Quenched large deviations for Glauber evolution with Kac interaction and Random Field

Olivier Benois, Mustapha Mourragui, Enza Orlandi, Ellen Saada, Livio Triolo

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Olivier Benois, [1] Mustapha Mourragui, [2] Enza Orlandi, [3] Ellen Saada, [4] Livio Triolo [5]

Abstract

We study a spin-flip model with Kac type interaction, in the presence of a random field given by i.i.d. bounded random variables. The system, spatially inhomogeneous, evolves according to a non conservative (Glauber) dynamics. We show an almost sure (with respect to the random field) large deviation principle for the empirical magnetizations of this process. The rate functional associated with the large deviation principle depends on the statistical properties of the external random field, it is lower semicontinuous with compact level sets.

1. Introduction

We consider interacting spin-flip systems, in dimension d, with Kac type interaction in the presence of a random field given by i.i.d. bounded random variables. Kac potentials J_γ are two-body interactions with range γ^{-1} and strength γ^d, where γ is a dimensionless scaling parameter. When $\gamma \to 0$, i.e. very long range compared with the inter particle distance, the strength of the interaction becomes very weak, but in such a way that the total interaction between one particle and all the others is finite. Kac potentials were introduced in [KUH], and then generalized in [LP], to present a rigorous derivation of the van der Waals theory of a gas-liquid phase transition. There has been in the last decades an increasing interest in them. Indeed they induce the intermediate scale of interaction γ^{-1} (called mesoscopic) between the microscopic (lattice) one and a macroscopic one much bigger than the latter. They are suitable to interpolate not only between short and long range interactions, but, scaling space and time as functions of γ, one can hope to obtain more insights into the physics of the model. Recently they have been considered as models to describe social interactions and more general complex social systems, see for example [CDS] and references therein.

There has been several results on Kac Ising spin systems (without random field) in equilibrium and in non equilibrium statistical mechanics. We refer for a survey to the book [P]. The papers [C], [CE] were among the first dealing with dynamics issues. They considered spin systems in a torus evolving according to

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[1] Université de Rouen, LMRS, UMR 6085, Avenue de l'Université, BP. 12, 76801 Saint Etienne du Rouvray, France. Olivier.Benois@univ-rouen.fr

[2] Université de Rouen, LMRS, UMR 6085, Avenue de l'Université, BP. 12, 76801 Saint Etienne du Rouvray, France. Mustapha.Mourragui@univ-rouen.fr

[3] Dipartimento di Matematica, Universitá di Roma Tre, L.go S.Murialdo 1, 00146 Roma, Italy. orlandi@mat.uniroma3.it

[4] CNRS, MAP5, UMR 8145, Université Paris Descartes, Sorbonne Paris Cité, 45 rue des Saints-Pères, 75270 Paris cedex 06, France.
ellen.saada@mi.parisdescartes.fr

[5] Dipartimento di Matematica, Universitá di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy. triolo@mat.uniroma2.it

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a reversible and non conservative (Glauber) dynamics, with Kac interactions. In [C] the long time analysis of the spin system is studied, using large deviations techniques. In [CE] the main results are the infinite particle limits of the non-critical and critical fluctuation processes. In [DOPT] and [KS] (see also references therein), a complete description of the development and motion of interfaces (long time behaviour) has been derived: it is governed by the law of *motion by mean curvature*.

A natural extension of this analysis is its application to disordered systems. One of the simplest prototype models is obtained by adding a random magnetic field to an interacting spin system. Equilibrium statistical properties of these systems have been widely studied in the last decades, see [Bo] for a survey of results in this direction. The case of Kac type interaction has been investigated in $d = 1$ by [COP], [COPV] and [OP].

In this paper we study a reversible, nonconservative (Glauber) dynamics of ± 1 valued spins, interacting via a Kac potential and under the influence of an external random field. We assume the latter given by i.i.d. random variables taking values $a_i \in I\!R$ with probability p_i, for $i = 1, \dots, N$, with N a fixed integer. We do not require the Kac potential to be positive (that is we do not restrict the model to the ferromagnetic case).

Our main result is a quenched large deviation principle, almost sure with respect to the random field, for the empirical random magnetizations of this spin-flip process. The rate functional associated with the large deviation principle, which depends on the distribution of the random field, is lower semicontinuous, positive, with compact level sets. In contrast with the non random case studied in [C], the magnetization m of our spin model is not of mean field type. Nevertheless, this difficulty is overcome by coloring the sites according to the random external force, so that the colored magnetizations become a mean field system. The large deviation rate functional is then obtained via a contraction principle from the rate functional associated with the large deviation principle of the empirical *colored* magnetizations m_i (i.e. the magnetization over the sites where the random field takes value a_i), $i = 1, \dots, N$; we have $m = \sum_{i=1}^N m_i$. As usual, the rate functional is determined by two distinct types of large deviations of the same order. The first one corresponds to large deviations from the initial state, the second one to the stochasticity of the evolution. Suppose $\mathcal{A} = \{\pi^\gamma(\cdot, dr) \simeq v(\cdot, r)dr, t \in [0, T]\}$ where $\pi^\gamma(\cdot, dr)$ is the local magnetization density, $\simeq$ denotes closeness in some norm and v is a profile different from the solution of the nonlinear macroscopic equation giving the law of large numbers. We need to modify the measure of the process over the magnetization profiles so that event $\mathcal{A}$ becomes typical. One possible choice is to drive the spin system by weak, slowly varying, space-time dependent external forces. This is the standard choice for spin systems evolving according to non conservative (Glauber) evolutions without randomness involved, or to conserved (Kawasaki) evolutions with gradient type interactions. For conserved non gradient systems, the force must be configuration dependent (see [Sp], p. 248), to take into account that for these systems the response in the current to an external force field is partially delayed. Namely, when an external random field is added to the Hamiltonian, in the conservative, non gradient case (see [MO]), one needs to take the external force weakly dependent on the field randomness. In the non conservative case, it turns out that the external force strongly depends on the field randomness. In other words, in dynamics with a conserved quantity, there is less freedom in choosing the class of perturbations than in non conservative dynamics.

We distinguish between sites where the random field takes different values; on each of them we take a deterministic space-time dependent external force. This allows to write the rate functional associated to the large deviation principle in a closed form with respect to the local *colored* magnetization. We carried out explicitly the computations for a couple ($N = 2$); the general case follows. The simplest case to have in mind is $a_1 = 1$, $a_2 = -1$, $p_1 = p_2 = 1/2$ and $J \geq 0$; then, when β (which is proportional to the inverse temperature) is such that $\beta \geq \beta_c = (\int J(r)\, dr)^{-1}$, interesting phenomena appear when studying the long time behaviour of the spin system. This is related to the fact that the underlying spin systems at equilibrium undergo to phase transition, even in one dimension in the limit $\gamma \downarrow 0$. In this paper we will study the dynamics of the spin system for finite time: in this regime, the evolution does not depend crucially on the value of β. We will then set $\beta = 1$.

The random Curie-Weiss model (RCW), which describes a mean field interaction, has given rise to many results on short and long time dynamics. In [DD], short time dynamics has been studied. More precisely the large deviations for the empirical measures in the product space of magnetization trajectories and realizations of the random field are given. From this result one could derive annealed large deviations for the RCW but not quenched ones. In [MP] and [FMP], long time dynamics, convergence to equilibrium when the random field takes only the two values $\pm\epsilon$ are considered. In [BEGK], the RCW model is analyzed when the random field takes finitely many values, as an example of the use of the potential theoretical approach to metastability. Furthermore, in [BBI], the previous results are extended to continuous distributions of the field, and precise asymptotics of metastable characteristics are derived.

There are no available results for short and long time dynamics of the random field Kac model. We make here a first step in addressing this problem.

In Section 2 we present the model, main definitions and results. In Section 3 we define the rate functional associated to the large deviation principle, we exhibit different representations for it, and we give its main properties (lower semicontinuity, compactness). There, we follow the scheme of [C], Section III, but working with the couple (m_1, m_2) induces intricate computations. Since the spins have value ± 1, the local and colored magnetizations are always between $+1$ and -1. A consequence of the randomness is that the functional becomes infinite for the colored particle system at the boundary $p_i \le |m_i| \le 1$ $(i = 1, 2)$ of the coupled magnetization. Thus these boundaries are not rare enough in the large deviations regime, and we have to deal with this lack of regularity. This is different from the non random case [C], where the boundary is reduced to the two values ± 1 of the magnetization. A preliminary step to derive the large deviation principle (LDP) is the hydrodynamic behavior for the colored particle process, sketched in Section 4. The class of time dependent, random perturbations needed to derive the LDP lower bound is introduced in Section 5, where the perturbed process is studied. In Section 6 we derive the upper bound and in Section 7 the lower bound of the LDP. The lower bound is obtained first for trajectories that are smooth is space and time, and outside the boundaries $p_i \le |m_i| \le 1$ $(i = 1, 2)$. Then it is extended to a larger class $\mathcal{P}$ of paths, by smoothing by successive steps trajectories with a finite rate functional, using techniques introduced in [QRV]. In this context, $\mathcal{P}$ consists in trajectories absolutely continuous with respect to the Lebesgue measure and absolutely continuous in time. Then, in order for the usual martingale technique to be effective to obtain the upper bound, we need to show that the process concentrates on $\mathcal{P}$. To this aim, we introduce an energy functional via an exponential martingale which excludes the trajectories not in $\mathcal{P}$ (in the spirit of [QRV], [MO], [FLM]). The appendices (Sections 8 and 9) gather the most technical proofs.

2. The model and the main results

The space of configurations: Let Λ be the d-dimensional torus of diameter 1. For $0 < \gamma < 1$ such that $\gamma^{-1} \in I\!\!N$, $\Lambda_\gamma = Z\!\!\!Z^d / \gamma^{-1} Z\!\!\!Z^d$ is the d-dimensional discrete torus of diameter γ^{-1}. We denote by $\mathcal{S}_\gamma \equiv \{-1, +1\}^{\Lambda_\gamma}$ the configuration space and by $\sigma = (\sigma(x))_{x \in \Lambda_\gamma}$ a spin configuration, where for each $x \in \Lambda_\gamma$, $\sigma(x) \in \{-1, 1\}$.

The disorder: It is described by a collection of i.i.d. random variables $\alpha = \{\alpha(x), x \in Z\!\!\!Z^d\}$ taking two values, i.e. $\alpha(x) \in \{a_1, a_2\}$. The corresponding product measure on $\Omega = \{a_1, a_2\}^{Z\!\!\!Z^d}$ is denoted by $I\!\!P$ (and $I\!\!E$ is the expectation with respect to $I\!\!P$),

$$I\!\!P\{\alpha(x) = a_i\} = p_i, \quad i = 1, 2. \tag{2.2}$$

For γ^{-1} an odd integer, α induces in a natural way a random field on Λ_γ, also denoted by α.

The Kac potential: We consider a pair interaction among particles given by a Kac potential of the form

$$J_\gamma(x,y) \equiv \gamma^d J(\gamma(x-y)), \quad (x,y) \in \Lambda_\gamma \times \Lambda_\gamma, \tag{2.3}$$

where $J : \Lambda \to I\!R$ is a symmetric function, that is $J(r) = J(-r)$, such that $\int J(r)dr = 1$ (normalization). The interaction J might have any sign. Denote by $\mathcal{C}(\Lambda)$ (resp. $\mathcal{C}^1(\Lambda)$, $\mathcal{C}^2(\Lambda)$) the space of continuous (resp. continuously differentiable, twice continuously differentiable) real functions on Λ. We assume $J \in \mathcal{C}^1(\Lambda)$.

The Energy: Given a realization α of the magnetic field, define for all γ, $\theta > 0$, $\sigma \in \mathcal{S}_\gamma$, the Hamiltonian

$$H^{\gamma,\alpha}(\sigma) = -\frac{1}{2} \sum_{(x,y)\in\Lambda_\gamma\times\Lambda_\gamma} J_\gamma(x,y)\sigma(x)\sigma(y) - \theta \sum_{x\in\Lambda_\gamma} \alpha(x)\sigma(x), \tag{2.4}$$

and the Gibbs measure $\mu^{\gamma,\alpha,\beta}$ associated to $H^{\gamma,\alpha}$ at inverse temperature β, with normalization constant $Z^{\gamma,\alpha,\beta}$:

$$\mu^{\gamma,\alpha,\beta}(\sigma) = \frac{1}{Z^{\gamma,\alpha,\beta}} \exp\left[-\beta H^{\gamma,\alpha}(\sigma)\right].$$

The Glauber dynamics: Denote by σ^x the configuration obtained from σ by flipping the spin at site x:

$$\sigma^x(z) = \begin{cases} -\sigma(x) & \text{if} \quad z = x, \\ \sigma(z) & \text{otherwise,} \end{cases}$$

so that the energy difference resulting from a spin flip at x is

$$H^{\gamma,\alpha}(\sigma^x) - H^{\gamma,\alpha}(\sigma) = 2\sigma(x)\left[(J_\gamma \star \sigma)(x) + \theta\alpha(x)\right], \tag{2.5}$$

where without loss of generality we have assumed $J(0) = 0$, and we define the discrete convolution $\star$ between function J_γ and a configuration σ by

$$(J_\gamma \star \sigma)(x) = \gamma^d \sum_{y\in\Lambda_\gamma} J(\gamma(x-y))\sigma(y). \tag{2.6}$$

We consider a Markovian evolution on $\mathcal{S}_\gamma$, whose generator $\mathcal{L}^{\gamma,\alpha}$ acts on cylinder functions f as

$$\mathcal{L}^{\gamma,\alpha} f(\sigma) = \sum_{x\in\Lambda_\gamma} c_x^{\gamma,\alpha}(\sigma)[f(\sigma^x) - f(\sigma)], \tag{2.7}$$

where, for $x \in \Lambda_\gamma$,

$$c_x^{\gamma,\alpha}(\sigma) = \frac{\exp[-(\beta/2)(H^{\gamma,\alpha}(\sigma^x) - H^{\gamma,\alpha}(\sigma))]}{2\cosh[(\beta/2)(H^{\gamma,\alpha}(\sigma^x) - H^{\gamma,\alpha}(\sigma))]}. \tag{2.8}$$

Then $\mathcal{L}^{\gamma,\alpha}$ viewed as an operator on $L^2(\mu^{\gamma,\alpha,\beta})$ is self-adjoint. Since temperature is kept fixed in all the paper and does not play any role we set for simplicity $\beta = 1$. We fix a time $T > 0$, and we will study the process $(\sigma_t)_{t\in[0,T]}$ with infinitesimal generator given in (2.7).

The measure spaces: Let $\mathcal{M}_1$ be the set of signed Borel measures μ on the Borel σ-field of Λ with total variation norm bounded by 1. We equip $\mathcal{M}_1$ with the weak τ^* topology induced by $\mathcal{C}(\Lambda)$ via $<\mu, G> = \int G d\mu$ (for $G \in \mathcal{C}(\Lambda)$). We denote by $\rho(\cdot,\cdot)$ the distance which makes $(\mathcal{M}_1, \tau^*)$ a metrizable compact space, see [Bill]: that is, given $(H_k)_{k\in I\!N}$ a dense subset in the unit ball of $\mathcal{C}(\Lambda)$ for $\mu_i \in \mathcal{M}_1$, $i = 1, 2$,

$$\rho(\mu_1, \mu_2) = \sum_{k\geq 0} 2^{-k} |<\mu_1 - \mu_2, H_k>|. \tag{2.9}$$

Let $0 < q \leq 1$, and

$$\mathcal{M}_q^{ac} = \left\{ \mu \in \mathcal{M}_1 : \mu << \lambda \quad \text{and} \quad \left| \frac{d\mu}{d\lambda} \right| \leq q \quad \lambda - a.s. \right\}, \tag{2.10}$$

where λ is the Lebesgue measure on Λ. We identify $\mu \in \mathcal{M}_q^{ac}$ with its Radon-Nikodym derivative $\dfrac{d\mu}{d\lambda}$, and, by an abuse of notation, we write $< \mu, G > = < \dfrac{d\mu}{d\lambda}, G >$. Since $\mathcal{M}_q^{ac}$ is a closed ball of $\mathcal{M}_1$, it is τ^* compact.

If $\sigma \in \mathcal{S}_\gamma$ we define the *empirical measure* $\pi^\gamma(\sigma) \in \mathcal{M}_1$ by

$$\pi^\gamma(\sigma)(dr) = \gamma^d \sum_{x \in \Lambda_\gamma} \sigma(x) \delta_{\gamma x}(dr), \tag{2.11}$$

where $\delta_{\gamma x}$ is the Dirac measure concentrated on point γx. Remark that if we denote by $\mu * G$ the convolution of a measure μ and a function G over Λ, namely $(\mu * G)(r') = \int_\Lambda G(r' - r)\, \mu(dr)$, then we can rewrite

$$(J_\gamma \star \sigma)(x) = (\pi^\gamma(\sigma) * J)(\gamma x). \tag{2.12}$$

We denote by $D([0,T], \mathcal{M}_1)$ (resp. $D([0,T], \mathcal{S}_\gamma)$) the space of functions from $[0,T]$ to $\mathcal{M}_1$ (resp. to $\mathcal{S}_\gamma$) that are right continuous with left limits, endowed with the Skorohod topology, see [Bill].

The initial condition: Let $(\sigma^\gamma)_\gamma$ be a sequence of configurations such that $\pi^\gamma(\sigma^\gamma)$ converges when $\gamma \to 0$ in the weak topology to the measure $m_0 \lambda$, for a continuous function $m_0 : \Lambda \to [-1, 1]$. This means that

$$\lim_{\gamma \to 0} \rho(\pi^\gamma(\sigma^\gamma), m_0 \lambda) = 0. \tag{2.13}$$

We denote by $P_{\sigma^\gamma}^{\gamma;\alpha}$ the law (and by $E_{\sigma^\gamma}^{\gamma;\alpha}$ the expectation) of the process $(\sigma_t)_{t \in [0,T]}$ on $D([0,T], \mathcal{S}_\gamma)$ starting at time $t = 0$ from the deterministic initial configuration σ^γ, and by $Q_{\sigma^\gamma}^{\gamma;\alpha}$ the law on $D([0,T], \mathcal{M}_1)$ of the corresponding empirical measure process $(\pi_t^\gamma)_{t \in [0,T]}$, where π_t^γ stands for $\pi^\gamma(\sigma_t)$.

We first obtain the "law of large numbers".

Theorem 2.2 *Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). For all $t \geq 0$,*

$$\lim_{\gamma \to 0} \rho(\pi_t^\gamma, m(t, \cdot)\lambda) = 0, \qquad I\!P\text{-a.s.}, \tag{2.14}$$

where $m(\cdot, \cdot)$ is the unique weak solution of

$$\begin{cases} \partial_t m(t,r) = -m(t,r) + \displaystyle\sum_{i=1,2} p_i \tanh\left[(J * m(t,\cdot))(r) + a_i \theta\right] \\ m(0, \cdot) = m_0(\cdot). \end{cases} \tag{2.15}$$

Furthermore, for all $G \in \mathcal{C}^{0,1}([0,T] \times \Lambda)$ (that is, continuous in its first variable, and continuously differentiable in its second variable), $\delta > 0$,

$$\lim_{\gamma \to 0} P_{\sigma^\gamma}^{\gamma;\alpha} \left[\sup_{t \in [0,T]} |< \pi_t^\gamma, G(t, \cdot) > - < m(t, \cdot), G(t, \cdot) >| \geq \delta \right] = 0. \tag{2.16}$$

By an abuse of notation we write from now on $(J * m)(t, r)$ instead of $(J * m(t, \cdot))(r)$.

Remark 2.3 . The Cauchy problem (2.15) in this setup is well posed with a unique global solution, because the right hand side of (2.15) is uniformly Lipschitz, and because the set $\{m \in L^\infty(\Lambda) : \|m\|_\infty \leq 1\}$ is left invariant, since $|\tanh z| \leq 1$ for all z. Furthermore the solution is differentiable in time.

Next we state the *quenched* large deviation principle for $Q^{\gamma;\alpha}_{\sigma^\gamma}$. Different choices of initial conditions could be treated as well. The only difference would be an extra term to add to the rate functional associated with the large deviation principle $\widetilde{I}_{m_0}(\cdot)$, taking into account the deviation from the initial profile at time $t = 0$. The functional $\widetilde{I}_{m_0}(\cdot)$ depends on the distribution of the random field but not on its realization; it is obtained through a contraction principle, as explained in the introduction. Its explicit formulation relies on several intermediate steps. Let

$$\mathcal{D}(\widetilde{I}_{m_0}) = \{\pi \in D([0,T], \mathcal{M}_1) : \widetilde{I}_{m_0}(\pi) < \infty\}. \tag{2.17}$$

Theorem 2.4 *Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). For all closed subsets $\mathcal{F} \subset D([0,T], \mathcal{M}_1)$ and open subsets $\mathcal{A} \subset D([0,T], \mathcal{M}_1)$, we have*

$$\limsup_{\gamma \to 0} \gamma^d \log Q^{\gamma;\alpha}_{\sigma^\gamma}(\mathcal{F}) \leq -\inf_{\pi \in \mathcal{F}} \widetilde{I}_{m_0}(\pi), \quad I\!P - a.s., \tag{2.18}$$

$$\liminf_{\gamma \to 0} \gamma^d \log Q^{\gamma;\alpha}_{\sigma^\gamma}(\mathcal{A}) \geq -\inf_{\pi \in \mathcal{A}} \widetilde{I}_{m_0}(\pi), \quad I\!P - a.s. \tag{2.19}$$

The functional $\widetilde{I}_{m_0}(\cdot)$, defined in (2.28) below, is non-negative for $\pi \in D([0,T], \mathcal{M}_1)$, lower semicontinuous with compact level sets and, see Definition 3.1 later on,

$$\mathcal{D}(\widetilde{I}_{m_0}) \subset \{m \in \mathcal{C}([0,T], \mathcal{M}_1^{ac}) : m(t,.) \quad \textit{absolutely continuous for} \quad t \in [0,T]\}.$$

The colored particle system: To derive the rate functional associated with the large deviation principle we introduce *random* empirical measures $\overline{\pi}^\gamma = \left(\pi_1^\gamma, \pi_2^\gamma\right)$. For $\alpha \in \Omega, x \in \Lambda_\gamma$, $i = 1,2$, set

$$\alpha_i(x) = 1\!\!1_{\{\alpha(x) = a_i\}}, \tag{2.20}$$

$$\pi_i^\gamma(\sigma)(dr) = \gamma^d \sum_{x \in \Lambda_\gamma} \alpha_i(x) \sigma(x) \delta_{\gamma x}(dr). \tag{2.21}$$

Though we do not write it explicitly, $\pi_i^\gamma(\sigma) \in \mathcal{M}_1$ depends on the randomness. Moreover the knowledge of $\pi_i^\gamma(\sigma)$ for $i = 1,2$ determines $\pi^\gamma(\sigma) = \pi_1^\gamma(\sigma) + \pi_2^\gamma(\sigma)$. We denote by $\overline{Q}^{\gamma;\alpha}_{\sigma^\gamma}$ the law on $D([0,T], \mathcal{M}_1 \times \mathcal{M}_1)$ of the empirical measure process $(\overline{\pi}^\gamma_t)_{t \in [0,T]} = (\pi^\gamma_{1,t}, \pi^\gamma_{2,t})_{t \in [0,T]}$ under $P^{\gamma;\alpha}_{\sigma^\gamma}$. We denote, for $\overline{G} = (G_1, G_2) \in \left(\mathcal{C}(\Lambda)\right)^2$,

$$< \overline{\pi}^\gamma_t, \overline{G} > = \sum_{i=1,2} \gamma^d \sum_{x \in \Lambda_\gamma} G_i(\gamma x) \alpha_i(x) \sigma_t(x) \tag{2.22}$$

and, for $\overline{m} = (m_1, m_2) \in (L^\infty(\Lambda))^2$, by an abuse of notation,

$$< \overline{m}, \overline{G} > = < (m_1 \lambda, m_2 \lambda), \overline{G} > = \sum_{i=1}^2 \int_\Lambda G_i(r) m_i(r)\, dr. \tag{2.23}$$

Theorem 2.5 *Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). For all $t \in [0,T]$, $\delta > 0$ and $\overline{G} \in \left(\mathcal{C}^1(\Lambda)\right)^2$,*

$$\lim_{\gamma \to 0} P^{\gamma;\alpha}_{\sigma^\gamma}\left[\left| < \overline{\pi}^\gamma_t, \overline{G} > - < \overline{m}(t,\cdot), \overline{G} > \right| \geq \delta\right] = 0 \quad I\!P - a.s. \ ,$$

where $\overline{m}=(m_1,m_2)$ *is the unique weak solution of*

$$\begin{cases} \partial_t m_i(t,r) = -m_i(t,r) + p_i \tanh\left[\beta((J*m)(t,r)+a_i\theta)\right], \\ m = m_1+m_2; \quad m_i(0,\cdot)=p_i m_0(\cdot), \quad i=1,2. \end{cases} \tag{2.24}$$

Remark 2.6 . Similarly to Remark 2.3, the Cauchy problem (2.24) in this setup is well posed with a unique global solution; here, the set $\{\overline{m}\in(L^\infty(\Lambda))^2 : \|m_i\|_\infty \le p_i, i=1,2\}$ is left invariant. The solution is differentiable in time. The case $J\ge 0$, $a_1=1$, $a_2=-1$, $p_1=p_2=1/2$ is analyzed in [COP4].

