remains continuous in this region. However, the existence of maxima in the behavior of the NP thermodynamic quantities is expected in this interval, since they should "die" at high temperatures. The NP pressure is a part of the full GM pressure, which should be a smoothly growing function of T throughout the whole temperature range. All other full thermodynamic quantities are smoothly growing functions of T only below and above T_c. Within our approach they are expected to have discontinuities of different character at T_c, as mentioned above (apart from the gluon condensate). Following the authors of Ref. [53], we shall refer to the GM in the transition region as a "mixed phase". The changes in the dynamical structure of the GM just in this region will determine the nature of the phase transition at T_c.

(iii). The temperatures interval starting at T_c. This interval itself can be clearly subdivided into two intervals, since in the integrals (31)-(32) $\omega_{eff} = 3.75T_c = 1$ GeV:
(a) The moderate-temperatures $(1-3.75)T_c$, when the NP effects are still significant and
(b) The high-temperatures starting at $3.75T_c$, when the NP effects become small.

Therefore, even above T_c, the GM can be understood as being in the two different forms. The first one can be considered as strongly coupled GM, where the NP effects are still important. The second one can be considered as weakly coupled GM, where the NP effects become already negligible. The typical temperature which is of about $3.75T_c = 1$ GeV for the GM, is too high to be reached even at LHC. For QCD matter this temperature should be substantially decreased, and therefore be accessible at RHIC, and especially at LHC. The general feature for the behavior of all the properly scaled NP thermodynamic quantities above T_c is their power-type fall off, while their full counterparts should show a power-type rise at high temperatures. Apparently, we may

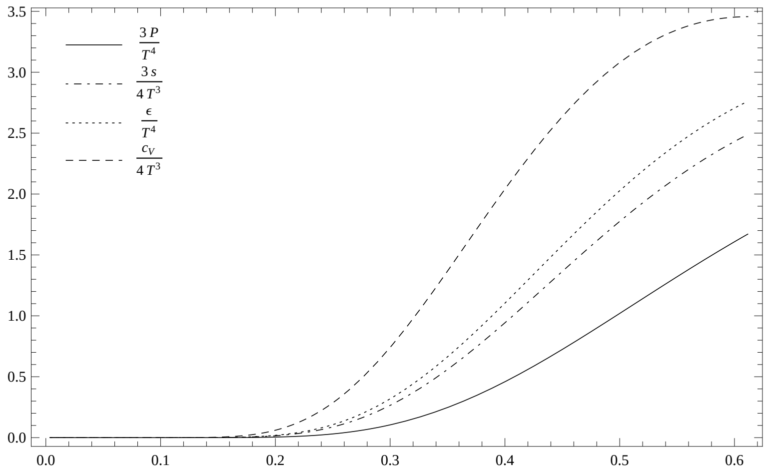

Figure 6. The NP gluon pressure, the entropy and energy densities, the heat capacity as functions of T/T_c in the low-temperatures region shown up to $0.6T_c$.

refer to the GM in the moderate temperature region as an "extended mixed phase". The behavior of all the full thermodynamic quantities in both mixed phases is governed by the NP pressure (30) and by the first term in the PT pressure (33). We shall refer to the GM in the high temperatures region as an "AF phase", since the behavior of all the full thermodynamic quantities will be determined by the second term in the PT pressure (33) and the free gluons contribution.

A few remarks in advance are here in order. From our discussion it clearly follows that the approximation of the gluon pressure (29) by its NP part (30) only (though exactly calculated in the present investigation) breaks down at T_c, when the NP pressure achieves its maximum. For the derivatives of the pressure, this approximation breaks down even earlier, starting from about $0.6T_c$, as discussed above. This explains the sign change in the interaction measure and the supersonic value of the speed of sound squared both at T_c as seen in Fig. 4. The rapid rise of conformity and the speed of sound squared above T_c in Fig. 4 is also unphysical. This also explains why the gluon condensate in Fig. 5 increases drastically above T_c. Taking into account the NP (30), PT (33), and free gluons contributions to the full GM pressure, all these unphysical effects will disappear in the full thermodynamic quantities, thus allowing them to achieve their corresponding SB limits. This will allow one to compare with lattice data above T_c [49, 52] as well.

