For a Data-Driven Interpretation of Rules wrt GMP Conclusions in Abductive Problems

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Abstract

Abductive reasoning is an explanatory process in which potential causes of an observation are unearthed. In its classical – crisp – version it offers little lattitude for discovery of new knowledge. Placed in a fuzzy context, abduction can explain observations which did not, originally, exactly match the expected conclusions. Studying the effects of slight modifications through the use of linguistic modifiers was, therefore, of interest in order to describe the extent to which observations can be modified yet still explained and, possibly, create new knowledge. We will concentrate on the formal definition of fuzzy abduction given by Mellouli and Bouchon-Meunier. Our results will be shown to be incompatible with established theories. We will show where this incompatibility comes from and derive from it a selection of fuzzy implication, based on observable data.

Key words: Abductive reasoning, fuzzy inference, fuzzy implications, Generalised Modus Ponens

1 Introduction

Abductive reasoning, or inference to the best explanation, constructs a set of likely explanations for an observation, given the current knowledge of the world. It has been used in various fields such as medical diagnosis [1,2], fault diagnosis [3,4], query refinement [5], natural text comprehension [6,7] or even

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Preprint submitted to Journal of Uncertain Systems 28 May 2009

web browsing [8] where context relative hypotheses are required to interpret observations.

These fields are generally imprecise and uncertain either because of human intervention or because the data in question simply is not crisp. These imprecision and uncertainty of data or knowledge would justify the use of a fuzzy formalism in themselves. The added benefit of a fuzzy formalism is that the observed data need not necessarily match the knowledge exactly.

Therefore we think that the study of linguistic modifiers in abductive reasoning is essential in preserving added information. For instance, if an appendicitis implies a pain in the lower abdomen, shouldn't an observed *severe* pain in the abdomen lead to conclude to a peritonitis rather than, at best, the original appendicitis?

We limit our study to the formal description of fuzzy abductive reasoning given by Mellouli and Bouchon-Meunier in [9,10] which generalises classical abduction by inversing the Generalised Modus Ponens (GMP). We will start this paper with a reminder of these results and their origin in Section 2.3. In Section 2.4 we briefly summarise the different types of rules, as specified by Dubois and Prade [11,12], in particular those for which Mellouli and Bouchon-Meunier have chosen to reverse the GMP. This initial segmentation of the problem will set the context of our work.

This context is 'fuzzy implication dependent'. Therefore, Sections 3 and 4 introduce our first results using different linguistic modifiers and Gödel's and Łukasiewicz's implications. One of these results is shown to be in contradiction with GMP consequences.

Section 5 sets off with an explanation of the original inconsistency and moves on to explain how we may use this result to classify rules and give them an 'a posteriori' interpretation. To do this we introduce, in Section 5.2, the implication and GMP operators we will focus on and revise some results on the GMP.

We conclude this paper with our perspectives on future works on the GMP for abductive reasoning, on a different semantic classification of fuzzy implications and on our continuing study of the fuzzification of abductive reasoning, for other operators, complex rules and hypotheses classification.

2 Problem setting

2.1 Abduction

Suppose you know that 'If it rains, then the ground is wet' and you wake up to find that the ground is indeed wet. What abduction suggests is that it might have rained. Formally, a crisp abduction scheme is represented in Table 1. What it says is that, given your knowledge of the world, p is a suitable explanation for an observed event q. It also says that any rule which concludes on the observation q offers a candidate explanation in its premise.

Rule:	If p, then q
Observation:	q
Explanation:	p is a possible explanation for observation q

Table 1
General crisp abduction scheme

Suppose, now, you also knew that 'If the automatic watering system goes off, then the ground is wet'. Then you would have two possible explanations - setting aside their conjunction, for the time being - with no way of establishing which was more valid. This would be the case, as we have already seen, for any rule concluding on wet ground. The problem of ranking candidate explanations is a complicated and usually task specific problem in itself, and not one we will address in this work. We will suppose, from here on out, that we have one observation and study the construction of one explanation for it. Should more than one rule match the observation, we would apply the same method to generate all potential explanations and only then determine which, in our context, was the most satisfactory.

We will also, at this time, set aside the problem of complex conclusions. Suppose that 'If it rains, then the ground is wet and there are clouds' and you observe, once again, the ground to be wet. You could either condition the explanation of this observation to the observation of clouds (i.e. knowing the ground is wet, if there are clouds, then it may have rained), or you could use this 'missing' (often referred to as 'abduced') information to subsequently rank generated explanations. The described crisp scheme for abduction obviously offers little alternatives, since observable elements either match, do not match or have not yet been observed. We will see that this is different in our studied fuzzy abduction scheme.

2.2 Fuzzy Abduction and Rule Types

In the crisp context abduction provides, for a given observation, the premises of the rules to which it was a conclusion. This is relatively forthcoming from the fact that if the premise were true, then the conclusion *must* be true. This does not mean, obviously, that the premise *is* true, but that it qualifies as an adequate hypothesis, as a sufficient condition for the observation to be true.

The adaptation of this process to approximate reasoning is complicated by the fact that, in general, a rule does not entail a single, fixed conclusion. A fuzzy conclusion is conditionned by, amongst other things, the shape of the observation. This is also a benefit to abductive reasoning, as observations do not need to match known conclusions exactly. However, identifying potential explanations in a fuzzy context still needs to find premises whose truth would maximise the chances of making the observation. One way of doing this is to look at the fuzzy inference scheme. Just like crisp abduction tries to find a hypothesis H such that the observation O follows from H's combination with known facts or theory T (i.e. $T \cup H \models O$), the abduction scheme we will focus on will build constraints on the rule's original premise, so that the observation is indeed a consequence of our hypothesis, as shown in Table 2.

Resrictions on studied rules

We will, in the rest of this paper, consider rules of the type 'If u is A, then v is B', where A (respectively B) is a fuzzy label on variable u (respectively v) defined on an universe of discourse U (respectively V). The membership functions for either label will be denoted $f_A(u)$ and $f_B(v)$, respectively. Observations in the abduction context will be denoted B' and their explanations either A' or A_G depending on the context and as specified. Conversely, in the deductive context observations will be denoted A' and their consequences B'.

Rule:	If u is A, then v is B
Observation:	v is B'
Explanation:	What A' would be a satisfactory explanation for this observation?

Table 2
General fuzzy abduction scheme

2.3 Reversing the Generalised Modus Ponens

Mellouli and Bouchon-Meunier's approach of abduction [9,10] aims at finding conditions on premise A so that observation B' is satisfied. To do this,

they choose to reverse the Generalised Modus Ponens (GMP), the fuzzy inference model, firstly because it ensures that the conditions on A entail B', and secondly because it gives a mathematical expression of said conditions. Obviously, due to the large number of fuzzy implication and GMP operators at hand, one cannot consider such a task as a unique problem. Mellouli and Bouchon-Meunier therefore considered the different classes of fuzzy implications as described by Dubois and Prade in [11,12], as well as by Mas et alii in [13], and presented in Table 3. In their works they reversed the GMP for two classes of implications: s-implications and r-implications. The results for s-implications gave an expression of $f_{A'}(u)$ the conditionned premise's membership function. They chose not to delve into an in-depth study of this result, because it offered no immediate difficulty. On the other hand, the reversal of the GMP for r-implications – denoted I_R in Equation 1 – resulted in the definition of a 'maximal explanation A_G' such that any adequate explanation A' should be included in A_G.
This maximal explanation A_G is given by:

$$\forall u \in U, f_{A_G}(u) = \inf_{v \in V} I_R(I_R(f_A(u), f_B(v)), f_{B'}(v)) \tag{1}$$

The authors, then studied the effects of linguistic modifiers on their maximal explanation. They showed that some modifiers were preserved in the maximal explanation using Gödel's implication and Zadeh's GMP operator. The interest of linguistic modifiers on abductive reasoning is rather similar to their influence in conventional inference. Indeed, since in a fuzzy rule-base every rule is triggered to the amount – possibly null – to which its premise matches an observation, studying modifications of the observation offers an idea of what conclusions can be drawn. Similarly, in an abduction context, one may generalise from the effects of modified observations and thus build conditions on the observations and on the hypotheses. Mellouli and Bouchon-Meunier studied a particular class of modifiers (viz. uncertain expansive modifiers) and, then only using Gödel's implication. We wished to see if we could generalise their results to other types of modifiers and other implication/t-norm pairs. This article is an extension of our previous work [14].