To derive Theorem 2.9 below, we need a stronger type of convergence:

Corollary 2.7 *For all* $\overline{G}\in\left(\mathcal{C}^{0,1}([0,T]\times\Lambda)\right)^2$, $\delta>0$,

$$\lim_{\gamma\to 0} P^{\gamma,\alpha}_{\sigma^\gamma}\left[\sup_{t\in[0,T]}\left| <\overline{\pi}^\gamma_t,\overline{G}(t,\cdot)> - <\overline{m}(t,\cdot),\overline{G}(t,\cdot)>\right|\ge\delta\right]=0.$$

Remark 2.8 . Theorem 2.5 and Corollary 2.7 imply Theorem 2.2 since if $\overline{G}=(G,G)$,

$$<\overline{\pi}^\gamma_t,\overline{G}> = <\pi^\gamma_{1,t},G> + <\pi^\gamma_{2,t},G> = <\pi^\gamma_t,G>.$$

Next theorem states the large deviation principle for the colored particle system. Theorem 2.4 is based on this important intermediate result, interesting for itself.

Theorem 2.9 *Assume* $(\sigma^\gamma)_\gamma, m_0$ *satisfy (2.13). We have, for all open subset* $\overline{\mathcal{A}}$ *and closed subset* $\overline{\mathcal{F}}$ *in* $D([0,T],\mathcal{M}_1\times\mathcal{M}_1)$,

$$\liminf_{\gamma\to 0}\gamma^d\log\overline{Q}^{\gamma,\alpha}_{\sigma^\gamma}(\overline{\mathcal{A}}) \ge -\inf_{\overline{\pi}\in\overline{\mathcal{A}}} I_{m_0}(\overline{\pi}), \quad I\!\!P-a.s. \tag{2.25}$$

$$\limsup_{\gamma\to 0}\gamma^d\log\overline{Q}^{\gamma,\alpha}_{\sigma^\gamma}(\overline{\mathcal{F}}) \le -\inf_{\overline{\pi}\in\overline{\mathcal{F}}} I_{m_0}(\overline{\pi}), \quad I\!\!P-a.s. \tag{2.26}$$

where

$$I_{m_0}(\overline{\pi}) = \begin{cases} I_0(\overline{\pi}) & \text{if} \quad \pi_i(0,\cdot)=p_i m_0(\cdot)\lambda, \quad i=1,2, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.27}$$

and $I_0(\cdot)$, *defined in (3.6) below, is lower semicontinuous with compact level sets.*

Define, for a path $\pi\in D([0,T],\mathcal{M}_1)$,

$$\widetilde{I}_{m_0}(\pi) = \inf\Big(I_{m_0}(\overline{\pi}),\ \overline{\pi}=(\pi_1,\pi_2), \pi_i\in D([0,T],\mathcal{M}_1), i=1,2,\ \pi_1+\pi_2=\pi\Big). \tag{2.28}$$

Since the map $(\pi_1,\pi_2)\mapsto\pi_1+\pi_2$ is continuous in $D([0,T],\mathcal{M}_1\times\mathcal{M}_1)$, by the contraction principle, see [V], [DZ], Theorem 2.9 proves Theorem 2.4. Therefore in the following sections we will focus on the colored particle system.

3. Rate functional

In this section we define the rate functional $I_0(\cdot)$ of the colored particle system and state its main properties. Proofs of the latter, quite technical, are carried out in Section 8. Heuristics to define $I_0(\cdot)$ consists in finding, for any path $\overline{\phi}$ on $[0,T]$ smooth enough, an exponential change of probability under which the process $(\sigma_t)_{t\in[0,T]}$ is uniformly close to $\overline{\phi}$ on $[0,T]$. When there exists some potential $\overline{V}(t,\cdot)$, $t\in[0,T]$ smooth enough for $\overline{\phi}$ to be the solution of *a perturbed equation* (obtained by the law of large numbers from the process $(\sigma_t)_{t\in[0,T]}$, see (5.5) later on), then $I_0(\cdot)$ is related to the Radon-Nikodym derivative of the distribution of $(\sigma_t)_{t\in[0,T]}$ with respect to the distribution of the original process, see Theorem 5.4. In the general case, i.e. when there is no such $\overline{V}(t,\cdot)$, we are still able to provide an explicit representation of $I_0(\cdot)$ (this is similar to the results of [C]). We will then show that this representation of $I_0(\cdot)$ is equivalent to the usual definition of the rate functional, given through the macroscopic functional associated to the Radon-Nikodym derivative, see (3.8). We start by specifying the functional spaces on which we will define $I_0(\cdot)$. For $(p_1,p_2)\in[0,1]^2$, we identify the set

$$B_{p_1,p_2}=\{\overline{u}=(u_1,u_2): u_i\in L^\infty(\Lambda),\ \|u_i\|_\infty\le p_i,\ i=1,2\} \tag{3.1}$$

with $\mathcal{M}^{ac}_{p_1}\times\mathcal{M}^{ac}_{p_2}$, see (2.10), and extend the distance ρ (see (2.9)) to elements of $\mathcal{M}^{ac}_{p_1}\times\mathcal{M}^{ac}_{p_2}$ by

$$\rho(\overline{\mu},\overline{\nu})=\sum_{i=1,2}\rho(\mu_i,\nu_i). \tag{3.2}$$

Definition 3.1 *Let $\mathcal{AC}([0,T],B_{1,1})\subset\mathcal{C}([0,T],B_{1,1})$ be the subset of absolutely continuous functions $\overline{\phi}=(\phi_1,\phi_2)$, that is, for $j=1,2$: for all $t'\in[0,T]$, $t\in[t',T]$, there exists $\dot\phi_j\in L^1([0,T]\times\Lambda)$ such that*

$$\phi_j(t)(r)-\phi_j(t')(r)=\int_{t'}^t\dot\phi_j(s,r)\,ds\,,\quad \lambda-a.s.$$

By an abuse of notation, from now on we write $\phi_j(t,r)$ instead of $\phi_j(t)(r)$.

To write $I_0(\cdot)$, we start by defining, for each $t\in[0,T]$, the following functionals, in which time is kept fixed, therefore we omit to write it. For $\overline{\pi}=(\pi_1,\pi_2)\in\mathcal{M}_1\times\mathcal{M}_1$ (we write $\pi=\pi_1+\pi_2$), $\overline{\mu}=(\mu_1,\mu_2)\in\mathcal{M}_1\times\mathcal{M}_1$ and $\overline{V}=(V_1,V_2)\in(L^\infty(\Lambda))^2$ denote

$$\begin{aligned}F_{\overline{V}}(\overline{\mu},\overline{\pi})=&\sum_{i=1,2}<\mu_i,\tanh(\pi*J+a_i\theta)\sinh(2V_i)+\cosh(2V_i)-1>\\&-\sum_{i=1,2}<\pi_i,\tanh(\pi*J+a_i\theta)[\cosh(2V_i)-1]+\sinh(2V_i)>,\end{aligned} \tag{3.3}$$

and for $\overline{u}=(u_1,u_2)\in B_{1,1}$, $\overline{g}\in\big(L^1(\Lambda)\big)^2$,

$$\Gamma_{\overline{V}}(\overline{u})=F_{\overline{V}}\big((p_1\lambda,p_2\lambda),(u_1\lambda,u_2\lambda)\big), \tag{3.4}$$

$$\mathcal{H}^*(\overline{u},\overline{g})=\sup_{\overline{V}\in(L^\infty(\Lambda))^2}[<\overline{V},\overline{g}>-\frac{1}{2}\Gamma_{\overline{V}}(\overline{u})]. \tag{3.5}$$

The function $\overline{g}\to\mathcal{H}^*(\overline{u},\overline{g})$ is convex. Next lemma ensures that $\mathcal{H}^*(\overline{u},\cdot)$ is the Fenchel-Legendre transform of $\Gamma_{(\cdot)}(\overline{u})$ when $\overline{u}\in B_{p_1,p_2}$, and we will derive in that case an explicit formula for $\mathcal{H}^*(\overline{u},\overline{g})$.

Lemma 3.2 *As a function of $\overline{V}\in(L^\infty(\Lambda))^2$, $\Gamma_{\overline{V}}(\overline{u})$ is convex differentiable for $\overline{u}\in B_{p_1,p_2}$.*

Definition 3.3*The dynamical rate functional $I_0: D([0,T],\mathcal{M}_1\times\mathcal{M}_1)\to I\!R\cup\{\infty\}$ is given by*

$$I_0(\overline{\pi})=\begin{cases} I_0(\overline{\phi})=\displaystyle\int_0^T\mathcal{H}^*(\overline{\phi}(s,\cdot),\dot{\overline{\phi}}(s,\cdot))ds, & \textit{for } \overline{\pi}=(\phi_1\lambda,\phi_2\lambda),\overline{\phi}=(\phi_1,\phi_2)\in\mathcal{AC}([0,T],B_{1,1}),\\ \infty\quad \textit{otherwise.}\end{cases} \tag{3.6}$$

To derive properties of the rate functional associated with the large deviation principle it is convenient to have different representations of I_0. To this aim let $\overline{V} = (V_1, V_2) \in (L^\infty([0,T] \times \Lambda))^2$. We define, for $\overline{\pi} \in D([0,T], \mathcal{M}_1 \times \mathcal{M}_1)$ (cf. (3.4)),

$$K_{\overline{V}}(\overline{\pi}) = \begin{cases} K_{\overline{V}}(\overline{\phi}), & \text{for } \overline{\pi} = (\phi_1\lambda, \phi_2\lambda),\ \overline{\phi} = (\phi_1, \phi_2) \in \mathcal{AC}([0,T], B_{1,1}), \\ \infty & \text{otherwise,} \end{cases} \tag{3.7}$$

where

$$K_{\overline{V}}(\overline{\phi}) = \int_0^T < \overline{V}(s,\cdot), \dot{\overline{\phi}}(s,\cdot) > ds - \frac{1}{2}\int_0^T \Gamma_{\overline{V}(s,\cdot)}(\overline{\phi}(s,\cdot))ds,$$

$$J_0(\overline{\pi}) = \sup_{\overline{V} \in (L^\infty([0,T]\times\Lambda))^2} K_{\overline{V}}(\overline{\pi}). \tag{3.8}$$

$$J_1(\overline{\pi}) = \begin{cases} J_1(\overline{\phi}) = \displaystyle\int_0^T \int_\Lambda \mathcal{H}(\overline{\phi}(t,r), \dot{\overline{\phi}}(t,r))\, dr\, dt, & \text{for } \overline{\pi} = (\phi_1\lambda, \phi_2\lambda),\ \overline{\phi} = (\phi_1, \phi_2) \in \mathcal{AC}([0,T], B_{1,1}), \\ \infty & \text{otherwise,} \end{cases} \tag{3.9}$$

where for $\overline{u} = (u_1, u_2) \in B_{1,1}$, $\overline{g} = (g_1, g_2) \in (L^1([0,T] \times \Lambda))^2$, $(t,r) \in [0,T] \times \Lambda$,

$$\mathcal{H}(\overline{u}, \overline{g})(t,r) = \mathcal{H}(\overline{u}(t,r), \overline{g}(t,r)) = \sum_{i=1}^2 H_i(\overline{u}, g_i)(t,r), \tag{3.10}$$

$$H_i(\overline{u}, g_i)(t,r) = \sup_{v_i \in I\!R} \left\{ g_i(t,r)v_i - \frac{1}{2} B_i(\overline{u}(t,r), v_i) \right\}, \quad i = 1, 2, \tag{3.11}$$

$$\begin{aligned} B_i(\overline{u}(t,r), v_i) =& (p_i - u_i(t,r)) \frac{e^{(J*u)(t,r)+a_i\theta}}{2\cosh[(J*u)(t,r) + a_i\theta]} \left[e^{2v_i} - 1\right] \\ &+ (p_i + u_i(t,r)) \frac{e^{-[(J*u)(t,r)+a_i\theta]}}{2\cosh[(J*u)(t,r) + a_i\theta]} \left[e^{-2v_i} - 1\right]. \end{aligned} \tag{3.12}$$

When $\overline{u}(t,r) \in [-p_1, p_1] \times [-p_2, p_2]$, $\sum_{i=1}^2 B_i(\overline{u}(t,r), \cdot)$ is convex so that $\mathcal{H}(\overline{u}(t,r), \cdot)$ is its Fenchel-Legendre transform. We now give an explicit representation of $\mathcal{H}(\cdot,\cdot)$. To simplify notations denote

$$\begin{aligned} &A_i = A_i(u,\theta)(t,r) = (J*u)(t,r) + a_i\theta, \\ &R_i = R_i(\overline{u}, g_i, \theta)(t,r) = \sqrt{(g_i(t,r)\cosh[A_i(u,\theta)(t,r)])^2 + p_i^2 - u_i^2(t,r)}, \\ &D_i = D_i(\overline{u}, g_i, \theta)(t,r) = g_i(t,r)\cosh[A_i(u,\theta)(t,r)] + R_i(\overline{u}, g_i, \theta)(t,r). \end{aligned} \tag{3.13}$$

Note that $D_i(\overline{u}, g_i, \theta)(t,r) \geq 0$ regardless of the sign of $g_i(t,r)$. When (t,r) is kept fixed we omit to write it. The function $\text{sgn} : I\!R \to I\!R$ is given by

$$\text{sgn}(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } \quad x \neq 0, \\ 0 & \text{if } \quad x = 0. \end{cases} \tag{3.14}$$

Proposition 3.4

(a) If $|u_1| > p_1$ or $|u_2| > p_2$, then $\mathcal{H}(\overline{u}, \overline{g}) = +\infty$.
(b) For $i = 1, 2$, when $|u_i| < p_i$, then

$$H_i(\overline{u}, g_i) = \frac{g_i}{2}\left(\log \frac{D_i}{p_i - u_i} - A_i\right) + \frac{p_i}{2} - \frac{u_i}{2}\tanh A_i - \frac{R_i}{2\cosh A_i}. \tag{3.15}$$

(c) For $i = 1, 2$, when either ($u_i = p_i$ and $g_i \le 0$) or ($u_i = -p_i$ and $g_i \ge 0$), then

$$H_i(\overline{u}, g_i) = 1\!\!1_{\{g_i \neq 0\}} \frac{|g_i|}{2}\left(\log\left\{\frac{|g_i|\cosh A_i}{p_i e^{-\mathrm{sgn}(u_i)A_i}}\right\} - 1\right) + p_i \frac{e^{-\mathrm{sgn}(u_i)A_i}}{2\cosh A_i}. \tag{3.16}$$

(d) For $i = 1, 2$, when either ($u_i = p_i$ and $g_i > 0$) or ($u_i = -p_i$ and $g_i < 0$), then $H_i(\overline{u}, g_i) = +\infty$.

The following proposition shows that the order of supremum and the integrals can be reversed. In particular we can compute the supremum for each point $(t, r) \in [0, T] \times \Lambda$.

Proposition 3.5 *For $\overline{\pi} = (\phi_1\lambda, \phi_2\lambda)$, $\overline{\phi} = (\phi_1, \phi_2) \in \mathcal{AC}([0,T], B_{1,1})$, we have $I_0(\overline{\pi}) = J_0(\overline{\pi}) = J_1(\overline{\pi})$. Furthermore if $\overline{\phi} \in \mathcal{AC}([0,T], B_{1,1}) \setminus \mathcal{AC}([0,T], B_{p_1,p_2})$, then $I_0(\overline{\phi}) = +\infty$.*

Proof. This follows and extends the proof in [C], p. 171, Properties III(a). By their respective Definitions (3.6), (3.8), (3.9) (see also (3.5), (3.7), (3.10)), we have $J_0(\overline{\pi}) \le I_0(\overline{\pi}) \le J_1(\overline{\pi})$. We now prove that we have equalities. In all cases, for $i = 1, 2$, we denote by ϑ_i the value of v_i that realizes the extremum of $H_i(\overline{u}, g_i)$. From Proposition 3.4, ϑ_i belongs to $I\!R \cup \{+\infty, -\infty\}$. Let $\vartheta_i^m = \mathrm{sgn}(\vartheta_i) \times [|\vartheta_i| \wedge m]$ and b_i^m be the corresponding (finite) value of $H_i(\overline{u}, g_i)$. Then as $m \to \infty$, $\vartheta_i^m \to \vartheta_i$ and $b_i^m \to H_i(\overline{u}, g_i) \in I\!R^+ \cup \{\infty\}$. According to the case we consider, either a_i and/or b_i are finite, and there is no problem, or $b_i = +\infty$ thus $b_i^m > 0$ for m large enough, or, when $\overline{u} \in B_{p_1,p_2}$, b_i^m is non-negative because ϑ_i^m is between 0 and ϑ_i, and $v_i \mapsto B_i(\overline{u}, v_i)$ is a convex function. Therefore in all cases we apply Fatou's Lemma to get

$$J_1(\overline{\pi}) \le \liminf_{m\to\infty} \int_0^T \int_\Lambda \sum_{i=1,2} b_i^m(t, r)\, drdt$$

which is smaller than $J_0(\overline{\pi})$, whence the result. Notice that this implies that $I_0(\overline{\pi})$ is infinite when $\overline{u} \notin B_{p_1,p_2}$.
□

Next we characterize the finite energy trajectories.

Proposition 3.6 *Take $(t, r) \in [0, T] \times \Lambda$.*
(a) Let $\overline{u}$ be such that for $i = 1, 2$, $|u_i| < p_i$. There exist positive constants K_1,K_2 and C such that

$$\begin{aligned}&\mathcal{H}(\overline{u}, \overline{g})(t, r)\\ &\le \sum_{i=1,2} \frac{|g_i|}{2}\left[(\log|g_i|)^+ + 1\!\!1_{\{g_i>0\}}\left(\log\frac{1}{p_i - u_i}\right)^+ + 1\!\!1_{\{g_i<0\}}\left(\log\frac{1}{p_i + u_i}\right)^+ + K_i\right](t, r) + C\end{aligned} \tag{3.17}$$

$$\mathcal{H}(\overline{u}, \overline{g})(t, r) \ge \sum_{i=1,2} \frac{|g_i|}{2}\left[\log|g_i| - K_i\right](t, r) - C. \tag{3.18}$$

(b) $I_0(\overline{\phi}) < \infty$ if and only if for $i = 1, 2$, $\dot{\phi}_i \log|\dot{\phi}_i|$, $\dot{\phi}_i \log\frac{1}{p_i - \phi_i} 1\!\!1_{\{\dot{\phi}_i>0\}}$, $\dot{\phi}_i \log\frac{1}{p_i + \phi_i} 1\!\!1_{\{\dot{\phi}_i<0\}}$ belong to $L^1([0,T] \times \Lambda)$.

Proposition 3.7 *(1) The functional $I_0(\cdot)$ is lower semicontinuous on $D([0,T],B_{1,1})$.*

(2) The set $D_{L_0}=\{\overline{\pi};I_0(\overline{\pi})\le L_0\}$ is compact in $D([0,T],B_{1,1})$ for all $L_0>0$.

(3) $I_0(\overline{\phi})\ge 0$, and $I_0(\overline{\phi})=0$ if and only if $\overline{\phi}$ is the solution of equation (2.24).

4. Hydrodynamic behavior for the colored system

In this section, we prove Theorem 2.5, through a by now standard scheme. Nevertheless, we detail it since many of its parts will also appear in the following sections.