7.1. The mass gap and a "fuzzy bag" term

On account of the first of Eqs. (28) and Eq. (31), it is instructive to re-write the NP gluon pressure (30) equivalently as follows:

$$P_{NP}(T) = \Delta^2 T^2 - \frac{6}{\pi^2}\Delta^2 P_1'(T) + \frac{16}{\pi^2} T[P_2(T) + P_3(T) - P_4(T)], \qquad (46)$$

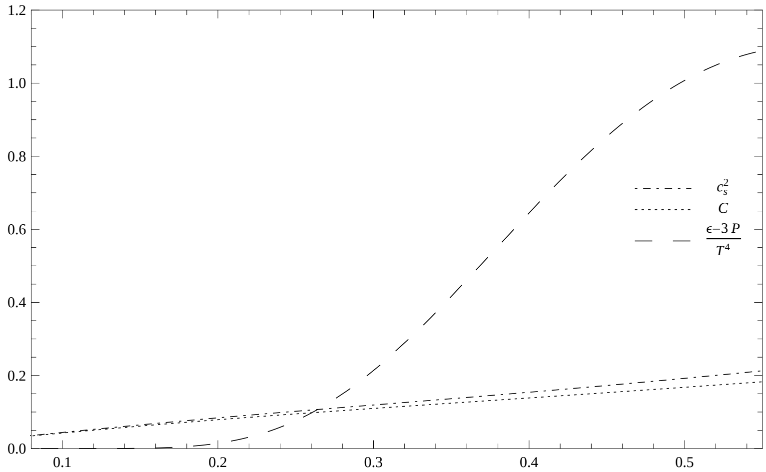

Figure 7. The NP velocity of sound, conformity and the trace anomaly relation as functions of T/T_c. They are shown in the low-temperatures region up to $0.6T_c$.

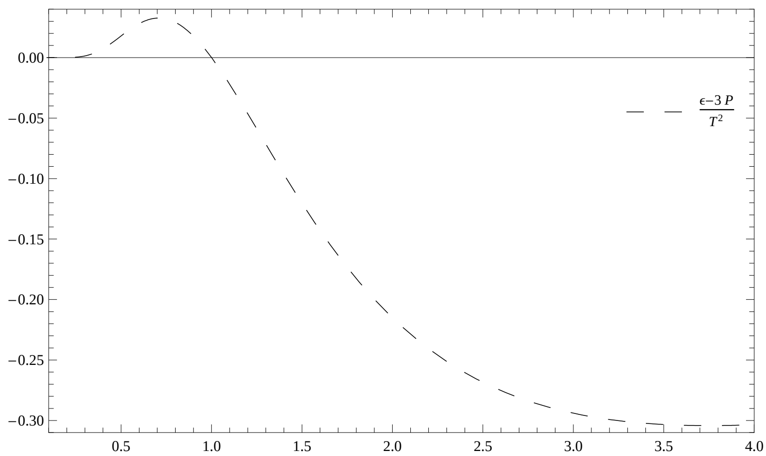

Figure 8. The NP trace anomaly relation scaled by T^2 in GeV^2 units (see Eq. (48)) as a function of T/T_c. It clearly approaches a finite constant in the high temperature limit (compare with Fig. 4).

where

$$P_1'(T) = \int_0^{\omega_{eff}} \mathrm{d}\omega \frac{\omega}{e^{\beta\omega} - 1}. \tag{47}$$

It scales as $T\omega_{eff}$ in the $T \to \infty$ limit, while the integrals $P_2(T)$, $P_3(T)$ and $P_4(T)$ remain unchanged, see Eqs. (32). Let us emphasize the explicit presence of the NP mass gap term in Eq. (46), namely $\Delta^2 T^2$. It will be present in the full EoS as well, since in the PT part of the gluon pressure the mass gap contribution is logarithmical suppressed (see Eq. (33)), and hence its contribution cannot be canceled. This is in full agreement with the so-called "fuzzy bag" model [52, 54, 55, 56] and with the massless Boltzmann stringy model [57]. The presence of the mass gap Δ^2 in our approach from the very beginning naturally explains the properties of such models.

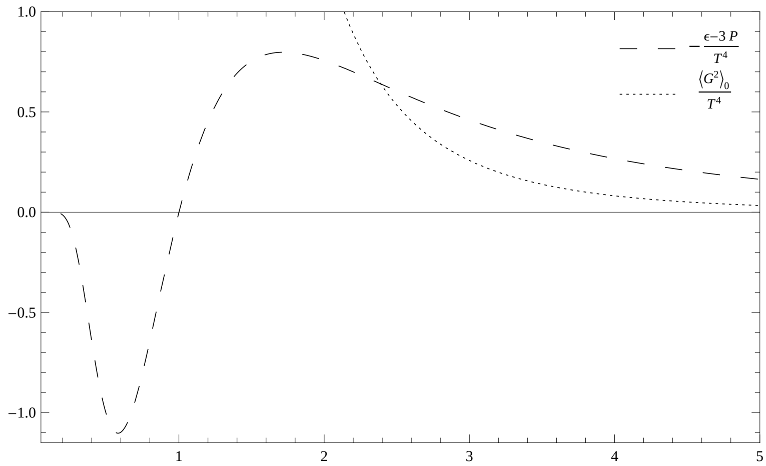

Figure 9. The zero temperature gluon condensate and the NP trace anomaly relation with minus sign, as it enters Eq. (49). Both scaled by T^4 and shown as functions of T/T_c.

