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Probabilities of Majority and Minority Violation in Proportional Representation

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Abstract

Apportionment methods are used in proportional representation systems for rounding electoral vote proportions to integer numbers of seats in parliament. Many criteria are known to judge this adjustment process and to decide whether a specific method is suitable for practical use. As in general it is not possible to fulfill all desired criteria together, it is necessary to consider probabilities of violated criteria for specific apportionment methods. This letter demonstrates how to calculate the probability of such violations for the important majority and minority criteria by means of a recently developed geometric-combinatorial approach. Results are given for several popular apportionment methods.

Key words: majority criterion, minority criterion, stationary divisor methods, Hamilton method, apportionment method, geometric-combinatorial approach

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1 Introduction

In proportional representation systems the electoral vote proportions have to be mapped to specific seat allocations. The latter are integer numbers, whereas the vote proportions are almost continuous quantities by comparison. Therefore, the apportionment process involves some adjustment of the fractional numbers of seats that would arise if literal calculation were possible. In order to decide for or against a specific apportionment method several criteria are of interest (e.g. Niemeyer and Wolf, 1984, Kopfermann, 1991, Niemeyer, 1998, Heinrich and Schwingenschlögl, 2006). Amongst them the following seem to be obvious:

1.) A majority of votes implies a majority of seats in parliament.
2.) A minority of votes implies a minority of seats in parliament.

However, these criteria are violated by many popular methods.

Let the ordered probability simplex $S^\ell_{\geq}$ be the set of all ordered non-negative weight vectors $w = (w_1, \ldots, w_\ell)^t$ summing to one,

$$S^\ell_{\geq} := \left\{ w \in [0, 1]^\ell : \sum_{i=1}^\ell w_i = 1, w_1 \geq \ldots \geq w_\ell \right\}.$$  \hspace{1cm} (1)

We interpret $w_i$ as the share of votes for the $i$-th largest of the $\ell$ competing parties, $i = 1, \ldots, \ell$. For a given house size $M$, that is the number of seats to be allocated among the parties, the possible seat allocations $m = (m_1, \ldots, m_\ell)^t$ form the grid set

$$G^\ell_{\geq}(M) := \left\{ m \in \mathbb{N}_0^\ell : \sum_{i=1}^\ell m_i = M, m_1 \geq \ldots \geq m_\ell \right\}.$$  \hspace{1cm} (2)
An apportionment method \( A \) maps a weight vector \( \mathbf{w} \) into a seat allocation vector \( \mathbf{m} \),

\[
A : S^\ell_\geq \rightarrow G^\ell_\geq(M). \tag{3}
\]

Note that rounding the weights \( w_i \) individually does not generally result in a feasible apportionment method as the side condition \( \sum_{i=1}^\ell m_i = M \) is not enforced automatically (e.g. Happacher, 2001, Section 1).

In this letter we address some of the most popular apportionment methods. The \textit{q-stationary divisor methods}, with parameter \( q \in [0,1] \), allocate to the weight vector \( \mathbf{w} \) the integer vector \( A(\mathbf{w}) \) via

\[
A_i(\mathbf{w}) = r_q \left( \frac{w_i}{D} \right) \quad \text{for all } i \in \{1, \ldots, \ell\}. \tag{4}
\]

The divisor \( D \in (0, \infty) \) is chosen such that \( A(\mathbf{w}) \in G^\ell_\geq(M) \). For a real number \( x \geq 0 \), the \textit{rounding function} \( r_q(x) \) is defined to be the smallest integer larger than \( x \) if the fractional part of \( x \) is larger than \( q \), and the largest integer smaller than \( x \) otherwise. Rounding up \((q = 0)\), standard rounding \((q = 0.5)\), and rounding down \((q = 1)\) result in the apportionment methods of Adams, Webster/Sainte-Lague, and Jefferson/d’Hondt, respectively.