2.4 Fuzzy rule classification

Table 3 gives an overview of the most usual fuzzy-implication classes, their boolean origin and their expression as functions of membership functions, where $\top$ represents a t-norm and $\perp$ a t-conorm.

From these formally different classes of implications, Dubois and Prade define semantic classes of operators. They give each class their interpretation of its boolean counterpart. Mellouli and Bouchon-Meunier concentrate on two of

Class	Boolean equivalent	General form
s-implication (strong implication)	$p \to q \Leftrightarrow \neg p \vee q$	$I(a,b) = \bot(\neg a, b)$
r-implication (residual implication)	$p \wedge (\neg p \vee q) = p \wedge q \Rightarrow p \wedge (p \to q) \leqslant q$	$I(a,b) = \sup\{w \in [0,1] \mid \top(a,w) \leqslant b\}$
ql-implication (quantum-logic implication)	$\neg p \vee q = \neg p \vee (p \wedge q) = \neg(p \wedge \neg(p \wedge q))$	$I(a,b) = \neg\top(a, \neg\top(a,b))$
t-implication (t-norm implication)	none, doesn't preserve boolean implication	$I(a,b) = \top(a,b)$

Table 3
Common classes of fuzzy implications and their boolean counterparts

these semantic classes, namely: 'certainty rules' built from s-implications ('The more u is A the more certain v lies in B', e.g. 'The younger a man, the more likely he is single') and 'gradual rules' from r-implications ('The more u is A the more v is B', e.g. 'The more typical the symptoms, the more likely the diagnosis').

In [9], the authors study the effects of expansive modifiers on an observation, given a gradual rule and Gödel's implication. Mellouli and Bouchon-Meunier use their definition of the membership function of the maximal solution to this type of problem to prove that, in some cases, the original modifiers may be preserved.

Our aim here is to generalise these results to other hedges and implications. We have chosen to study classical power modifiers as defined by Zadeh [15,16], and translation modifiers considered by Bouchon-Meunier and Yao [17]. We will finally consider the particular cases, defined from these translations, of reinforcement hedges which contract both supports and cores (i.e. $f_{B'}(v) = \min(f_B(v+\varepsilon), f_B(v-\varepsilon))$) and their inverses which dilate them. These modifiers are semantically consistent with Zadeh's interpretation, yet their impact on the support and kernel of the original labels implies a shift in precision, both formally and intuitively.

Because we will, essentially, be studying gradual rules as did Bouchon-Meunier and Mellouli, we would like to introduce an illustrative example, to which we will refer throughout the paper. Suppose we have an expert built medical diagnosis rule system. Among these rules we have one about chickenpox, which states that 'From fourteen days before the first eruption of vesicles to seven days after, the subject is contagious' and that the subject is most contagious from two days before to five days after the breakout. Now, Figure 1 illustrates this rule. Note that the time periods in the premise are before (T_0), during (T_1) and after (T_2) the outbreak of the illness and that the subject is either not sick, contagious or healing. The other labels are given to show how our observations are 'measured'. Our rule only links the during stage to the contagious state. More importantly, the kernel of the 'during' (T_1) period corresponds to the maximally contagious stage. That is to say that, the more specifically in the during period a patient, the more specifically contagious.

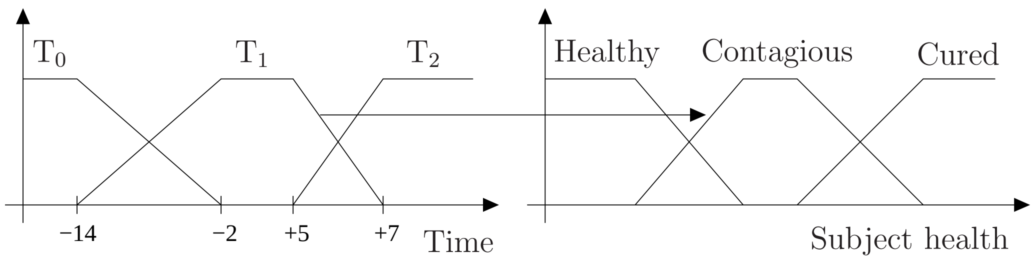


Fig. 1. Chickenpox contagiousness

2.5 Notation

In the rest of this document we will suppose that $B \subset V$ is a trapezoidal fuzzy subset of V, defined as $B = \langle s_{B_L}, k_{B_L}, k_{B_R}, s_{B_R} \rangle$ where $s_{B_L} \leqslant k_{B_L} \leqslant k_{B_R} \leqslant s_{B_R}$ and $Support(B) = (s_{B_L}, s_{B_R})$, $Kernel(B) = [k_{B_L}, k_{B_R}]$. We will also suppose $\inf\limits_{v \in V} f_B(v) = 0$ and $\sup\limits_{v \in V} f_B(v) = 1$, and make similar assumptions for $A \subset U$, $A = \langle s_{A_L}, k_{A_L}, k_{A_R}, s_{A_R} \rangle$.

3 Gödel Implication

We will start using Gödel's implication, I_G, the same way Mellouli and Bouchon-Meunier have. The results we will present have, however, been constructed using modifiers which introduce no uncertainty (i.e. $\inf\limits_{v \in V} f_{B'}(v) = 0$).

As a reminder, I_G is given by: $I_G(a, b) = \begin{cases} 1 \text{ if } a \leqslant b \\ b \text{ otherwise} \end{cases}$

3.1 Inexplicable observations

Our first result is a generalisation on the 'inexplicability' of certain modified observations. The original modifiers were shifting modifiers and Zadeh's 'very'. We shall see that the latter does not comply with the hypothetical v_0 introduced in the condition of Proposition 1, since $Support(B') \equiv Support(B)$.

However, we can still prove the same result because $\inf_{v\in V, f_{B'}(v)<f_B(v)\leqslant f_A(u)} f_{B'}(v) = 0$.

***Proposition* 1** *Given Gödel's implication and the definition for the maximal solution to a gradual rule abductive problem given in Formula 1, the gradual rule 'If u is A, then v is B' and an observation B' such that there exists a $v_0 \in V$ with $0 = f_{B'}(v_0) < f_B(v_0)$, then $A_G = \varnothing$, i.e. the maximal solution is empty and no explanation can be derived.*

Proof Suppose B' is such that there exists some $v_0 \in V$ with $0 = f_{B'}(v_0) < f_B(v_0)$. Let $B_- = \{v \in V | f_{B'}(v) < f_B(v)\}$ and $B_+ = V \setminus B_-$. Then

$$f_{B'}(v) \geqslant f_B(v), \forall v \in B_+ \text{ and } f_{B'}(v) < f_B(v), \forall v \in B_-$$

$$\begin{aligned} f_{A_G}(u) &= \inf_{v\in V} I_G(I_G(f_A(u), f_B(v)), f_{B'}(v)) \\ &= \min(M, N) \end{aligned}$$

where

$$\begin{aligned} M &= \inf\nolimits_{v\in V, f_B(v)\geqslant f_A(u)} I_G(I_G(f_A(u), f_B(v)), f_{B'}(v)) \\ &= \inf\nolimits_{v\in V, f_B(v)\geqslant f_A(u)} I_G(1, f_{B'}(v)) \\ &= \inf\nolimits_{v\in V, f_B(v)\geqslant f_A(u)} f_{B'}(v) \end{aligned}$$

and

$$\begin{aligned} N &= \inf\nolimits_{v\in V, f_B(v)\leqslant f_A(u)} I_G(I_G(f_A(u), f_B(v)), f_{B'}(v)) \\ &= \inf\nolimits_{v\in V, f_B(v)\leqslant f_A(u)} I_G(f_B(v), f_{B'}(v)) \\ &= \min(N_1, N_2) \end{aligned}$$

where

$$\begin{aligned} N_1 &= \inf\nolimits_{v\in B_+, f_B(v)\leqslant f_A(u)} I_G(f_B(v), f_{B'}(v)) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} N_2 &= \inf\nolimits_{v\in B_-, f_B(v)\leqslant f_A(u)} I_G(f_B(v), f_{B'}(v)) \\ &= \inf\nolimits_{v\in B_-, f_B(v)\leqslant f_A(u)} f_{B'}(v) \\ &= 0 \end{aligned}$$

Thus, $\boldsymbol{f_{A_G}(u) = 0, \forall u \in U,}$ **and** $\boldsymbol{A_G = \varnothing}$

This first result shows that inclusion, even partial, in the expected conclusion renders an observation inexplicable. Indeed, since the maximal explanation is empty, all potential explanations are included in the empty-set and are therefore empty. Therefore, if we have a patient whose condition is extremely specific of the contagious stage of our illness (see Figure 2), we cannot explain our observation.