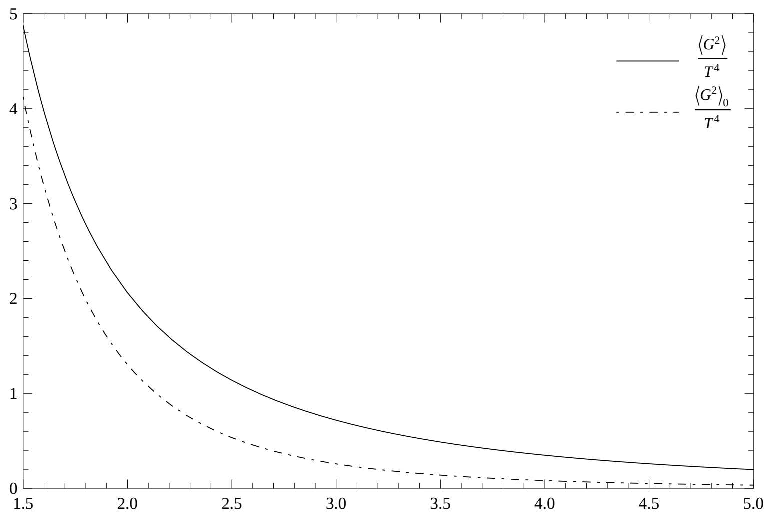

Figure 10. The NP gluon condensate scaled by T^4 as a function of T/T_c. The zero temperature gluon condensate scaled by T^4 is also shown. At temperatures above $1.5T_c$ they go down as explicitly shown by Eq. (49).

The NP trace anomaly relation scaled by T^2, t.e.,

$$\frac{\epsilon_{NP}(T) - 3P_{NP}(T)}{T^4} \times T^2 = \frac{\epsilon_{NP}(T) - 3P_{NP}(T)}{T^2} \longrightarrow -const., \; T \to \infty \quad (48)$$

is shown in Fig. 8. It clearly goes to a finite constant (having the dimensions of Δ^2) in the high temperature limit, as it should in accordance with the previous discussion and the thermal lattice QCD results [49, 52, 55, 58].

This effect immediately shows up in the NP gluon condensate (39). Scaled by T^4, analytically it becomes

$$\frac{< G^2 >_T}{T^4} = \frac{< G^2 >_0}{T^4} - \frac{[\epsilon_{NP}(T) - 3P_{NP}(T)]}{T^2} \times \frac{1}{T^2} \sim \frac{const.}{T^2}, \ T \to \infty. \quad (49)$$

The difference in the fall off at high temperatures between the zero temperature gluon condensate and minus trace anomaly relation (as it enters Eq. (49)), both scaled by T^4, is clearly seen in Fig. 9. Evidently, this difference is due to the asymptotic behavior of Eq. (48). At temperatures below T_c there is no difference between $< G^2 >_T /T^4$ and $< G^2 >_0 /T^4$, which is as it should be (see, for example again Fig. 5). At temperatures above T_c the difference between these two quantities is explicitly determined by Eq. (49) and is clearly shown in Fig. 10. The presence of the mass gap contribution in Eq. (46), and hence in the trace anomaly relation (38) scaled by T^4, explains that the way it goes down is not $1/T^4$ [49, 58] but, as follows from our consideration, it decreases as $1/T^2$.

The inclusion of the PT contribution $[\epsilon_{PT}(T) - 3P_{PT}(T)]/T^4$ in Eq. (49) will produce a rather small numerical correction at high temperatures due to the SB limit (42) for the trace anomaly. The numerical results presented in Table 1 confirm this statement. It shows rather good numerical agreement between our NP analytical trace anomaly, as it enters Eq. (49), and its lattice counterpart at hight temperatures starting already from $2T_c$ [49].

In conclusion, let us note that in lattice QCD at non-zero temperatures there is a possibility to distinguish between the magnetic (m) and electric (e) parts of the gluon condensate (see, for example Refs. [49, 59, 60] and references therein). In Refs. [59, 60] and Ref. [49], such a gauge-invariant field-strength correlators in pure YM theory have been calculated at finite temperatures around T_c and above T_c, respectively. Following Ref. [60], we can formally write $< G^2 >_T = < G_e^2 >_T + < G_m^2 >_T$, but at this stage we do not know how to calculate these two parts separately within our approach. There is therefore as yet no possibility to compare with lattice data. In any case, such an analytical investigation at zero and non-zero temperatures is beyond the scope of the present investigation.

Table 1. The NP analytical and lattice trace anomalies at high temperatures.

