Statistics of geometric random simplicial complexes

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STATISTICS OF GEOMETRIC RANDOM SIMPLICIAL COMPLEXES

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Abstract. Given a Poisson process on a d-dimensional torus, its random geometric simplicial complex is the complex whose vertices are the points of the Poisson process and simplices are given by the Čech complex associated to the coverage of each point. We compute explicitly the variance of number of k-simplices as well as the variance of the Euler's characteristic. The solution strategy used to compute the second moment can be used to compute analytically the third moment and allows to stablish a conjecture for the nth moment. We apply concentration inequalities on the results of homology and the moments of the Euler's characteristics to find bounds for the for the coverage probability

1. Motivation

As technology goes on [1, 2, 3], one can expect a wide expansion of the so-called *sensor networks*. Such networks represent the next evolutionary step in building, utilities, industrial, home, agriculture, defense and many other contexts [4].

These networks are built upon a multitude of small and cheap sensors which are devices with limited transmission capabilities and power. Each sensor monitors a region around itself by measuring some environmental quantities (e.g., temperature, humidity), detecting intrusion, etc, and broadcasts its collected informations to other sensors or to a central node. The question of whether information can be shared among the whole network often is of crucial importance.

Many researches have recently been dedicated to this problem considering a variety of situations. It is possible to categorize three main scenarios: those where it is possible to choose the position of each sensor, those where sensors are arbitrarily deployed in the target region with the control of a central station and those where the sensor locations are random in a decentralized system.

The problem of the first scenario is that, in many cases, placing the sensors is impossible or implies a high cost. Sometimes this impossibility comes from the fact that the cost of placing each sensor is too large and sometimes the network has an inherent random behavior (like in the ad hoc case, where users move). In addition, this policy cannot take into account the configuration of the network in the case of failure of some sensor.

The drawback of the second scenario is a higher unity cost of sensors, since each one has to communicate with the central station. Besides, the central station itself increases the cost of the whole system. Moreover, if sensors are supposed to know their positions, an absolute positioning system has to be included in each sensor, making their hardware even more complex and then more expensive.

It is thus important to investigate the third scenario: randomly located sensors, no central station. Actually, if we can predict some characteristics of the topology of a random network, the number of sensors (or, as well, the power supply of them) can be *a priori* determined such that a given network may operate with high

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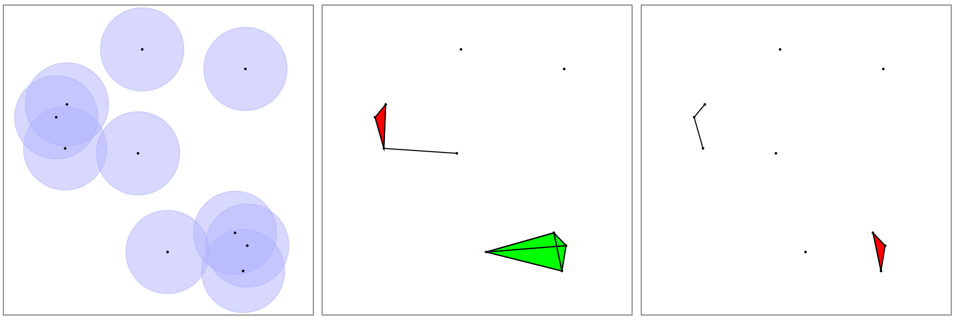


a) b) c)

FIGURE 1. a) Sensors and their coverage; b) simplicial complex representation when sensors are monitoring the region; c) simplicial complex representation when sensors are communicating amoung them.

probability. For instance, we can choose the mean number of sensors such that, if they are randomly deployed, there is more than 99% of probability the target region to be completely covered.

Usually, sensors are deployed in the plane or in the ambient space, thus mathematically speaking, one has to deal with configurations in R^2, R^3 or a mainfold. The recent works of Ghrist and his collaborators [5, 6] show how, in any dimension, algebraic topology can be used to compute the coverage of a given configuration of sensors. Trying to pursue their work for random settings, [7] has completely solved this problem in one dimension, without using, however, the sofisticated tools of algebraic topology. Due the fact that we cannot order the points in R^d, it is not possible to find as much results as found for the one-dimensional case, but the results in this work holds for any dimension d.

The principal idea of the problem is that each sensor can control some environmental information (such as temperature, pression, presence of an intruder, etc) around them. The homology of the coverage of this network, as shown in [5], can be represented by a simplicial complex. A simplicial complex is a generalization of a graph, so, while we represent a graph with points and edges, a simplicial complex can be represented by points, edges, filled triangles, filled tetraedrons and so on. Almost all the work considers the radius of monitoring ϵ but a different interpretation can be done if the sensors are communicating amoung them. In this case, we suppose that sensors have a power suply allowing them to transmit theirs ID's and, at the same time, sensors have receivers which can identify the transmitted ID's of other sensors above a threshold power. The sensors, knowing mutually the ID's of the close neighbors, are considered connected, creating an information network. The problem remains analogous as the previous one, except that we substitute the coverage radius ϵ by a communication one of $\epsilon/2$. We can see examples of simplicial complexes representations given by sensors communicating amoung them or monitoring a region in Fig. 1.

In this work we consider that sensors are points of a Poisson point processus. This assumption reflects the fact that, due the lack of control of the sensors positioning, only a random fraction of the availables sensors will actually lie in the target region or some sensors may shut down by running out of energy, moreover, the position of each sensor, a priori, does not interact with the positions other sensors. Instead of using the Euclidean norm, we use the maximum norm along this paper. We consider this for three reasons: this norm represents a superior and an inferior limits for the

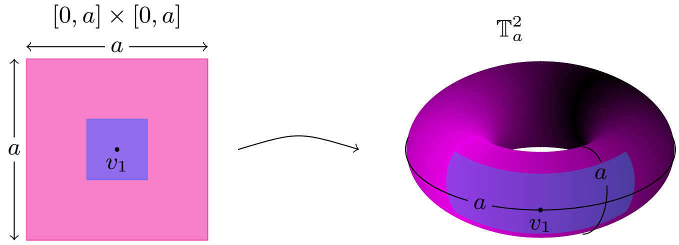


Figure 2. Ilustration of the coverage of a point and the region where points can lie, in the 2 dimensional case

euclidean norm (we can inscribe and circunscribe a circle with two squares); due to the random interactions with the environment (causing shadowing and fading), even the euclidean norm cannot capture with precision the real behavior of this kind of sensor networks, so we choose the norm that allows us to simplify the calculations; using the maximum norm, the Cech complex become equal to the Rips-Vietoris complexes, which allows us to apply directly some results of the algebraic topology. Finally, we assume that sensors lie over d-torus. We justify this choice for three reasons: it avoids the border effects, it helps to determine weather or not a sensor network in the d-box is completely covered and if ϵ is small compared to a, the calculations for all parameters in the d-torus are a good approximation for the $[0,a]^d$ box. The coverage of a point and the region where points can lie in are illustrated in Fig. 1, representing the case where a point is deployed over a plan.