The \textit{quota method of greatest remainders} (Hamilton/Hare) operates in two stages. In the first stage, the proportions \( w_i M \) are rounded down to their integer parts \( \bar{m}_i \). In the unlikely case that all \( w_i M \) are integers the discrepancy \( \delta := M - \sum_{i=1}^\ell \bar{m}_i \in \mathbb{N}_0 \) vanishes and we have \( \mathbf{m} := (\bar{m}_1, \ldots, \bar{m}_\ell) \). If there is a positive discrepancy, the fractional parts \( w_i M - \bar{m}_i \) are ranked in the second stage (where ties are broken arbitrarily). Then the vector \( \mathbf{m} \) is obtained by setting \( m_i = \bar{m}_i + 1 \) for the \( \delta \) largest remainders and \( m_i = \bar{m}_i \) for the
\(\ell - \delta\) smallest remainders. A survey on apportionment methods is given in Kopfermann (1991) or Balinski and Young (2001).

## 2 Results

<table>
<thead>
<tr>
<th>method</th>
<th>(\ell)</th>
<th>(M)</th>
<th>(P_+(A, \ell))</th>
<th>(P_-(A, \ell))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamilton</td>
<td>3 even</td>
<td>(\frac{12M - 8}{8M^2})</td>
<td>(\frac{12M - 16}{8M^2})</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3 odd</td>
<td>(\frac{M - 1}{8M^2})</td>
<td>(\frac{M + 1}{8M^2})</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4 even</td>
<td>(\frac{12M^2 - 16M + 8}{8M^3})</td>
<td>(\frac{12M^2 + 16M - 48}{8M^3})</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4 odd</td>
<td>(\frac{M^2 - 2M + 1}{8M^3})</td>
<td>(\frac{M^2 + 2M - 1}{8M^3})</td>
<td></td>
</tr>
<tr>
<td>Adams</td>
<td>3 even</td>
<td>(\frac{3M - 6}{3(M - 1)(M - 2)})</td>
<td>(\frac{3(M - 2)}{3(M - 1)(M - 2)})</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3 odd</td>
<td>(\frac{M - 1}{3(M - 1)(M - 2)})</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4 even</td>
<td>(\frac{4M^2 - 22M + 28}{3(M - 1)(M - 2)(M - 3)})</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4 odd</td>
<td>(\frac{2M^2 - 8M + 6}{3(M - 1)(M - 2)(M - 3)})</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Jefferson</td>
<td>3 even</td>
<td>(\frac{M + 2}{3(M + 1)(M + 2)})</td>
<td>(\frac{3M - 2}{3(M + 1)(M + 2)})</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3 odd</td>
<td>0</td>
<td>(\frac{M + 1}{3(M + 1)(M + 2)})</td>
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<tr>
<td></td>
<td>4 even</td>
<td>(\frac{4M^2 + 22M - 20}{3(M + 1)(M + 2)(M + 3)})</td>
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<td></td>
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<tr>
<td></td>
<td>4 odd</td>
<td>0</td>
<td>(\frac{2M^2 + 8M + 6}{3(M + 1)(M + 2)(M + 3)})</td>
<td></td>
</tr>
</tbody>
</table>

Table 1

Probabilities of violated majority \(P_+(A, \ell)\) and minority \(P_-(A, \ell)\) criteria for several apportionment methods in systems of \(\ell = 3\) and \(\ell = 4\) parties.
Probabilities of majority and minority violation in proportional representation can be evaluated by means of a geometric-combinatorial approach, as discussed in Section 3. Details of the calculation are addressed in Section 4. Table 1 summarizes results for the apportionment methods introduced before, the quota method of greatest remainders and the divisor methods with rounding up and down. For the former method, parts of the findings already appear in a previous paper by Niemeyer and Wolf (1984).

The formulas of table 1 suggest that the probabilities of violated majority and minority criteria generally decay of order \(1/M\) for growing house size \(M\).