	Numbers	
T/T_c	Ours, Fig. 9	Lattice, Ref. [49]
2	0.762	0.860
3	0.453	0.475
4	0.252	0.255
5	0.166	0.166

7.2. The NP dynamical structure of the GM. A brief description

The NP dynamical structure of the GM within our approach is highly non-trivial. In the whole temperature range we have the two different massive gluonic excitations ω' and $\bar{\omega}$ with the effective masses $m'_{eff} = 1.17$ GeV and $\bar{m}_{eff} = 0.585$ GeV, respectively. Both effective masses are due to the mass gap Δ^2, which is responsible for the large-scale structure of the QCD ground state. It is dynamically generated by the nonlinear interaction of massless gluon modes [43, 45, 46]. The first massive excitations can be interpreted as the glueballs, since m'_{eff} is comparable to the masses of scalar glueballs (though it is lighter than the pseudoscalar ones) [61]. The second one $\bar{m}_{eff}$ might be identified with an effective gluon mass of about $(500 - 800)$ MeV, which arises in different approaches (see again the above-mentioned review [61] and references therein). We also have the two different massless gluonic excitations ω: the NP massless gluons, propagating according to the integral $P_1(T)$ in Eq. (31), and almost SB massless gluons, propagating according to the integral $P_2(T)$ in Eqs. (32). We stress that the integral (31) should be multiplied by $(6/\pi^2)\Delta^2$, and all other integrals (32) are to be multiplied by $(16/\pi^2)T$, when one speaks about different NP contributions to the NP pressure (30).

It is worth emphasizing once more that all these NP gluonic excitations are of dynamical origin, and only such can be accounted for within our approach. Whether the PT part (33) of the gluon pressure (29) will introduce a new massless or even massive gluonic excitations (or, equivalently, an effective gluonic degrees of freedom [16, 62]) is an open question at this stage. In our opinion, however, they may be really present in the full GM EoS. In addition to the mass gap the PT part (33) depends on the asymptotic scale parameter of QCD. The combination of these two fundamental scale parameters may lead to the creation of new massive gluonic excitations around T_c, for example, pseudoscalar glueballs [60, 61, 63]. In any case, we should definitely have the SB free massless gluons far away from T_c. However, one thing should be made perfectly clear. Because of these and a possible new massive excitations in the GM, the behavior of all the thermodynamic variables in the moderate temperatures region $(1 - 3.75)T_c$ will be substantially different from the behavior of a gas of free massless gluons. This our qualitative yet conclusion is in agreement with recent lattice data and quasi-particle approaches both cited above. The recent lattice data will be reproduced quantitatively after the evaluation of the NP (30), PT (33), and free gluon contributions to the full GM pressure.

The existence of the SB-type relations (43)-(44) indicates that in the transition region near T_c a dramatic increase in the number of effective gluonic degrees of freedom will appear (for example, the glueballs will begin rapidly to dissolve). This effect will be reflected in the strong increase of the thermodynamic variables near T_c, in agreement with recent lattice results [60, 63, 64]. This will lead to drastic changes in the structure of the GM. A change in this number is enough to generate pressure gradients, but not enough to affect the pressure itself. It varies slowly and therefore remains continuous in this region. At the same time, the pressure gradients, such as the energy and entropy densities, etc., should undergo sharp changes in their behavior. The penetration of the PT part (33) of the gluon pressure below T_c should not go deeply into the transition region $(0.6-1)T_c$, since the INP effective charge is logarithmical suppressed. However, it should go sufficiently deep in order to eliminate "non-physical" maxima and other effects, created by the derivatives of $P_{NP}(T)$ in the transition region. Instead of "non-physical" maxima, the full thermodynamic quantities (apart from the pressure) should have discontinuities at T_c, as follows from our approach. This has also been established by the thermal QCD lattice cited above.

Of course, not all the glueballs will be dissolved in the transition region. Some of them will remain above T_c, along with other massive and massless gluonic excitations, forming thus a possible mixed and extended mixed phases around T_c [53]. After the evaluation of the PT part (33), which is also "contaminated" by the NP contributions, we hope that the NP physics of the mixed phases will be well understood in the GM. At very high temperatures, starting at $3.75T_c$ the NP effects become very small, and the structure of the GM will be mainly determined by the SB relations (40)-(42) between all the important thermodynamic quantities.

8. Conclusions

The effective potential approach for composite operators [26] has been generalized to non-zero temperatures in order to derive analytical EoS for pure $SU(3)$ YM fields, shown in Eqs. (29)-(33). In its NP part (30) there is no dependence on the coupling constant, only the dependence on the mass gap, which is responsible for the large-scale structure of the QCD ground state. A key element of this work is the generalization of the expression for the bag constant at zero temperature [27] to non-zero temperatures. The NP part (30) of the gluon pressure (29) has been exactly evaluated, while its PT part (33) has been left undetermined at this stage, since it requires a separate investigation.

Our main quantitative and qualitative results in this investigation, which will not be changed (or only slightly changed) after the inclusion of the PT part (33), are:

(i). In Eq. (29) the confining dynamics at non-zero temperatures (15) is taken into account through the T-dependent bag constant (22) and the mass gap Δ^2.