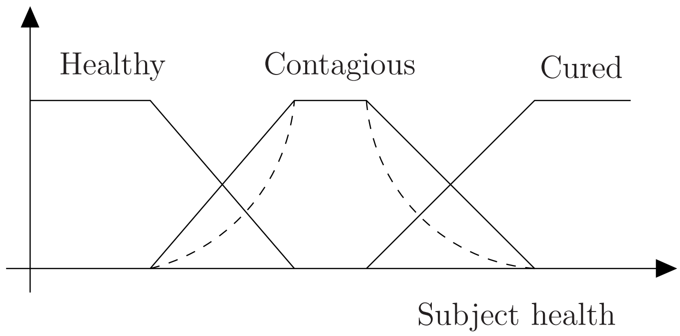


Fig. 2. Very specifically contagious subject

3.2 Zadeh's original 'approximately' modifier

Our observed 'approximately B' is such that:

$$\forall v \in V, f_{B'}(v) = f_B(v)^\alpha, 0 < \alpha < 1$$

so

$$f_{B'}(v) \geqslant f_B(v), \forall v \in V$$

$$
\begin{aligned}
f_{A_G}(u) &= \inf_{v \in V} I_G(I_G(f_A(u), f_B(v)), f_B(v)^\alpha) \\
&= \min(M, N)
\end{aligned}
$$

where

$$
\begin{aligned}
M &= \inf_{v \in V, f_A(u) \leqslant f_B(v)} I_G(1, f_B(v)^\alpha) \\
&= \inf_{v \in V, f_A(u) \leqslant f_B(v)} f_B(v)^\alpha \\
&= f_A(u)^\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
N &= \inf_{v \in V, f_A(u) > f_B(v)} I_G(I_G(f_A(u), f_B(v)), f_B(v)^\alpha) \\
&= \inf_{v \in V, f_A(u) > f_B(v)} I_G(f_B(v), f_B(v)^\alpha) = 1
\end{aligned}
$$

Therefore, $\boldsymbol{f_{A_G}(u) = f_A(u)^\alpha, \forall u \in U}$

We interpret this as 'If the observation is *approximately* B, then the explanation is, *at most, approximately* A'.

3.3 An expansive modifier inflating Kernel and Support

Our observed categorisation B' is such that:

$$\forall v \in V, f_{B'}(v) = \max(f_B(v+\varepsilon), f_B(v-\varepsilon))$$

therefore we have:

$$f_{B'}(v) \geqslant f_B(v), \forall v \in V$$

$\forall v \in V$, let $f_{B_+}(v) = \dfrac{v - s_{B_L}}{k_{B_L} - s_{B_L}}$ and $f_{B_-}(v) = \dfrac{s_{B_R} - v}{s_{B_R} - k_{B_R}}$.

Let also $\varepsilon_1 = \dfrac{\varepsilon}{k_{B_L} - s_{B_L}}$ and $\varepsilon_2 = \dfrac{\varepsilon}{s_{B_R} - k_{B_R}}$, then we may write, $\forall v \in V$:

$$\begin{aligned}
f_B(v) &= \max(0, \min(1, \min(f_{B_+}(v), f_{B_-}(v)))) \\
f_B(v+\varepsilon) &= \max(0, \min(1, \min(f_{B_+}(v) + \varepsilon_1, f_{B_-}(v) - \varepsilon_2))) \\
f_B(v-\varepsilon) &= \max(0, \min(1, \min(f_{B_+}(v) - \varepsilon_1, f_{B_-}(v) + \varepsilon_2)))
\end{aligned}$$

therefore, $\forall v \in V$:

$$\max(f_B(v-\varepsilon), f_B(v+\varepsilon)) = \max(0, \min(1, \min(f_{B_+}(v) + \varepsilon_1, f_{B_-}(v) + \varepsilon_2)))$$

since $\varepsilon_1 \geqslant 0$ and $\varepsilon_2 \geqslant 0$

$$\begin{aligned}
f_{A_G}(u) &= \inf_{v \in V} I_G(I_G(f_A(u), f_B(v)), f_{B'}(v)) \\
&= \min(M, N)
\end{aligned}$$

where

$$\begin{aligned}
M &= \inf\nolimits_{v \in V, f_A(u) > f_B(v)} I_G(I_G(f_A(u), f_B(v)), f_{B'}(v)) \\
&= \inf\nolimits_{v \in V, f_A(u) > f_B(v)} I_G(f_B(v), f_{B'}(v)) = 1
\end{aligned}$$

and

$$
\begin{aligned}
N &= \inf_{v \in V, f_A(u) \leqslant f_B(v)} I_G(I_G(f_A(u), f_B(v)), f_{B'}(v)) \\
&= \inf_{v \in V, f_A(u) \leqslant f_B(v)} I_G(1, f_{B'}(v)) \\
&= \inf_{v \in V, f_A(u) \leqslant f_B(v)} \max(f_B(v+\varepsilon), f_B(v-\varepsilon)) \\
&= \inf_{v \in V, f_A(u) \leqslant f_B(v)} \max(0, \min(1, \min(f_{B_+}(v)+\varepsilon_1, f_{B_-}(v)+\varepsilon_2))) \\
&= \max(0, \min(1, \min(f_A(u)+\varepsilon_1, f_A(u)+\varepsilon_2))) \\
&= \min(1, f_A(u) + \min(\varepsilon_1, \varepsilon_2))
\end{aligned}
$$

Therefore, $\boldsymbol{f_{A_G}(u) = \min(1, f_A(u) + \min(\varepsilon_1, \varepsilon_2)), \forall u \in U}$

Which we understand as 'If the observation is *around* B, then it is relatively certain that the explanation is, *at most, around* A'. Figure 3 shows an illustration of the maximal explanation of this type of expansive modifier. If the patient is probably contagious but possibly not ill, then he may be in the outbreak stage.

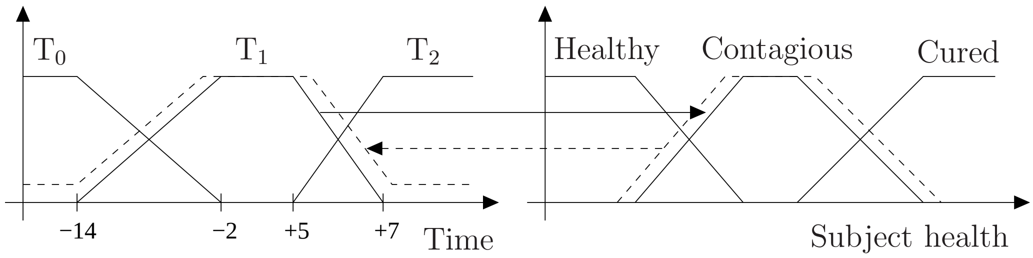


Fig. 3. An expansive modifier inflating *Kernel* and *Support* and its maximal explanation, using Gödel's implication

Proposition 1 states that no observation which is locally more precise than the original conclusion may be explained, in this context (i.e. gradual rule based abductive reasoning using Gödel's implication). Obviously, reinforcement modifiers fall into this category when they contract *Support*(B) and we have already discussed what happens when they do not. Shifting modifiers, criticised by de Cock and Kerre in [18,19] because they don't preserve the 'semantic entailment [1] ', are also inexplicable due to their partial inclusion in the expected conclusion, even though the shift in their relative preciseness is less obvious.

We have also shown, confirming Mellouli and Bouchon-Meunier's results, that an expansive modifier is *nearly* preserved through abduction.

[1] By semantic entailment, de Cock and Kerre mean:
extremely(A) $\subseteq$ very(A) $\subseteq A \subseteq$ more_or_less(A) $\subseteq$ about(A)

4 Łukasiewicz Implication

We will now continue our study of modified observations using a different implication operator, namely Łukasiewicz's implication given by:
$I_L(a, b) = \min(1 - a + b, 1)$.