We first highlight that throughout the paper, one of the key ingredients to deal with the randomness of the interaction will be the following applications of the ergodic theorem and strong law of large numbers. For all function h on Λ_γ, integer l, we denote by $h^{(l)}$ the averaged function

$$h^{(l)}(x)=\frac{1}{(2l+1)^d}\sum_{y\in\Lambda_\gamma,|y-x|\le l}h(y)\ ,\quad x\in\Lambda_\gamma. \tag{4.1}$$

Lemma 4.1 (ergodic theorem for local functions) *Let $\Theta(\alpha)$ be a bounded measurable cylinder function on Ω and $G\in\mathcal{C}(\Lambda)$. Then, for almost any disorder configuration α,*

$$\lim_{\gamma\to 0}\gamma^d\sum_{x\in\Lambda_\gamma}G(\gamma x)\tau_x\Theta(\alpha)=I\!E\big[\Theta\big]\int_\Lambda G(r)dr\,.$$

Proof. Write

$$\gamma^d\sum_{x\in\Lambda_\gamma}G(\gamma x)\tau_x\Theta(\alpha)=\gamma^d\sum_{x\in\Lambda_\gamma}G(\gamma x)\big[\tau_x\Theta(\alpha)-I\!E[\Theta]\big]+I\!E[\Theta]\gamma^d\sum_{x\in\Lambda_\gamma}G(\gamma x).$$

For any $l\in I\!N$, by the regularity of G,

$$\Big|\gamma^d\sum_{x\in\Lambda_\gamma}G(\gamma x)\big[\tau_x\Theta(\alpha)-I\!E[\Theta]\big]\Big|\le\|G\|_\infty\gamma^d\sum_{x\in\Lambda_\gamma}\big|\left(\tau.\Theta(\alpha)\right)^{(l)}(x)-I\!E\big[\Theta\big]\big|+\epsilon(\gamma l),$$

where $\lim_{s\to 0}\epsilon(s)=0$. Keeping l fixed, by the ergodic theorem,

$$\lim_{\gamma\to 0}\gamma^d\sum_{x\in\Lambda_\gamma}\big|\left(\tau.\Theta(\alpha)\right)^{(l)}(x)-I\!E[\Theta]\big|=I\!E\big[\big|\left(\tau.\Theta\right)^{(l)}(x)-I\!E[\Theta]\big|\big].$$

The law of large numbers (letting $l\to\infty$) gives the result. □

We introduce (cf. [K]), for $i=1,2$ and $\delta>0$,

$$A_{l,\delta}(x,i)=\Big\{\alpha\in\Omega:\Big|\alpha_i^{(l)}(x)-I\!E(\alpha_i(x))\Big|\le\delta\Big\},\quad x\in\Lambda_\gamma, \tag{4.2}$$

$$\mathcal{E}_i(\delta,l,\gamma,\alpha)=\gamma^d\sum_{x\in\Lambda_\gamma}1\!\!1_{A^c_{l,\delta}(x,i)}(\alpha). \tag{4.3}$$

Lemma 4.2 *For any $\delta>0$, for $i=1,2$, $\lim_{l\to\infty}\lim_{\gamma\to 0}\mathcal{E}_i(\delta,l,\gamma,\alpha)=0,\quad I\!P-a.s.$*

Proof. Applying Lemma 4.1 to the function $\Theta=1\!\!1_{A^c_{l,\delta}(0,i)}$ gives $\lim_{\gamma\to 0}\mathcal{E}_i(\delta,l,\gamma,\alpha)=I\!P(A^c_{l,\delta}(0,i)),\quad I\!P-$ *a.s.* Then by the strong law of large numbers $\lim_{l\to\infty}I\!P(A^c_{l,\delta}(0,i))=0$. □

In the following it is convenient to define the random discrete measures $\overline{\lambda}^\gamma(\alpha) = (\lambda_1^\gamma(\alpha), \lambda_2^\gamma(\alpha))$, where

$$\lambda_i^\gamma(\alpha) = \gamma^d \sum_{x\in\Lambda_\gamma} \alpha_i(x)\delta_{\gamma x}, \;\; i=1,2, \qquad \lambda^\gamma = \lambda_1^\gamma(\alpha) + \lambda_2^\gamma(\alpha) = \gamma^d \sum_{x\in\Lambda_\gamma} \delta_{\gamma x}. \tag{4.4}$$

Proof of Theorem 2.5 We follow the general scheme introduced in [KL] chap. 4. We have to show:

(i) For any α, the sequence $(\overline{Q}^{\gamma,\alpha}_{\sigma^\gamma})_\gamma$ is tight.

(ii) Any limit point $\overline{Q}^\alpha$ of $(\overline{Q}^{\gamma,\alpha}_{\sigma^\gamma})_\gamma$ is $I\!P$-a.s. concentrated on measures $(\overline{\pi}_t)_{t\in[0,T]} \in \mathcal{C}([0,T], \mathcal{M}^{ac}_{p_1}\times\mathcal{M}^{ac}_{p_2})$.

(iii) For $I\!P$-a.s. α, any limit point $\overline{Q}^\alpha$ of $(\overline{Q}^{\gamma,\alpha}_{\sigma^\gamma})_\gamma$ is concentrated on trajectories $(\overline{\pi}_t)_{t\in[0,T]}$ such that $\overline{\pi}_t(dr) = \overline{m}(t,r)dr$, where the density $\overline{m}$ is a weak solution of (2.24).

(iv) Equation (2.24) has a unique weak solution.

For (ii) we use that the spins are finite-valued (cf. [KL]). Namely, fix $G\in\mathcal{C}(\Lambda)$,

$$\sup_{0\le t\le T} \big|< \pi^\gamma_{i,t}, G >\big| \le \gamma^d \sum_{x\in\Lambda_\gamma} |G(\gamma x)|\alpha_i(x), \quad i=1,2,$$

because there is at the most one spin per site and $\alpha_i(x)\ge 0$. As in the case without random field, the application $(\pi_{i,t})_{t\in[0,T]} \mapsto \sup_{t\in[0,T]} < \pi_{i,t}, G >$ is continuous in the weak topology. Thus by weak convergence and Lemma 4.1 (by the independence of the r.v. α's) all limits points are concentrated on trajectories $(\pi_{i,t})_{t\in[0,T]}$ such that

$$|< \pi_{i,t}, G >| \le \int_\Lambda |G(r)|p_i dr, \quad I\!P - a.s.$$

Point (iv) is derived similarly to the proof of the Cauchy-Lipschitz theorem. For Points (i) and (iii), let $\overline{G} = (G_1,G_2) \in \big(\mathcal{C}^{1,0}([0,T]\times\Lambda)\big)^2$. For $\overline{\pi}\in D([0,T],\mathcal{M}_1\times\mathcal{M}_1)$, let

$$\ell_t(\overline{\pi},\overline{G}) = < \overline{\pi}_t, \overline{G}(t,\cdot) > - < \overline{\pi}_0, \overline{G}(0,\cdot) > - \int_0^t < \overline{\pi}_s, \partial_s\overline{G}(s,\cdot) > \, ds. \tag{4.5}$$

We have, for $x\in\Lambda_\gamma$,

$$\begin{aligned} \mathcal{L}^{\gamma,\alpha}(\sigma(x)) &= -\sigma(x) + \sigma(x)(1-2c_x^{\gamma,\alpha}(\sigma)) \\ &= -\sigma(x) + \tanh[(J_\gamma\star\sigma)(x) + \theta\alpha(x)]. \end{aligned} \tag{4.6}$$

The $P^{\gamma,\alpha}_{\sigma^\gamma}$-martingale $\overline{N}^{\overline{G}}_\gamma \equiv (\overline{N}^{\overline{G}}_\gamma(t))_{t\in[0,T]}$ with respect to the natural filtration associated to $(\sigma_t)_{t\in[0,T]}$ (cf. (2.20)) given by

$$\begin{aligned} \overline{N}^{\overline{G}}_\gamma(t) &= \ell_t(\overline{\pi}^\gamma,\overline{G}) - \gamma^d \int_0^t \sum_{i=1,2}\sum_{x\in\Lambda_\gamma} G_i(s,\gamma x)\alpha_i(x)\mathcal{L}^{\gamma,\alpha}(\sigma_s(x))\, ds \\ &= \ell_t(\overline{\pi}^\gamma,\overline{G}) + \int_0^t < \overline{\pi}^\gamma_s, \overline{G}(s,\cdot) > ds - \sum_{i=1,2}\int_0^t < \lambda_i^\gamma(\alpha), G_i(s,.)\tanh[\pi^\gamma_s * J + a_i\theta] > ds\;, \end{aligned} \tag{4.7}$$

has quadratic variation

$$< \overline{N}^{\overline{G}}_\gamma, \overline{N}^{\overline{G}}_\gamma > (t) = -2\gamma^{2d} \sum_{i=1,2}\sum_{x\in\Lambda_\gamma} \alpha_i(x) G_i^2(s,\gamma x) \int_0^t \big\{ -1 + \sigma_s(x)\tanh[(J_\gamma\star\sigma_s)(x) + a_i\theta]\big\} ds. \tag{4.8}$$

Hence, for any $\alpha \in \Omega$, since tanh is a smooth function and $(J_\gamma \star \sigma)(x) + a_i\theta$ is uniformly bounded in x, σ, by Doob's inequality,

$$\lim_{\gamma\to 0} P^{\gamma;\alpha}_{\sigma^\gamma}\Big(\sup_{t\in[0,T]} \big|\overline{N}^{\overline{G}}_\gamma(t)\big| > \delta\Big) \le \lim_{\gamma\to 0} \frac{1}{\delta^2} E^{\gamma;\alpha}_{\sigma^\gamma}\Big(\big|\overline{N}^{\overline{G}}_\gamma(T)\big|^2\Big) \le \lim_{\gamma\to 0} \frac{1}{\delta^2} C(\overline{G},T)\gamma^d = 0. \tag{4.9}$$

Bound (4.9) yields Point (i), by Prohorov's criterion. Point (iii) will consist in identification of the limit. To obtain a closed form for the limiting equation, we only need to average over the disorder, that is to replace in the limit $\gamma \to 0$ the random discrete measures $\lambda_1^\gamma(\alpha)$ and $\lambda_2^\gamma(\alpha)$ by their expectations $p_1\lambda$ and $p_2\lambda$ with respect to the environment. Denote

$$\widetilde{\ell}_t(\overline{\pi}^\gamma,\overline{G}) \;=\; \int_0^t \Big\{\big\langle \overline{\pi}^\gamma_s, \overline{G}(s,\cdot)\big\rangle - \sum_{i=1}^2 \big\langle p_i\lambda^\gamma, G_i(s,\cdot)\tanh(\pi^\gamma_s * J + a_i\theta)\big\rangle\Big\}\, ds. \tag{4.10}$$

Putting together (4.7), (4.9), and applying Lemma 4.3 below, we get that for all subsequences

$$\liminf_{k\to+\infty} Q^{\gamma_k,\alpha}_{\sigma^{\gamma_k}}\Big(\sup_{t\le T}\Big|\ell_t(\overline{\pi}^{\gamma_k},\overline{G}) + \widetilde{\ell}_t(\overline{\pi}^{\gamma_k},\overline{G})\Big| > \frac{\delta}{2}\Big) = 0. \tag{4.11}$$

Denoting $m = m_1 + m_2$, for $\overline{m} = (m_1, m_2)$, see (2.24), this gives, for almost any α,

$$\begin{aligned} &Q^\alpha\Big(\sup_{t\le T}\Big|\sum_{i=1}^2 \int_\Lambda \Big[\{G_i(t,r)m_i(t,r) - G_i(0,r)m_i(0,r)\} - \int_0^t \partial_s G_i(s,r)m_i(s,r)\,ds\Big]\,dr \\ &+ \sum_{i=1,2}\int_0^t\int_\Lambda [G_i(s,r)m_i(s,r) - I\!E(\alpha_i(0))\tanh[(J*m)(s,r) + a_i\theta]]\; drds\Big| > \frac{\delta}{2}\Big) \\ &= Q^\alpha\Big(\sup_{t\le T}\Big|\ell_t(\overline{m},\overline{G}) + \widetilde{\ell}_t(\overline{m},\overline{G})\Big| > \frac{\delta}{2}\Big) = 0, \end{aligned} \tag{4.12}$$

where we set by an abuse of notation

$$\ell_t(\overline{m},\overline{G}) = <\overline{m}(t,\cdot),\overline{G}(t,\cdot)> - <\overline{m}(0,\cdot),\overline{G}(0,\cdot)> - \int_0^t <\overline{m}(s,\cdot),\partial_s\overline{G}(s,\cdot)>ds, \tag{4.13}$$

$$\widetilde{\ell}_t(\overline{m},\overline{G}) \;=\; \int_0^t \Big\{\big\langle \overline{m}(s,\cdot),\overline{G}(s,\cdot)\big\rangle - \sum_{i=1}^2 \big\langle p_i\lambda, G_i(s,\cdot)\tanh(J*m(s,\cdot) + a_i\theta)\big\rangle\Big\}ds. \tag{4.14}$$

This leads to identification of the limit (iii), that is to equation (2.24). □

Lemma 4.3 *For $i = 1,2$, $G_i \in \mathcal{C}^{1,0}([0,T]\times\Lambda)$, there exists a positive function ϵ on $I\!R_+$ with $\lim_{s\to 0}\epsilon(s) = 0$ such that for all $l \in I\!N\setminus\{0\}$, $\delta > 0$, and $\mathcal{E}_i(\delta,l,\gamma,\alpha)$ defined in (4.3), the quantity*

$$\Delta_i^\gamma(\alpha,\sigma,T) = \int_0^T \Big| < \lambda_i^\gamma(\alpha) - p_i\lambda, G_i(s,.)\tanh[\pi^\gamma_s * J + a_i\theta] > \Big|\, ds \tag{4.15}$$

satisfies

$$\Delta_i^\gamma(\alpha,\sigma,T) \le \frac{\delta}{2}T\|G_i(s,\cdot)\|_1 + 2T\|G_i(s,\cdot)\|_\infty \mathcal{E}_i(\delta,l,\gamma,\alpha) + \epsilon(\gamma l)T + \epsilon(\gamma)T, \tag{4.16}$$

$$\lim_{\gamma\to 0}\Delta_i^\gamma(\alpha,\sigma,T)=0,\qquad I\!P-a.s. \tag{4.17}$$

Proof. We introduce averages over large microscopic boxes of size l but small w.r.t. the range γ^{-1} of the interaction (l will go to infinity but after the limit $\gamma\to 0$). To keep notation readable, the function ϵ may vary from one line to another but keeping the same property that $\lim_{s\to 0}\epsilon(s)=0$. Since J and tanh are uniformly Lipschitz, and G_i is uniformly continuous (in space), there are constants $c_1>0$, $c_2>0$ such that, see (4.1), for $x\in\Lambda_\gamma$,

$$\sup_{\sigma\in\mathcal{S}_\gamma}\left|(\pi^\gamma(\sigma)*J)(\gamma x)-\big((\pi^\gamma(\sigma)*J)(\gamma .)\big)^{(l)}(x)\right|\le c_1\gamma l,$$

$$\sup_{\sigma\in\mathcal{S}_\gamma}\left|\tanh[\beta((\pi^\gamma(\sigma)*J)(\gamma x)+a_i\theta)]-\big(\tanh[(\pi^\gamma(\sigma)*J)(\gamma .)+a_i\theta]\big)^{(l)}(x)\right|\le c_2\gamma l.$$

$$\sup_{0\le s\le T}\left|G_i(s,\gamma x)-\big(G_i(s,\gamma .)\big)^{(l)}(x)\right|\le\epsilon(\gamma l).$$

Recalling notation (4.4), by summation by parts we get

$$\int_0^T\left|<\lambda_i^\gamma(\alpha)-\lambda_i^\gamma(\alpha^{(l)}),G_i(s,.)\tanh[\pi_s^\gamma*J+a_i\theta]>\right|\,ds\le\epsilon(\gamma l)T$$

and by uniform continuity or Lipschitz condition

$$\int_0^T|<p_i\lambda^\gamma-p_i\lambda,G_i(s,.)\tanh[\pi_s^\gamma*J+a_i\theta]>|\,ds\le\epsilon(\gamma)T.$$

Therefore, we have

$$\Delta_i^\gamma(\alpha,\sigma,T)\le\epsilon(\gamma l)T+\epsilon(\gamma)T+\int_0^T\left|<\lambda_i^\gamma(\alpha^{(l)})-p_i\lambda^\gamma,G_i(s,\cdot)\tanh[\pi_s^\gamma*J+a_i\theta]>\right|\,ds.$$

To derive (4.16), we take into account definitions (4.2), (4.3), and that $|\alpha_i^{(l)}(x)-p_i|\le 2$, to write

$$\begin{aligned}
&|<\lambda_i^\gamma(\alpha^{(l)})-p_i\lambda^\gamma,G_i(s,\cdot)\tanh[\pi_s^\gamma*J+a_i\theta]>|=\\
&\left|\gamma^d\sum_{x\in\Lambda_\gamma}G_i(s,\gamma x)\tanh[(J_\gamma\star\sigma_s)(x)+a_i\theta]\left[1\!\!1_{A_{l,\delta}(x,i)}(\alpha)+1\!\!1_{A^c_{l,\delta}(x,i)}(\alpha)\right]\left(\alpha_i^{(l)}(x)-I\!E(\alpha_i(x))\right)\right|\\
&\le\gamma^d\sum_{x\in\Lambda_\gamma}|G_i(s,\gamma x)\tanh[(J_\gamma\star\sigma_s)(x)+a_i\theta]|\left[\delta+2\,1\!\!1_{A^c_{l,\delta}(x,i)}(\alpha)\right]\\
&\le\delta\|G_i(s,\cdot)\|_1+2\|G_i(s,\cdot)\|_\infty\mathcal{E}_i(\delta,l,\gamma,\alpha).
\end{aligned}$$

Applying Lemma 4.2 to (4.16) we get (4.17). □

Proof of Corollary 2.7: First notice that applying Lebesgue dominated convergence Theorem in the time integral, Theorem 2.5 implies that for any $G_i\in\mathcal{C}^{0,1}([0,T]\times\Lambda)$ we have

$$\lim_{\gamma\to 0}P^{\gamma,\alpha}_{\sigma^\gamma}\left[\int_0^T\left|<\pi^\gamma_{i,s},G_i(s,\cdot)>-<m_i(s,\cdot),G_i(s,\cdot)>\right|\,ds\ge\delta\right]=0. \tag{4.18}$$

Now remark that integrating in time (2.24),

$$< m_i(t,\cdot), G_i(t,\cdot) >=< m_i(0,\cdot), G_i(0,\cdot) > + \int_0^t < m_i(s,\cdot), \partial_s G_i(s,\cdot) - G_i(s,\cdot) > \, ds$$
$$+ p_i \int_0^t < \tanh[(J*m)(s,\cdot) + a_i\theta], G_i(s,\cdot) > \, ds.$$

Introducing the martingale $\overline{N}^G_\gamma$, see (4.7), and using (4.9) we get

$$P^{\gamma,\alpha}_{\sigma^\gamma}\left[\sup_{t\in[0,T]} \big|< \pi^\gamma_{i,t}, G_i(t,\cdot) >- < m_i(t,\cdot), G_i(t,\cdot) >\big| \geq \delta\right] \leq A_\gamma + B_\gamma + C_\gamma + \epsilon(\gamma),$$

with $\lim_{\gamma\to 0}\epsilon(\gamma) = 0$ and

$$A_\gamma = P^{\gamma,\alpha}_{\sigma^\gamma}\left[\int_0^T \big|< \pi^\gamma_{i,s} - m_i(s,\cdot), \partial_s G_i(s,\cdot) - G_i(s,\cdot) >\big| \, ds \geq \frac{\delta}{4}\right],$$
$$C_\gamma = P^{\gamma,\alpha}_{\sigma^\gamma}\left[\big|< \pi^\gamma_{i,0} - m_i(0,\cdot), G_i(0,\cdot) >\big| \geq \frac{\delta}{4}\right],$$

$$\begin{aligned} B_\gamma &= P^{\gamma,\alpha}_{\sigma^\gamma}\left[\int_0^T \big|< \lambda^\gamma_i(\alpha), G_i(s,\cdot)\tanh[\pi^\gamma_s * J + a_i\theta] > - < p_i\lambda, G_i(s,\cdot)\tanh[(J*m)(s,\cdot) + a_i\theta] >\big| \, ds \geq \frac{\delta}{4}\right] \\ &\leq P^{\gamma,\alpha}_{\sigma^\gamma}\left[\int_0^T |< \lambda^\gamma_i(\alpha) - p_i\lambda, G_i(s,\cdot)\tanh[\pi^\gamma_s * J + a_i\theta] >| \, ds \geq \frac{\delta}{8}\right] \\ &\quad + P^{\gamma,\alpha}_{\sigma^\gamma}\left[\int_0^T \big|< p_i\lambda, G_i(s,\cdot)\big(\tanh[\pi^\gamma_s * J + a_i\theta] - \tanh[(J*m_i)(s,\cdot) + a_i\theta]\big) >\big| \, ds \geq \frac{\delta}{8}\right]. \end{aligned} \tag{4.19}$$

From (2.13) and (4.18), $\lim_{\gamma\to 0} A_\gamma = \lim_{\gamma\to 0} C_\gamma = 0$. For B_γ, from Lemma 4.3, the limit when $\gamma \to 0$ of the first term in the right hand side of (4.19) is equal to zero; the second term vanishes from (4.18) since the function tanh is Lipschitz continuous. □

5. The perturbed dynamics and Radon-Nikodym derivative

The general strategy to derive the large deviation principle prescribes to find a family of mean one positive martingales that can be expressed as functions of the empirical measures. Following [DV], the relevant martingales are obtained as Markovian perturbations of the original process. In this section we define a class of time dependent, random external potentials, *the perturbations*, to which we can associate a trajectory $(m(t,\cdot))_{t\in[0,T]}$ smooth in time. We show the law of large numbers for the empirical measures of the dynamics associated to these perturbations and derive the Radon-Nikodym derivative of the perturbed process with respect to the unperturbed one.

Given a realization α of the magnetic field, $\overline{V} = (V_1, V_2) \in \big(\mathcal{C}^{1,0}([0,T]\times\Lambda)\big)^2$, let

$$V(t,\gamma x,\alpha(x)) = \sum_{i=1,2} \alpha_i(x) V_i(t,\gamma x) \tag{5.1}$$

be the full external random perturbation for the magnetization trajectories $\pi^\gamma(\sigma)$ (not colored). As pointed out in the introduction this perturbation strongly depends on the randomness. It is therefore convenient to consider a Glauber evolution for the colored particle system, associated to the Hamiltonian obtained by summing up (2.4) and

$$H^{\overline{V},\gamma,\alpha}(\sigma) = -\sum_{x\in\Lambda_\gamma}\sum_{i=1,2}\alpha_i(x)V_i(t,\gamma x)\sigma(x). \tag{5.2}$$

To this aim we define time dependent rates, for all $x \in \Lambda_\gamma$, $\sigma \in \mathcal{S}_\gamma$,

$$c_x^{\overline{V},\gamma,\alpha}(\sigma,t) = e^{-\sigma(x)2V(t,\gamma x,\alpha(x))}c_x^{\gamma,\alpha}(\sigma) = \frac{e^{-\sigma(x)[(J_\gamma\star\sigma)(x)+\theta\alpha(x)+2V(t,\gamma x,\alpha(x))]}}{2\cosh[(J_\gamma\star\sigma)(x)+\theta\alpha(x)]}. \tag{5.3}$$

Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). We denote by $P_{\sigma^\gamma}^{\overline{V},\gamma,\alpha}$ the law (and by $E_{\sigma^\gamma}^{\overline{V},\gamma,\alpha}$ the expectation) of the corresponding inhomogeneous Glauber process $(\sigma_t)_{t\in[0,T]}$ on $\mathcal{S}_\gamma$, that is the unique probability measure on $D([0,T],\mathcal{S}_\gamma)$ with initial condition σ^γ under which $f(\sigma_t) - f(\sigma_0) - \int_0^t \mathcal{L}_s^{\overline{V},\gamma,\alpha}(f)(\sigma_s)ds$ is a martingale w.r.t. the canonical filtration, for all cylinder function f, where

$$\mathcal{L}_s^{\overline{V},\gamma,\alpha}(f)(\sigma_s) = \sum_{x\in\Lambda_\gamma} c_x^{\overline{V},\gamma,\alpha}(\sigma_s,s)[f(\sigma_s^x) - f(\sigma_s)]. \tag{5.4}$$

Let $\overline{Q}_{\sigma^\gamma}^{\overline{V},\gamma,\alpha}$ be the law of the corresponding empirical measures.