One of the main results of this paper is the explicit expression for the variance of the number of k-simplices, N_k, the covariance between N_k and N_l and the variance of the Eulers's characteristic, χ, in such complex, which allows us to apply concentrations inequalities. For $d \geq 2$, χ is expressed by a power serie and if $d = 1$, it is possible to find its closed-form expression. A complex closed-form expression for the third order central moment of N_k is explicitely calculated and using the same strategy solution of this case, it is possible to find an expression that allows the computation of the moments of any order of the number of simplices.

The paper is organized in the following way: Section 2 presents the preliminaires of the topological tools used in the paper as well as the tools used from the Malliavin calculus; the calculations of the second order moments as well an application of the concentration inequlities are presented in the Section 3; in section 4, we present generalization of the procedure to find the second order moments allowing the computation of the n-th moment of the number of k-simplices.

2. Algebraic Topology

For further reading on topology, see [9, 10, 11]. Graphs can be generalized to more generic topological objects known as simplicial complexes. While graphs model binary relations, simplicial complexes represent higher order relations. Given a set of points V, a k-simplex is an unordered subset $\{v_0, v_1, \cdots, v_k\}$ where $v_i \in V$ and $v_i \neq v_j$ for all $i \neq j$. The faces of the k-simplex $\{v_0, v_1, \cdots, v_k\}$ are defined as all the $(k-1)$-simplices of the form $\{v_0, \cdots, v_{j-1}, v_{j+1}, \cdots, v_k\}$ with $0 \leq j \leq k$. A simplicial complex is a collection of simplices which is closed with respect to the inclusion of faces, i.e., if $\{v_0, v_1, \cdots, v_k\}$ is a k-simplex then all its faces are in the set of $(k-1)$-simplices.

Given $\mathcal{U} = (U_v, v \in \mathfrak{T})$ a collection of open sets, the Čech complex of $\mathcal{U}$ denoted by $\mathcal{C}(\mathcal{U})$, is the abstract simplicial complex whose k-simplices correspond to $(k+1)$-tuples of distinct elements of $\mathcal{U}$ that have non empty intersection, so $\{v_0, v_1, \cdots, v_k\}$ is a k-simplex if and only if $\bigcap_{i=0}^k U_{v_k} \neq \emptyset$.

One can define an orientation for a simplicial complex by defining an order on vertices. A change in the orientation corresponds to a change in the sign of the coefficient as

$$[v_0, \cdots, v_i, \cdots, v_j, \cdots, v_k] = -[v_0, \cdots, v_j, \cdots, v_i, \cdots, v_k].$$

For each integer k, $C_k(X)$ is the vector space spanned by the set of oriented k-simplices of X. The boundary map ∂_k is defined to be the linear transformations $\partial_k : C_k \to C_{k-1}$ which acts on basis elements $[v_0, \cdots, v_k]$ via

$$\partial_k[v_0, \cdots, v_k] = \sum_{i=0}^{k} (-1)^k [v_0, \cdots, v_{i-1}, v_{i+1}, \cdots, v_k].$$

Examples of such operations are given in Fig. 3.

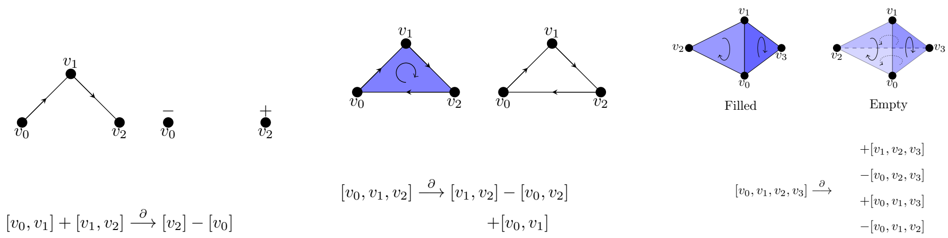


FIGURE 3. Examples of boundary maps. From left to right. An application over 1-simplices. Over a 2-simplex. Over a 3-simplex, turning a filled tetrahedron to an empty one.

This map gives rise to a chain complex: a sequence of vector spaces and linear transformations

$$\cdots \xrightarrow{\partial_{k+2}} C_{k+1}(X) \xrightarrow{\partial_{k+1}} C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X).$$

A standard result then asserts that for any integer k,

$$\partial_k \circ \partial_{k+1} = 0.$$

If one defines

$$Z_k = \ker \partial_k \text{ and } B_k = \mathrm{im}\partial_{k+1},$$

this induces that $B_k \subset Z_k$.

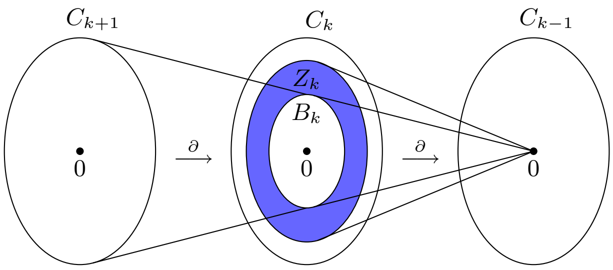


FIGURE 4. A chain complex showing the sets C_k, Z_k and B_k.

The k-dimensional homology of X, denoted $H_k(X)$ is the quotient vector space,

$$H_k(X) = \frac{Z_k(X)}{B_k(X)}.$$

and the k-th Betti number of X is its dimension:

$$\beta_k = \dim H_k = \dim Z_k - \dim B_k.$$

Let s_k be the number of k-simplices in a simplicial complex X. The well known topological invariant named Euler characteristic for X, denoted by $\chi(X)$, is an integer defined by:

$$\chi(X) = \sum_{i=0}^{\infty}(-1)^i \beta_i.$$

A well known theorem states that this is also given by:

$$\chi(X) = \sum_{i=0}^{\infty}(-1)^i s_i.$$

The simplicial complexes we consider are of a special type. They can be considered as a generalization of geometric random graphs.