3 Geometric-combinatorial approach

In order to analyze the average behaviour of apportionment methods with respect to a variety of properties the geometric-combinatorial approach has turned out to be very successful (Schwingenschlögl et al., 2004, 2005, 2006a, 2006b, 2007a, 2007b). It is based on a decomposition of the probability simplex

\[ S^\ell := \{ w \in [0, 1]^\ell : \sum_{i=1}^\ell w_i = 1 \} \]

into rounding polytopes

\[ P(m) := \text{cl}\{ w \in S^\ell : A(w) = m \}, \]

where \(\text{cl}\) denotes set closure, and explicit knowledge about the vertices of these rounding polytopes.

Let \(0_\ell\) and \(1_\ell\) denote the row vectors in \(\mathbb{R}^\ell\) with all components equal to zero or one, respectively, and let \(R(m) := \{ i : m_i \neq 0 \}\) and \(r(m) := |R(m)|\) for \(m \in G^\ell(M) := \{ m \in \mathbb{N}_0^\ell : \sum_{i=1}^\ell m_i = M \}\). In Drton and Schwingenschlögl (2004) it is shown that the quota method of greatest remainders and the \(q\)-stationary divisor methods then give rise to rounding polytopes with \(2^\ell - 2^{\ell-r(m)} - 1\) vertices \(v(\lambda)\), induced by the vectors \(\lambda \in \{0, 1\}^\ell \setminus \{0_\ell, 1_\ell\}\) with \(\lambda_j = 0\) for
some index \( j \in R(m) \). If \( r(m) = 1 \), then \( v(0) \) is an additional vertex. For the quota method of greatest remainders the components of these vertices are, for \( i = 1, \ldots, \ell \),

\[
v_i(\lambda) = \begin{cases} 
\frac{1}{\ell} \left( m_i + 1 - \frac{e(\lambda)}{\ell - z(\lambda)} \right) & \text{if } \lambda_i = 1, \\
\frac{1}{\ell} \left( m_i - \frac{e(\lambda)}{\ell - z(\lambda)} \right) & \text{if } \lambda_i = 0 \text{ and } i \in R(m), \\
0 & \text{if } \lambda_i = 0 \text{ and } i \not\in R(m),
\end{cases}
\]

where \( z(\lambda) := |\{ i \not\in R(m) : \lambda_i = 0 \}| \) and \( e(\lambda) := |\{ 1 \leq i \leq \ell : \lambda_i = 1 \}| \). For the \( q \)-stationary divisor methods we have the components, for \( i = 1, \ldots, \ell \),

\[
v_i(\lambda) = \begin{cases} 
\frac{m_i + q}{c(\lambda)} & \text{if } \lambda_i = 1, \\
\frac{m_i + q - 1}{c(\lambda)} & \text{if } \lambda_i = 0 \text{ and } i \in R(m), \\
0 & \text{if } \lambda_i = 0 \text{ and } i \not\in R(m),
\end{cases}
\]

with the normalization \( c(\lambda) = M + \ell q - |\{ i : \lambda_i = 0 \}| \).

Average properties of apportionment methods can be determined by summation over the possible seat allocations \( m \in G_\geq^\ell (M) \), taking into account the probabilities \( P(A(w) = m| w_1 \geq \cdots \geq w_\ell ) \) of the seat allocations to be realized. For the apportionment methods under consideration the latter probabilities turn out to depend only on the number \( r \) of parties actually obtaining at least one seat, thus on the number of non-vanishing components of \( m \). The distribution of the seat allocation vector \( A(w) \) is known in detail from Drton and Schwingenschlöggl (2005, Section 2). As a consequence, it is useful to
decompose the grid set \( G^\ell_\geq (M) \) into disjoint subsets, for \( r = 1, \ldots, \ell \),

\[
K_r(M) := \left\{ m \in G^\ell_\geq (M) : m_r > 0 = m_{r+1} \right\}.
\] (7)

Summation over \( G^\ell_\geq (M) \) now reduces to the combinatorial problem of determining the so-called apportionment polynomials, for \( r = 1, \ldots, \ell \),

\[
A^{(r)}(M) := \sum_{m \in K_r(M)} \frac{1}{b_r(m)},
\] (8)

where \( b_r(m) \) denotes the number of permutations of the first \( r \) components of \( m \) leaving the seat allocation vector invariant. Explicit expressions for these polynomials can be obtained recursively in the spirit of Drton and Schwingschlägl (2004, Theorem 3).