4.1 Expansive modifiers and Łukasiewicz

Once again, Theorem 1 derives from the generalisation of results on expansive modifiers presenting no uncertainty. We will see how this result affects our view on fuzzy abduction.

***Theorem* 1** *Given Łukasiewicz's implication, a gradual-rule abductive problem and an observation B' such that $f_{B'}(v) \geqslant f_B(v), \forall v \in V$ and supposing $\exists v_0 \in V$ such that $f_{B'}(v_0) = f_B(v_0) = 0$, then the maximal explanation is $A_G = A$.*

Proof

$$\begin{aligned} f_{A_G}(u) &= \inf_{v \in V} I_L(I_L(f_A(u), f_B(v)), f_{B'}(v)) \\ &= \inf_{v \in V} I_L(\min(1 - f_A(u) + f_B(v), 1), f_{B'}(v)) \\ &= \inf_{v \in V} \min(1 - \min(1 - f_A(u) + f_B(v), 1) + f_{B'}(v), 1) \\ &= \min(M, N) \end{aligned}$$

where

$$\begin{aligned} M &= \inf_{v \in V, f_B(v) \geqslant f_A(u)} \min(1 - \min(1 - f_A(u) + f_B(v), 1) + f_{B'}(v), 1) \\ &= \inf_{v \in V, f_B(v) \geqslant f_A(u)} \min(f_{B'}(v), 1) \\ &= \inf_{v \in V, f_B(v) \geqslant f_A(u)} f_{B'}(v) \end{aligned}$$

and

$$\begin{aligned} N &= \inf_{v \in V, f_B(v) \leqslant f_A(u)} \min(1 - \min(1 - f_A(u) + f_B(v), 1) + f_{B'}(v), 1) \\ &= \inf_{v \in V, f_B(v) \leqslant f_A(u)} \min(f_A(u) - f_B(v) + f_{B'}(v), 1) \\ &= \inf_{v \in V, f_B(v) \leqslant f_A(u) \leqslant 1 + f_B(v) - f_{B'}(v)} f_A(u) - f_B(v) + f_{B'}(v) \\ &= f_A(u) + \inf_{v \in V, f_B(v) \leqslant f_A(u) \leqslant 1 + f_B(v) - f_{B'}(v)} f_{B'}(v) - f_B(v) \end{aligned}$$

Since $\forall v \in V, f_{B'}(v) - f_B(v) \geqslant 0 = f_{B'}(v_0) - f_B(v_0)$ and since $\forall u \in U, f_A(u) \geqslant 0 = f_B(v_0)$ we show that $N = f_A(u) + f_{B'}(v_0) - f_B(v_0) = f_A(u)$. Also, $M \geqslant f_A(u)$ because $f_{B'}(v) \geqslant f_B(v)$ and $\inf_{v \in V, f_B(v) \geqslant f_A(u)} f_B(v) = f_A(u)$.

Therefore we have proved that $M \geqslant N$ and that:

$$\boldsymbol{f_{A_G}(u) = f_A(u), \forall u \in U}$$

This particular result is problematic in that it is in obvious contradiction with Mellouli and Bouchon-Meunier's results on the general shape of explanations with respect to the relative inclusion of the observation in the expected conclusion (i.e. if $B_1 \supset B_2$, then $A_{G_1} \supset A_{G_2}$).

In addition to this, since A_G is the maximal explanation of B', any A' which adequately explains the observation should be included in A_G. Studies on the GMP have shown that, given the original rule 'If u is A, then v is B', no such $A' \subset A$ can entail a $B' \supset B$.

Section 5 of this paper will show why this is the case and how we may use this result.

4.2 Zadeh's 'very' modifier and the like

Our observed categorisation B' is such that:

$$\forall v \in V, f_{B'}(v) = f_B(v)^\alpha, \alpha > 1$$

consequently we have:

$$f_{B'}(v) \leqslant f_B(v), \forall v \in V$$

$$\begin{aligned}
f_{A_G}(u) &= \inf_{v \in V} I_L(I_L(f_A(u), f_B(v)), f_B(v)^\alpha) \\
&= \inf_{v \in V} I_L(\min(1 - f_A(u) + f_B(v), 1), f_B(v)^\alpha) \\
&= \inf_{v \in V} \min(1 - \min(1 - f_A(u) + f_B(v), 1) + f_B(v)^\alpha, 1) \\
&= \min(M, N)
\end{aligned}$$

where

$$\begin{aligned}
M &= \inf\nolimits_{v \in V, f_B(v) \geqslant f_A(u)} \min(f_B(v)^\alpha, 1) \\
&= \inf\nolimits_{v \in V, f_B(v) \geqslant f_A(u)} f_B(v)^\alpha \\
&= f_A(u)^\alpha
\end{aligned}$$

and

$$\begin{aligned} N &= \inf_{v\in V, f_B(v)<f_A(u)} \min(f_A(u) - f_B(v) + f_B(v)^\alpha, 1) \\ &= \inf_{v\in V, f_B(v)<f_A(u)} f_A(u) - f_B(v) + f_B(v)^\alpha \\ &= f_A(u) + \inf_{v\in V, f_B(v)<f_A(u)} f_B(v)^\alpha - f_B(v) \\ &= \min(N_1, N_2) \end{aligned}$$

Let $f_\alpha(v) = f_B(v)^\alpha - f_B(v), \forall v \in V$. We know that $f_\alpha(v) \leqslant 0$ everywhere, and furthermore that $f_\alpha(v) < 0, \forall v \in V$ such that $0 < f_B(v) < 1$. Therefore there is a γ such that: $\gamma^\alpha - \gamma = \inf\limits_{v\in V} f_\alpha(v)$ and $f_\alpha(v)$ decreases for all v such that $f_B(v) \leqslant \gamma$ and increases for all v such that $f_B(v) \geqslant \gamma$. Therefore, we may write:

$$\begin{aligned} N_1 &= f_A(u) + \inf_{v\in V, f_B(v)<f_A(u)<\gamma} f_B(v)^\alpha - f_B(v) \\ &= f_A(u)^\alpha \end{aligned}$$

and

$$\begin{aligned} N_2 &= f_A(u) + \inf_{v\in V, f_B(v)<f_A(u), f_A(u)\geqslant\gamma} f_B(v)^\alpha - f_B(v) \\ &= f_A(u) - \gamma + \gamma^\alpha \end{aligned}$$

$$\text{therefore, } \boldsymbol{f_{A_G}(u) = \begin{cases} f_A(u) - \gamma + \gamma^\alpha & \textbf{if } f_A(u) \geqslant \gamma = \dfrac{1}{\alpha^{1/(\alpha-1)}} \\ f_A(u)^\alpha & \textbf{otherwise} \end{cases}}$$

We observe, here, the introduction of an imprecision in the de-normalisation of f_{A_G}. We may interpret this as 'if B' is *very* B', then it is '$\gamma - \gamma^\alpha$ unlikely that A_G be *very* A'.

In this case our initial intuition, though somewhat tamed down, seems to be confirmed. Indeed, in our original example, if we were to observe a 'very strong pain in the lower abdomen' we might conclude on a '$\frac{1}{4}$ uncertain serious case of appendicitis', or 'three quarters of a chance of peritonitis'.