Definition 1. *Given ω a finite set of points on the torus. For $\epsilon > 0$, we define $\mathcal{U}_\epsilon(\omega) = \{B_{d_\infty}(v, \epsilon), v \in \omega\}$ and $\mathcal{C}_\epsilon(\omega) = \mathcal{C}(\mathcal{U}_\epsilon(\omega))$, where $B_{d_\infty}(x, r) = \{y \in \mathbb{T}_d^a, \|x-y\|_\infty < r\}$.*

Theorem 1. *Suppose $\epsilon < a/4$. Then $\mathcal{C}_\epsilon(\omega)$ has the same homotopy type as $\mathcal{U}_\epsilon(\omega)$. In particular they have the same Betti numbers.*

Proof. This will follow from the so-called nerve lemma of Leray, as stated in [12, Theorem 7.26] or [13, Theorem 10.7]. One only needs to check that any non-empty intersection of sets $B_{d_\infty}(v, \epsilon)$ is contractible.

Consider such a non-empty intersection, and let x be a point contained in it. Then, since $\epsilon < a/4$, the ball $B_{d_\infty}(x, 4\epsilon)$ can be identified with a cube in the Euclidean space. Then each $B_{d_\infty}(v, \epsilon)$ containing x is contained in $B_{d_\infty}(x, 4\epsilon)$, hence also becomes a cube with this identification, hence convex. Then the intersection of these convex sets is convex, hence contractible. □

Definition 2. *Let ω be a finite set of points in $\mathbb{T}_d^a$. For any $\epsilon > 0$, the Rips-Vietoris complex of ω, $\mathcal{R}_\epsilon(\omega)$, is the abstract complex whose k-simplices correspond to unordered $(k+1)$-tuples of points in ω which are pairwise within distance less than ϵ of each other.*

Lemma 2. *For the torus $\mathbb{T}_a^d$ equipped with the product distance d_∞, $\mathcal{R}_\epsilon(\omega)$ has the homotopy type of the Čech complex $\mathcal{C}_{2\epsilon}(\omega)$*

The proof is given in [5] in a slightly different context, but it is easy to check that it works here as well. It must be pointed out that Čech and Rips-Vietoris simplicial complexes can be defined similarly for any distance on $\mathbb{T}_a^d$ but it is only for the product distance that the homotopy type of both complexes coincides.

By Lemma 2, k points are forming a $(k-1)$-simplex whenever they are two-by-two closer than 2ϵ from each other. We define along the paper $h(v_1, \cdots, v_k)$ as

$$\begin{aligned} h(v_1, \cdots, v_k) &= h_k(v_1, \cdots, v_k) \\ &= \prod_{1 \le i < j \le k} \mathbf{1}_{[\|v_i - v_j\| < 2\epsilon]}, \end{aligned}$$

which determines if a set of k distinct ordered points generates a $(k-1)$-simplex.

Proposition 3. *Let $\omega \in \mathbb{T}_a^d$ be a set of points, generating the simplicial complex $\mathcal{C}_\epsilon(\omega)$. Then, if $i > d$, $\beta_i(\omega) = 0$.*

Proof. By Theorem 1, $\mathcal{C}_\epsilon(\omega)$ has the same homology as $\mathcal{U}_\epsilon(\omega)$. But $\mathcal{U}_\epsilon(\omega)$ is an open manifold of dimension d, so its Betti numbers $\beta_i(\omega)$ vanish for $i > d$, see for example [14, Theorem 22.24]. □

Proposition 4. *Let $\omega \in \mathbb{T}_a^d$ be a set of points, generating the simplicial complex $\mathcal{C}_\epsilon(\omega)$. There are only two possible values for the d-th Betti number of $\mathcal{C}_\epsilon(\omega)$:*
i) $\beta_d = 0$, or
ii) $\beta_d = 1$.
If the second holds, then we also have $\chi(\mathcal{C}_\epsilon(\omega)) = 0$.

Proof. By Theorem 1, $\mathcal{C}_\epsilon(\omega)$ has the same homology as $\mathcal{U}_\epsilon(\omega)$. Now, $\mathcal{U}_\epsilon(\omega)$ is an open submanifold of the torus, so there are only two possibilities:
i) $\mathcal{U}_\epsilon(\omega)$ is a strict open submanifold, hence non-compact
ii) $\mathcal{U}_\epsilon(\omega) = \mathbb{T}_a^d$.
In the first case, $\beta_d(\omega) = 0$ by [14, Corollary 22.25]. In the second case $\mathcal{C}_\epsilon(\omega)$ has same homology as the torus, hence $\beta_d(\omega) = 1$ and $\chi(\omega) = 0$. □

2.1. **Application to sensor networks.** We now interpret the topological properties of simplicial complexes in terms of connectivity and coverage. In terms of coverage in a network, a 0-simplex represents a single sensor and the existence of a k-simplex means that the $(k+1)$ points of this simplex are covering the convex hull containing those points. We can see in Figure 1, examples of some simplices and their interpretation in terms of sensor networks.

In a very intuitive fashion, the number of k-simplices itself shows some tendency in the network: if in two networks with identical number of sensors, one of them has more 1-simplices than the other, this first one has a tendency to be more connected; by the same reason, if a network has more 2-simplices than another one, the region on the first case tends to be more strongly covered.

In a more sophisticated way, Theorem 1 formalizes that, in order to determine coverage of sensors, it suffices interpret them as Čech complexes. Unfortunately, a moment of thought shows that constructing the Čech complex cannot be done by pairwise only communications between sensors. Thus, the only complex that can be computed this way is the Rips-Vietoris complex.

An interpretation to Euler characteristic is given by Proposition 4, where we see that $\chi = 0$ is a necessary condition to have a complete coverage of the torus, and $\beta_d = 1$ is a necessary and sufficient condition. This could in turn translate into conditions for coverage in $[0, a]^d$ when considered as embedded in Euclidean space (i.e., not as a torus), but then one needs to be careful about border effects. For example, one can say that $\beta_d = 1$ is a sufficient condition for coverage of $[\epsilon, a - \epsilon]^d$.

3. Stochastic Model

3.1. **Poisson point process.** To characterize the randomness of the system, we consider that the set of points is represented by a Poisson point process ω with intensity λ in a Polish space Y. The space of configurations on Y, is the set of locally finite simple point measures (cf [15]):

$$\Omega^Y = \left\{ \omega = \sum_{k=0}^{n} \delta(x_k) \ : \ (x_k)_{k=0}^{k=n} \subset Y, \ n \in \mathbb{N} \cup \{\infty\} \right\},$$

where $\delta(x)$ denotes the Dirac measure at $x \in Y$. Simple measure means that $\omega(\{x\}) \leq 1$ for any $x \in Y$. Locally finite means that $\omega(K) < \infty$ for any compact K