(ii). Other NP effects are also taken into account via the YM part (27).

(iii). The mass gap Δ^2 or, equivalently, ω_{eff} is the only one independent input scale parameter needed to calculate the NP thermodynamic quantities.

(iv). The presence of the four different types of massive and massless gluonic excitations of the NP origin.

(v). The characteristic temperature $T_c = 266.5$ MeV is a temperature at which the maximum of the NP part of the gluon pressure is achieved.

(vi). The low-temperatures region up to $0.6T_c$ is under control. The exponential fall off in the $T \to 0$ limit of all the main NP thermodynamic quantities (see Fig. 6) is expected to be preserved for their full counterparts as well.

(vii). The behavior of all the thermodynamic quantities in the transition region $(0.6-1)T_c$ depends on how deeply the PT part (33) penetrates this region.

(viii). The existence of the SB-type relations (43)-(44) at T_c shows that the structure of the GM below and above T_c may be really rather different.

(ix). Since the NP entropy and energy densities satisfy them, we expect for their full counterparts to have jump discontinuities at T_c, while the full pressure remains continuous.

(x). Since the NP heat capacity does not satisfy them, we expect for its full counterpart to have an essential discontinuity at T_c, while the full speed of sound squared is to be zero at T_c.

(xi). In the moderate temperatures region $(1-3.75)T_c$ the NP vacuum effects are still significant.

(xii). Because of this, our qualitative prediction is that in this region the behavior of all the thermodynamic variables will be substantially different from the behavior of a gas of free massless gluons.

(xiii). All the full thermodynamic quantities, therefore, approach their SB limits rather slowly.

(xiv). A possible understanding of new forms of the GM below and above $3.75T_c$ as strong and weak coupling GM, respectively, has to be pointed out.

(xv). The existence of the mass gap term Δ^2T^2 in the NP pressure (46), which remains in the gluon pressure (29) as well.

(xvi). Because of the mass gap term Δ^2T^2 the full trace anomaly and the gluon condensate will go down as $1/T^2$ at high temperatures, and not as $1/T^4$.

In the subsequent paper (part II) we shall evaluate the PT part (33) of the gluon pressure (29), as well as include the free gluons contribution. It will make it possible to establish the order of the phase transition at T_c. We will be able to compare our numerical results with thermal QCD lattice calculations [49, 52] at high temperatures above T_c and in the transition region [52, 60, 63] as well. Only after completion of this programme, we will include the quark degrees of freedom in order to derive the NP QGP EoS within our formalism. For example, this will allow one to confirm a possible existence of Quarkyonic Matter (QM) [65, 66] and of a triple point in the QCD phase diagram (see Ref. [67] and references therein). Finally, let us note that recently [68] the pressure and the trace anomaly relation both scaled by T^4 have been calculated in 3d "electrostatic QCD" in apparent contradiction with the 4d lattice data [49, 52]. It is worth noting once more that our calculations are in a good agreement with them (see Table 1 and discussion above).

Acknowledgments

This paper is dedicated to the memory of the late Prof. J. Zimanyi, who has initiated this investigation. Support by HAS-JINR Scientific Agreement (P. Levai) is to be acknowledged. We would like to thank L. Csernai, T. Biró, T. Csorgö, P. Van, G. Barnaföldi, V. Skokov and especially C. Wilkin and S. Pochybova for useful discussions, comments, remarks and help.

Appendix A. The summation of the thermal logarithms

In the second terms of Eqs. (19) and (25) the summation over the Matsubara frequencies can be done explicitly [7], as follows:

$$\sum_{n=-\infty}^{+\infty} \frac{1}{\omega^2+\omega_n^2} = \sum_{n=-\infty}^{\infty} \frac{1}{\omega^2+(2\pi T)^2 n^2} = \left(\frac{\beta}{2\pi}\right)^2 \sum_{n=-\infty}^{+\infty} \frac{1}{n^2+(\beta\omega/2\pi)^2}$$
$$= \left(\frac{\beta}{2\pi}\right)^2 \frac{2\pi^2}{\beta\omega}\left(1+\frac{2}{e^{\beta\omega}-1}\right) = \frac{\beta}{2\omega}\left(1+\frac{2}{e^{\beta\omega}-1}\right). \quad \text{(A.1)}$$

In terms of the above-introduced parameters, the sums in Eq. (19) containing the corresponding logarithms look like:

$$\sum_{n=-\infty}^{+\infty} \ln[3\Delta^2+\omega^2+\omega_n^2] = \ln\omega'^2 + 2\sum_{n=1}^{\infty} \ln(2\pi/\beta)^2[n^2+(\beta\omega'/2\pi)^2] \quad \text{(A.2)}$$

and

$$\sum_{n=-\infty}^{+\infty} \ln[\omega^2+\omega_n^2] = \ln\omega^2 + 2\sum_{n=1}^{\infty} \ln(2\pi/\beta)^2[n^2+(\beta\omega/2\pi)^2]. \quad \text{(A.3)}$$