4.3 A translation modifier explained by Łukasiewicz

Suppose we were to observe a B' such that:

$$\forall v \in V, f_{B'}(v) = f_B(v + \varepsilon)$$

Let $V_{B1} = (k_{B_R} - \varepsilon, s_{B_R})$ and $V_{B2} = V \setminus V_{B1}$, then $f_{B'}(v) < f_B(v), \forall v \in V_{B1}$ and $f_{B'}(v) \geqslant f_B(v), \forall v \in V_{B2}$.

$$\begin{aligned}
f_{A_G}(u) &= \inf_{v \in V} I_L(I_L(f_A(u), f_B(v)), f_B(v+\varepsilon)) \\
&= \inf_{v \in V} I_L(\min(1 - f_A(u) + f_B(v), 1), f_B(v+\varepsilon)) \\
&= \inf_{v \in V} \min(1 - \min(1 - f_A(u) + f_B(v), 1) + f_B(v+\varepsilon), 1) \\
&= \min(M, N)
\end{aligned}$$

where

$$M = \inf_{v \in V, f_B(v) > f_A(u)} f_B(v+\varepsilon) = \min(M_1, M_2)$$

with

$$\begin{aligned}
M_1 &= \inf_{v \in V_{B1}, f_B(v) > f_A(u)} f_B(v+\varepsilon) \\
&= \inf_{v \in V_{B1}, f_B(v) > f_A(u)} \max(0, \min(1, \min(f_{B_+}(v) + \varepsilon_1, f_{B_-}(v) - \varepsilon_2))), \text{ see § 3.3} \\
&= \max(f_A(u) - \varepsilon_2, 0)
\end{aligned}$$

and

$$M_2 = \inf_{v \in V_{B2}, f_B(v) > f_A(u)} f_B(v+\varepsilon) = f_A(u) + \gamma \geqslant f_A(u), \text{ since } f_B(v+\varepsilon) \geqslant f_B(v)$$

and

$$\begin{aligned}
N &= \inf_{v \in V, f_B(v) \leqslant f_A(u)} \min(f_A(u) - f_B(v) + f_B(v+\varepsilon), 1) \\
&= \inf_{v \in V, f_B(v) \leqslant f_A(u) \leqslant 1 + f_B(v) - f_B(v+\varepsilon)} f_A(u) - f_B(v) + f_B(v+\varepsilon) \\
&= f_A(u) + \inf_{v \in V, f_B(v) \leqslant f_A(u) \leqslant 1 + f_B(v) - f_B(v+\varepsilon)} f_B(v+\varepsilon) - f_B(v) \\
&= \max(f_A(u) - \varepsilon_2, 0)
\end{aligned}$$

therefore, $\boldsymbol{f_{A_G}(u) = \max(f_A(u) - \varepsilon_2, 0), \forall u \in U}$

Similarly, we show that given an observation B' shifted to the left, i.e. $f_{B'}(v) = f_B(v - \varepsilon)$, we conclude on the maximal hypothesis A_G such that:

$$\boldsymbol{f_{A_G}(u) = \max(f_A(u) - \varepsilon_1, 0), \forall u \in U}$$

Figure 4 shows that should the subject be somewhere between contagious and cured, than she is in the second stage of her illness to a lesser degree than expected and never fully. We understand this denormalisation as a doubt in the membership.

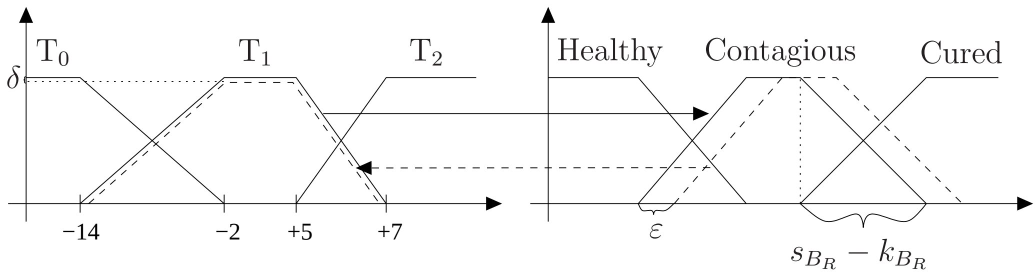


Fig. 4. Introduction of some doubt by Łukasiewicz and a translation, where $\delta = \frac{\varepsilon}{s_{B_R} - k_{B_R}}$

4.4 Contraction of Support and Kernel using modifiers

Our observed categorisation B' is such that:

$$\forall v \in V, f_{B'}(v) = \min(f_B(v+\varepsilon), f_B(v-\varepsilon))$$

Consequently

$$f_{B'}(v) \leqslant f_B(v), \forall v \in V$$

$$\begin{aligned}
f_{A_G}(u) &= \inf_{v\in V} I_L(I_L(f_A(u), f_B(v)), f_{B'}(v)) \\
&= \inf_{v\in V} I_L(\min(1 - f_A(u) + f_B(v), 1), f_{B'}(v)) \\
&= \inf_{v\in V} \min(1 - \min(1 - f_A(u) + f_B(v), 1) + \min(f_B(v+\varepsilon), f_B(v-\varepsilon)), 1) \\
&= \min(M, N)
\end{aligned}$$

where

$$\begin{aligned}
M &= \inf_{v\in V, f_B(v) > f_A(u)} \min(f_B(v+\varepsilon), f_B(v-\varepsilon)) \\
&= \min(\inf_{v\in V, f_B(v) > f_A(u)} f_B(v+\varepsilon), \inf_{v\in V, f_B(v) > f_A(u)} f_B(v-\varepsilon)) \\
&= \min(\max(f_A(u) - \varepsilon_2, 0), \max(f_A(u) - \varepsilon_1, 0)), \text{ see § 4.3} \\
&= \max(f_A(u) - \max(\varepsilon_2, \varepsilon_1), 0)
\end{aligned}$$

and

$$\begin{aligned} N &= \inf_{v\in V, f_B(v)\leqslant f_A(u)} \min(f_A(u) - f_B(v) + \min(f_B(v+\varepsilon), f_B(v-\varepsilon)), 1) \\ &= \inf_{v\in V, f_B(v)\leqslant f_A(u)\leqslant 1+f_B(v)-f_{B'}(v)} f_A(u) - f_B(v) + \min(f_B(v+\varepsilon), f_B(v-\varepsilon)) \\ &= f_A(u) + \inf_{v\in V, f_B(v)\leqslant f_A(u)\leqslant 1+f_B(v)-f_{B'}(v)} \min(f_B(v+\varepsilon), f_B(v-\varepsilon)) - f_B(v) \\ &= \max(f_A(u) - \max(\varepsilon_2, \varepsilon_1), 0) \end{aligned}$$

Therefore, $\boldsymbol{f_{A_G}(u) = \max(f_A(u) - \max(\varepsilon_2, \varepsilon_1), 0), \forall u \in U}$

Figure 5 shows how the denormalised maximal explanation of shifted observation persists in their combination. In this case, if the patient is definitely ill and probably contagious, then he cannot be completely in the outbreak stage.

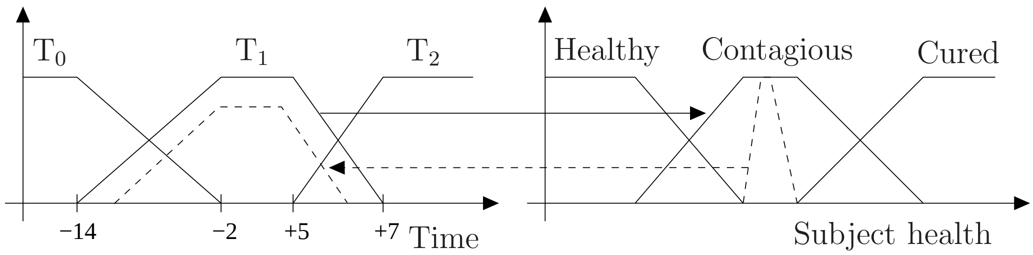


Fig. 5. The $\min(f_B(v+\varepsilon), f_B(v-\varepsilon))$ reinforcement modifier and its de-normalised explanation

We have seen how translation modifiers introduce a de-normalised maximal solution, as does their combination, using Łukasiewicz's implication. We have also proved that the extension of Mellouli and Bouchon-Meunier's formal results on abduction sometimes generates incoherent results. Next section will trace this incoherence's origin and will introduce a way of using this to give a semantically consistent interpretation of a rule set.

5 Classification with respect to observations

Theorem 1 concluded that, in extending Mellouli and Bouchon-Meunier's results, we come upon an incoherence. This section will show why this is not acceptable yet how we may still use it.

5.1 *Origin of inconsistency*

We know that given a gradual-rule abductive problem, Łukasiewicz's implication and an observation such that $B' \supset B$ and $\inf_{v \in V} f_{B'}(v) = 0$, then $A_G = A$. Our problem is that this result is:

- Inconsistent with general results on the Generalised Modus Ponens:
  · if $A' \subseteq A$, then $B' = B$
  · if $A' \supset A$, then $B' \supset B$
- Inconsistent with previous results on abduction
  · if $B'_1 \subset B'_2$, then $A_{G_1} \subset A_{G_2}$, here if $B'_1 = B$, then $A_{G_1} = A$
- Inconsistent with Łukasiewicz as a residual and strong implication
  · Any satisfactory explanation A' is such that $A' \subset A_G$
  · r-implication : $A_G = A$
  · s-implication : $A' = U$

We must therefore reconsider our result in view of these observations. We can trace the inconsistency back to our original observation. Indeed, since given Łukasiewicz's implication the general expression of a GMP conclusion B' is $f_{B'}(v) = \max(\sup_{u \in U, f_A(u) \leqslant f_B(v)} f_{A'}(u), f_B(v) + \sup_{u \in U, f_A(u) \geqslant f_B(v)} f_{A'}(u) - f_A(u))$, we can show that if $B' \supset B$, then $\inf_{v \in V} f_{B'}(v) > 0$.