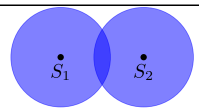
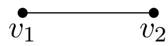
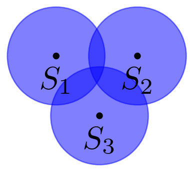
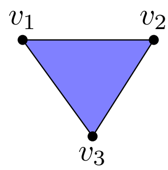
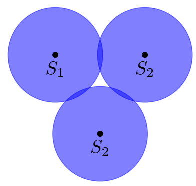
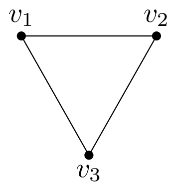
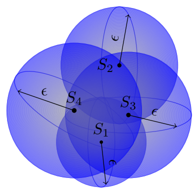
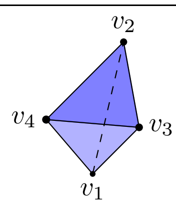
Sensor network coverage	Čech complex representation
	
	
	
	
	

TABLE 1. Topological representation of the coverage of a sensor network. Each node v represents a sensor. From top to bottom, the highest order simplex is a vertex, an edge, a triangle, three edges, a tetrahedron.

of Y. It is often convenient to identify an element ω of Ω^Y with the set corresponding to its support, i.e., $\sum_{k=0}^n \delta(x_k)$ is identified with the unordered set $\{x_1, \cdots, x_n\}$. For $A \in \mathcal{B}(Y)$, we have $\delta(x_k)(A) = \mathbf{1}_{[x_k \in A]}$, so

$$\omega(A) = \sum_{x_k \in \omega} \mathbf{1}_{[x_k \in A]} = \int_A \mathrm{d}\omega(x),$$

counts the number of atoms in A. The configuration space Ω^Y is endowed with the vague topology and its associated σ-algebra denoted by $\mathcal{F}^Y$. Since ω is a Poisson point process of intensity λ:

i) For any A, $\omega(A)$ is a random variable of parameter $\lambda S(A)$, i.e.,

$$\mathbf{P}(\omega(A) = k) = e^{-\lambda S(A)} \frac{(\lambda S(A))^k}{k!}.$$

ii) For $A' \in \mathcal{B}(Y)$, for any disjoints A, A', the random variables $\omega(A)$ and $\omega(A')$ are independent.

Define $\Delta_n = \{(x_1, \cdots, x_n) \in Y^n \mid x_i \neq x_j, \forall i \neq j\}$. Let $f(x_1, \cdots, x_n)$ be a measurable function and let $F(\omega)$ be a random variable given by

$$F(\omega) = \sum_{\substack{x_i \in \omega \cap A, 1 \leq i \leq n \\ x_i \neq x_j \ if \ i \neq j}} f(x_1, \cdots, x_n) = \int_{A \cap \Delta_n} f(x_1, \cdots, x_n) \, \mathrm{d}\omega(x_1) \cdots \, \mathrm{d}\omega(x_n),$$

A well known property of the Poisson point processes [16] states that

$$\mathbf{E}_\lambda [F(\omega)] = \int_A f(x_1, \cdots, x_n) \, \mathrm{d}\lambda(x_1) \cdots \, \mathrm{d}\lambda(x_n).$$

A real function $f : Y^n \to \mathbb{R}$ is called symmetric if

$$f(x_{\sigma(1)}, \cdots, x_{\sigma(n)}) = f(x_1, \cdots, x_n)$$

for all permutations σ of $\mathfrak{S}_n$.The space of symmetric square integrable random variables is denoted by $L^2(\lambda)^{\circ n}$. For $f \in L^2(\lambda)^{\circ n}$, the multiple Poisson stochastic integral $I_n(f_n)$ is then defined as

$$I_n(f_n)(\omega) = \int_{\Delta^n} f_n(x_1, \cdots, x_n)(\, \mathrm{d}\omega(x_1) - \, \mathrm{d}\lambda(x_1)) \cdots (\, \mathrm{d}\omega(x_n) - \, \mathrm{d}\lambda(x_n)).$$

If $f_n \in L^2(\lambda)^{\circ n}$ and $g_m \in L^2(\lambda)^{\circ m}$, the isometry formula

$$\mathbf{E}_\lambda [I_n(f_n) I_m(g_m)] = n! \mathbf{1}_{[m=n]} \langle f_n, g_m \rangle_{L^2(\lambda)^{\circ n}} \tag{1}$$

holds true (see [15]). Furthermore, we have:

Theorem 5. *Every random variable $F \in L^2(\Omega^Y, \mathbf{P})$ admits a (unique) Wiener-Poisson decomposition of the type*

$$F = \mathbf{E}_\lambda [F] + \sum_{n=1}^{\infty} I_n(f_n),$$

where the series converges in $L^2(\mathbf{P})$ and, for each $n \geq 1$, the kernel f_n is an element of $L^2(\lambda)^{\circ n}$. Moreover, we have the isometry

$$\| F - \mathbf{E}_\lambda [F] \|^2_{L^2(\lambda)^{\circ n}} = \sum_{n=1}^{\infty} n! \| f_n \|^2_{L^2(\mathbb{R}_+)^{\circ n}}. \tag{2}$$

For $f_n \in L^2(\lambda)^{\circ n}$ and $g_m \in L^2(\lambda)^{\circ m}$, we define $f_n \otimes_k^l g_m$, $0 \leq l \leq k$, to be the function:

$$(y_{l+1}, \cdots, y_n, x_{k+1}, \cdots, x_m) \longmapsto \int_{Y^l} f_n(y_1, \cdots, y_n) g_m(y_1, \cdots, y_k, x_{k+1}, \cdots, x_m) \, \mathrm{d}\lambda(y_1) \ldots \, \mathrm{d}\lambda(y_l). \tag{3}$$

We denote by $f_n \circ_k^l g_m$ the symmetrization in $n + m - k - l$ variables of $f_n \otimes_k^l g_m$, $0 \leq l \leq k$. This leads us to the next proposition, shown in [15]:

Proposition 6. *For $f_n \in L^2(\lambda)^{\circ n}$ and $g_m \in L^2(\lambda)^{\circ m}$, we have*

$$I_n(f_n) I_m(g_m) = \sum_{s=0}^{2(n \wedge m)} I_{n+m-s}(h_{n,m,s}),$$

where

$$h_{n,m,s} = \sum_{s \leq 2i \leq 2(s \wedge n \wedge m)} i! \binom{n}{i} \binom{m}{i} \binom{i}{s-i} f_n \circ_i^{s-i} g_m$$

belongs to $L^2(\lambda)^{\circ n+m-s}$, $0 \leq s \leq 2(m \wedge n)$.

In what follows, given $f \in L^2(\lambda)^{\circ q}$ $(q \geq 2)$ and $t \in Y$, we denote by $f(*, t)$ the function on Y^{q-1} given by $(x_1, \cdots, x_{q-1}) \longmapsto f(x_1, \cdots, x_{q-1}, t)$.