It is convenient to introduce the notations:

$$L(\omega') = \sum_{n=1}^{\infty} \ln[n^2+(\beta\omega'/2\pi)^2] = \sum_{n=1}^{\infty} \ln n^2 + \sum_{n=1}^{\infty} \ln\left[1-\frac{x'^2}{n^2\pi^2}\right] \quad \text{(A.4)}$$

and similarly, letting $\omega' \to \bar{\omega}$

$$L(\omega) = \sum_{n=1}^{\infty} \ln[n^2+(\beta\omega/2\pi)^2] = \sum_{n=1}^{\infty} \ln n^2 + \sum_{n=1}^{\infty} \ln\left[1-\frac{x^2}{n^2\pi^2}\right]. \quad \text{(A.5)}$$

In these expressions we introduced the following notations:

$$x'^2 = -\left(\frac{\beta\omega'}{2}\right)^2, \quad x^2 = -\left(\frac{\beta\omega}{2}\right)^2. \quad \text{(A.6)}$$

So the difference $L(\omega') - L(\omega)$ becomes

$$L(\omega') - L(\omega) = \sum_{n=1}^{\infty} \ln\left[1-\frac{x'^2}{n^2\pi^2}\right] - \sum_{n=1}^{\infty} \ln\left[1-\frac{x^2}{n^2\pi^2}\right]$$
$$= \ln\sin x' - \frac{1}{2}\ln x'^2 - \ln\sin x + \frac{1}{2}\ln x^2, \quad \text{(A.7)}$$

or, equivalently,

$$L(\omega') - L(\omega) = -\frac{1}{2}\ln\left(\frac{x'^2}{x^2}\right) + \ln\left(\frac{\sin x'}{\sin x}\right). \quad \text{(A.8)}$$

From the relation (A.6) it follows that

$$x' = \pm i\left(\frac{\beta\omega'}{2}\right), \quad x = \pm i\left(\frac{\beta\omega}{2}\right), \tag{A.9}$$

so eq. (A.8) finally becomes

$$L(\omega') - L(\omega) = -\frac{1}{2}\ln\left(\frac{\omega'^2}{\omega^2}\right) + \frac{1}{2}\beta(\omega' - \omega) + \ln\left(\frac{1 - e^{-\beta\omega'}}{1 - e^{-\beta\omega}}\right). \tag{A.10}$$

Appendix B. The scale-setting scheme

Let us note that ω_{eff}, which appears first in the integral (21), is the only free parameter of our approach. In frequency-momentum space it is

$$\omega_{eff} = \sqrt{q_{eff}^2 - \omega_c^2}, \tag{B.1}$$

where we introduce the "constant" Matsubara frequency ω_c, which is always positive. Hence ω_{eff} is always less or equal to the q_{eff} of four-dimensional QCD, i.e., $\omega_{eff} \leq q_{eff}$. One can then conclude that q_{eff} is a very good upper limit for possible values of ω_{eff}. In this connection, let us recall that the bag constant B_{YM} at zero temperature has been successfully calculated at a scale $q_{eff}^2 = 1$ GeV^2, in fair agreement with other phenomenological quantities such as gluon condensate [27]. So ω_{eff} is fixed as follows:

$$\omega_{eff} = \sqrt{q_{eff}^2} = 1 \text{ GeV}. \tag{B.2}$$

The mass gap squared Δ^2, also calculated at this scale, has a value [27]

$$\Delta^2 = 0.4564 \text{ GeV}^2, \quad \Delta = 0.6756 \text{ GeV}. \tag{B.3}$$

The effective gluon masses, defined in the relations (20) and (26), then become

$$m'_{eff} = \sqrt{3}\Delta = 1.17 \text{ GeV}, \quad \bar{m}_{eff} = \frac{\sqrt{3}}{2}\Delta = 0.585 \text{ GeV}. \tag{B.4}$$

The above-mentioned gluon condensate at zero temperature $< G^2 >_0$ calculated at the scale (B.2) and at the same confining effective charge (13) in Ref. [27] is

$$< G^2 >_0 \equiv \langle 0|\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle = 0.1052 \text{ GeV}^4. \tag{B.5}$$

We need this value for the calculation of the temperature-dependent gluon condensate $< G^2 >_T$ via Eq. (39). Multiplying the right-hand-side of the relation (B.5) by $(4\alpha_s/\pi)$, where $\alpha_s = 0.1187$ [69], it numerically becomes

$$\langle 0|\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle \approx 0.016 \text{ GeV}^4, \tag{B.6}$$

It is in a good agreement with its phenomenological value, $< G^2 >_{ph} \equiv \langle 0|(\alpha_s/\pi)G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle \approx 0.012$ GeV^4, which can be changed by a factor of ~ 2, as mentioned in Ref. [41] (see also Ref. [70] and references therein). In this connection, we recall [27] that the quark contribution to the bag constant, and hence to the gluon condensate, is approximately an order of magnitude less than the pure YM one. So the above-mentioned agreement of our YM value (B.6) with the phenomenological value is impressive.