Theorem 5.1 *Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). For all $t \in [0,T]$, $\overline{G} = (G_1,G_2) \in \big(\mathcal{C}^1(\Lambda)\big)^2$, and $\delta > 0$,*

$$\lim_{\gamma\to 0}\overline{Q}_{\sigma^\gamma}^{\overline{V},\gamma,\alpha}\left[\Big|< \overline{\pi}_t^\gamma, \overline{G} > - < \overline{m}^{\overline{V}}(t,\cdot), \overline{G} >\Big| \geq \delta\right] = 0, \qquad I\!P - a.s.,$$

where $\overline{m}^{\overline{V}} = (m_1^{\overline{V}}, m_2^{\overline{V}})$ is the solution of, for $i = 1,2$,

$$\begin{cases} \partial_t m_i(t,r) = \{-m_i(t,r) + p_i\tanh[(J*m)(t,r) + a_i\theta + 2V_i(t,r)]\}\,\dfrac{\cosh[(J*m)(t,r)+a_i\theta+2V_i(t,r)]}{\cosh[(J*m)(t,r)+a_i\theta]}, \\ m_i(0,\cdot) = p_i m_0(\cdot), \quad m = m_1 + m_2. \end{cases} \tag{5.5}$$

Remark 5.2 . For existence and uniqueness of the solution $\overline{m}^{\overline{V}} \in (\mathcal{C}([0,T],L^\infty(\Lambda)))^2$, we refer to Remark 2.6. Notice that the set $\{\overline{m} \in (L^\infty(\Lambda))^2 : \|m_i\|_\infty \leq p_i, i = 1,2\}$ is still left invariant.

Proof. We proceed as for Theorem 2.5. We use

$$1 = \frac{(1-\sigma_s(x))}{2} + \frac{(1+\sigma_s(x))}{2} = 1\!\!1_{\{\sigma_s(x)=-1\}} + 1\!\!1_{\{\sigma_s(x)=1\}}. \tag{5.6}$$

For $i \in \{1,2\}$ we have

$$\begin{aligned} &\mathcal{L}_s^{\overline{V},\gamma,\alpha}(\alpha_i(x)\sigma(x)) = -2\alpha_i(x)\sigma(x)c_x^{\overline{V},\gamma,\alpha}(\sigma,s) \\ &= -\alpha_i(x)e^{-\sigma(x)2V_i(s,\gamma x)}2\sigma(x)\frac{e^{-\sigma(x)[(J_\gamma\star\sigma)(x)+a_i\theta]}}{2\cosh[J_\gamma\star\sigma)(x)+a_i\theta]} \\ &= -\alpha_i(x)\left[\frac{(\sigma(x)+1)}{2}\frac{e^{-[2V_i(s,\gamma x)+(J_\gamma\star\sigma)(x)+a_i\theta]}}{\cosh[(J_\gamma\star\sigma)(x)+a_i\theta]} + \frac{(\sigma(x)-1)}{2}\frac{e^{[2V_i(s,\gamma x)+(J_\gamma\star\sigma)(x)+a_i\theta]}}{\cosh[(J_\gamma\star\sigma)(x)+a_i\theta]}\right] \\ &= -\alpha_i(x)\sigma(x)\frac{\cosh[2V_i(s,\gamma x)+(J_\gamma\star\sigma)(x)+a_i\theta]}{\cosh[(J_\gamma\star\sigma)(x)+a_i\theta]} - \alpha_i(x)\frac{\sinh[2V_i(s,\gamma x)+(J_\gamma\star\sigma)(x)+a_i\theta]}{\cosh[(J_\gamma\star\sigma)(x)+a_i\theta]}. \end{aligned} \tag{5.7}$$

$\square$

We have the analogous result to Corollary 2.7:

Corollary 5.3 *For all* $\overline{G} = (G_1, G_2) \in \left(\mathcal{C}^1(\Lambda)\right)^2$, *and* $\delta > 0$,

$$\lim_{\gamma\to 0} \overline{Q}^{\overline{V},\gamma,\alpha}_{\sigma^\gamma} \left[\sup_{t\in[0,T]} \left| < \overline{\pi}^\gamma_t, \overline{G} > - < \overline{m}^{\overline{V}}(t,\cdot), \overline{G} > \right| \geq \delta \right] = 0.$$

Theorem 5.4 *Let* $\overline{V} = (V_1, V_2) \in \left(\mathcal{C}^{1,0}([0,T]\times\Lambda)\right)^2$. *The Radon-Nikodym derivative is given by*

$$\frac{dP^{\overline{V},\gamma,\alpha}_{\sigma^\gamma}}{dP^{\gamma,\alpha}_{\sigma^\gamma}}(\sigma_{[0,T]}) = \exp\Big\{\gamma^{-d}\Big(\ell_T(\overline{\pi}^\gamma(\sigma), \overline{V}) - \frac{1}{2}\int_0^T F_{\overline{V}(s)}(\overline{\lambda}^\gamma(\alpha), \overline{\pi}^\gamma_s)\, ds\Big)\Big\}, \tag{5.8}$$

where ℓ_T *was defined in (4.5),* $\overline{\lambda}^\gamma(\alpha)$ *in (4.4),* $F_{\overline{V}(s)}(\cdot,\cdot)$ *in (3.3), and we have abbreviated* $\sigma_{[0,T]} = (\sigma_t)_{t\in[0,T]}$.

Proof. The Radon-Nikodym derivative associated with rates (5.3) is given by (see [HS] or [KL], Appendix 1, Proposition 7.3)

$$\begin{aligned}
\frac{dP^{\overline{V},\gamma,\alpha}_{\sigma^\gamma}}{dP^{\gamma,\alpha}_{\sigma^\gamma}}(\sigma_{[0,t]}) &= \exp\Big\{ - H^{\overline{V},\gamma,\alpha}(\sigma_t) + H^{\overline{V},\gamma,\alpha}(\sigma_0) \\
&\quad - \int_0^t \exp\Big\{H^{\overline{V},\gamma,\alpha}(\sigma_s)\Big\}\big(\partial_s + \mathcal{L}_\gamma\big)\exp\Big\{ - H^{\overline{V},\gamma,\alpha}(\sigma_s)\Big\}\, ds\Big\} \\
&= \exp\Big\{\ell_t(\overline{\pi}^\gamma(\sigma), \overline{V}) - \int_0^t \sum_{x\in\Lambda_\gamma} c^{\gamma,\alpha}_x(\sigma_s)\left[e^{-2\sigma_s(x)V(s,\gamma x,\alpha(x))} - 1\right]\, ds\Big\},
\end{aligned}$$

because of (4.5), (5.2). To get (5.8), we use trigonometric formulas to write (remember (2.8), (2.20), (5.7))

$$\begin{aligned}
&2\gamma^d \sum_{x\in\Lambda_\gamma} c^{\gamma,\alpha}_x(\sigma_s)\left[e^{-2\sigma_s(x)V(s,\gamma x,\alpha(x))} - 1\right] \\
&\quad= 2\gamma^d \sum_{x\in\Lambda_\gamma}\left\{\frac{(1-\sigma_s(x))}{2} + \frac{(1+\sigma_s(x))}{2}\right\}\left\{\sum_{i=1,2}\alpha_i(x)\right\} c^{\gamma,\alpha}_x(\sigma_s)\left[e^{-2\sigma_s(x)V(s,\gamma x,\alpha(x))} - 1\right] \\
&\quad= \gamma^d \sum_{i=1,2}\sum_{x\in\Lambda_\gamma}\frac{(1-\sigma_s(x))}{2}\alpha_i(x)\frac{\exp[(J_\gamma\star\sigma_s)(x) + a_i\theta]}{\cosh[(J_\gamma\star\sigma_s)(x) + a_i\theta]}\left[e^{2V_i(s,\gamma x)} - 1\right] \\
&\qquad + \gamma^d \sum_{i=1,2}\sum_{x\in\Lambda_\gamma}\frac{(1+\sigma_s(x))}{2}\alpha_i(x)\frac{\exp[-(J_\gamma\star\sigma_s)(x) - a_i\theta]}{\cosh[(J_\gamma\star\sigma_s)(x) + a_i\theta]}\left[e^{-2V_i(s,\gamma x)} - 1\right] \\
&\quad= \gamma^d \sum_{i=1,2}\sum_{x\in\Lambda_\gamma}\alpha_i(x)\left\{\cosh[2V_i(s,\gamma x)] - 1 + \tanh[(J_\gamma\star\sigma_s)(x) + a_i\theta]\sinh[2V_i(s,\gamma x)]\right\} \\
&\qquad - \gamma^d \sum_{i=1,2}\sum_{x\in\Lambda_\gamma}\alpha_i(x)\sigma_s(x)\left\{\tanh[(J_\gamma\star\sigma_s)(x) + a_i\theta](\cosh[2V_i(s,\gamma x)] - 1) + \sinh[2V_i(s,\gamma x)]\right\} \\
&\quad= F_{\overline{V}(s)}(\overline{\lambda}^\gamma(\alpha), \overline{\pi}^\gamma_s).
\end{aligned}$$

$\square$

Note that the Radon-Nikodym derivative depends on the randomness through $\overline{\pi}^\gamma$ and $\overline{\lambda}^\gamma(\alpha)$. By next proposition, which is proved in Appendix B, we can replace $\overline{\lambda}^\gamma(\alpha)$ in $F_{\overline{V(s)}}(\cdot,\overline{\pi}^\gamma_s)$ with $(p_1\lambda^\gamma, p_2\lambda^\gamma)$, making an error which goes uniformly (for all $\sigma \in \mathcal{S}_\gamma$ and $I\!P$-a.s.) to zero as $\gamma \to 0$.

Proposition 5.5 *Let $\overline{V} = (V_1, V_2) \in \big(\mathcal{C}^{1,0}([0,T]\times\Lambda)\big)^2$. There exists a positive function ϵ on $I\!R_+$ with $\lim_{s\to 0}\epsilon(s) = 0$ such that for any $\delta > 0$, $l \in I\!N\setminus\{0\}$, we have*

$$\left|\int_0^T \Big[F_{\overline{V(s)}}(\overline{\lambda}^\gamma(\alpha),\overline{\pi}^\gamma_s) - F_{\overline{V(s)}}((p_1\lambda^\gamma, p_2\lambda^\gamma),\overline{\pi}^\gamma_s)\Big]\, ds\right| \le \epsilon(\gamma l)T + TC(V_1,V_2)[\delta + \sum_{i=1,2}\mathcal{E}_i(\delta,l,\gamma,\alpha)]$$

where the positive constant $C(V_1,V_2)$ depends on the L^∞ norm of (V_1,V_2).

6. Upper Bound

In this section we investigate the upper bound of the large deviation principle for compact sets and then closed sets of the topological space $D([0,T),\mathcal{M}_1\times\mathcal{M}_1)$. Notice that in [C] the result was stated for closed sets in $\mathcal{C}([0,T),\mathcal{M}_1\times\mathcal{M}_1)$. We first prove exponential tightness, so that it is enough to derive the upper bound of the large deviation principle for compact subsets. The strategy then follows the martingale approach introduced by [DV]: we need to show that trajectories which are not absolutely continuous with respect to the Lebesgue measure and not absolutely continuous in time can be neglected in the large deviations regime. To exclude these "bad" paths, as in [FLM], we introduce an energy functional via an exponential martingale. With this we prove an upper bound with an auxiliary rate functional which is infinite on the set of bad trajectories.

Proposition 6.1 *For any $\ell \ge 1$, there exists a compact subset $\overline{K}_\ell \subset D([0,T],\mathcal{M}_1\times\mathcal{M}_1)$ such that for any $\sigma_\gamma \in \mathcal{S}_\gamma$,*

$$\limsup_{\gamma\to 0}\gamma^d \log \overline{Q}^{\gamma,\alpha}_{\sigma^\gamma}(\overline{K}^c_\ell) \le -\ell.$$

The proof is standard, however the main lines are recalled in Appendix B.

For $\overline{\pi} \in D\big([0,T],\mathcal{M}_1\times\mathcal{M}_1\big)$, $\overline{G} = (G_1,G_2) \in (\mathcal{C}^{1,0}([0,T]\times\Lambda))^2$ denote

$$\begin{aligned}\mathbf{J}_{\overline{G}}(\overline{\pi}) \;&=\; \ell_T(\overline{\pi},\overline{G}) \;+\; \widetilde{\ell}_T(\overline{\pi},\overline{G})\\ &\quad - 2\sum_{i=1}^2\int_0^T \Big\{\big\langle p_i\lambda, G_i^2(s,\cdot)\big\rangle - \big\langle \pi_{i,s}, G_i^2(s,\cdot)\tanh(\pi_s * J + a_i\theta)\big\rangle\Big\}\, ds\,,\end{aligned} \tag{6.1}$$

for ℓ_T, $\tilde{\ell}_T$ given in (4.13), (4.14). We define the auxiliary rate functional $\mathcal{J} : D\big([0,T],\mathcal{M}_1\times\mathcal{M}_1\big) \to \overline{I\!R}$ as

$$\mathcal{J}(\overline{\pi}) = \begin{cases} \displaystyle\sup_{\overline{G}\in(\mathcal{C}^{1,0}([0,T]\times\Lambda))^2}\Big(\mathbf{J}_{\overline{G}}(\overline{\pi})\Big) & \text{if } \overline{\pi}\in D([0,T],\mathcal{M}_1^{ac}\times\mathcal{M}_1^{ac})\,,\\ +\infty & \text{otherwise}\,.\end{cases} \tag{6.2}$$

Lemma 6.2 *For all $\overline{\pi} \in D\big([0,T],\mathcal{M}_1\times\mathcal{M}_1\big)$, if $\mathcal{J}(\overline{\pi}) < \infty$, then $\overline{\pi} \in \mathcal{C}\big([0,T],\mathcal{M}_1^{ac}\times\mathcal{M}_1^{ac}\big)$.*

Proof. Fix $\overline{\pi} \in D\big([0,T], \mathcal{M}_1 \times \mathcal{M}_1\big)$ such that $\mathcal{J}(\overline{\pi}) < \infty$. By definition of $\mathcal{J}(\cdot)$, $\overline{\pi} \in D\big([0,T], \mathcal{M}_1^{ac} \times \mathcal{M}_1^{ac}\big)$. Let $\overline{g} = (g_1, g_2) \in (\mathcal{C}(\Lambda))^2$ and $0 \le s < t < T$. For each $\delta > 0$, let $\psi_{s,t}^{\delta} : [0,T] \to I\!R$ be the function given by

$$\psi_{s,t}^{\delta}(\tau) = \begin{cases} 0 \quad \text{if} \quad 0 \le \tau \le s \ \text{ or } \ t \le \tau \le T\,, \\ \dfrac{\tau - s}{\delta} \quad \text{if} \quad s \le \tau \le s + \delta\,, \\ 1 \quad \text{if} \quad s + \delta \le \tau \le t - \delta\,, \\ \dfrac{t - \tau}{\delta} \quad \text{if } t - \delta \le \tau \le t\,. \end{cases} \tag{6.3}$$

Denote $\overline{G}^{\delta}(\tau, r) = \psi_{s,t}^{\delta}(\tau)\overline{g}(r)$. Since $\overline{G}^{\delta}$ can be approximated by functions in $(\mathcal{C}^{1,0}([0,T] \times \Lambda))^2$, considering $\dfrac{\overline{G}^{\delta}}{\sqrt{t-s}}$ as a test function and performing the limit $\delta \to 0$, we obtain

$$\begin{aligned} \sqrt{t-s}\ \lim_{\delta \to 0} \mathbf{J}_{\frac{\overline{G}^{\delta}}{\sqrt{t-s}}}(\overline{\pi}) &= < \overline{\pi}_t, \overline{g} > - < \overline{\pi}_s, \overline{g} > \\ &+ \int_s^t \Big\{ < \overline{\pi}_{\tau}, \overline{g} > - \sum_{i=1}^{2} < p_i \lambda, g_i \tanh(\pi_{\tau} * J + a_i \theta) > \Big\} d\tau \\ &- 2 \frac{1}{\sqrt{t-s}} \sum_{i=1}^{2} \int_s^t \Big\{ < p_i \lambda, g_i^2 > - < \pi_{i,\tau}, g_i^2 \tanh(\pi_{\tau} * J + a_i \theta) > \Big\} d\tau\,. \end{aligned} \tag{6.4}$$

Since $\sqrt{t-s}\ \lim_{\delta \to 0} \mathbf{J}_{\frac{\overline{G}^{\delta}}{\sqrt{t-s}}}(\overline{\pi}) \le \sqrt{t-s}\ \mathcal{J}(\overline{\pi})$, we get

$$\begin{aligned} \Big| < \overline{\pi}_t, \overline{g} > - < \overline{\pi}_s, \overline{g} > \Big| &\le C_0 (t-s) \sum_{i=1}^{2} \Big\{ \|g_i\|_1 + \frac{1}{\sqrt{t-s}} \|g_i\|_2^2 \Big\} + \sqrt{t-s}\mathcal{J}(\overline{\pi}) \\ &= C_0\,(t-s) \sum_{i=1}^{2} \|g_i\|_1 + \sqrt{t-s} \Big\{ C_0 \sum_{i=1}^{2} \|g_i\|_2^2 + \mathcal{J}(\overline{\pi}) \Big\}\,, \end{aligned}$$

for some positive constant C_0. This implies that $\overline{\pi} \in \mathcal{C}\big([0,T], \mathcal{M}_1^{ac} \times \mathcal{M}_1^{ac}\big)$. $\square$

To prove next Lemma, we will use the following characterization of absolutely continuous functions, see [DS].

Proposition 6.3 *A function $\overline{\phi}$ belongs to $\mathcal{AC}([0,T], B_{1,1})$ if and only if: for all $\epsilon > 0$, there exists $\Delta > 0$ such that for all integer $k > 0$, rectangles $A_1, \ldots, A_k$ of Λ and $\{(s_i, t_i),\ 1 \le i \le k\}$ nonempty disjoint intervals of $[0,T]$,*

$$\sum_{i=1}^{k} |t_i - s_i| \lambda(A_i) < \Delta \ \Rightarrow \ \sum_{i=1}^{k} \Big| \int_{A_i} (\phi_j(t_i, r) - \phi_j(s_i, r))\ dr \Big| < \epsilon\,, \quad j = 1, 2\,.$$

Lemma 6.4 *Let $\overline{\pi} = (\phi_1(s,r)dr, \phi_2(s,r)dr) \in D\big([0,T], \mathcal{M}_1^{ac} \times \mathcal{M}_1^{ac}\big)$ such that $\mathcal{J}(\overline{\pi}) < \infty$, then*
(a) for $i = 1, 2$, $F_i(s,r) := \big[p_i - \phi_i(s,r) \tanh(\pi_s(r) * J + a_i \theta)\big] \ge 0$ for almost all $(s,r) \in [0,T] \times \Lambda$,
(b) $(\phi_1, \phi_2) \in \mathcal{AC}\big([0,T], B_{1,1}\big)$.

Proof. (a) Taking as a test function $AG(\cdot,\cdot)$, for all $\overline{G}=(G_1,G_2)\in(\mathcal{C}^{1,0}([0,T]\times\Lambda))^2$ and $A>0$, we obtain from Definition (6.2) of the rate function $\mathcal{J}$,

$$-\sum_{i=1}^{2}\int_0^T\int_\Lambda G_i^2(s,r)\,F_i(s,r)\,dr\,ds\;\le\;-\frac{1}{2A}\Big\{\ell_T(\pi,\overline{G})\;+\;\widetilde{\ell}_T(\pi,\overline{G})\Big\}\;+\;\frac{1}{2A^2}\mathcal{J}(\pi)\,. \tag{6.5}$$

Letting $A\uparrow\infty$, we get

$$\sum_{i=1}^{2}\int_0^T\int_\Lambda G_i^2(s,r)\,F_i(s,r)drds\;\ge\;0\,.$$

Since $\overline{G}$ is arbitrary, we conclude that $F_i(s,r)\ge 0$ for $1\le i\le 2$, almost everywhere.
(b) We show the absolute continuity in time for ϕ_1. The proof for ϕ_2 is similar. We apply the characterization of $\mathcal{AC}\big([0,T],B_{1,1}\big)$ given in Proposition 6.3. For all positive integer k let $\{A_i\;,\;1\le i\le k\}$ be rectangles of Λ and $\{(s_i,t_i)\;,\;\;1\le i\le k\}$ be nonempty disjoint intervals of $[0,T]$. For $i=1,\dots,k$, denote

$$\eta_i:=\mathrm{sgn}\Big(\int_{A_i}[\phi_1(t_i,r)-\phi_1(s_i,r)]\;dr\Big)\,,$$

For each $1\le i\le k$, $0<\delta<\dfrac{1}{4}\min\limits_{1\le i\le k}\big(t_i-s_i\big)$, we set, see (6.3),

$$V_1(t,r)=\sum_{i=1}^{k}\eta_i\times\psi^\delta_{s_i,t_i}(t)\times \mathbb{1}_{A_i}(r),\quad V_2(t,r)=0,\quad \overline{V}=(V_1,V_2). \tag{6.6}$$

Since V_1 can be approximated by functions in $\mathcal{C}^{1,0}([0,T]\times\Lambda)$, proceeding as in (6.4), we obtain for any $b>0$, see (6.5),

$$\begin{aligned}\sum_{i=1}^{k}\eta_i\Big\{\int_{A_i}[\phi_1(t_i,r)-\phi_1(s_i,r)]\;dr\Big\}\;\le\;&-\sum_{i=1}^{k}\int_{s_i}^{t_i}<\eta_i\,\mathbb{1}_{A_i},\big[\phi_1(s,\cdot)-p_1\tanh(\pi_s*J+a_1\theta)\big]>\;ds\\&+\;2b\sum_{i=1}^{k}\int_{s_i}^{t_i}\int_\Lambda|\eta_i|\,\mathbb{1}_{A_i}(r)\,F_1(s,r)\,drds\;+\;\frac{\mathcal{J}(\pi)}{b}\,.\end{aligned}$$

Minimizing over b yields

$$\begin{aligned}\sum_{i=1}^{k}\Big|\int_{A_i}[\phi_1(t_i,r)-\phi_1(s_i,r)]\;dr\Big|\;\le\;&2\sum_{i=1}^{k}(t_i-s_i)\lambda\big(A_i\big)\\&+\;2\sqrt{2\mathcal{J}(\pi)}\Big(\sum_{i=1}^{k}\int_{s_i}^{t_i}\int_\Lambda|\eta_i|\,\mathbb{1}_{A_i}(r)\,F_1(s,r)\,drds\Big)^{1/2}\\&\le\;2\sum_{i=1}^{k}(t_i-s_i)\lambda\big(A_i\big)\;+\;4\sqrt{\mathcal{J}(\pi)}\sqrt{\sum_{i=1}^{k}(t_i-s_i)\lambda\big(A_i\big)}\,.\end{aligned} \tag{6.7}$$

For all $\varepsilon>0$ denote $\Delta=\min\big(\varepsilon/4\,,\,\varepsilon^2/(64\mathcal{J}(\pi))\big)$. It follows from (6.7) that $\sum_{i=1}^k(t_i-s_i)\lambda\big(A_i\big)\le\Delta$ implies $\sum_{i=1}^k\Big|\int_{A_i}[\phi_1(t_i,r)-\phi_1(s_i,r)]\;dr\Big|\le\varepsilon$. This concludes the proof. $\square$

For $\overline{\pi} \in D\big([0,T], \mathcal{M}_1 \times \mathcal{M}_1\big)$, ℓ_T as in (4.13) and $F_{\overline{V}(s,.)}$ defined in (3.3) let

$$\widehat{J}_{\overline{V}}(\overline{\pi}) \;=\; \ell_T(\overline{\pi}, \overline{V}) - \frac{1}{2}\int_0^T F_{\overline{V}(s,.)}\big((p_1\lambda, p_2\lambda), \overline{\pi}_s\big)\, ds, \tag{6.8}$$

$$\widehat{J}(\overline{\pi}) = \sup_{\overline{V} \in (\mathcal{C}^{1,0}([0,T]\times\Lambda))^2} \Big\{ \widehat{J}_{\overline{V}}(\overline{\pi}) \Big\}. \tag{6.9}$$

Remark that when $\overline{\pi} = (\phi_1\lambda, \phi_2\lambda)$, with $\overline{\phi} = (\phi_1, \phi_2) \in \mathcal{AC}([0,T], B_{1,1})$, $\widehat{J}$ coincides with the functional $I_0 = J_0 = J_1$ (cf. Proposition 3.5). The proof of the upper bound of the large deviation principle relies on the following proposition.