Definition 3. *Let* $\operatorname{Dom} D$ *be the the set of random variables* $F \in L^2(P)$ *admitting a chaotic decomposition such that*

$$\sum_{n=1}^{\infty} qq!\|f_n\|^2 < \infty.$$

Let D *be defined by*

$$\begin{aligned} D \ :\ \operatorname{Dom} D &\to L^2(\Omega^Y \times Y, P \times \lambda) \\ F = \mathbf{E}_\lambda[F] + \sum_{n\geq 1} I_n(f_n) &\longmapsto D_t F = \sum_{n\geq 1} nI_{n-1}(f_n(*, t)). \end{aligned}$$

It is known, cf. [17]*, that we also have*

$$D_t F(\omega) = F(\omega \cup \{t\}) - F(\omega),\ dP \times dt \text{ a.e..}$$

Definition 4. *The Ornstein-Uhlenbeck operator* L *is given by*

$$LF = -\sum_{n=1}^{\infty} nI_n(f_n),$$

whenever $F \in \operatorname{Dom} L$*, given by those* $F \in L^2P$ *such that their chaotic expansion verifies*

$$\sum_{n=1}^{\infty} q^2 q!\|f_n\|^2 < \infty.$$

Note that $\mathbf{E}_\lambda[LF] = 0$*, by definition and* (1).

Definition 5. *For* $F \in L^2(\mathbf{P})$ *such that* $\mathbf{E}_\lambda[F] = 0$*, we may define* L^{-1} *by*

$$L^{-1}F = -\sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n).$$

Combining Stein's method and Malliavin calculus yields the following theorem, see [8]:

Theorem 7. *Let* $F \in \operatorname{Dom} D$ *be such that* $\mathbf{E}_\lambda[F] = 0$ *and* $Var(F) = 1$*. Then,*

$$\begin{aligned} d_W(F, \mathcal{N}(0,1)) \leq \mathbf{E}_\lambda &\left[\left| 1 - \int_Y [D_t F \times D_t L^{-1} F] \ d\lambda(t) \right| \right] \\ &+ \int_Y \mathbf{E}_\lambda \left[|D_t F|^2 \left| D_t L^{-1} F \right| \right] \ d\lambda(t). \end{aligned}$$

Another result from the Malliavin calculus used in this work is the following one, quoted from [15]:

Theorem 8. *Let* $F \in \operatorname{Dom} D$ *be such that* $DF \leq K$*, a.s., for some* $K \geq 0$ *and* $\|DF\|_{L^\infty(\Omega, L^2(Y))} < \infty$*. Then*

$$\mathbf{P}(F - \mathbf{E}_\lambda[F] \geq x) \leq \exp\left(-\frac{x}{2K} \log\left(1 + \frac{xK}{\|DF\|_{L^\infty(\Omega, L^2(Y))}} \right) \right). \tag{4}$$

Proposition 9. *Let* X *a compact subset of* $\mathbf{R}^d$ *and consider the map* $\tau \ :\ X \to Y$ *as* $x_i = ky_i$ *for* $x_i \in X$*,* $y_i \in Y$ *and* k *a positive real constant. Denote by* $\tau_*\omega$ *the image measure of* ω *by* τ*, i.e.,* $\tau_* \ :\ \Omega^X \to \Omega^Y$ *maps*

$$\omega = \sum_{i=1}^{\infty} \delta(x_i) \quad \text{to} \quad \tau_* \sum_{i=1}^{\infty} \delta(kx_i).$$

The application $\tau_* \ :\ \Omega^X \to \Omega^Y$ *maps the Poisson measure* λ *on* Ω^X *to the Poisson measure* $\lambda_\tau = \lambda / k^d$ *on* Ω^Y*. Moreover, if* ϵ_τ *is the distance in* Y *such that two* *points will be connected, the homology of the two simplicial complexes* $\mathcal{C}_\epsilon(\omega)_{\omega\in\mathbb{T}^d_{[a]}}$ *and* $\mathcal{C}_{\epsilon_\tau}(\tau_*\omega)_{\tau_*\omega\in\mathbb{T}^d_{[ak]}}$ *are the same for any* k *if* $\lambda_\tau = \lambda/k^d$ *and* $\epsilon_\tau = k\epsilon$.

Proof. A slightly changing on Propositions 6.1.7 and 6.1.8 of [15] is enough to show that τ_* maps the Poisson measure λ on Ω^X to the Poisson measure $\lambda_\tau = \lambda/k^d$ on Ω^Y. Then, it suffices to realize that for $x_i \in X$ and for $y_i \in Y$:

$$\begin{aligned} h(x_1,\cdots,x_k) &= \prod_{1\le i<j\le k} \mathbf{1}_{[\|x_i-x_j\|<2\epsilon]} \\ &= \prod_{1\le i<j\le k} \mathbf{1}_{[\|kx_i-kx_j\|<2k\epsilon]}, \end{aligned}$$

hence

$$h(y_1,\cdots,y_k) = \prod_{1\le i<j\le k} \mathbf{1}_{[\|y_i-y_j\|<2\epsilon_\tau]},$$

which concludes the proof. □

4. Second order moments

4.1. **Number of k-simplices.** We use all the definitions of the previous section.. The number of $(k-1)$-simplices can be counted by the expression:

$$N_k = \sum_{\substack{v_1,\dots,v_k\in\omega \\ v_i\neq v_j \text{ if } i\neq j}} \frac{h(v_1,\cdots,v_k)}{k!} = \int_{\Delta^k} \frac{h(v_1,\cdots,v_k)}{k!} d\omega(x_1)\cdots d\omega(x_k) \tag{5}$$

Lemma 10. *We can rewrite* N_k *as*

$$N_k = \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\lambda^{k-i} I_i\left(\int_{(\mathbb{T}^d_a)^i} h(x_1,\dots,x_k)\ dx_1\dots\ dx_{k-i}\right).$$

Proof. We have that

$$\int_{\Delta^k} h(x_1,\cdots,x_k)(\ \mathrm{d}\omega(x_1)-\lambda\ \mathrm{d}x_1)\dots(\ \mathrm{d}\omega(x_i)-\lambda\ \mathrm{d}x_i)\lambda\ \mathrm{d}x_{i+1}\dots\lambda\ \mathrm{d}x_k$$
$$= \sum_{j=0}^{i}(-1)^j\binom{i}{j}\int_{\Delta^k} h(x_1,\cdots,x_k)\ \mathrm{d}\omega(x_1)\dots\ \mathrm{d}\omega(x_j)\lambda\ \mathrm{d}x_{j+1}\dots\lambda\ \mathrm{d}x_k.$$