4 Probabilities of violation

An example for a violated majority criterion is given in Figure 1, which shows for a system of \( \ell = 3 \) parties both the probability simplex and the grid set of possible seat allocations, for the divisor method with rounding up \((q = 0)\). The marked set of weight vectors represents a majority of votes for the largest party. Still, because the divisor method with rounding up leads to the seat allocation \( m = (2, 2, 1)^t \), there is no majority of seats. In the following we aim at calculating the probability that the majority or minority criterion is not fulfilled. Therefore, we introduce the set of weight vectors violating the majority criterion

\[
V_+(A, \ell) := \left\{ w \in S^\ell_\geq : w_1 > \frac{1}{2} \text{ and } A(w_1) \leq \frac{M}{2} \right\},
\] (9)
Fig. 1. Set of weight vectors $V_+(A, \ell)$ violating the majority criterion for the divisor method with rounding up ($M = 5$, $\ell = 3$).

and the set of weight vectors violating the minority criterion

$$V_-(A, \ell) := \left\{ \mathbf{w} \in S^\ell : w_1 < \frac{1}{2} \text{ and } A(w_1) \geq \frac{M}{2} \right\}.$$  \hfill (10)

Assuming a uniform distribution of the weight vectors $\mathbf{w}$ on $S^\ell$, we are interested in calculating the probabilities

$$P_+(A, \ell) := P(\mathbf{w} \in V_+(A, \ell) | w_1 \geq \ldots \geq w_\ell)$$ \hfill (11)

$$P_-(A, \ell) := P(\mathbf{w} \in V_-(A, \ell) | w_1 \geq \ldots \geq w_\ell).$$ \hfill (12)

To that end, we have to distinguish whether the house size $M$ is even or odd.

For $M$ even, under the condition that $w_1 = 1/2$ implies $A(w_1) \leq M/2$ we have

$$P_+(A, \ell) = P \left( w_1 > \frac{1}{2} \mid w_1 \geq \ldots \geq w_\ell \right)$$

$$= \sum_{\mathbf{m} \in G^\ell_+(M), \text{ } m_1 \geq \frac{M}{2}} P(A(\mathbf{w}) = \mathbf{m} | w_1 \geq \ldots \geq w_\ell),$$ \hfill (13)

and under the condition that $w_1 = 1/2$ implies $A(w_1) \geq M/2$ we have
\[
P_-(A, \ell) = \sum_{m \in G^+_5(M), \ m_1 \geq \frac{M}{2}} P(A(w) = m | w_1 \geq \ldots \geq w_\ell) - P \left( w_1 > \frac{1}{2} \mid w_1 \geq \ldots \geq w_\ell \right). \tag{14}
\]

The quota method of greatest remainders fulfills both conditions, since the definition of \( z(\lambda) \) and \( e(\lambda) \) implies \( z(\lambda) + e(\lambda) \leq \ell \), and therefore
\[
\frac{e(\lambda)}{\ell - z(\lambda)} \leq 1. \tag{15}
\]

By (5), this guarantees that a weight vector \( w \) with \( w_1 = 1/2 \) leads to a seat allocation \( m \) with \( m_1 = M/2 \).