Proposition 6.5 *Let $\overline{K}$ be a compact set of $D\big([0,T], \mathcal{M}_1 \times \mathcal{M}_1\big)$. For any $0 < b < 1$,*

$$\limsup_{\gamma\to 0} \gamma^d \log \overline{Q}^{\gamma,\alpha}_{\sigma^\gamma}(\overline{K}) \le -\frac{1}{1+b} \inf_{\overline{\pi}\in\overline{K}} \Big[\widehat{J}(\overline{\pi}) \;+\; b\mathcal{J}(\overline{\pi}) \Big].$$

Proof. For $\varepsilon > 0$, $\mu \in \mathcal{M}_1$, $g \in \mathcal{C}(\Lambda)$ denote by ι_ε the approximation of the identity

$$\iota_\varepsilon(r) = \frac{1}{(2\varepsilon)^d} 1\!\!1_{\big\{ \big[-\varepsilon,\varepsilon\big]^d \big\}}(r), \quad r \in \Lambda$$

and by $\mu * \iota_\varepsilon$ the measure defined by $\big\langle \mu * \iota_\varepsilon, g \big\rangle = \big\langle \mu, g * \iota_\varepsilon \big\rangle$. It is absolutely continuous with respect to the Lebesgue measure with density

$$\frac{d(\mu * \iota_\varepsilon)}{d\lambda}(r) \;=\; \langle \mu, \iota_\varepsilon(r - \cdot)\rangle\;, \qquad r \in \Lambda\,.$$

In general, we can only bound this density by $\|\iota_\varepsilon\|_\infty$ which is of order ε^{-d}. Nevertheless, in the case of the empirical measure, we have

$$|\langle \pi^\gamma_s, \iota_\varepsilon(r-\cdot)\rangle| \;=\; \left| \frac{\gamma^d}{(2\varepsilon)^d} \sum_{x\,:\,\gamma x \in [r-\varepsilon, r+\varepsilon]} \sigma_s(x) \right| \;\le\; 1\,, \qquad \text{for almost all} \quad 0 \le s \le T\,,$$

which means that $\pi^\gamma * \iota_\varepsilon \in \mathcal{M}_1^{ac}$, when $0 < \gamma < \varepsilon$. Furthermore for any $\overline{\pi} \in D\big([0,T], \mathcal{M}_1 \times \mathcal{M}_1\big)$, denote by $\overline{\pi}_s * \iota_\varepsilon := \big(\pi_{1,s} * \iota_\varepsilon\;,\; \pi_{2,s} * \iota_\varepsilon\big)$, $0 \le s \le T$ the trajectory in $D\big([0,T], \mathcal{M}_1^{ac} \times \mathcal{M}_1^{ac}\big)$.

Fix a function $\overline{G} \in (\mathcal{C}^{1,0}([0,T]\times\Lambda))^2$. Consider the mean one exponential martingale $\big(\overline{\mathcal{Z}}^{\overline{G},\gamma}_t\big)_{t\ge 0}$

$$\overline{\mathcal{Z}}^{\overline{G},\gamma}_t = \exp\left\{ \gamma^{-d} \overline{N}^{\overline{G}}_\gamma(t) - \frac{\gamma^{-2d}}{2} \langle \overline{N}^{\overline{G}}_\gamma, \overline{N}^{\overline{G}}_\gamma \rangle(t) \right\}\,,$$

where the martingale $\big(\overline{N}^{\overline{G}}_\gamma(t)\big)_{t\ge 0}$ and its quadratic variation $\big(\langle \overline{N}^{\overline{G}}_\gamma, \overline{N}^{\overline{G}}_\gamma \rangle(t)\big)_{t\ge 0}$ were given in (4.7) and (4.8). Using the same arguments as in Proposition 5.5, by smoothness of $\overline{G}$ and $\pi^\gamma * J$, a spatial summation by parts and Taylor expansion permit to rewrite the martingale $\overline{\mathcal{Z}}^{\overline{G},\gamma}_t$ as

$$\overline{\mathcal{Z}}^{\overline{G},\gamma}_t = \exp\Big\{ \gamma^{-d} \mathbf{J}_{\overline{G}}(\overline{\pi}^\gamma * \iota_\varepsilon) \;+\; \gamma^{-d} r(\overline{G}, \gamma, \varepsilon, l, \delta, \alpha) \Big\}\,, \tag{6.10}$$

where $0<\gamma,\varepsilon,\delta<1$, l is a positive integer. Here and in the sequel, $r(\overline{G},\gamma,\varepsilon,l,\delta,\alpha)$ (resp. $r(\overline{G},\overline{V},\gamma,\varepsilon,l,\delta,\alpha)$ later on) stands for some random variable satisfying

$$\limsup_{\delta\to 0}\limsup_{l\to\infty}\limsup_{\varepsilon\to 0}\limsup_{\gamma\to 0} r(\overline{G},\gamma,\varepsilon,l,\delta,\alpha)=0,\quad I\!\!P-\text{a-e}\,. \tag{6.11}$$

Let $\overline{K}$ be a compact set of $D\big([0,T],\mathcal{M}_1\times\mathcal{M}_1\big)$. By Hölder inequality,

$$\begin{aligned}\gamma^d\log\overline{Q}^{\gamma,\alpha}_{\sigma^\gamma}(\overline{K}) &= \gamma^d\log E^{Q^{\gamma,\alpha}_{\sigma^\gamma}}\Big[\mathbb{1}_{\overline{K}}(\overline{\pi}^\gamma)\big(\overline{\mathcal{Z}}^{G,\gamma}_t\big)^{\frac{b}{1+b}}\times\big(\overline{\mathcal{Z}}^{G,\gamma}_t\big)^{\frac{-b}{1+b}}\Big]\\ &\le \frac{b}{1+b}\gamma^d\log E^{Q^{\gamma,\alpha}_{\sigma^\gamma}}\Big[\mathbb{1}_{\overline{K}}(\overline{\pi}^\gamma)\overline{\mathcal{Z}}^{G,\gamma}_t\Big]\;+\;\frac{1}{1+b}\gamma^d\log E^{Q^{\gamma,\alpha}_{\sigma^\gamma}}\Big[\mathbb{1}_{\overline{K}}(\overline{\pi}^\gamma)\big(\overline{\mathcal{Z}}^{G,\gamma}_t\big)^{-b}\Big]\\ &\le \frac{1}{1+b}\gamma^d\log E^{Q^{\gamma,\alpha}_{\sigma^\gamma}}\Big[\mathbb{1}_{\overline{K}}(\overline{\pi}^\gamma)\big(\overline{\mathcal{Z}}^{G,\gamma}_t\big)^{-b}\Big]\,.\end{aligned} \tag{6.12}$$

We now exclude paths whose densities are not absolutely continuous with respect to the Lebesgue measure. Fix a sequence $\{F_k : k\ge 1\}$ of smooth nonnegative functions dense in $\mathcal{C}(\Lambda)$ for the uniform topology. For $k\ge 1$, $\varrho>0$ and $\delta>0$, let

$$D_{k,\varrho}=\Big\{\overline{\pi}\in D([0,T],\mathcal{M}_1\times\mathcal{M}_1):\,0\le<|\pi_{i,t}|,F_k>\le\int_\Lambda F_k(x)\,dx\;+\;C_k\varrho\;,\;0\le t\le T, i=1,2\Big\}\,,$$

where $C_k=C(\|\nabla F_k\|_\infty)$ is a constant depending on the gradient ∇F_k of F_k. The sets $D_{k,\varrho}$, $k\ge 1$ are closed subsets of $D([0,T],\mathcal{M}_1\times\mathcal{M}_1)$, as well as

$$E_{m,\varrho}\;=\;\bigcap_{k=1}^m D_{k,\varrho}\;,\quad m\ge 1\,.$$

Note that the empirical measure $\overline{\pi}^\gamma$ belongs to $E_{m,\varrho}$ for γ sufficiently small. We have that

$$D([0,T],\mathcal{M}_1^{ac}\times\mathcal{M}_1^{ac})=\cap_{n\ge 1}\cap_{m\ge 1}E_{m,1/n}. \tag{6.13}$$

Fix $0<b<1$. For $\overline{G},\overline{V}\in(\mathcal{C}^{1,0}([0,T]\times\Lambda))^2$, $\varepsilon>0$ and $m,n\in Z\!\!\!Z_+$, let $\widehat{\mathbf{J}}^{b,m,n}_{\overline{V},\overline{G},\varepsilon}:D([0,T],\mathcal{M}_1\times\mathcal{M}_1)\to I\!\!R\cup\{\infty\}$ be the functional given by

$$\widehat{\mathbf{J}}^{b,m,n}_{\overline{V},\overline{G},\varepsilon}(\overline{\pi})=\begin{cases}\widehat{J}_{\overline{V}}(\overline{\pi}*\iota_\varepsilon)+b\mathbf{J}_G(\overline{\pi}*\iota_\varepsilon) & \text{if}\quad\overline{\pi}\in E_{m,\frac{1}{n}}\,,\\ \qquad\qquad+\infty & \text{otherwise}\,.\end{cases} \tag{6.14}$$

It is lower semicontinuous because so is $\overline{\pi}\mapsto\widehat{J}_{\overline{V}}(\overline{\pi}*\iota_\epsilon)+b\mathbf{J}_G(\overline{\pi}*\iota_\epsilon)$, and because $E_{m,1/n}$ is closed.

We now return to inequality (6.12). By Proposition 5.5, the exponential martingale $\mathcal{M}^{\overline{V},\gamma}_t$ defined by the Girsanov formula (5.8) satisfies

$$\mathcal{M}^{\overline{V},\gamma}_T\;:=\;\frac{dP^{\overline{V},\gamma,\alpha}_{\sigma^\gamma}}{dP^{\gamma,\alpha}_{\sigma^\gamma}}(\sigma_{[0,T]})\;=\;\exp\Big\{\gamma^{-d}\widehat{J}_{\overline{V}}(\overline{\pi}^\gamma*\iota_\varepsilon)\;+\;\gamma^{-d}r(\overline{V},\gamma,\varepsilon,l,\delta,\alpha)\Big\}. \tag{6.15}$$

We rewrite (6.12) as

$$\gamma^d\log\overline{Q}^{\gamma,\alpha}_{\sigma^\gamma}(\overline{K})\le\frac{1}{1+b}\gamma^d\log E^{Q^{\gamma,\alpha}_{\sigma^\gamma}}\Big[\mathbb{1}_{\overline{K}}(\overline{\pi}^\gamma)\mathcal{M}^{\overline{V},\gamma}_T\times\big(\mathcal{M}^{\overline{V},\gamma}_T\big)^{-1}\times\big(\overline{\mathcal{Z}}^{G,\gamma}_t\big)^{-b}\Big]\,.$$

Since $\mathcal{M}_T^{\overline{V},\gamma}$ is a mean one positive martingale, taking into account (6.10) and (6.15) and optimizing over $\overline{\pi}$ in $\overline{K}$, we obtain, for all positive integers m, n,

$$\begin{aligned}
\limsup_{\gamma\to 0} \gamma^d \log \overline{Q}_{\sigma^\gamma}^{\gamma,\alpha}(\overline{K}) &\le \frac{1}{1+b} \sup_{\overline{\pi}\in\overline{K}\cap E_{m,\frac{1}{n}}} \Big\{ -\widehat{J}_{\overline{V}}(\overline{\pi} * \iota_\epsilon) \; - \; b\mathbf{J}_G(\overline{\pi} * \iota_\epsilon)\Big\} \\
&\quad + \limsup_{\gamma\to 0} r(\overline{V}, \overline{G}, \gamma, \varepsilon, l, \delta, \alpha) \\
&= \frac{1}{1+b} \sup_{\overline{\pi}\in\overline{K}} \Big\{ -\widehat{\mathbf{J}}^{b,m,n}_{\overline{V},\overline{G},\varepsilon}(\overline{\pi})\Big\} + \limsup_{\gamma\to 0} r(\overline{V}, \overline{G}, \gamma, \varepsilon, l, \delta, \alpha)\, .
\end{aligned}$$

Optimizing the previous expression with respect to $\overline{V}$, $\overline{G}$, $\varepsilon, l, \delta, m, n$, taking into account (6.11), we get

$$\limsup_{\gamma\to 0} \gamma^d \log \overline{Q}_{\sigma^\gamma}^{\gamma,\alpha}(\overline{K}) \le \inf_{\overline{V},\overline{G},\varepsilon,l,\delta,m,n} \Big\{ \frac{1}{1+b} \sup_{\overline{\pi}\in\overline{K}} \Big\{ -\widehat{\mathbf{J}}^{b,m,n}_{\overline{V},\overline{G},\varepsilon}(\overline{\pi})\Big\}\Big\}\, . \tag{6.16}$$

Since $\overline{K}$ is compact and $\overline{\pi} \mapsto \dfrac{1}{1+b} \sup_{\overline{\pi}\in\overline{K}} \Big\{ -\widehat{\mathbf{J}}^{b,m,n}_{\overline{V},\overline{G},\varepsilon}(\overline{\pi})\Big\}$ is lower semi-continuous for all $\overline{V}$, $\overline{G}$ and $\varepsilon, l, \delta, m, n$, we may apply the arguments presented in [V], Lemma 11.3 to exchange the supremum with the infimum. In this way we obtain that the right hand side of (6.16) is bounded above by

$$\sup_{\overline{\pi}\in\overline{K}} \inf_{\overline{V},\overline{G},\varepsilon,l,\delta,m,n} \Big\{ -\frac{1}{1+b}\widehat{\mathbf{J}}^{b,m,n}_{\overline{V},\overline{G},\varepsilon}(\overline{\pi})\Big\}\, .$$

By (6.13) we have

$$\limsup_{\varepsilon\to 0}\limsup_{m\to\infty}\limsup_{n\to\infty} \widehat{\mathbf{J}}^{b,m,n}_{\overline{V},\overline{G},\varepsilon}(\overline{\pi}) \; := \; \widehat{\mathbf{J}}^{b}_{\overline{V},\overline{G}}(\overline{\pi}) = \begin{cases} \widehat{J}_{\overline{V}}(\overline{\pi}) + b\mathbf{J}_G(\overline{\pi}) & \text{if} \quad \overline{\pi} \in D([0,T], \mathcal{M}_1^{ac}\times\mathcal{M}_1^{ac}), \\ \qquad\qquad +\infty & \text{otherwise}\;. \end{cases}$$

By (6.2) and (6.9) we have that $\sup_{\overline{V},\overline{G}} \Big\{\widehat{\mathbf{J}}^{b}_{\overline{V},\overline{G}}(\overline{\pi})\Big\} = \widehat{J}(\overline{\pi}) + b\mathcal{J}(\overline{\pi})$. Therefore,

$$\begin{aligned}
\limsup_{\gamma\to 0} \gamma^d \log \overline{Q}_{\sigma^\gamma}^{\gamma,\alpha}(\overline{K}) &\le \sup_{\overline{\pi}\in\overline{K}} \inf_{\overline{V},\overline{G}} \Big\{ -\frac{1}{1+b}\widehat{\mathbf{J}}^{b}_{\overline{V},\overline{G}}(\overline{\pi})\Big\} \\
&= -\frac{1}{1+b} \inf_{\overline{\pi}\in\overline{K}} \sup_{\overline{V},\overline{G}} \Big\{\widehat{\mathbf{J}}^{b}_{\overline{V},\overline{G}}(\overline{\pi})\Big\} \; = \; -\frac{1}{1+b} \inf_{\overline{\pi}\in\overline{K}} \Big\{\widehat{J}(\overline{\pi}) + b\mathcal{J}(\overline{\pi})\Big\}\, .
\end{aligned} \tag{6.17}$$

□

Proof of the upper bound. Let $\overline{K}$ be a compact set of $D\big([0,T], \mathcal{M}_1\times\mathcal{M}_1\big)$. By Proposition 6.5, if $\mathcal{J} \equiv +\infty$ on $\overline{K}$, then the upper bound of the large deviation principle is satisfied. Otherwise, there exists $\overline{\pi}\in\overline{K}$ such that $\mathcal{J}(\overline{\pi}) < \infty$. By semicontinuity of the functional $\overline{\pi}\mapsto\mathcal{J}(\overline{\pi})$, we obtain from (6.17) for any $0 < b < 1$,

$$\limsup_{\gamma\to 0} \gamma^d \log \overline{Q}_{\sigma^\gamma}^{\gamma,\alpha}(\overline{K}) \; \le \; -\frac{1}{1+b} \inf_{\overline{\pi}\in\overline{K}\,,\mathcal{J}(\overline{\pi})<\infty} \widehat{J}(\overline{\pi}) - \frac{b}{1+b} \inf_{\overline{\pi}\in\overline{K}} \mathcal{J}(\overline{\pi})\, .$$

Letting $b\to 0$, we get

$$\limsup_{\gamma\to 0} \gamma^d \log \overline{Q}_{\sigma^\gamma}^{\gamma,\alpha}(\overline{K}) \le -\inf_{\overline{\pi}\in\overline{K}\,,\mathcal{J}(\overline{\pi})<\infty} \big\{\widehat{J}(\overline{\pi})\big\} \le -\inf_{\overline{\pi}\in\overline{K}} \big\{I_0(\overline{\pi})\big\}.$$

For the last inequality we used Lemma 6.4. By Proposition 6.1 the proof of the upper bound of the large deviation principle is completed. □

7. Lower Bound

We first get in Lemma 7.1 a lower estimate for the probability of a neighborhood of suitable trajectories. We perform the computation with the uniform metric on the time interval $[0, T]$ defined as following: for μ and ν in $D([0, T], \mathcal{M}_1)$ and $\rho(\cdot, \cdot)$ defined in (2.9),

$$\rho_{[0,T]}(\mu, \nu) = \sup_{t \in [0,T]} \rho(\mu_t, \nu_t) \quad \text{and} \quad \rho_{[0,T]}(\overline{\mu}, \overline{\nu}) = \sum_{i=1,2} \rho_{[0,T]}(\mu_i, \nu_i). \tag{7.1}$$

Taking into account that if $d^S_{[0,T]}(\mu, \nu)$ denotes the Skorohod distance, then

$$d^S_{[0,T]}(\mu, \nu) \leq \rho_{[0,T]}(\mu, \nu) \qquad (7.2),$$

the result holds for the Skorohod topology as well, see [Bill].

To conclude the proof of the lower bound of the large deviation principle in Theorem 2.9, it will remain to show that all $\overline{\pi}$'s such that $I_0(\overline{\pi}) < \infty$ can be approximated by a sequence $(\overline{\pi}_n)_n$ of smooth trajectories, for which Lemma 7.1 holds with $\lim_{n\to\infty} I_0(\overline{\pi}_n) = I_0(\overline{\pi})$. For this, in Lemma 7.2 we prove that any trajectory $\overline{m}$ smooth enough and far away from the boundaries $(\pm p_1, \pm p_2)$ is associated to a function $\overline{V}(\cdot, \cdot)$.
Then, given $\overline{m}_0 \in B_{p_1,p_2}$, denote by $\overline{R}(t, \cdot)$, $t \in [0, T]$ the solution of (2.24) with $\overline{R}(0, \cdot) = \overline{m}_0(\cdot)$: for $i = 1, 2$,

$$R_i(t, \cdot) = e^{-t} m_i(0, \cdot) + p_i \int_0^t e^{-(t-s)} \tanh[(J * R)(s, \cdot) + a_i \theta] \, ds, \tag{7.3}$$

where $R = R_1 + R_2$. It is continuously differentiable in time, actually it is $\mathcal{C}^\infty$ in time for $t \geq t_0 > 0$, and there exists δ_i which depends on T such that $|R_i(t, \cdot)| \leq p_i - \delta_i$ for $t \in [t_0, T]$. Namely, since $|\tanh z| \leq 1 - d$, for $|z| \leq K(\beta, \theta)$ with $1 > d = d(\beta, \theta) > 0$, we have, for $t \in [0, T]$,

$$|R_i(t, \cdot)| \leq e^{-t} m_i(0, \cdot) + p_i(1 - d) \int_0^t e^{-(t-s)} \, ds \leq p_i[1 - d(1 - e^{-t})]. \tag{7.4}$$

Recall that $I_0(\overline{R}) = 0$, see (3) of Proposition 3.7. Define the sets:

$$\mathcal{C}_0 = \mathcal{C}_0(m_0) = \{\overline{\phi} \in \mathcal{AC}([0, T], B_{p_1,p_2}) : \overline{\phi}(0) = \overline{m}_0, I_0(\overline{\phi}) < \infty\}, \tag{7.5}$$

$$\mathcal{C}_1 = \{\overline{\phi} \in \mathcal{C}_0 : \exists\, 0 < \eta < T, \overline{\phi}(t) = \overline{R}(t), t \in [0, \eta]\}, \tag{7.6}$$

$$\mathcal{C}_2 = \{\overline{\phi} \in \mathcal{C}_1 : \forall \eta \in (0, T], \exists \delta_i = \delta_i(\overline{\phi}) > 0, i = 1, 2 : \|\phi_i(t)\|_\infty \leq p_i - \delta_i, \quad t \in [\eta, T]\}, \tag{7.7}$$

$$\mathcal{C}_3 = \{\overline{\phi} \in \mathcal{C}_2 : \phi_i \in \mathcal{C}^2((0, T], B_{p_1,p_2}), i = 1, 2, \ \phi_i(t) \in \mathcal{C}(\Lambda), \forall t \in (0, T]\}. \tag{7.8}$$

By construction $\mathcal{C}_3 \subset \mathcal{C}_2 \subset \mathcal{C}_1 \subset \mathcal{C}_0$. By Lemma 7.2 below we can associate a function $\overline{V}$ to $\overline{\phi} \in \mathcal{C}_3$. To extend the lower bound, we show that for $i \in \{1, 2, 3\}$, $\mathcal{C}_i$ *is* $(\rho_{[0,T]}, I_0)$*-dense* in $\mathcal{C}_{i-1}$, that is, for all $\overline{\phi} \in \mathcal{C}_{i-1}$ there exists a sequence $(\overline{\phi}_n)_n \subset \mathcal{C}_i$ such that

$$\lim_{n\to\infty} \rho_{[0,T]}(\overline{\phi}_n, \overline{\phi}) = 0, \quad \lim_{n\to\infty} I_0(\overline{\phi}_n) = I_0(\overline{\phi}). \tag{7.9}$$

This method has been inspired by a similar strategy in [QRV].