Thus, after some algebrism with the binomial factors, we have

$$\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\int_{\Delta^k} h(x_1,\cdots,x_k)\ \mathrm{d}\omega(x_1)\dots\ \mathrm{d}\omega(x_j)\lambda\ \mathrm{d}x_{j+1}\dots\lambda\ \mathrm{d}x_k$$
$$= k!\int_{\Delta^k} h(x_1,\cdots,x_k)\ \mathrm{d}\omega(x_1)\dots\ \mathrm{d}\omega(x_k) = N_k,$$

concluding the proof. □

Definition 6. *Let* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *be two simplices with common vertices. For* $L \in \mathcal{P}(\{1,2\})$, *let us denote* m_L *the number of vertices belonging exactly to the list* L *of simplices.*

Then $M = m_{12}+m_1+m_2$ *is the total number of vertices and* $\mathcal{J}_2$ *represents the integral on these two simplices:*

$$\mathcal{J}_2(m_{12},m_1,m_2) = \int_{\Delta^{m_{12}+m_1}}\int_{\Delta^{m_{12}+m_2}} h_{m_{12}+m_1}h_{m_{12}+m_2}\ dx_1\dots\ dx_M.$$

with $x_1,\cdots,x_M$ *being the* M *vertices.*

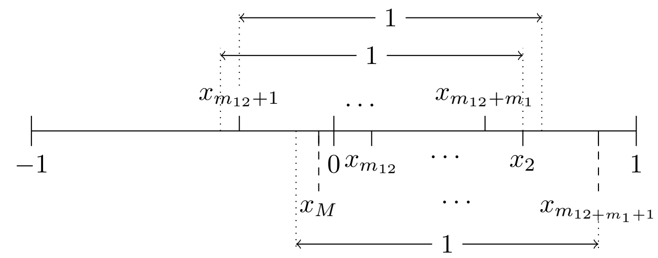


FIGURE 5. Example of relative positions of the points

Lemma 11. *For $d = 1$ and $\epsilon = 1/2$, we have*

$$\mathcal{J}_2(m_{12}, m_1, m_2) = m_{12} + m_1 + m_2 + \frac{2m_1m_2}{m_{12}+1}. \tag{6}$$

Proof. Let us split the integration domain of $\mathcal{J}_2$ in two domains S_1 and S_2 corresponding to the cases:

(1) All the vertices are connected with each other, thus there is only one simplex. The integral on S_1 is simply the number of points in the simplex: $M = m_{12} + m_1 + m_2$.
(2) There are at least two vertices at distance $d > 1$, which leads to two simplices. By symetry we can choose to order the m_L vertices for each $L \in \mathcal{P}(\{1,2\})$ from lowest to greatest or the opposite and choose which simplex is on which side of the axis. Thus we have the integral on S_2 equal to $2m_{12}!m_1!m_2!\mathcal{A}$, with $\mathcal{A}$ an integral whose calculation is detailed below.

We choose to enumerate the vertices of the simplexes such that:

- $x_1, \cdots, x_{m_{12}}$ are the m_{12} common vertices.
- $x_{m_{12}+1}, \cdots, x_{m_{12}+m_1}$ are the m_1 vertices of only $\mathcal{C}_1$.
- $x_{m_{12}+m_1+1}, \cdots, x_M$ are the m_2 vertices of only $\mathcal{C}_2$.

Without loss of generality we can choose the origin to be x_1. The vertices are now order as described in Fig. 5:

$$\begin{gathered} 0 \le x_{m_{12}} \le x_{m_{12}-1} \le \cdots \le x_2 \le 1, \\ -1 \le x_2 - 1 \le x_{m_{12}+1} \le x_{m_{12}+2} \le \cdots \le x_{m_{12}+m_1} \le x_{m_{12}+1} + 1, \\ x_{m_{12}+1} \le x_{m_{12}+m_1+1} - 1 \le x_M \le x_{M-1} \le \cdots \le x_{m_{12}+m_1+1} \le 1, \end{gathered}$$

Let us denote $J_a(f)(x) = \int_a^x f(u)\,\mathrm{d}u$ then we write the composition $J_a^{(2)}(f)(x) = \int_a^x \int_a^u f(v)\,\mathrm{d}v\,\mathrm{d}u$. We also denote $m = m_{12} + 1$ and $n = m_{12} + m_1 + 1$, then we have:

$$\mathcal{A} = \int_0^1 J_0^{(m_{12}-2)}(\mathbf{1})(x_2) \int_{x_2-1}^0 -J_{x_m+1}^{(m_1-1)}(\mathbf{1})(x_m) \int_{x_m+1}^1 J_{x_n-1}^{(m_2-1)}(\mathbf{1})(x_n)\,\mathrm{d}x_n\,\mathrm{d}x_m\,\mathrm{d}x_2.$$

We easily find that:

$$\begin{aligned} J_0^{(m_{12}-2)}(\mathbf{1})(x_2) &= \frac{x_2^{m_{12}-2}}{(m_{12}-2)!}, \\ -J_{x_m+1}^{(m_1-1)}(\mathbf{1})(x_m) &= \frac{1}{(m_1-1)!}, \\ J_{x_n-1}^{(m_2-1)}(\mathbf{1})(x_n) &= \frac{1}{(m_2-1)!}. \end{aligned}$$

Thus we have:

$$
\begin{aligned}
A &= \frac{1}{(m_{12}-2)!(m_1-1)!(m_2-1)!}\int_0^1 x_2^{m_{12}-2}\int_{x_2-1}^0 -x_m \,\mathrm{d}x_m\,\mathrm{d}x_2 \\
&= \frac{1}{(m_1-1)!(m_2-1)!(m_{12}+1)!},
\end{aligned}
$$

concluding the proof. □

Theorem 12. *Let $\epsilon \le a/6$. Then, the covariance between the number of $(k-1)$-simplices, N_k, and the number of $(l-1)$-simplices, N_l, for $l \le k$ is given by*

$$
\begin{aligned}
&Cov_\lambda [N_k, N_l] \\
&= \sum_{i=0}^{l-1} \frac{1}{i!(k-l+i)!(l-i)!}(\lambda(2\epsilon)^d)^{k+i}\left(\frac{a}{2\epsilon}\right)^d\left(k+i+2\frac{i(k-l+i)}{l-i+1}\right)^d.
\end{aligned}
\tag{7}
$$