References

[1] McLerran L 2008 *J. Phys. G: Nucl. Part. Phys.* **35** 104001
[2] Weiner R M 2006 *Int. Jour. Mod. Phys.* E **15** 37
[3] Csernai L P 1994 *Introduction to relativistic Heavy Ion Collisions* (J. Wiley and Sons)
[4] Quark-Gluon Plasma, Contributions from the RBRC Workshop, 14-15, May, 2004, organized by M. Gyulassy, L. McLerran and W. Busza, edited by D. Rischke and G. Levin 2005 *Nucl. Phys.* A **750** 30
[5] Alam J, Chattopadhyay S, Nayak T, Sinha B and Yiyogi Y P (ed.) 2008 *Proc. Quark Matter J. Phys. G: Nucl. Part. Phys.* **35** issue 10
[6] Kajantie K, Lane M, Rummukainen K and Schroder Y 2003 *Phys. Rev.* D **67** 105008
[7] Kapusta J I and Gale C 2006 *Finite-Temperature Field Theory* (Cambridge University Press)
[8] Braaten E and Pisarski R D 1990 *Nucl. Phys.* B **337** 569
[9] Letessier J and Rafelski J 2003 *Phys. Rev.* C **67** 031902 (*Preprint* hep-ph/0301099)
[10] Gelfand I M and Shilov G E 1964 *Generalized Functions, Vol.I* (AP)
[11] Aoki Y, Fodor Z, Katz S D and Szabo K K 2006 *J. High Energy Phys.* **0601** 089 (*Preprint* hep-lat/0510084)
[12] Schmidt C, Fodor Z and Katz S D 2005 *PoS LAT2005* 163 (*Preprint* arXiv:hep-lat/0510087)
[13] Karsch F 2008 *J. Phys. G: Nucl. Part. Phys.* **35** 104096 (*Preprint* arXiv:0804.4148 [hep-lat]), 2007 *PoS CPOD07* 026 (*Preprint* arXiv:0711.0656 [hep-lat]), 2007 *PoS LAT2007* 015 (*Preprint* arXiv:0711.0661 [hep-lat])
[14] Cheng M et al. 2008 *Phys. Rev* D **77** 014511 (*Preprint* arXiv:0710.0354 [hep-lat])
[15] Gupta R et al. 2008 *PoS LATTICE2008* 170 (*Preprint* arXiv:0810.1764 [hep-lat])
[16] Levai P and Heinz U 1998 *Phys. Rev.* C **57** 1879 (*Preprint* hep-ph/9710463)
[17] Biró T S, Shanenko A A and Toneev V D 2003 *Physics of Atomic Nuclei* **66** 982
[18] Szabo K K and Toth A I 2003 *J. High Energy Phys.* **0306** 008 (*Preprint* hep-ph/0302255)
[19] Peshier A, Kampfer B, Pavlenko O P and Soff G 1996 *Phys. Rev.* D **54** 2399
[20] Peshier A, Kampfer B and Soff G 2000 *Phys. Rev.* C **61** 045203 (*Preprint* hep-ph/9911474)
[21] Bluhm M, Kampfer B and Soff G 2005 *Phys. Lett.* B **620** 131 (*Preprint* hep-ph/0411106)
[22] Thaler M A, Schneider R A and Weise W 2004 *Phys. Rev.* C **69** 035210
[23] Ratti C, Roessner S, Thaler M A and Weise W 2007 *Eur. Phys. J.* C **49** 213 (*Preprint* hep-ph/0609218)
[24] Ivanov Y B, Skokov V V and Toneev V D 2005 *Phys. Rev.* D **71** 014005 (*Preprint* hep-ph/0410127)
[25] Cassing W 2007 *Nucl. Phys.* A **791** 365
[26] Cornwall J M, Jackiw R and Tomboulis E 1974 *Phys. Rev.* D **10** 2428
[27] Barnaföldi G G and Gogokhia V 2010 *J. Phys. G: Nucl. Part. Phys.* **37** 025003 (*Preprint* arXiv:0708.0163 [hep-ph], [hep-th])
[28] Leonidov A, Redlich K, Satz H, Subonen E and Weber G 1994 *Phys. Rev.* D **50** 4657
[29] Schafer T 2000 *Nucl. Phys.* B **575** 269
[30] Rajaraman R 1975 *Phys. Rep.* C **21** 227
[31] Callan C, Dashen R, Gross D 1979 *Phys. Rev.* D **20** 3279
[32] Golub G.H., van Loan C.F 1996 *Matrix Computations* (J. Hopkins University Press, Baltimore)
[33] Rubakov V A 1999 *Classical Gauge Fields* (Moscow: Editorial YRSS)
[34] Confinement, Duality, and Nonperturbative Aspects of QCD, edited by P. van Baal, NATO ASI, Series B: Physics, vol. 368 (Plenum, New York, 1997)
[35] Non-Perturbative QCD, Structure of the QCD Vacuum, edited by K-I. Aoki, O. Miymura and T. Suzuki, Prog. Theor. Phys. Suppl. 131 (1998) 1
[36] Marciano W and Pagels H 1978 *Phys. Rep.* C **36** 137
[37] Chodos A, Jaffe R L, Johnson K, Torn C B and Weisskopf V F 1974 *Phys. Rev.* D **9** 3471
[38] DeGrand T, Jaffe R L, Johnson K and Kiskis J 1975 *Phys. Rev.* D **12** 2060
[39] Shuryak E V 1984 *Phys. Rep.* **115** 151