Lemma 7.1 *Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). Let $\delta > 0$ and $\overline{\mu} = \overline{m}^{\overline{V}}\lambda$, where $\overline{m}^{\overline{V}}$ is the solution of (5.5) for $\overline{V} = (V_1, V_2) \in \left(\mathcal{C}^{1,0}([0,T]\times\Lambda)\right)^2$ and $m_i^{\overline{V}}(0,\cdot) = p_i m_0(\cdot)$ for $i = 1,2$. Then we have, for $\mathcal{V}_\delta(\overline{\mu}) = \{\overline{\mu}' \in D([0,T], \mathcal{M}_1\times\mathcal{M}_1) : \rho_{[0,T]}(\overline{\mu},\overline{\mu}') < \delta\}$, and I_{m_0} given in (2.27),*

$$\liminf_{\gamma\to 0} \gamma^d \log \overline{Q}^{\gamma,\alpha}_{\sigma^\gamma}(\mathcal{V}_\delta(\overline{\mu})) \geq -I_{m_0}(\overline{\mu}), \quad I\!\!P - a.s.$$

Proof. We introduce the perturbed process. By Jensen inequality we get

$$\log \overline{Q}^{\gamma,\alpha}_{\sigma^\gamma}(\mathcal{V}_\delta(\overline{\mu})) \geq E^{\overline{V},\gamma,\alpha}_{\sigma^\gamma}\left[1\!\!\mathrm{I}_{\mathcal{V}_\delta(\overline{\mu})}(\overline{\pi}^\gamma_{[0,T]}) \log \frac{dP^{\gamma,\alpha}_{\sigma^\gamma}}{dP^{\overline{V},\gamma,\alpha}_{\sigma^\gamma}}(\sigma_{[0,T]})\right] \left(\overline{Q}^{\overline{V},\gamma,\alpha}_{\sigma^\gamma}(\mathcal{V}_\delta(\overline{\mu}))\right)^{-1} + \log \overline{Q}^{\overline{V},\gamma,\alpha}_{\sigma^\gamma}(\mathcal{V}_\delta(\overline{\mu})).$$

By Corollary 5.3, $\lim_{\gamma\to 0} \overline{Q}^{\overline{V},\gamma,\alpha}_{\sigma^\gamma}(\mathcal{V}_\delta(\overline{\mu})) = 1$. By Lebesgue dominated convergence Theorem,

$$\liminf_{\gamma\to 0} \gamma^d \log \overline{Q}^{\gamma,\alpha}_{\sigma^\gamma}(\mathcal{V}_\delta(\overline{\mu})) \geq \liminf_{\gamma\to 0} E^{\overline{V},\gamma,\alpha}_{\sigma^\gamma}\left[\gamma^d \log \frac{dP^{\gamma,\alpha}_{\sigma^\gamma}}{dP^{\overline{V},\gamma,\alpha}_{\sigma^\gamma}}(\sigma_{[0,T]})\right].$$

By Radon-Nikodym formula, see Theorem 5.4, and Proposition 5.5 we have

$$\begin{aligned}\gamma^d \log \frac{dP^{\gamma,\alpha}_{\sigma^\gamma}}{dP^{\overline{V},\gamma,\alpha}_{\sigma^\gamma}}(\sigma_{[0,T]}) &\geq -\ell_T\left(\overline{\pi}^\gamma(\sigma), \overline{V}\right) + \frac{1}{2}\int_0^T \Gamma_{\overline{V}(s,\cdot)}(\overline{\pi}^\gamma_s)\, ds \\ &- \epsilon(\gamma l)T - TC(V_1,V_2)[\delta + \sum_{i=1,2} \mathcal{E}_i(\delta, l, \gamma, \alpha)].\end{aligned}$$

From Theorem 5.1, recalling the definition of $K_{\overline{V}}(\cdot)$ given in (3.8), we get that for any l,

$$\liminf_{\gamma\to 0} \gamma^d \log \overline{Q}^{\gamma,\alpha}_{\sigma^\gamma}(\mathcal{V}_\delta(\overline{\mu})) \geq -K_{\overline{V}}(\overline{\mu}) - TC(V_1,V_2)[\delta + \lim_{\gamma\to 0}\sum_{i=1,2} \mathcal{E}_i(\delta, l, \gamma, \alpha)],$$

which yields the result letting $l \to \infty$ by Lemma 4.2 and Proposition 3.5. $\square$

Lemma 7.2 *Given $\overline{m} = (m_1, m_2) \in \left(\mathcal{C}^{2,0}([0,T]\times\Lambda)\right)^2$, with, for $i = 1,2$, $|m_i(t,r)| < p_i$, for all $t \in [0,T]$, $r \in \Lambda$, there exists $\overline{V} = (V_1,V_2) \in \left(\mathcal{C}^{1,0}([0,T]\times\Lambda)\right)^2$ such that $\overline{m} = \overline{m}^{\overline{V}}$ is the solution of (5.5). For $(t,r) \in (0,T]\times\Lambda$,*

$$\begin{aligned}&2V_i(t,r) = \\ &\log\left\{\partial_t m_i(t,r)\cosh\left[(J*m)(t,r) + a_i\theta\right] + \sqrt{\left(\partial_t m_i(t,r)\cosh\left[(J*m)(t,r)+a_i\theta\right]\right)^2 + p_i^2 - m_i^2(t,r)}\right\} \\ &- \left[(J*m)(t,r) + a_i\theta\right] - \log\{p_i - m_i(t,r)\},\end{aligned} \tag{7.10}$$

and for $t = 0$ we set $\lim_{t\to 0} V_i(t,r) = V_i(0,r)$.

Proof. By (5.5), for $t \in (0,T]$, we determine $\overline{V}(t,\cdot) = (V_1(t,\cdot), V_2(t,\cdot))$ with $V_i \in \mathcal{C}^{1,0}((0,T]\times\Lambda)$ for $i = 1,2$, such that $\overline{m} = \overline{m}^{\overline{V}}$. Namely, for $(t,r) \in (0,T]\times\Lambda$, denoting $A_i = (J*m)(t,r) + a_i\theta$, $Y_i = (\cosh A_i)\partial_t m_i(t,r)$, $Z_i = -m_i(t,r)$, (5.5) is written as

$$Y_i = Z_i \cosh\left[A_i + 2V_i(t,r)\right] + p_i \sinh\left[A_i + 2V_i(t,r)\right].$$

We multiply both sides by $X_i = e^{2V_i(t,r)}$ and obtain

$$e^{A_i}(Z_i + p_i)X_i^2 - 2Y_i X_i + e^{-A_i}(Z_i - p_i) = 0.$$

Its positive solution is

$$X_i = \frac{Y_i + \sqrt{Y_i^2 - (Z_i^2 - p_i^2)}}{e^{A_i}(Z_i + p_i)},$$

which gives (7.10). Note that $\overline{V} = (V_1, V_2)$ has the same spatial regularity as $\overline{m}$, namely the argument of the square root is always strictly positive. □

Corollary 7.3 *If $\overline{m}$ is solution of (2.24) then $\overline{V} = 0$ in (7.10).*

Remark 7.4 . Lemma 7.2 could have been stated requiring $\overline{m} \in \mathcal{AC}([0,T], B_{1,1})$. In this case one would get $\overline{V} \in \big(L^1([0,T], \mathcal{C}(\Lambda))\big)^2$. We prefer to obtain more regularity in time for $\overline{V}$, so that uniformity and other technical needs become straightforward.

Lemma 7.5 *$\mathcal{C}_1$ is $(\rho_{[0,T]}, I_0)$-dense in $\mathcal{C}_0$.*

Proof. Fix $\overline{m} \in \mathcal{C}_0$. Let $\overline{R}(t,\cdot)$, $t \in [0,T]$, be the solution of (2.24) with initial datum $\overline{R}(0,\cdot) = \overline{m}_0(\cdot)$. For any $\eta \in (0,T)$, define

$$\overline{m}^\eta(t,\cdot) = \begin{cases} \overline{R}(t,\cdot) & \text{for} \quad t \in [0,\eta], \\ \overline{R}(2\eta - t,\cdot) & \text{for} \quad t \in (\eta, 2\eta], \\ \overline{m}(t - 2\eta,\cdot) & \text{for} \quad t \in (2\eta, T]. \end{cases}$$

We have $\overline{m}^\eta \in \mathcal{C}_1$ for any $0 < \eta < T$ and $\lim_{\eta\to 0} \rho_{[0,T]}(\overline{m}^\eta, \overline{m}) = 0$. Since I_0 is lower semicontinuous it remains to show

$$\lim_{\eta \to 0} I_0(\overline{m}^\eta) \le I_0(\overline{m}). \tag{7.11}$$

We split $[0,T]$ into $[0,2\eta]$ and $[2\eta,T]$ in the integration. We have that

$$\int_{2\eta}^{T} \int_\Lambda \mathcal{H}(\overline{m}^\eta, \dot{\overline{m}}^\eta)(t,r)\, dr dt = \int_0^{T-2\eta} \int_\Lambda \mathcal{H}(\overline{m}, \dot{\overline{m}})(t,r)\, dr dt \le I_0(\overline{m}).$$

Next we show that

$$\lim_{\eta\to 0} \int_0^{2\eta} \int_\Lambda \mathcal{H}(\overline{m}^\eta, \dot{\overline{m}}^\eta)(t,r)\, dr dt = 0.$$

Since $\overline{m}^\eta = \overline{R}$ for $t \in [0,\eta]$ solves (2.24), by (3) of Proposition 3.7,

$$\int_0^\eta \int_\Lambda \mathcal{H}(\overline{R}, \dot{\overline{R}})(t,r)\, dr dt = 0. \tag{7.12}$$

Since the profile $\overline{m}^\eta$ in $(\eta, 2\eta]$ is the profile in $(0,\eta]$ backwards in time, we have

$$\int_\eta^{2\eta} \int_\Lambda \mathcal{H}(\overline{m}^\eta, \dot{\overline{m}}^\eta)(t,r)\, dr dt = \int_0^\eta \int_\Lambda \mathcal{H}(\overline{R}, -\dot{\overline{R}})(t,r)\, dr dt.$$

Because $\overline{R}$ solves (2.24) and for $t > 0$, $|R_i(t,\cdot)| \le p_i - \delta_i$, for $i = 1,2$, $\mathcal{H}(\overline{R}, \dot{\overline{R}})$ belongs to $L^1([0,T] \times \Lambda)$, as well as $\mathcal{H}(\overline{R}, -\dot{\overline{R}})$, see explicit formula (3.15). By dominated convergence,

$$\lim_{\eta\to 0} \int_\eta^{2\eta} \int_\Lambda \mathcal{H}(\overline{m}^\eta, \dot{\overline{m}}^\eta)(t,r)\, dr dt = 0.$$

In this way we prove (7.11). $\square$

Lemma 7.6 *$\mathcal{C}_2$ is $(\rho_{[0,T]}, I_0)$-dense in $\mathcal{C}_1$.*

Proof. Let $\overline{m} \in \mathcal{C}_1$ and $\eta \in (0,T)$ so that $\overline{m}(t,\cdot) = \overline{R}(t,\cdot)$ for $t \in [0,\eta]$. By (7.4), $\|m_i(\eta,\cdot)\|_\infty \le p_i - \delta_i$ for some $\delta_i > 0$ and $i = 1,2$. Define

$$m_i^n(t,r) = \begin{cases} m_i(t,r) & \text{for } t \in [0,\eta], \\ m_i(\eta,r) + \left(1 - \dfrac{1}{n}\right)(m_i(t,r) - m_i(\eta,r)) & \text{for } t \in (\eta,T]. \end{cases} \tag{7.13}$$

By construction and from (7.12), $I_0(\overline{m}^n) = \displaystyle\int_\eta^T \int_\Lambda \mathcal{H}(\overline{m}^n(t,r), \frac{\partial \overline{m}^n}{\partial t}(t,r))\, drdt$. Moreover, since $I_0(\overline{m}) < \infty$, by Proposition 3.4 we have $\|m_i(t)\|_\infty \le p_i$ for $t \in [\eta,T]$, then

$$\|m_i^n(t)\|_\infty \le p_i - \frac{\delta_i}{n}, \quad \forall t \in [\eta,T]. \tag{7.14}$$

Hence $\overline{m}^n \in \mathcal{C}_2$ for all n. Furthermore $\lim_{n\to\infty} m_i^n(t,r) = m_i(t,r)$ and $\frac{\partial m_i^n}{\partial t}(t,r) = (1 - \frac{1}{n})\frac{\partial m_i}{\partial t} \to \frac{\partial m_i}{\partial t}(t,r)$ for almost all $(t,r) \in [\eta,T] \times \Lambda$. Then, by Proposition 3.4, $\mathcal{H}(\overline{m}^n(t,r), \frac{\partial \overline{m}^n}{\partial t}(t,r))$ is given by (3.15), while $\mathcal{H}(\overline{m}(t,r), \frac{\partial \overline{m}}{\partial t}(t,r))$ is given either by (3.15) when $|m_i(t,r)| < p_i$, or, when $|m_i(t,r)| = p_i$, by (3.16), or is infinite. We hence check that pointwise

$$\lim_{n\to\infty} \mathcal{H}(\overline{m}^n(t,r), \frac{\partial \overline{m}^n}{\partial t}(t,r)) = \mathcal{H}(\overline{m}(t,r), \frac{\partial \overline{m}}{\partial t}(t,r)).$$

To apply the Lebesgue dominated convergence Theorem we give an upper bound, uniformly with respect to n, of $|\mathcal{H}(\overline{m}^n, \frac{\partial \overline{m}^n}{\partial t})(t,r)|$ (see also [C] p. 174). For that we combine (3.17) with the facts that,

$$\{(t,r) : \frac{\partial m_i^n}{\partial t} > 0\} = \{(t,r) : \frac{\partial m_i}{\partial t} > 0\},$$

and on the set $\{(t,r) : m_i^n(t,r) \ge p_i - \delta_i\}$ we have $m_i(t,r) - m_i(0,r) \ge 0$ and $p_i - m_i^n(t,r) \ge p_i - m_i(t,r)$. To get shorter notation, we denote for $\overline{\phi}, \overline{\psi} \in \mathcal{AC}([0,T], B_{p_1,p_2})$

$$\Upsilon(\phi_i,\psi_i) = \mathbb{I}_{\{\dot\phi_i > 0;\, p_i - \delta_i \le \psi_i\}} \left(\log \frac{1}{p_i - \psi_i}\right)^+ + \mathbb{I}_{\{\dot\phi_i < 0;\, -p_i + \delta_i \ge \psi_i\}} \left(\log \frac{1}{p_i + \psi_i}\right)^+.$$

We have

$$
\begin{aligned}
&2\mathcal{H}(\overline{m}^n, \frac{\partial \overline{m}^n}{\partial t})(t,r) \\
&\le \sum_{i=1,2} |\dot m_i| \Big[(\log|\dot m_i|)^+ + \Upsilon(\dot m_i, m_i^n) + K_i\Big](t,r) \\
&+ \sum_{i=1,2} |\dot m_i| \left[1\!\!1_{\{\dot m_i>0;\, m_i^n<p_i-\delta_i\}} \left(\log \frac{1}{p_i - m_i^n}\right)^+ + 1\!\!1_{\{\dot m_i<0;\, m_i^n>-p_i+\delta_i\}} \left(\log \frac{1}{p_i + m_i^n}\right)^+ \right](t,r) + C \\
&\le \sum_{i=1,2} |\dot m_i| \Big[(\log|\dot m_i|)^+ + \Upsilon(\dot m_i, m_i) + K_i\Big](t,r) \\
&+ \sum_{i=1,2} |\dot m_i| \left[1\!\!1_{\{\dot m_i>0;\, m_i^n<p_i-\delta_i\}} \log \frac{1}{\delta_i} + 1\!\!1_{\{\dot m_i<0;\, m_i^n>-p_i+\delta_i\}} \log \frac{1}{\delta_i} \right](t,r) + C \\
&\le \sum_{i=1,2} |\dot m_i| \left[(\log|\dot m_i|)^+ + 1\!\!1_{\{\dot m_i>0\}} \left(\log \frac{1}{p_i - m_i}\right)^+ + 1\!\!1_{\{\dot m_i<0\}} \left(\log \frac{1}{p_i + m_i}\right)^+ + K_i + \log \frac{1}{\delta_i}\right](t,r) + C.
\end{aligned}
$$

Since by assumption $I_0(\overline{m}) < \infty$, by Proposition 3.6, part (b), the above upper bound is integrable. By Lebesgue dominated convergence Theorem we then have

$$\lim_{n\to\infty} I_0(\overline{m}^n) = I_0(\overline{m}). \tag{7.15}$$

Obviously $\overline{m}^n \to \overline{m}$ in the metric (3.2). $\square$

Lemma 7.7 *$\mathcal{C}_3$ is $(\rho_{[0,T]}, I_0)$-dense in $\mathcal{C}_2$.*

Proof. Take $\overline{\psi} \in \mathcal{C}_2$. To get more regularity we convolve with a smooth kernel the function both in time and space. To perform the convolution in time we extend the definition of $\overline{\psi}$ to $[T, T+1]$ by setting, for each $s \in [0,1]$, if $\overline{u} = (\overline{u}_1, \overline{u}_2)$ is the solution of equation (2.24) with initial condition $\overline{\psi}(T,\cdot)$,

$$\overline{\psi}(T+s, r) = \overline{u}(s,r). \tag{7.16}$$

Since $\overline{\psi} \in \mathcal{C}_2$ there exist δ_i, $i=1,2$, such that $|\psi_i(T,r)| \le p_i - \delta_i$. It follows from (7.3) that $\overline{\psi}_i(T+s,r) \le p_i - \tilde\delta_i$ for all $s \in [0,1]$, for some $\tilde\delta_i$ smaller than δ_i. In the following we will denote it always by δ_i. Denote by $\theta_s\overline{\psi}$ the time translation of $\overline{\psi}$, $(\theta_s\overline{\psi})(t,r) = \overline{\psi}(t+s,r)$ for $(t,r) \in [0,T]\times\Lambda$. Let Φ_{ϵ_1} be a smooth non-negative kernel, $\Phi_{\epsilon_1} \in \mathcal{C}^\infty(\Lambda)$ with support in a ball of radius ϵ_1 and integral one which we use as spatial mollifier. For $\epsilon_0 > 0$, let Ψ_{ϵ_0} be the $\mathcal{C}^\infty(I\!R)$ non-negative temporal mollifier with support $[0,\epsilon_0]$ and integral one. Set $\epsilon \equiv (\epsilon_0, \epsilon_1)$, $\epsilon \downarrow 0$ stands for $\epsilon_0 \downarrow 0$ and $\epsilon_1 \downarrow 0$. Let $\eta > 0$ be such that $\overline{\psi}(t,\cdot) = \overline{R}(t,\cdot)$ for $t \in [0, 3\eta]$. Let $\chi_1(t)$, $\chi_2(t)$ be a $\mathcal{C}^2$ partition of the unity enjoying the properties:

$$
\begin{cases}
\chi_1(t) = 1 \quad \text{for} \quad t \in [0,\eta], & \chi_1(t) = 0 \quad \text{for} \quad t \in [2\eta, T],\\
\chi_2(t) = 0 \quad \text{for} \quad t \in [0,\eta], & \chi_2(t) = 1 \quad \text{for} \quad t \in [2\eta, T],\\
\chi_1(t) + \chi_2(t) = 1, \qquad \forall t \in [0,T]. &
\end{cases}
$$

Let

$$\psi_i^\epsilon(t,\cdot) = \chi_1(t)\psi_i(t,\cdot) + \chi_2(t)\int_{I\!R} \Psi_{\epsilon_0}(s)(\Phi_{\epsilon_1} * \theta_s\psi_i)(t,\cdot)\, ds. \tag{7.17}$$

By construction $\psi_i(\cdot,\cdot)$, $i=1,2$, satisfies all the regularity requirements to be in $\mathcal{C}_3$. Furthermore, since $|\psi_i(t,r)| \le p_i - \delta_i, \delta_i > 0$, for all $\epsilon > 0$ and $t \in [0,T]$, we still have that

$$|\psi_i^\epsilon(t,r)| \le p_i - \delta_i, \qquad i = 1,2, \tag{7.18}$$

therefore $\overline{\psi}^\epsilon \in \mathcal{C}_3$. Moreover

$$\lim_{\epsilon\to 0} \rho_{[0,T]}(\overline{\psi}^\epsilon, \overline{\psi}) = 0.$$

Since I_0 is lower semicontinuous, see Proposition 3.7, (1), it is enough to prove

$$\lim_{\epsilon\to 0} I_0(\overline{\psi}^\epsilon) \le I_0(\overline{\psi}). \tag{7.19}$$

By using the expression (3.9) of I_0, see Proposition 3.5, we have

$$I_0(\overline{\psi}^\epsilon) - I_0(\overline{\psi}) = \int_0^T \int_\Lambda \left[\mathcal{H}(\overline{\psi}^\epsilon, \frac{\partial \overline{\psi}^\epsilon}{\partial t})(t,x) - \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t,x) \right] dxdt. \tag{7.20}$$

We split the time integral into 3 pieces: (i) a first integral on $[0,\eta]$, which is equal to 0 by definition (7.17) of $\overline{\psi}^\epsilon$; (ii) a second one on $[\eta, 2\eta]$, treated in Lemma 7.9 below; (iii) a third one on $[2\eta, T]$, that we now analyze. Notice that for $t \ge 2\eta$, see (7.17), $\chi_1(t) = 0$ and $\chi_2(t) = 1$, therefore $\psi_i^\epsilon(t,\cdot)$ reduces to a convex combination, and we exploit that $\mathcal{H}(\overline{m}, \overline{a})$ is convex with respect to $\overline{a}$. Then, for $t \ge 2\eta$, by Jensen inequality we obtain

$$\mathcal{H}(\overline{\psi}^\epsilon, \frac{\partial \overline{\psi}^\epsilon}{\partial t})(t,x) \le \int_{I\!R} \Psi_{\epsilon_0}(s) \int_\Lambda \Phi_{\epsilon_1}(y) \mathcal{H}(\overline{\psi}^\epsilon(t,x), \frac{\partial \overline{\psi}}{\partial t}(t+s, x-y))\, dyds. \tag{7.21}$$