Similarly, concerning Proposition 1, since using Gödel's implication we get $f_{B'}(v) = \max(\sup_{u \in U, f_B(v) \geqslant f_A(u)} f_{A'}(u), f_B(v))$, we know that $B' \supseteq B$ and our condition is impossible.

Even if the two implications we have studied cannot reach the observations we suggest, these do not seem intuitively incoherent as such. It is therefore possible that there exists some implication which may generate one or the other via Generalised Modus Ponens. If one or more rule's expected conclusion differs only slightly from this observation, the only coherent implication operators will be amongst these.

Furthermore, we claim that, given a feasible observation and a set of rules, we can categorise the set of implications to be used. Since a given observation will match only part of the conclusions in the rule-set, we offer a categorisation of a rule system coherent with observed data. Indeed, in most cases the semantic interpretation of a rule will be given a priori, even if the rule is learnt, and an implication operator chosen regardless of its potential inconsistency with the data. Our approach aims at building entailment consistent rule-subsets, interpreting these with respect to the observed data and giving them the semantic interpretation of the corresponding implication-subset [11,12].

To do this, we need to classify the shapes which may be reached via GMP for each implication and consistent GMP-operator. This type of study has been led in the past yet, since their use was to be different, the results are neither sufficiently precise nor general. Classical studies of the GMP have typically looked at what a precise observation in a given fuzzy premise will generate or at very local modifications [20]. The problem here is that we need to rule out, or accept, a given shape for an implication. So we need to extend the existing results to be certain that no unexpected case is overlooked.

5.2 Examples of GMP conclusions

5.2.1 Foreword

Before we present our study of GMP conclusions with respect to the fuzzy subsets they entail, we think it wise to remind the reader of the general expression of the GMP conclusion and of the expressions of the fuzzy implications we will study, and their classification.

For a fuzzy rule of the type 'If u is A, then v is B' and an observation A', the expected conclusion is given by:

$$f_{B'}(v) = \sup_{u \in U} \top(f_{A'}(u), I(f_A(u), f_B(v)))$$

where A and A' are fuzzy subsets of U, B and B' fuzzy subsets of V, I some fuzzy implication and $\top$ an adequate (i.e. the crisp limit cases are preserved by the joint use of I and $\top$) Generalised Modus Ponens operator, or t-norm.

We will study the fuzzy implications and their respective GMP operators as given by [20] outlined in Table 4, where the GMP operators are given in Table 5.

5.2.2 Reichenbach

A rule used with Reichenbach's implication and Łukasiewicz's GMP operator will conclude on something of the form:

$$f_{B'}(v) = \sup_{u \in U} \max(0, f_{A'}(u) + f_A(u) \times (f_B(v) - 1))$$

from which we draw the following constraints on all conclusions B':

- If $Kernel(A') \cap \overline{Support(A)} \neq \varnothing$, then $B' = V$
- If $A' \supseteq A$, then $B' \supseteq B$ and $\inf_{v \in V} f_{B'}(v) \geqslant \sup_{u \in \overline{Support(A)}} f_{A'}(u)$

Implication	Expression	Class	Compatible t-norm(s)
Reichenbach	$I_R(a,b) = 1 - a + a \times b$	s-implication	Łukasiewicz
Willmott	$I_W(a,b) = max(1-a, min(a,b))$	Ql-implication	Łukasiewicz
Mamdani	$I_M(a,b) = \min(a,b)$	t-implication	Zadeh, Łukasiewicz, Goguen
Rescher-Gaines	$I_{RG}(a,b) = \begin{cases} 1 \text{ if } a \leqslant b \\ 0 \text{ otherwise} \end{cases}$	r-implication	Zadeh, Łukasiewicz, Goguen
Kleene-Dienes	$I_{KD}(a,b) = max(1-a, b)$	s-implication	Łukasiewicz
Gödel	$I_G(a,b) = \begin{cases} 1 \text{ if } a \leqslant b \\ b \text{ otherwise} \end{cases}$	r-implication	Zadeh, Łukasiewicz, Goguen
Goguen	$I_{Gn}(a,b) = \begin{cases} min(b/a, 1) \text{ if } a \neq 0 \\ 1 \text{ otherwise} \end{cases}$	r-implication	Łukasiewicz, Goguen
Łukasiewicz	$I_L(a,b) = \min(1-a+b, 1)$	r- & s-implication	Łukasiewicz

Table 4
Fuzzy implications, classes and assorted GMP operators

Operator	Expression
Łukasiewicz	$\top(a,b) = \max(0, a+b-1)$
Zadeh	$\top(a,b) = \min(a,b)$
Goguen	$\top(a,b) = a \times b$

Table 5
Fuzzy GMP operators

- If $A' \subset A$ and $Kernel(A') \cap Kernel(A) \neq \varnothing$, then $B' = B$
- Otherwise, if $A' \subset A$, then $B' \subset B$

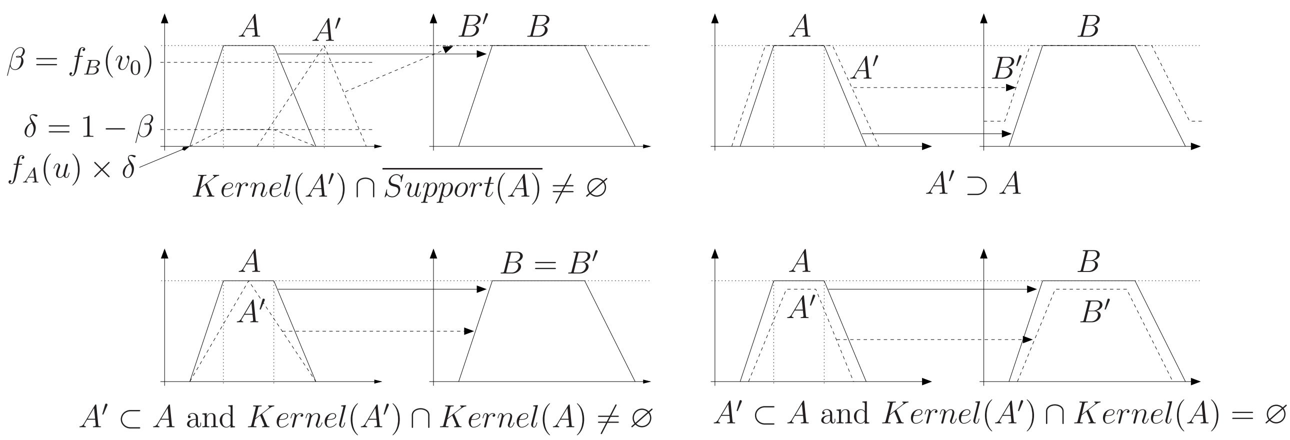


Fig. 6. GMP conclusions with Reichenbach's implication

5.2.3 Willmott

With Łukasiewicz's t-norm for GMP operator, the conclusion of a fuzzy inference given Willmott's implication is:

$$f_{B'}(v) = \max(f_B(v), \sup_{u \in U} f_{A'}(u) - f_A(u))$$

which gives us:

- $f_{B'}(v) \geqslant f_B(v), \forall v \in V$
- $f_{B'}(v) \geqslant \sup\limits_{u \in U} f_{A'}(u) - f_A(u), \forall v \in V$
- Therefore, $\inf\limits_{v \in V} f_{B'}(v) \geqslant \sup_{u \in \overline{Support(A)}} f_{A'}(u)$