[40] Chanowitz M S and Sharpe S 1983 *Nucl. Phys.* B **222** 211
[41] Shifman M A, Vainshtein A I and Zakharov V I 1979 *Nucl. Phys.* B **147** 385
[42] Novikov V A, Shifman M A, Vainshtein A I and Zakharov V I 1981 *Nucl. Phys.* B **191** 301
[43] Gogokhia V 2009 *Int. J. Theor. Phys.* **48** 3061 (*Preprint* arXiv:0806.0247 [hep-th], [hep-ph])
[44] Jaffe A and Witten E, Yang-Mills Existence and Mass Gap, *http://www.claymath.org/prize-problems/*, *http://www.arthurjaffe.com*
[45] Gogokhia V 2009 *Int. J. Theor. Phys.* **48** 3470 (*Preprint* arXiv:0904.2266 [hep-ph], [hep-th])
[46] Gogokhia V 2009 *Int. J. Theor. Phys.* **48** 3449 (*Preprint* arXiv:0907.0082 [hep-ph], [hep-th])
[47] Dolan L and Jakiw R 1974 *Phys. Rev.* D **9** 3320
[48] H. Leutwyler, in Proceedings of the Conference $QCD - 20\ YEARS\ LATER$ (ed. by P.M. Zerwas and H.A. Kastrup) World Scientific 1993, p. 693
[49] Boyd G, Engels J, Karsch F, Laermann E, Legeland C, Lutgemeier M and Petersson B 1996 *Nucl. Phys.* B **469** 419 (*Preprint* hep-lat/9602007)
[50] Fukugita M, Okawa M and Ukawa A 1990 *Nucl. Phys.* B **337** 181
[51] Iwasaki Y et al. 1992 *Phys. Rev.* D **46** 4657
[52] Panero M 2009 *Phys. Rev. Lett.* **103** 232001 (*Preprint* arXiv:0907.3719 [hep-lat])
[53] Gyulassy M and McLerran L 2005 *Nucl. Phys.* A **750** 30
[54] Pisarski R D 2006 *Phys. Rev.* D **74** 121703(R)
[55] Pisarski R D 2007 *Prog. Theor. Phys. Suppl.* **168** 276 (*Preprint* hep-ph/0612191)
[56] Andreev O 2007 *Phys. Rev.* D **76** 087702
[57] Biró T S and Cleymans J 2008 *Phys. Rev.* C **78** 034902
[58] Shuryak E 2009 *Prog. Part. Nucl. Phys.* **62** 48 (*Preprint* arXiv:0807.3033 [hep-ph])
[59] D'Elia M, Di Giacomo A and Meggiolaro E 2003 *Phys. Rev.* D **67** 114504
[60] Buisseret F 2009 *Preprint* arXiv:0912.0678 [hep-ph]
[61] Mathieu V, Kochelev A K, Vento V 2009 *Int. J. Mod. Phys.* E **18** 1 (*Preprint* arXiv:0810.4453 [hep-ph])
[62] Flechsig F, Rebhan A K and Schulz H 1995 *Phys. Rev.* D **52** 2994
[63] Meyer H B, 2009 *Phys. Rev.* D **80** 051502(R)
[64] Ishii N and Suganuma H 2003 *Eur. Phys. J.* A **17** 77
[65] McLerran L and Pisarski R D 2007 *Nucl. Phys* A **796** 83
[66] Glozman L Ya and Wagenbrunn R F 2008 *Phys. Rev* D **77** 054027
[67] Andronic A et al. 2009 *Preprint* arXiv:0911.4806 [hep-ph]
[68] Hietanen A, Kajantie K, Laine M, Rummukainen K and Schroder Y 2009 *Phys. Rev.* D **79** 045018
[69] Particle data group, 2006 *J. Phys. G: Nucl. Part. Phys.* **33** 1
[70] Narison S 1996 *Phys. Lett.* B **387** 162