For all $s \in [0,1]$, $s < T$, we have

$$\begin{aligned}
&\int_{2\eta+s}^T \int_\Lambda \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t,x)\, dxdt = \int_{2\eta}^T \int_\Lambda \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t+s,x)\, dxdt - \int_{T-s}^T \int_\Lambda \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t+s,x)\, dxdt \\
&= \int_{2\eta}^T \int_\Lambda \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t+s,x)\, dxdt \\
&= \int_{I\!R} \Psi_{\epsilon_0}(s) \int_{2\eta}^T dt \int_\Lambda \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t+s,x)\, dxds \\
&= \int_{I\!R} \Psi_{\epsilon_0}(s) \int_{2\eta}^T \int_\Lambda dy \Phi_{\epsilon_1}(y) \int_\Lambda \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t+s,x-y)\, dxdtds,
\end{aligned} \tag{7.22}$$

where the first equality comes from a change of variables, the second one from the definition of $\overline{\psi}$ in $[T, T+1]$ (see (7.16)), the third one from $\int_\Lambda dy\Phi_{\epsilon_1}(y) = 1$ and $\int_\Lambda dx \mathcal{H}(\overline{\psi}, \frac{\partial\overline{\psi}}{\partial t})(t, x-y) = \int_\Lambda dx \mathcal{H}(\overline{\psi}, \frac{\partial\overline{\psi}}{\partial t})(t,x)$, and the last one from $\int_{I\!R} ds \Psi_{\epsilon_0}(s) = 1$. Therefore

$$\begin{aligned}
&\int_{2\eta}^T \int_\Lambda \left[\mathcal{H}(\overline{\psi}^\epsilon, \frac{\partial \overline{\psi}^\epsilon}{\partial t})(t,x) - \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t,x) \right] dxdt \\
&= \int_{2\eta}^T \int_\Lambda \mathcal{H}(\overline{\psi}^\epsilon, \frac{\partial \overline{\psi}^\epsilon}{\partial t})(t,x)\, dxdt - \int_{2\eta}^{2\eta+s} \int_\Lambda \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t,x)\, dxdt - \int_{2\eta+s}^T \int_\Lambda \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t,x)\, dxdt \\
&\le \int_{I\!R} \Psi_{\epsilon_0}(s) \int_{2\eta}^T \int_\Lambda \Phi_{\epsilon_1}(y) \int_\Lambda \mathcal{H}(\overline{\psi}^\epsilon(t,x), \frac{\partial \overline{\psi}}{\partial t}(t+s,x-y))\, dxdydtds \\
&- \int_{I\!R} \Psi_{\epsilon_0}(s) \int_{2\eta}^T \int_\Lambda \Phi_{\epsilon_1}(y) \int_\Lambda \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t+s,x-y)\, dxdydtds.
\end{aligned} \tag{7.23}$$

The inequality holds by (7.21), and because $\mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t,x) \ge 0$. Finally we use (7.22). To estimate the last difference in (7.23) we add and subtract to it the term

$$\int_{I\!R} ds \Psi_{\epsilon_0}(s) \int_{2\eta}^{T} dt \int_{\Lambda} \Phi_{\epsilon_1}(y) dy \int_{\Lambda} dx \mathcal{H}(\overline{\psi}(t,x), \frac{\partial \overline{\psi}}{\partial t}(t+s, x-y)),$$

which gives

$$\int_{2\eta}^{T} \int_{\Lambda} \left[\mathcal{H}(\overline{\psi}^{\epsilon}, \frac{\partial \overline{\psi}^{\epsilon}}{\partial t})(t,x) - \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t,x) \right] dx dt \le W_1 + W_2,$$

where

$$\begin{aligned} W_1 &= \int_{I\!R} \Psi_{\epsilon_0}(s) \int_{2\eta}^{T} \int_{\Lambda} \Phi_{\epsilon_1}(y) \int_{\Lambda} \left[\mathcal{H}(\overline{\psi}^{\epsilon}(t,x), \frac{\partial \overline{\psi}}{\partial t}(t+s, x-y)) - \mathcal{H}(\overline{\psi}(t,x), \frac{\partial \overline{\psi}}{\partial t}(t+s, x-y)) \right] dx dy dt ds, \\ W_2 &= \int_{I\!R} \Psi_{\epsilon_0}(s) \int_{2\eta}^{T} \int_{\Lambda} \Phi_{\epsilon_1}(y) \int_{\Lambda} \left[\mathcal{H}(\overline{\psi}(t,x), \frac{\partial \overline{\psi}}{\partial t}(t+s, x-y)) - \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t+s, x-y) \right] dx dy dt ds. \end{aligned} \tag{7.24}$$

Taking into account Lemma 7.8 below we get the result. □

The proofs of the next two Lemmas are postponed to Appendix B.

Lemma 7.8

$$\lim_{\epsilon \to 0} |W_i| = 0, \qquad i = 1, 2.$$

Lemma 7.9

$$\lim_{\epsilon \to 0} \int_{\eta}^{2\eta} \int_{\Lambda} \left[\mathcal{H}(\overline{\psi}^{\epsilon}, \frac{\partial \overline{\psi}^{\epsilon}}{\partial t})(t,x) - \mathcal{H}(\overline{\psi}, \frac{\partial \overline{\psi}}{\partial t})(t,x) \right] dx dt = 0.$$

8. Appendix A

In this Appendix we give the proofs of the properties of the rate functional stated in Section 3.

Proof of Lemma 3.2 The differentiability of $\Gamma_{\overline{V}}(\overline{u})$ in $(L^{\infty}(\Lambda))^2$ is easily verified. For the convexity we compute first the Hessian of $\Gamma_{\overline{V}}(\overline{u})$ with respect to V_1 and V_2. Since the Hessian is a diagonal matrix, it is enough to study separately the convexity with respect to V_1 and V_2, we do it for V_1. For $r \in \Lambda$, we set $V_1(r) = x, u_1(r) = m$ and denote by

$$f_1(x) = (p_1 \tanh \vartheta - m) \sinh(2x) + (p_1 - m \tanh \vartheta)(\cosh(2x) - 1)$$

the integrand term in $\Gamma_{\overline{V}}(\overline{u})$ which depends only on $\overline{V}$ with ϑ varying in some bounded interval of $I\!R$, $x \in I\!R$, $|m| \le p_1$. We then study the sign of the second derivative of f_1.

$$\frac{1}{4} f_1''(x) = \cosh(2x)[p_1 - m \tanh \vartheta] + \sinh(2x)[p_1 \tanh \vartheta - m] \,.$$

Notice that $p_1 - m \tanh \vartheta \ge 0$, and $p_1 - m \tanh \vartheta \ge p_1 \tanh \vartheta - m \ge -(p_1 - m \tanh \vartheta)$. Since $\cosh(2x) > |\sinh(2x)|$ and $|\tanh \vartheta| < 1$ when ϑ varies in a bounded interval we obtain that f_1 is convex. □

Proof of Proposition 3.4 Recalling (3.12), for $i = 1, 2$, denote

$$\begin{aligned} F_i(v_i) &= g_i v_i - \frac{1}{2} B_i(\overline{u}, v_i) \\ &= g_i v_i - (p_i - u_i)\frac{e^{A_i}}{4\cosh A_i}[e^{2v_i} - 1] - (p_i + u_i)\frac{e^{-A_i}}{4\cosh A_i}[e^{-2v_i} - 1]. \end{aligned} \tag{8.8}$$

Hence

$$\frac{\partial F_i}{\partial v_i} = g_i - (p_i - u_i)\frac{e^{A_i}}{2\cosh A_i}e^{2v_i} + (p_i + u_i)\frac{e^{-A_i}}{2\cosh A_i}e^{-2v_i}. \tag{8.9}$$

First assume that $u_i > p_i$. By (8.8), since $-(p_i - u_i) > 0$, we have $\lim_{v_i \to +\infty} F_i(v_i) = +\infty$. In the same way, we get $\lim_{v_i \to -\infty} F_i(v_i) = +\infty$ if $u_i < -p_i$ because then $-(p_i + u_i) > 0$. Therefore, (a) holds.

For the remaining cases, we exploit that for $\overline{u} \in B_{p_1,p_2}$, the function $v_i \mapsto B_i(\overline{u}, v_i)$ is convex differentiable on $I\!R$.

(b) To compute the Legendre transform of $B_i(\overline{u}, v_i)$, when $|u_i| < p_i$, by (8.9), the maximum in (3.11) is obtained for (remember (3.13))

$$e^{2v_i} = e^{-A_i}\frac{D_i}{p_i - u_i}, \qquad \text{hence } v_i = \frac{1}{2}\left(\log\frac{D_i}{p_i - u_i} - A_i\right). \tag{8.10}$$

Inserting (8.10) in (8.8) we have

$$H_i(\overline{u}, g_i) = \frac{g_i}{2}\left[\log\frac{D_i}{p_i - u_i} - A_i\right] + p_i\frac{e^{A_i} + e^{-A_i}}{4\cosh A_i} - u_i\frac{e^{A_i} - e^{-A_i}}{4\cosh A_i} - \frac{1}{4\cosh A_i}\left[D_i + \frac{p_i^2 - u_i^2}{D_i}\right],$$

which yields (3.15) since, using (3.13), we write

$$D_i + \frac{p_i^2 - u_i^2}{D_i} = g_i\cosh A_i + R_i + \frac{(p_i^2 - u_i^2)\,(g_i\cosh A_i - R_i)}{(g_i\cosh A_i)^2 - (g_i\cosh A_i)^2 - p_i^2 + u_i^2} = 2R_i.$$

(c) When $u_i = p_i$ (resp. $u_i = -p_i$),

$$\frac{\partial F_i}{\partial v_i} = g_i + p_i\frac{e^{-\mathrm{sgn}(u_i)A_i}}{\cosh A_i}e^{-2\mathrm{sgn}(u_i)v_i}$$

and to solve $\dfrac{\partial F_i}{\partial v_i} = 0$ (that is to find a finite extremum) we need $g_i < 0$ (resp. $g_i > 0$), namely

$$g_i = -p_i\frac{e^{-\mathrm{sgn}(u_i)A_i}}{\cosh A_i}e^{-2\mathrm{sgn}(u_i)v_i}.$$

Inserting this value in (8.8) we get (3.16) when $g_i \neq 0$.

When $u_i = p_i$ (resp. $u_i = -p_i$) and $g_i = 0$, (8.8) becomes

$$F_i(v_i) = p_i\frac{e^{-\mathrm{sgn}(u_i)A_i}}{2\cosh A_i}[1 - e^{-2\mathrm{sgn}(u_i)v_i}].$$

It is an increasing (resp. decreasing) function with a finite maximal limit:

$$\lim_{v_i \to +\infty} F_i(v_i) = p_i\frac{e^{-\mathrm{sgn}(u_i)A_i}}{2\cosh A_i} = H_i(\overline{u}, g_i), \quad \text{resp.} \quad \lim_{v_i \to -\infty} F_i(v_i) = p_i\frac{e^{-\mathrm{sgn}(u_i)A_i}}{2\cosh A_i} = H_i(\overline{u}, g_i).$$

(d) When $u_i = p_i$ and $g_i > 0$ (resp. $u_i = -p_i$ and $g_i < 0$), (8.8) becomes

$$F_i(v_i) = g_i v_i + p_i \frac{e^{-\mathrm{sgn}(u_i)A_i}}{2\cosh A_i}[1 - e^{-2\mathrm{sgn}(u_i)v_i}].$$

Hence

$$\lim_{v_i \to +\infty} F_i(v_i) = +\infty = H_i(\overline{u}, g_i), \qquad \text{resp.} \quad \lim_{v_i \to -\infty} F_i(v_i) = +\infty = H_i(\overline{u}, g_i).$$

□

Proof of Proposition 3.6 We use the explicit representation of $\mathcal{H}(\cdot,\cdot)$ given in Proposition 3.4.

(a) We give an upper bound of expression (3.15). The difficulty comes from the term

$$F(\overline{u}, g_i, \theta) = g_i \log \frac{D_i(\overline{u}, g_i, \theta)}{p_i - u_i}, \tag{8.11}$$

where $D_i(\overline{u}, g_i, \theta)$ is defined in (3.13). Let $-\overline{u} = (-u_1, -u_2)$. We have

$$\begin{aligned} F(-\overline{u}, -g_i, \theta) &= g_i \log \frac{(p_i + u_i)\left\{g_i \cosh[(J*u) - a_i\theta] + \sqrt{(g_i \cosh[(J*u) - a_i\theta])^2 + p_i^2 - u_i^2}\right\}}{-(g_i \cosh[(J*u) - a_i\theta])^2 + (g_i \cosh[(J*u) - a_i\theta])^2 + p_i^2 - u_i^2} \\ &= g_i \log \frac{g_i \cosh[(J*u) - a_i\theta] + \sqrt{(g_i \cosh[(J*u) - a_i\theta])^2 + p_i^2 - u_i^2}}{p_i - u_i} \\ &= F(\overline{u}, g_i, -\theta). \end{aligned}$$

We write

$$F(\overline{u}, g_i, \theta) = F(\overline{u}, g_i, \theta)\mathbb{1}_{\{g_i \geq 0\}} + F(-\overline{u}, -g_i, -\theta)\mathbb{1}_{\{g_i < 0\}}. \tag{8.12}$$

Hence it suffices to estimate $F(\overline{u}, g_i, \theta)$ for $g_i > 0$ and $\theta \in I\!R$. We get

$$F(\overline{u}, g_i, \theta) \leq |g_i| \left\{ \log D_i(\overline{u}, g_i, \theta) + \left(\log \frac{1}{p_i - u_i}\right)^+ \mathbb{1}_{\{g_i > 0\}} + \left(\log \frac{1}{p_i + u_i}\right)^+ \mathbb{1}_{\{g_i < 0\}} \right\}.$$

We obtain (3.17) by the upper bound $D_i(\overline{u}, g_i, \theta) \leq 2|g_i| + 1$. The lower bound (3.18) is obtained as in [C], p. 171. We rely on formulas (3.11), (3.12). Since $e^{\beta a} \leq e^{\beta |a|}$, there exists a constant C such that

$$B_i(\overline{u}, v_i) \leq 2C[e^{2|v_i|} - 1]$$

$$2p_i \geq 2C \geq \max\left\{(p_i - u_i)\frac{e^{A_i}}{2\cosh A_i}, (p_i + u_i)\frac{e^{-A_i}}{2\cosh A_i}\right\}.$$

Then

$$H_i(\overline{u}, g_i) \geq \sup_{v_i \in I\!R} \left\{ g_i v_i - C\left[e^{2|v_i|} - 1\right]\right\} = \max\left\{\frac{|g_i|}{2}\left[\log \frac{|g_i|}{2C} - 1\right] + C, 0\right\}.$$

(b) If (3.17) holds then $I_0(\overline{\phi}) < \infty$. For the converse, by (3.18), it is necessary to have $\dot{\phi}_i \log |\dot{\phi}_i| \in L^1([0,T] \times \Lambda)$. To conclude, notice that when $g_i > 0$, uniformly in $\theta \in I\!R$,

$$F(\overline{u}, g_i, \theta) - g_i \log\{g_i \cosh A_i\} \geq 2g_i \mathbb{1}_{\{g_i > 0\}} \log \frac{1}{p_i - u_i}.$$

$\square$

Proof of Proposition 3.7 For (1), (2) we refer to the similar proof of [C], Theorem III.4, p. 148 (indeed, the rate functional in infinite outside $\mathcal{C}([0,T],\mathcal{B}_{1,1})$). To show the first part of (3), notice that for $\overline{V}=0$ the r.h.s. of the argument of the sup in (3.5) is equal to zero. This implies that for $s\in[0,T]$, $\mathcal{H}^*(\overline{\phi}(s,\cdot),\dot{\overline{\phi}}(s,\cdot))\geq 0$ in (3.6), therefore $I_0(\overline{\pi})\geq 0$. For the second half of (3), we start by proving that if $I_0(\overline{\pi})=0$, then $\overline{\pi}=(\phi_1\lambda,\phi_2\lambda)$ with $\overline{\phi}=(\phi_1,\phi_2)\in\mathcal{AC}([0,T],B_{1,1})$ is the solution of equation (2.24). From Proposition 3.5, we know that $J_0(\overline{\pi})=0$ (see (3.7), (3.8)), that is, for any $\overline{V}=(V_1,V_2)\in(L^\infty([0,T]\times\Lambda))^2$, we have

$$\int_0^T <\overline{V}(s,\cdot),\dot{\overline{\phi}}(s,\cdot)>ds\leq\frac{1}{2}\int_0^T\Gamma_{\overline{V}(s,\cdot)}(\overline{\phi}(s,\cdot))\,ds.$$

Now take $V_2=0$ and ηV_1 instead of V_1, where $\eta>0$. Denote $\phi=\phi_1+\phi_2$, then recalling definitions (3.3) and (3.4), we get

$$\begin{aligned}&2\eta\int_0^T <V_1(s,\cdot),\dot{\phi}_1(s,\cdot)>ds\\&\leq p_1\int_0^T <\tanh(\phi(s,\cdot)*J+a_1\theta)\sinh(2\eta V_1(s,\cdot))+\cosh(2\eta V_1(s,\cdot))-1>ds\\&\quad-\int_0^T<\phi_1(s,\cdot)\Big(\tanh(\phi(s,\cdot)*J+a_1\theta)[\cosh(2\eta V_1(s,\cdot))-1]+\sinh(2\eta V_1(s,\cdot))\Big)>ds.\end{aligned}$$

Using Taylor expansion in η when $\eta\to 0$, dividing by η and letting $\eta\to 0$, we obtain

$$\int_0^T<V_1(s,\cdot),\dot{\phi}_1(s,\cdot)>ds\leq p_1\int_0^T<\tanh(\phi(s,\cdot)*J+a_1\theta)V_1(s,\cdot)>ds-\int_0^T<\phi_1(s,\cdot)V_1(s,\cdot)>ds.$$

Since all terms in the previous expression are linear in V_1, we may change V_1 into $-V_1$ to obtain the converse inequality. Then, exchanging the roles of indices 1 and 2, we have, for $i=1,2$,

$$\int_0^T<V_i(s,\cdot),\dot{\phi}_i(s,\cdot)>ds=p_i\int_0^T<\tanh(\phi(s,\cdot)*J+a_i\theta)V_i(s,\cdot)>ds-\int_0^T<\phi_i(s,\cdot)V_i(s,\cdot)>ds.$$

This means that $\overline{\phi}$ is the (unique) weak solution of (2.24), since by definition (2.27) of the rate functional the initial condition is fulfilled.

For the reverse, we prove that if $\overline{\phi}\in\mathcal{AC}([0,T],B_{1,1})$ is the solution of equation (2.24), then $\overline{\pi}=(\phi_1\lambda,\phi_2\lambda)$ is such that $J_1(\overline{\pi})=0$; hence, by Proposition 3.5, $I_0(\overline{\pi})=0$. We insert equation (2.24) into the explicit representation (3.15). Namely if $\overline{\pi}$ solves (2.24) then, by Corollary 7.3,

$$\log\frac{D_i}{p_i-\phi_i}-A_i=0,\tag{8.13}$$

$$R_i=D_i-(\partial_t\phi_i)\cosh A_i=e^{A_i}(p_i-\phi_i)+(\phi_i-p_i\tanh A_i)\cosh A_i.$$

Hence

$$p_i-\phi_i\tanh A_i-\frac{R_i}{\cosh A_i}=(p_i-\phi_i)(1+\tanh A_i-\frac{e^{A_i}}{\cosh A_i})=0.\tag{8.14}$$

By (8.13), (8.14), the right hand side of (3.15) is equal to zero, which completes the proof of (3). $\square$

9. Appendix B

This appendix is devoted to proofs postponed from Sections 5, 6 and 7.

Proof of Proposition 5.5 Let $s \in [0,T]$,

$$C(V_1,V_2) = \sum_{i=1,2} C(V_i) = \sum_{i=1,2} \sup_{s\in[0,T]} \sup_{r\in\Lambda} \big(\sinh^2[V_i(s,r)] + |\sinh[2V_i(s,r)]|\big).$$

Then

$$\Big| F_{\overline{V}(s)}(\overline{\lambda}^\gamma(\alpha), \overline{\pi}^\gamma_s) - F_{\overline{V}(s)}((p_1\lambda^\gamma, p_2\lambda^\gamma), \overline{\pi}^\gamma_s) \Big| \leq \left| \frac{\gamma^d}{2} \sum_{x\in\Lambda_\gamma} [\alpha_i(x) - p_i]\, \mathcal{B}_i(x,\sigma,s) \right|,$$

with

$$\mathcal{B}_i(x,\sigma,s) = \cosh[2V_i(s,\gamma x)] - 1 + \tanh[(J_\gamma \star \sigma_s)(x) + a_i\theta] \sinh[2V_i(s,\gamma x)]$$

$$|\mathcal{B}_i(x,\sigma,s)| \leq \sinh^2[V_i(s,\gamma x)] + |\sinh[2V_i(s,\gamma x)]| \leq C(V_i).$$

Take $l \in \mathbb{Z}$, $l \neq 0$. Since $\mathbb{E}[\alpha_i(x)] = p_i$ for all $x \in \Lambda_\gamma$,

$$\begin{aligned}\frac{\gamma^d}{2} \sum_{x\in\Lambda_\gamma} [\alpha_i(x) - p_i]\, \mathcal{B}_i(x,\sigma,s) &= \frac{\gamma^d}{2} \sum_{x\in\Lambda_\gamma} \frac{1}{(2l+1)^d} \sum_{|y|\leq l} [\mathcal{B}_i(x+y,\sigma,s) - \mathcal{B}_i(x,\sigma,s)]\, \alpha_i(x+y) \\ &\quad - \frac{\gamma^d}{2} \sum_{x\in\Lambda_\gamma} \mathcal{B}_i(x,\sigma,s) \left[\alpha_i^{(l)}(x) - \mathbb{E}[\alpha_i(x)]\right].\end{aligned} \tag{9.1}$$

Using uniform continuity as in the proof of Lemma 4.3, there exists a positive function ϵ on $\mathbb{R}_+$ with $\lim_{s\to 0} \epsilon(s) = 0$ (depending only on T, J and $\overline{V}$) such that the first term on the r.h.s. of (9.1) is bounded uniformly in α and σ; for the second term, let $\delta > 0$ and $\mathcal{E}_i(\delta,l,\gamma,\alpha)$ defined in (4.3). We conclude by

$$\left| \frac{\gamma^d}{2} \sum_{x\in\Lambda_\gamma} [\alpha_i(x) - p_i]\, \mathcal{B}_i(x,\sigma,s) \right| \leq \epsilon(\gamma l) + C(V_i)\, [\delta + \mathcal{E}_i(\delta,l,\gamma,\alpha)].$$

$\square$

Proof of Proposition 6.1 Consider a sequence of functions $\{H_k\}_{k\geq 1}$ in $\mathcal{C}^2(\Lambda)$ dense in $\mathcal{C}(\Lambda)$ for the uniform topology with $\|H_k\|_\infty \leq 1$. Denote for all integers $m \geq 1$, $\ell \geq 1$, and $\delta > 0$,

$$\mathcal{A}_{m,\delta,\ell} = \{\overline{\pi} \in D([0,T], \mathcal{M}_1 \times \mathcal{M}_1) : \inf_{\{t'_i\}} \max_i \sup_{t'_i \leq s < t'_{i+1}} \sum_{j=1}^m \frac{|<\overline{\pi}_t, H_j> - <\overline{\pi}_s, H_j>|}{4^j} \leq \frac{\ell+1}{m}\},$$

where the infimum extends over all positive integers K and all finite sets of points $\{t'_i, 0 \leq i \leq K\}$ satisfying $0 = t'_0 < t'_1 < \ldots < t'_K = T$, $t'_{i+1} - t'_i > \delta$. We first show that for $m \geq 1$ and for $\ell \geq 1$ there exists $\delta(m,\ell)$ and $\gamma_0(m,\ell)$ so that for all $\gamma \leq \gamma_0$,

$$\overline{Q}^{\gamma,\alpha}_{\sigma^\gamma}[\overline{\pi} \notin \mathcal{A}_{m,\delta(m,\ell),\ell}] \leq e^{-\frac{\ell+1}{\gamma^d}}.$$

This is done taking into account that

$$\begin{aligned}&\{\inf_{\{t'_i\}} \max_i \sup_{t'_i \leq s < t'_{i+1}} |<\overline{\pi}_t, H> - <\overline{\pi}_s, H>| \geq \frac{\ell+1}{m}\} \\ &\qquad \subset \cup_{k=0}^{\frac{T}{\delta}} \{ \sup_{k\delta \leq t < (k+1)\delta} |<\overline{\pi}_t, H> - <\overline{\pi}_{k\delta}, H>| \geq \frac{\ell+1}{4m}\},\end{aligned}$$

and estimating the right hand side as in [KL] (p. 271, after formula (4.6)). Then the construction of the compact $\overline{K}_\ell$ is obtained by a general procedure as explained in [Bill] and Section 8 of [QRV]. $\square$

Next lemma states some technical results needed in the extension of the lower bound of the large deviation principle.