For the \( q \)-stationary divisor methods we examine whether the first component of the vertex \( v(\lambda = (0, 1, \ldots, 1)^t) \) of the rounding polytope \( P(m = (M/2 + 1, m_2, \ldots, m_\ell)^t) \) is at least 1/2, see (6),
\[
\frac{\frac{M}{2} + 1 \cdot q - 1}{M + \ell q - 1} \geq \frac{1}{2} \iff q \leq \frac{1}{\ell - 2}. \tag{16}
\]

as well as whether the first component of the vertex \( v(\lambda = (1, 0, \ldots, 0)^t) \) of the rounding polytope \( P(m = (M/2 - 1, m_2, \ldots, m_\ell)^t) \) is at most 1/2,
\[
\frac{\frac{M}{2} - 1 \cdot q}{M + \ell q - (\ell - 1)} \leq \frac{1}{2} \iff q \geq \frac{\ell - 3}{\ell - 2}. \tag{17}
\]

For system size \( \ell = 3 \) we therefore can compute both \( P_+(A, \ell) \) and \( P_-(A, \ell) \), particularly for the divisor methods with rounding up \( (q = 0) \) and down \( (q = 1) \). For \( \ell = 4 \), one can calculate \( P_+(A, \ell) \) for the divisor method with rounding up, and \( P_-(A, \ell) \) for the divisor method with rounding down.

Turning to the case \( M \) odd, we have the relation
\[ P_x(A, \ell) - P_-(A, \ell) = P \left( w_1 > \frac{1}{2} \mid w_1 \geq \ldots \geq w_\ell \right) \]
\[ - \sum_{m \in G_x^+ (M), \quad m_1 > \frac{1}{2}} P(A(w) = m \mid w_1 \geq \ldots \geq w_\ell), \]

which we can evaluate only if \( P_+(A, \ell) = 0 \) or \( P_-(A, \ell) = 0 \); except for the quota method of greatest remainders where we have additional geometrical insight.

For the \( q \)-stationary divisor methods we examine whether the first component of the vertex \( v(\lambda = (0, 1, \ldots, 1)^\ell) \) of the rounding polytope \( P(m = ((M + 1)/2, m_2, \ldots, m_\ell)^\ell) \) is at least 1/2, see (6),
\[ \frac{M+1}{2} + q - 1 \geq \frac{1}{2} \iff q = 0, \]

as well as whether the first component of the vertex \( v(\lambda = (1, 0, \ldots, 0)^\ell) \) of the rounding polytope \( P(m = ((M - 1)/2, m_2, \ldots, m_\ell)^\ell) \) is at most 1/2,
\[ \frac{M-1}{2} + q \leq \frac{1}{2} \iff q = 1. \]

For any system size, it follows that \( P_+(A, \ell) = 0 \) for the divisor method with rounding up (\( q = 0 \)) and that \( P_+(A, \ell) = 0 \) for the divisor method with rounding down (\( q = 1 \)).

By integration we obtain
\[ P \left( w_1 > \frac{1}{2} \mid w_1 \geq \ldots \geq w_\ell \right) = \]
\[ \int_0^{\frac{1}{2}} dw_\ell \int_{w_\ell}^{1/2-w_\ell} dw_{\ell-1} \ldots \int_{w_3}^{1/2-w_3-\ldots-w_3} dw_2 \frac{1}{2} \int_0^{\frac{1}{2}} dw_\ell \int_{w_\ell}^{1/2-w_\ell} dw_{\ell-1} \ldots \int_{w_3}^{1/2-w_3-\ldots-w_3} dw_2, \]

10
which yields the value $3/4$ when $\ell = 3$, and the value $1/2$ when $\ell = 4$. Finally, the summations

$$
\sum_{m \in G^\ell_s(M),
\ \ m_1 \geq \frac{M}{2}} P(A(w) = m | w_1 \geq \ldots \geq w_\ell) \quad (22)
$$

$$
\sum_{m \in G^\ell_s(M),
\ \ m_1 \geq \frac{M}{2}} P(A(w) = m | w_1 \geq \ldots \geq w_\ell) \quad (23)
$$

are evaluated via the apportionment polynomials $A^{(r)}(M), r = 1, \ldots, \ell$, taking into account the additional condition $m_1 > \frac{M}{2}$ or $m_1 \geq \frac{M}{2}$.

References