5.2.4 Mamdani

We have studied the conclusions of Mamdani rules with the min, product or Łukasiewicz GMP operators and their membership functions are:

with Zadeh's min t-norm:

$$f_{B'}(v) = f_B(v)$$

with Goguen's product t-norm:

$$f_{B'}(v) = \max \begin{pmatrix} \sup_{u \in U, f_A(u) \leqslant f_B(v)} f_{A'}(u) \times f_A(u), \\ \sup_{u \in U, f_A(u) \geqslant f_B(v)} f_{A'}(u) \times f_B(v) \end{pmatrix}$$

with Łukasiewicz's t-norm:

$$f_{B'}(v) = \max \begin{pmatrix} 0, & \sup_{u \in U, f_A(u) \leqslant f_B(v)} f_{A'}(u) + f_A(u) - 1, \\ & \sup_{u \in U, f_A(u) \geqslant f_B(v)} f_{A'}(u) + f_B(v) - 1 \end{pmatrix}$$

which gives us:

- $B' \equiv B$ for Zadeh's GMP operator
- $f_{B'}(v) \leqslant f_B(v), \forall v \in V$ otherwise

5.2.5 Rescher-Gaines

Whatever the GMP operator (min, product or Łukasiewicz's t-norm), the conclusion of a fuzzy inference given Rescher-Gaines' implication is:

$$f_{B'}(v) = \sup_{u \in U, f_B(v) \geqslant f_A(u)} f_{A'}(u)$$

which gives us:

- $\inf\limits_{v\in V} f_{B'}(v) = \sup_{u\in\overline{Support(A)}} f_{A'}(u)$
- $\sup\limits_{v\in V} f_{B'}(v) = \sup\limits_{u\in U} f_{A'}(u)$
- If $Support(A') = Support(A)$ and $A' = m(A)$, then $B' = m(B)$

5.2.6 Kleene-Dienes

With Łukasiewicz's GMP operator conclusions are given by:

$$f_{B'}(v) = \max\begin{pmatrix} 0, & \sup_{u\in U, 1-f_A(u)\leqslant f_B(v)} f_{A'}(u) + f_B(v) - 1, \\ & \sup_{u\in U, 1-f_A(u)\geqslant f_B(v)} f_{A'}(u) - f_A(u) \end{pmatrix}$$

which gives us:

- $\inf\limits_{v\in V} f_{B'}(v) = \sup_{u\in\overline{Support(A)}} f_{A'}(u)$

5.2.7 Gödel

The conclusion of a fuzzy inference given Gödel's implication and the min GMP operator is:

$$f_{B'}(v) = \max(\sup_{u\in U, f_B(v)\geqslant f_A(u)} f_{A'}(u), f_B(v))$$

which means:

- $B' \supseteq B$
- $\inf\limits_{v\in V} f_{B'}(v) = \sup_{u\in\overline{Support(A)}} f_{A'}(u)$

With Łukasiewicz's t-norm we get:

$$f_{B'}(v) = \max\begin{pmatrix} f_B(v) + \sup_{u\in U, f_A(u)\geqslant f_B(v)} f_{A'}(u) - 1, \\ \sup_{u\in U, f_A(u)\leqslant f_B(v)} f_{A'}(u) \end{pmatrix}$$

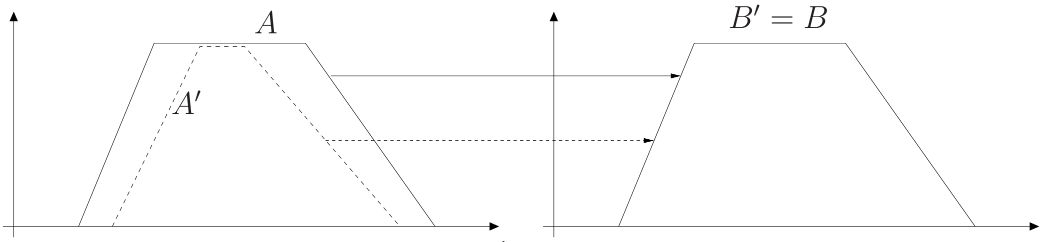
With Goguen's GMP operator we have:

$$f_{B'}(v) = \max\begin{pmatrix} \sup_{u\in U, f_A(u)\geqslant f_B(v)} f_{A'}(u) \times f_B(v), \\ \sup_{u\in U, f_A(u)\leqslant f_B(v)} f_{A'}(u) \end{pmatrix}$$

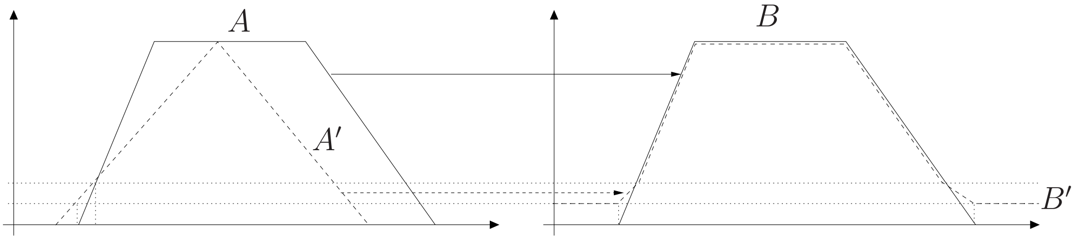
which means that for both t-norms we have:

- $\inf\limits_{v\in V} f_{B'}(v) \geqslant \sup_{u\in\overline{Support(A)}} f_{A'}(u)$

- If $Kernel(A') \cap Kernel(A) \neq \varnothing$, then $B' \supseteq B$
- Otherwise, if $A' \subset A$ and $Kernel(A') \cap Kernel(A) = \varnothing$ then $\sup\limits_{v \in V} f_{B'}(v) = \sup_{u \in U} f_{A'}(u)$



$B' \supseteq B$



$\inf\limits_{v \in V} f_{B'}(v) = \sup_{u \in \overline{Support(A)}} f_{A'}(u)$

Fig. 7. Essential properties of conclusions with Gödel's implication and Zadeh's t-norm

5.2.8 Goguen

Using Łukasiewicz's t-norm we get the following expression;

$$\begin{aligned} f_{B'}(v) &= \max\begin{pmatrix} \sup_{u \in U, f_A(u) \geqslant f_B(v), f_A(u) > 0} f_{A'}(u) + \frac{f_B(v)}{f_A(u)} - 1, \\ \sup_{u \in U, f_A(u) \leqslant f_B(v)} f_{A'}(u) \end{pmatrix} \\ &= \sup_{u \in U, f_A(u) \leqslant f_B(v)} f_{A'}(u) \end{aligned}$$

which implies:

- $\inf\limits_{v \in V} f_{B'}(v) \geqslant \sup_{u \in \overline{Support(A)}} f_{A'}(u)$
- $\sup\limits_{v \in V} f_{B'}(v) = \sup_{u \in U} f_{A'}(u)$.

When combined to Goguen's operator, we have:

$$f_{B'}(v) = \max\begin{pmatrix} f_B(v) \times \sup_{u \in U, f_A(u) \geqslant f_B(v), f_A(u) > 0} \frac{f_{A'}(u)}{f_A(u)}, \\ \sup_{u \in U, f_A(u) \leqslant f_B(v)} f_{A'}(u) \end{pmatrix}$$

which means:

- $\inf\limits_{v \in V} f_{B'}(v) \geqslant \sup_{u \in \overline{Support(A)}} f_{A'}(u)$

- If $Kernel(A') \cap Kernel(A) \neq \varnothing$, then $B' \supseteq B$
- Otherwise, if $A' \subset A$ and $Kernel(A') \cap Kernel(A) = \varnothing$ then $\sup\limits_{v \in V} f_{B'}(v) = \sup_{u \in U} f_{A'}(u)$

5.2.9 Łukasiewicz

The general expression of the conclusion of a fuzzy rule given Łukasiewicz's implication is:

$$f_{B'}(v) = \max \begin{pmatrix} f_B(v) + \sup_{u \in U, f_A(u) \geqslant f_B(v)} f_{A'}(u) - f_A(u), \\ \sup_{u \in U, f_A(u) \leqslant f_B(v)} f_{A'}(u) \end{pmatrix}$$

from which we see that:

- $\inf\limits_{v \in V} f_{B'}(v) \geqslant \sup_{u \in \overline{Support(A)}} f_{A'}(u)$
- If $A' \supset A$, then $B' \supset B$
- If $A' \subset A$ and $Kernel(A') \cap Kernel(A) \neq \varnothing$, then $B' = B$
- If $A' \subset A$ and $Kernel(A') \cap Kernel(A) = \varnothing$, then $B' \subset B$