Lemma 9.1 *For $(t,t') \in [0,T]^2$, $(x,y) \in \Lambda^2$, $\delta_0 > 0$, let $\tau \in \{t,t'\}$, $\zeta \in \{x,y\}$. We assume $(\overline{u}(\tau,\cdot),\overline{v}(\tau,\cdot)) \in (B_{p_1-\delta_0,p_2-\delta_0})^2$, $g_i(\tau,\zeta) \in I\!R$ and $h_i(\tau,\zeta) \in I\!R$, for $i=1,2$; we have*

$$|H_i(\overline{u}(t,x),g_i(t',y)) - H_i(\overline{v}(t,x),g_i(t',y))| \le |g_i(t',y)| \left(1+\frac{K}{\delta_0}\right) \left(\|v(t,\cdot)-u(t,\cdot)\|_1 + |u_i(t,x)-v_i(t,x)|\right), \tag{9.2}$$

$$|H_i(\overline{v}(t,x),g_i(t',y)) - H_i(\overline{v}(t',y),g_i(t',y))| \le |g_i(t',y)| \left(1+\frac{K}{\delta_0}\right) \left(|x-y| + |v_i(t,x)-v_i(t',y)|\right), \tag{9.3}$$

$$|H_i(\overline{u}(t,x),g_i(t,x)) - H_i(\overline{u}(t,x),h_i(t,x))| \le \left(K+\log\frac{1}{\delta_0} + K|g_i(t,x)|\right) |g_i(t,x)-h_i(t,x)|, \tag{9.4}$$

where the constant $K = K(J,\theta)$ may change from one occurrence to the other.

Proof. The assumptions enable to prove (9.2)–(9.4) by writing formula (3.15) for H_i, using (3.13) for A_i, D_i, R_i. The latter depend on $\overline{u}$ or $\overline{v}$, g_i or h_i, (t,x) or (t',y). In each computation, we stress the dependence on the involved quantities, writing e.g. $A_i(u)$ for (9.2), $R_i(g_i)$ for (9.4). Notice that, unlike in (3.15), those functions depend not only on (t,x), but on (t,x) and (t',y); this does not change the expression of H_i, since (3.15) was established pointwise in the proof of Proposition 3.4. In the intermediate computations, we omit to write $(t,x),(t',y)$.

We begin with auxiliary estimates. For (9.4), notice that $|(J*u)(t,x)| \le 1$ (since $\int J(r)\,dr = 1$), and

$$|A_i(u)| \le K(J,\theta). \tag{9.5}$$

When $|v_i| \le p_i - \delta_0$ and $g_i > 0$, we have $p_i^2 - v_i^2 = (p_i - |v_i|)(p_i+|v_i|) \ge p_i\delta_0$, hence

$$g_i K(J,\theta) + 1 \ge R_i(\overline{u},g_i) \ge \sqrt{g_i^2 + p_i\delta_0} \ge \max(g_i, \sqrt{p_i\delta_0}), \tag{9.6}$$

$$g_i K(J,\theta) + 1 \ge D_i(\overline{v},g_i) \ge g_i + \max(g_i,\sqrt{p_i\delta_0}). \tag{9.7}$$

For (9.2), we need

$$|(J*u)(t,x) - (J*v)(t,x)| \le \|J\|_\infty \|u(t,\cdot)-v(t,\cdot)\|_1 \tag{9.8}$$

and its consequences

$$\begin{aligned} &|v_i(t,x)\tanh[A_i(v)(t,x)] - u_i(t,x)\tanh[A_i(u)(t,x)]| \\ &\le |v_i(t,x)-u_i(t,x)||\tanh[A_i(v)(t,x)]| + |u_i(t,x)||\tanh[A_i(v)(t,x)] - \tanh[A_i(u)(t,x)]| \\ &\le |v_i(t,x)-u_i(t,x)| + \|J\|_\infty \|u(t,\cdot)-v(t,\cdot)\|_1. \end{aligned} \tag{9.9}$$

$$|\cosh[A_i(u)] - \cosh[A_i(v)]| \le K(J,\theta)\|u(t,\cdot)-v(t,\cdot)\|_1. \tag{9.10}$$

Respectively for (9.3), we need, since $|v(\cdot,\cdot)| \le 1$,

$$\begin{aligned} &|(J*v)(t,x) - (J*v)(t',y)| \le |(J*v)(t,x)-(J*v)(t,y)| + |(J*v)(t,y)-(J*v)(t',y)| \\ &\le \|J'\|_\infty |x-y| + |\int_\Lambda J(y-z)[v(t',z)-v(t,z)]\,dz| \\ &\le \|J'\|_\infty \left(|x-y| + \|v(t',\cdot)-v(t,\cdot)\|_1\right). \end{aligned} \tag{9.11}$$

as well as its consequence analogous to (9.9). The proofs of (9.2), (9.3) go along the same scheme. Namely all the estimates are done pointwise, they rely respectively on (9.7) to (9.11), the other changes being straightforward starting from expressions analogous to (9.12) below. Hence we detail only the proof of (9.2). We have

$$\begin{aligned}
&2\left[H_i(\overline{u}(t,x), g_i(t',y)) - H_i(\overline{v}(t,x), g_i(t',y))\right] \\
&= g_i(t',y)[(J*v)(t,x) - (J*u)(t,x)] + g_i(t',y)\left[\log\frac{D_i(\overline{u})(t,x)}{p_i - u_i(t,x)} - \log\frac{D_i(\overline{v})(t,x)}{p_i - v_i(t,x)}\right] \\
&+ v_i(t,x)\tanh[A_i(v)(t,x)] - u_i(t,x)\tanh[A_i(u)(t,x)] + \frac{R_i(\overline{v})(t,x)}{\cosh[A_i(v)(t,x)]} - \frac{R_i(\overline{u})(t,x)}{\cosh[A_i(u)(t,x)]}.
\end{aligned} \tag{9.12}$$

Next we show

$$\left|g_i(t',y)\left[\log\frac{D_i(\overline{u})(t,x)}{p_i - u_i(t,x)} - \log\frac{D_i(\overline{v})(t,x)}{p_i - v_i(t,x)}\right]\right| \le \frac{K}{\delta_0}|u_i(t,x)-v_i(t,x)|(1+|g_i(t',y)|)+|g_i(t',y)|\|u(t,.)-v(t,.)\|_1. \tag{9.13}$$

To this aim, see (8.11)–(8.12), it is enough to estimate, when $g_i(t',y) > 0$ and uniformly for $\theta \in I\!R$,

$$\begin{aligned}
&|F(\overline{u}(t,x), g_i(t',y), \theta) - F(\overline{v}(t,x), g_i(t',y), \theta)| = \left|g_i(t',y)\left[\log\frac{D_i(\overline{u})(t,x)}{p_i - u_i(t,x)} - \log\frac{D_i(\overline{v})(t,x)}{p_i - v_i(t,x)}\right]\right| \\
&\le g_i\left[\left|\log\frac{p_i - v_i}{p_i - u_i}\right| + \left|\log\frac{D_i(\overline{u})}{D_i(\overline{v})}\right|\right] \\
&\le g_i\left|\frac{u_i - v_i}{p_i - u_i}\right| + g_i\left|\frac{D_i(\overline{u}) - D_i(\overline{v})}{D_i(\overline{v})}\right|
\end{aligned} \quad , \tag{9.14}$$

because $|\log(1+a)| \le \log(1+|a|) \le |a|$. By (9.7), $g_i \le D_i(\overline{v})$. Using also (9.10) we get

$$\begin{aligned}
\left|g_i\frac{D_i(\overline{u}) - D_i(\overline{v})}{D_i(\overline{v})}\right| &\le \left|\frac{g_i^2\left(\cosh[A_i(u)] - \cosh[A_i(v)]\right)}{D_i(\overline{v})}\right| + \left|g_i\frac{R_i(\overline{u}) - R_i(\overline{v})}{D_i(\overline{v})}\right| \\
&\le g_i K(J,\theta)\|u(t,.) - v(t,.)\|_1 + |R_i(\overline{u}) - R_i(\overline{v})|.
\end{aligned} \tag{9.15}$$

To estimate the second term on the right hand side of (9.15), we apply (9.5), (9.6), (9.10) and obtain

$$\begin{aligned}
|R_i(\overline{u}) - R_i(\overline{v})| &= \left|\frac{[R_i(\overline{u})]^2 - [R_i(\overline{v})]^2}{R_i(\overline{u}) + R_i(\overline{v})}\right| \\
&\le \left|\frac{g_i^2\left(\cosh^2[A_i(u)] - \cosh^2[A_i(v)]\right)}{R_i(\overline{u}) + R_i(\overline{v})}\right| + \left|\frac{v_i^2 - u_i^2}{R_i(\overline{u}) + R_i(\overline{v})}\right| \\
&\le \frac{g_i^2 K(J,\theta)\|u(t,\cdot) - v(t,\cdot)\|_1}{R_i(\overline{u}) + R_i(\overline{v})} + \left|\frac{v_i^2 - u_i^2}{R_i(\overline{u}) + R_i(\overline{v})}\right| \\
&\le |g_i| K(J,\theta)\|u(t,\cdot) - v(t,\cdot)\|_1 + \frac{p_i}{\sqrt{p_i\delta_0}}|u_i - v_i|.
\end{aligned} \tag{9.16}$$

Combining (9.14), (9.16) we obtain (9.13). Next we estimate the last term of (9.12). Taking into account (9.10) and (9.16) we have

$$\begin{aligned}
&\left|\frac{R_i(\overline{v})}{\cosh[A_i(v)]} - \frac{R_i(\overline{u})}{\cosh[A_i(u)]}\right| = \left|\frac{R_i(\overline{v}) - R_i(\overline{u})}{\cosh[A_i(v)]} + R_i(\overline{u})\frac{\cosh[A_i(u)] - \cosh[A_i(v)]}{\cosh[A_i(v)]\cosh[A_i(u)]}\right| \\
&\le |R_i(\overline{v}) - R_i(\overline{u})| + R_i(\overline{u})\left|\cosh[A_i(u)] - \cosh[A_i(v)]\right| \\
&\le \sqrt{\frac{p_i}{\delta_0}}|u_i - v_i| + \left(K(J,\theta)|g_i| + K(J,\theta)[|g_i| + 1]\right)\|v(t,\cdot) - u(t,\cdot)\|_1.
\end{aligned} \tag{9.17}$$

Finally, combining (9.8), (9.9), (9.13), (9.17) yields (9.2).

We now derive (9.4) in a similar way:

$$
\begin{aligned}
&2\left[H_i(\overline{u}(t,x),g_i(t,x))-H_i(\overline{u}(t,x),h_i(t,x))\right]\\
&=[h_i(t,x)-g_i(t,x)]\left(A_i(u)(t,x)+\log\frac{1}{p_i-u_i(t,x)}\right)+g_i(t,x)\log D_i(\overline{u},g_i)(t,x)-h_i(t,x)\log D_i(\overline{u},h_i)(t,x)\\
&+\frac{R_i(\overline{u},h_i)(t,x)-R_i(\overline{u},g_i)(t,x)}{\cosh[A_i(u)(t,x)]}.
\end{aligned}
$$

We have, restricting ourselves to $g_i>0,h_i>0$, see (8.11)–(8.12), and using first $|\log(1+a)|\le|a|$ as in (9.14), then (9.7),

$$
\begin{aligned}
|g_i\log D_i(g_i)-h_i\log D_i(h_i)| &\le |g_i-h_i|\,|\log D_i(g_i)|+h_i\,|\log D_i(g_i)-\log D_i(h_i)|\\
&\le |g_i-h_i|\,|\log D_i(g_i)|+h_i\left|\frac{D_i(g_i)-D_i(h_i)}{D_i(h_i)}\right|\\
&\le |g_i-h_i|\left(|\log D_i(g_i)|+\frac{h_i\cosh[A_i(u)]}{D_i(h_i)}\right)+h_i\left|\frac{R_i(g_i)-R_i(h_i)}{D_i(h_i)}\right|\\
&\le |g_i-h_i|(g_i+1)K(J,\theta)+|R_i(g_i)-R_i(h_i)|.
\end{aligned}
$$

Then, as in (9.16),

$$
R_i(g_i)-R_i(h_i)=\frac{R_i^2(g_i)-R_i^2(h_i)}{R_i(g_i)+R_i(h_i)}=\frac{\left(g_i^2-h_i^2\right)(\cosh[A_i(u)])^2}{R_i(g_i)+R_i(h_i)}.
$$

Therefore, using that $|u_i|\le p_i-\delta_0$, and (9.6),

$$
2\,|H_i(\overline{u},g_i)-H_i(\overline{u},h_i)|\le|h_i-g_i|\left(K(J,\theta)+\log\frac{1}{\delta_0}+(g_i+1)K(J,\theta)+2[K(J,\theta)]^2\right).
$$

□

Proof of Lemma 7.8 We exploit that $\overline{\psi}\in\mathcal{AC}([0,T],B_{p_1,p_2})$ and $\overline{\psi}$ is differentiable in time in $(T,T+1]$ (see (7.16)), hence $\dfrac{\partial\psi_i}{\partial t}\in L^1([0,T+1]\times\Lambda)$. Therefore for $A>0$ and

$$
D_A=\{x\in\Lambda:\sup_{t\in[0,T+1]}\sum_{i=1}^{2}|\frac{\partial\psi_i}{\partial t}(t,x)|>A\}.
$$

we have for all $s\in[0,1]$,

$$
\lim_{A\to\infty}\sum_{i=1}^{2}\int_0^T\int_\Lambda|\frac{\partial\psi_i}{\partial t}(t+s,x)|1\!\!1_{D_A}(x)\,dxdt=0\,.
$$

By (9.2) of Lemma 9.1, we obtain, for $2\nu_0=\min\{\delta_1,\delta_2\}$, splitting $\Lambda=D_A\cup D_A^c$,

$$
\begin{aligned}
|W_1| &\le \int_{I\!R}\Psi_{\epsilon_0}(s)\int_\Lambda\Phi_{\epsilon_1}(y)\int_{2\eta}^{T}\int_\Lambda\sum_{i=1}^{2}|\frac{\partial\psi_i}{\partial t}(t+s,x-y)|(1\!\!1_{D_A}(x-y)+1\!\!1_{D_A^c}(x-y))(1+\frac{K}{\nu_0})\\
&\qquad\{|\psi_i^\epsilon(t,x)-\psi_i(t,x)|+\|\psi_i^\epsilon(t,\cdot)-\psi_i(t,\cdot)\|_1\}\,dxdtdyds\\
&\le A(1+\frac{K}{\nu_0})\sup_{t\in[0,T]}\sum_{i=1}^{2}\left(\int_\Lambda|\psi_i^\epsilon(t,x)-\psi_i(t,x)|\,dx+\|\psi_i^\epsilon(t,\cdot)-\psi_i(t,\cdot)\|_1\right)\\
&+4\sum_{i=1}^{2}\int_{2\eta}^{T}\int_\Lambda|\frac{\partial\psi_i}{\partial t}(t+s,x)|1\!\!1_{D_A}(x)\,dxdt,
\end{aligned}
$$

where we noticed that $\{|\psi_i^\epsilon(t,x)-\psi_i(t,x)|+\|\psi_i^\epsilon(t,\cdot)-\psi_i(t,\cdot)\|_1\}\le 4$. Letting first $\epsilon\to 0$ then $A\to\infty$, we get $\lim_{\epsilon\to 0}|W_1|=0$. Next we estimate W_2. We apply (9.3) of Lemma 9.1. More precisely

$$\left|\mathcal{H}(\overline{\psi}(t+s,x-y),\frac{\partial\overline{\psi}}{\partial t}(t+s,x-y))-\mathcal{H}(\overline{\psi}(t,x),\frac{\partial\overline{\psi}}{\partial t}(t+s,x-y))\right|$$
$$\le\sum_{i=1}^{2}\left|\frac{\partial\psi_i}{\partial t}(t+s,x-y)\right|\left(1+\frac{K}{\nu_0}\right)\{(|\psi_i(t+s,x-y)-\psi_i(t,x)|)+|y|\}.$$

As before take $A>0$ large enough, split $\Lambda=D_A\cup D_A^c$, to get

$$|W_2|\le A\left(1+\frac{K}{\nu_0}\right)\sum_{i=1}^{2}\int_{I\!R}\Psi_{\epsilon_0}(s)\int_{2\eta}^{T}\int_\Lambda\Phi_{\epsilon_1}(y)\int_\Lambda\{(|\psi_i(t+s,x-y)-\psi_i(t,x)|)+|y|\}\,dxdydtds$$
$$+C\sum_{i=1}^{2}\int_{2\eta}^{T}\int_\Lambda\int_{I\!R}\Psi_{\epsilon_0}(s)1\!\!1_{D_A}(x)\left|\frac{\partial\psi_i}{\partial t}(t+s,x)\right|\,dsdxdt.$$

Since

$$\lim_{\epsilon_1\to 0}\int_\Lambda\Phi_{\epsilon_1}(y)|y|dy=0\,,\quad\lim_{\epsilon_0\to 0}\int_{I\!R}x\Psi_{\epsilon_0}(s)\int_\Lambda|\psi_i(t+s,x-y)-\psi_i(t,x)|\,dxds=0,$$

letting $\epsilon\to 0$ and then $A\to\infty$ we obtain $\lim_{\epsilon\to 0}|W_2|=0$. □

Proof of Lemma 7.9 We have

$$\int_\eta^{2\eta}\int_\Lambda\left[\mathcal{H}(\overline{\psi}^\epsilon,\frac{\partial\overline{\psi}^\epsilon}{\partial t})(t,x)-\mathcal{H}(\overline{\psi},\frac{\partial\overline{\psi}}{\partial t})(t,x)\right]dxdt$$
$$=\int_\eta^{2\eta}\int_\Lambda\left[\mathcal{H}(\overline{\psi}^\epsilon,\frac{\partial\overline{\psi}^\epsilon}{\partial t})(t,x)-\mathcal{H}(\overline{\psi},\frac{\partial\overline{\psi}^\epsilon}{\partial t})(t,x)\right]dxdt \tag{9.18}$$
$$+\int_\eta^{2\eta}\int_\Lambda\left[\mathcal{H}(\overline{\psi},\frac{\partial\overline{\psi}^\epsilon}{\partial t})(t,x)-\mathcal{H}(\overline{\psi},\frac{\partial\overline{\psi}}{\partial t})(t,x)\right]dxdt.$$

The first term is estimated by applying (9.2) of Lemma 9.1. We have

$$\left|\int_\eta^{2\eta}\int_\Lambda\left[\mathcal{H}(\overline{\psi}^\epsilon,\frac{\partial\overline{\psi}^\epsilon}{\partial t})(t,x)-\mathcal{H}(\overline{\psi},\frac{\partial\overline{\psi}^\epsilon}{\partial t})(t,x)\right]dxdt\right|$$
$$\le\sum_{i=1,2}\int_\eta^{2\eta}\int_\Lambda\left(1+\frac{K}{\nu_0}\right)|\frac{\partial\psi_i^\epsilon(t,x)}{\partial t}|\left(\|\overline{\psi}^\epsilon(t,\cdot)-\overline{\psi}(t,\cdot)\|_1+|\psi_i(t,x)-\psi_i^\epsilon(t,x)|\right)dxdt, \tag{9.19}$$

where $\nu_0=\min\{\delta_1,\delta_2\}$. Note that

$$\frac{\partial\psi_i^\epsilon}{\partial t}(t,\cdot)=\chi_1(t)\frac{\partial\psi_i}{\partial t}(t,\cdot)+\chi_2(t)\int_{I\!R}\Psi_{\epsilon_0}(s)(\Phi_{\epsilon_1}*(\theta_s\frac{\partial\psi_i}{\partial t})(t,\cdot)+\chi_1'(t)[\psi_i(t,\cdot)-\psi_i^\epsilon(t,\cdot)]\,ds,\quad t\in[0,T],$$

where we denote $\chi_i'(t)=\frac{d}{dt}\chi_i(t)$, for $i=1,2$, and we use that $\chi_2'(t)=-\chi_1'(t)$. Since for $t\in(0,3\eta)$, $\overline{\psi}(t)=\overline{R}(t)$ solves (2.24) we have

$$\sup_{t\in[\eta,2\eta]}\sup_{x\in\Lambda}|\frac{\partial\psi_i^\epsilon}{\partial t}(t,\cdot)|\le\frac{6}{\eta}. \tag{9.20}$$

Therefore for all η letting $\epsilon \to 0$ the term in the right hand side of (9.19) goes to zero. For the second term in the r.h.s. of (9.18), applying (9.4) in Lemma 9.1 and taking into account (9.20), we get the result. $\square$

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