Proof. We want to evaluate $\mathbf{E}_\lambda[(N_k-\mathbf{E}_\lambda[N_k])(N_l-\mathbf{E}_\lambda[N_l])]$. By Lemma 10, this can be written as

$$
\mathbf{E}_\lambda\left[\frac{1}{k!}\sum_{i=1}^{k}\binom{k}{i}\lambda^{k-i}I_i\left(f_i^k\right)\frac{1}{l!}\sum_{i=1}^{l}\binom{l}{i}\lambda^{l-i}I_i\left(f_i^l\right)\right],
$$

where

$$
f_j^n = \int_{(\mathbb{T}_a^d)^j} h(v_1,\cdots,v_n)\,\mathrm{d}v_1\ldots\,\mathrm{d}v_{n-j}.
$$

Using the isometry formula, given by Eq. (1), we have

$$
\begin{aligned}
\mathrm{Cov}_\lambda[N_k,N_l] &= \frac{1}{k!l!}\sum_{i=1}^{l}\binom{k}{i}\binom{l}{i}\lambda^{k+l-2i}\mathbf{E}_\lambda\left[I_i\left(f_i^k\right)I_i\left(f_i^l\right)\right] \\
&= \frac{1}{k!l!}\sum_{i=1}^{l}\binom{k}{i}\binom{l}{i}\lambda^{k+l-2i}i!\langle f_i^k f_i^l\rangle_{L^2(\lambda)^{\circ i}} \\
&= \sum_{i=0}^{l-1}\frac{1}{i!(k-l+i)!(l-i)!}\lambda^{k-l+2i}\langle f_{l-i}^k f_{l-i}^l\rangle_{L^2(\lambda)^{\circ(l-i)}}.
\end{aligned}
\tag{8}
$$

Hence, we are reduced to compute

$$
\begin{aligned}
\langle f_j^k f_j^l\rangle_{L^2(\lambda)^{\circ(j)}} = \int_{(\mathbb{T}_a^d)^j}\Bigg(&\int_{(\mathbb{T}_a^d)^{l-j}} h(v_1,\cdots,v_l)\,\mathrm{d}v_{j+1}\ldots\,\mathrm{d}v_l \\
&\int_{(\mathbb{T}_a^d)^{k-j}} h(v_1,\cdots,v_k)\,\mathrm{d}v_{j+1}\ldots\,\mathrm{d}v_k\Bigg)\lambda\,\mathrm{d}v_1\ldots\lambda\,\mathrm{d}v_j.
\end{aligned}
$$

Since $a > \epsilon/6$,the integration region is convex (see Fig. 6) and we have

$$
\begin{aligned}
\langle f_j^k f_j^l\rangle_{L^2(\lambda)^{\circ(j)}} = &\int_{[0,a]^d}\lambda\,\mathrm{d}v_1\int_{([0,a]^d)^{k-1}} h(0,v_2,\cdots,v_k) \\
&\times h(0,v_2,\cdots,v_j,v_1',\cdots,v_{l-j}')\,\mathrm{d}v_{l-j}'\ldots\,\mathrm{d}v_1'\,\mathrm{d}v_k\ldots\,\mathrm{d}v_{j+1}\lambda\,\mathrm{d}v_j\ldots\lambda\,\mathrm{d}v_2.
\end{aligned}
$$

Moreover, if $v_i = (u_{i,1},\cdots,u_{i,d})$ and $v_i' = (u_{i,1}',\cdots,u_{i,d}')$ and we proceed to the following substitutions:

$$
\begin{aligned}
u_{i,1} &= 2\epsilon x_i \text{ if } 2 \le i \le j, \\
u_{i,1} &= 2\epsilon y_{k-j} \text{ if } j+1 \le i \le k, \\
u_{i,1}' &= 2\epsilon z_i \text{ if } 1 \le i \le l-j,
\end{aligned}
$$

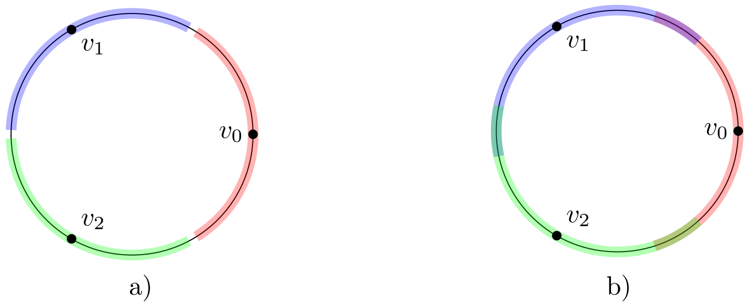


FIGURE 6. a) Maximum cover in $\mathbb{T}_a$ and $\epsilon = a/6$. The red region shows the cover of a point v_0, the blue region is the cover of v_1 and the greer region is the cover of v_2. b) Maximum cover in the same conditions of a) when $\epsilon = a/5$. In this case, we the three covers intersect each other pairwise, but there is no intersection of the three covers.

This results in a Jacobian $(2\epsilon)^{k+l-2i-1}$ and we recognize the integral to be exactely $\mathcal{J}_2(j, k-j, l-j)$ as defined in Definition 6. Thus, we have:

$$\langle f_j^k f_j^l \rangle_{L^2(\lambda)^{\circ(j)}} = \lambda^i a^d (2\epsilon)^{k+l-2i-1} \left(\mathcal{J}_2(j, k-j, l-j)\right)^d.$$

Finally, using Eq. (6) and Eq. (8) gives the result. $\square$

Remark. We remark that the possibility of writing $\mathrm{Var}(N_k)$ as Eq. (7) is due the fact that we use the maximum norm. This simplifies the calculations since we can treat each component individually. However, considering the Euclidean norm it is still possible to find analytically a closed-form expression for $\mathrm{Var}(N_k)$, but its calculation involves nasty integrals and a generic term cannot be found. When we consider the Rips-Vietoris complex in $\mathbb{T}_a^2$, the variance of the number of 1-simplices and 2-simplices are given by:

$$\mathbf{V}_\lambda\left[N_2\right] = \left(\frac{a}{2\epsilon}\right)^2 \left(\frac{\pi}{2}(4\lambda\epsilon^2)^2 + \pi^2(4\lambda\epsilon^2)^3\right),$$

and

$$\mathbf{V}_\lambda\left[N_3\right] = \left(\frac{a}{2\epsilon}\right)^2 \left((4\lambda\epsilon)^3\frac{\pi}{6}\left(\pi - \frac{3\sqrt{3}}{4}\right) + (4\lambda\epsilon^2)^4\pi\left(\frac{\pi^2}{2} - \frac{5}{12} - \frac{\pi\sqrt{3}}{2}\right)\right.$$
$$\left. + (4\lambda\epsilon^2)^5\frac{\pi^2}{4}\left(\pi - \frac{3\sqrt{3}}{4}\right)^2\right).$$

4.2. **Euler's characteristic.** Since we have an expression for the variance of the number of k-simplices, it is possible to calculate one for the Euler characteristic.