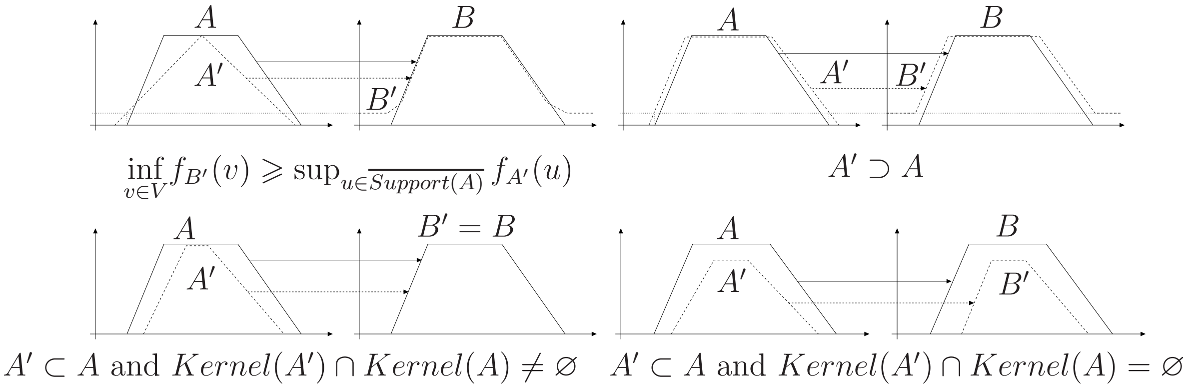


Fig. 8. Some properties of GMP conclusions given Łukasiewicz's implication

5.3 Summary

We will now review, in Table 6, the properties we have put forward in the previous paragraph, in order to suggest possible links between implications.

Property 1 is very specific to Mamdani's implication, whose claim to implication status is generally debatable and debated.

Similarly, Property 2 is only consistent with Rescher-Gaines' implication which is usually regarded as a crisp implication. However, it does show that all our modifiers could, potentially, have been observed, if the associated rules were interpreted with Rescher-Gaines.

Property		Implication	t-norm(s)
1	$B' \equiv B$	Mamdani	Zadeh
2	If $A' = m(A)$, then $B' = m(B)$	Rescher-Gaines	Zadeh, Goguen, Łukasiewicz
3	$B' \supseteq B$	Gödel	Zadeh
4	$B' \supseteq B$ if $Kernel(A') \cap Kernel(A) \neq \varnothing$	Gödel	Łukasiewicz
		Łukasiewicz	Łukasiewicz
5	Persistent denormalisation	Rescher-Gaines	Zadeh, Goguen, Łukasiewicz
		Gödel	Łukasiewicz
		Goguen	Goguen, Łukasiewicz
		Łukasiewicz	Łukasiewicz
6	$\inf_{v \in V} f_{B'}(v) = \sup_{u \in \overline{Support(A)}} f_{A'}(u)$	Rescher-Gaines	Zadeh, Goguen, Łukasiewicz
		Kleene-Dienes	Łukasiewicz
		Gödel	Zadeh
7	$\inf_{v \in V} f_{B'}(v) \geqslant \sup_{u \in \overline{Support(A)}} f_{A'}(u)$	Gödel	Łukasiewicz
		Goguen	Łukasiewicz
		Łukasiewicz	Łukasiewicz

Table 6
General properties of GMP conclusions

Property 3 shows, in passing, that the condition we had in Proposition 1 (i.e. B' such that $\exists v_0 \in V$ with $0 = f_{B'}(v_0) < f_B(v_0)$) was not possible, at least not using Zadeh's GMP operator. Similarly, note that none of our modifiers was applicable to Łukasiewicz's implication, because they all preserved $\inf\limits_{v \in V} f_{B'}(v)$ which is true if and only if $A' \equiv A$.

Property 5, named here 'Persistent denormalisation', means that if an observation is not normalised, then neither is the conclusion, e.g.
if $f_{A'}(u) = \max(f_A(u) - \varepsilon, 0)$, then $f_{B'}(v) = \max(f_B(v) - \varepsilon, 0)$. This property is, for at least two implications, related to property 4 in that it is true only under certain conditions. Note that it also gives conditions on A' for $B' \subset B$.

Properties 6 and 7, which are shared by most implication-t-norm couples, mean that if an observation were such that $Kernel(A') \cap \overline{Support(A)} \neq \varnothing$, then $B' = V$, i.e. $f_{B'}(v) = 1, \forall v \in V$. In the general case, though, no observation whose typical values lay outside the premise's support would be considered. Yet these two properties are also responsible for the uncertainties in the conclusions. Neither of these observations is ground-breaking but they do show what the conditions for having an uncertain conclusion are, and explain why the second modifier we studied in Section 3 (i.e.$\forall v \in V, f_{B'}(v) = \max(f_B(v + \varepsilon), f_B(v - \varepsilon)))$ was not compatible with Gödel's implication.

To illustrate our point, we will come back to our running example and our chickenpox rule. We will suppose this rule was given to us by an expert. To use this rule, we need to give it an interpretation so as to choose an implication operator. To do this, we try to undestand our expert's interpretation of the link between premise and conclusion. Suppose, now, we were to observe a denormalised class of specifically contagious patients, something like what we observed in Section 4.4. For our rule to accept such an observation, the rule would have to be attached to one of the implications exhibiting Property 5. Obviously there would still be a choice of sorts, but at least we would know that the implication was an r-implication and thus that the rule was a gradual one. This is what we refer to as 'data-driven' classification of the rules. The semantic interpretation of the rule may, of course, still be that given by Dubois and Prade, but the choice is coherent with the observations.

Another example can be found in colorimetric works, such as [21]. In this article, the authors introduce a search engine for images using a fuzzy definition of colours. The system defines basic colours and a toolbox of fuzzy modifiers. Each resulting colour, 'cornflower-blue' for instance, is a specialisation of its unmodified version. Therefore there is an implicit rule which states that 'If something is cornflower-blue, then it is blue'. The system also offers the user the possibility of having her personnal definitions of the colours learnt. So now, if the system learns a modified version of blue from the user, how does this new information help us qualify the original implicit rule? Well, should we suppose for a minute that the user has asked the system to learn her version of both blue and cornflower-blue, and that the modifiers for both are the same and introduce no uncertainty, then the only suitable implication operator is Rescher-Gaines, an r-implication. Therefore the implicit rule is a gradual one.

6 Conclusion

This paper's ambition was, originally, to extend formal fuzzy abductive results to different classes of implications and linguistic modifiers. While working on these results we noticed that the theory contradicted some established results. The explanation of these incoherences lay in the 'impossibility' of observing certain shapes. Yet these shapes did not seem incoherent with the data they were meant to represent. Tracing the incoherence of our results back to the 'observable' shapes of the selected fuzzy implications, we saw that observations were bound by the implication operator. To allow suspected 'data-coherent' observations we needed to find 'deduction-coherent' implications. Available studies of the Generalised Modus Ponens offered information on possible shapes, but did not allow us to definitely rule-out others. Therefore we had to generalise these results to conclude. We would like to extend this type of systematic analysis to other implications and their associated GMP operators, or t-norm.

Selecting an implication from the data meant we could interpret our rule-based knowledge using the semantic interpretation of the operators. We may well find that different rules, even though they are used in the same context, given with the same interpretation by the expert, belong to different implication classes and are therefore to be interpreted differently. Because the properties we use to qualify rules essentially come at the implication class level, we sometimes find it difficult to choose an implication. Indeed, we have seen that implications of different classes sometimes generate similar shapes. Conversely, some r-implications, for instance, do not accept the same modifications. Therefore, 'observation consistent' implications, which we use to classify our rule-base subsets according to the necessary properties, may have some semantic proximity. If not, their differences could entail as many potential interpretations. The properties we have laid out in this comparative study also seem connected to the choice of GMP-operator. This should be taken into account in the semantic interpretation processes. We plan to dedicate some time to these studies in the near future.

Finally, we have ongoing efforts on the general subject of fuzzy abduction. We are working on generalising the reversal of the Generalised Modus Ponens to other classes of implication operators and for complex rules. Concerning systems of potentially conflicting rules, the original method can be used for each rule. In this case, the next step is to rank the simultaneous explanation in order to choose the most satisfactory in the specific applicative context.

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