Theorem 13. *Let $\epsilon \leq a/6$. Then, the variance of the Euler characteristic in a d torus is:*

$$\mathbf{V}_\lambda\left[\chi\right] = \left(\frac{a}{2\epsilon}\right)^d \sum_{n=1}^{\infty} c_n^d (\lambda(2\epsilon)^d)^n,$$

where

$$c_n^d = \sum_{j=\lceil (n+1)/2 \rceil}^{n} \left[2 \sum_{i=n-j+1}^{j} \frac{(-1)^{i+j}}{(n-j)!(n-i)!(i+j-n)!} \left(n + \frac{2(n-i)(n-j)}{1+i+j-n} \right)^d \right. \\ \left. - \frac{1}{(n-j)!^2(2j-n)!} \left(n + \frac{2(n-j)^2}{1+2j-n} \right)^d \right].$$

Proof. The variance of χ is given by:

$$\begin{aligned} \mathbf{V}_\lambda[\chi] &= \mathbf{E}_\lambda\left[(\chi - \mathbf{E}_\lambda[\chi])^2\right] = \mathbf{E}_\lambda\left[\left(\sum_{k=1}^{\infty}(-1)^k N_k - \sum_{k=1}^{\infty}(-1)^k \mathbf{E}_\lambda[N_k]\right)^2\right] \\ &= \mathbf{E}_\lambda\left[\left(\sum_{k=1}^{\infty}(-1)^k (N_k - \mathbf{E}_\lambda[N_k]\right)^2\right] \\ &= \mathbf{E}_\lambda\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(-1)^{i+j}(N_i - \mathbf{E}_\lambda[N_i])(N_j - \mathbf{E}_\lambda[N_j])\right]. \end{aligned}$$

We remark that $N_i \leq \frac{N_1^i}{i!}$, so there is a constant c such that

$$\mathbf{E}_\lambda\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|(N_i - \mathbf{E}_\lambda[N_i])(N_j - \mathbf{E}_\lambda[N_j])|\right] \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{N_1^i}{i!}\frac{N_1^j}{j!}\right| \leq c\mathbf{E}_\lambda\left[e^{n_1}\right]^2 < \infty.$$

Thus the alternating serie converges absolutely allowing us to exchange the mean with the sums and we can write

$$\mathbf{V}_\lambda[\chi] = \sum_{i=1}^{\infty}(-1)^i \sum_{j=1}^{\infty}(-1)^j \mathrm{Cov}_\lambda[N_i, N_j].$$

The result follows by Eq. (7) and some tedious but straightforward algebra. □

Lemma 14. *Let n be a positive integer, then*

$$\sum_{j=1}^{n}\binom{n}{j}\left(\binom{j-1}{n-j-1} - \binom{j-1}{n-j}\right) = (-1)^n.$$

Proof. We first simplify the expression:

$$\sum_{j=1}^{n}\binom{n}{j}\left(\binom{j-1}{n-j-1} - \binom{j-1}{n-j}\right) = \sum_{j=1}^{n}\frac{2n-3j}{j}\binom{n}{j}\binom{j}{n-j},$$

Then, applying hypergeometric functions, we solve the sum:

$$\sum_{j=1}^{n}\frac{2n-3j}{j}\binom{n}{j}\binom{j}{n-j} = (-1)^n.$$

□

Theorem 15. *In one dimension, the expression of the variance of the Euler characteristic is:*

$$\mathbf{V}_\lambda[\chi] = a\left(\lambda e^{-2\lambda\epsilon} - 4\lambda^2\epsilon e^{-4\lambda\epsilon}\right).$$

Proof. If $d = 1$, according to Theorem 13:

$$\mathbf{V}_\lambda[\chi] = \frac{a}{2\epsilon}\sum_{n=1}^{\infty} c_n^1 (2\lambda\epsilon)^n, \tag{9}$$

and we define

$$\alpha_n = \sum_{j=\lceil\frac{n+1}{2}\rceil}^{n}\left[2\sum_{i=n-j+1}^{j}\frac{(-1)^{i+j}n}{(n-j)!(n-i)!(i+j-n)!} - \frac{n}{(n-j)!^2(2j-n)!}\right].$$

and $\beta_n = c_n^1 - \alpha$. It is well known that

$$\sum_{i=0}^{2j-n}(-1)^i\binom{j}{i} = (-1)^{2j-n-1}\binom{j-1}{2j-n},$$

using Stiffel's relation, we obtain:

$$\begin{aligned}
\alpha_n &= (-1)^n\frac{n}{n!}\sum_{j=\lceil\frac{n+1}{2}\rceil}^{n}\left[\binom{n}{j}2\sum_{i=0}^{2j-n}(-1)^i\binom{j}{i} + 2(-1)^n\binom{n}{j}\right]\\
&= \frac{1}{(n-1)!}\sum_{j=\lceil\frac{n+1}{2}\rceil}^{n}\left[2\binom{n}{j}\binom{j-1}{n-j-1} - \binom{n}{j}\binom{j}{n-j} - 2(-1)^n\binom{n}{j}\right]\\
&= \frac{1}{(n-1)!}\sum_{j=\lceil\frac{n+1}{2}\rceil}^{n}\left[\binom{n}{j}\left(\binom{j-1}{n-j} - \binom{j-1}{n-j-1}\right) - 2(-1)^n\binom{n}{j}\right]. \qquad (10)
\end{aligned}$$

The identity $\binom{n}{j} = \binom{n}{n-j}$ allows us to write that

$$\sum_{j=\lceil(n+1)/2\rceil}^{n}(-2(-1)^n)\binom{n}{j} = \sum_{j=0}^{n}\binom{n}{j} = 2^n, \quad n \text{ odd},$$

$$\sum_{j=\lceil(n+1)/2\rceil}^{n}(-2(-1)^n)\binom{n}{j} = \binom{n}{n/2} + \sum_{j=0}^{n}-\binom{n}{j} = -2^n + \binom{n}{n/2}, \quad n \text{ even}.$$

Since $\binom{j-1}{n-j} = 0$ for $j < \lceil\frac{n+1}{2}\rceil$, we have

$$\sum_{j=\lceil\frac{n+1}{2}\rceil}^{n}\binom{n}{j}\left(\binom{j-1}{n-j} - \binom{j-1}{n-j-1}\right) = \sum_{j=1}^{n}\binom{n}{j}\left(\binom{j-1}{n-j} - \binom{j-1}{n-j-1}\right)$$

for n odd and

$$\begin{aligned}
&\sum_{j=\lceil\frac{n+1}{2}\rceil}^{n}\binom{n}{j}\left(\binom{j-1}{n-j} - \binom{j-1}{n-j-1}\right)\\
&\qquad = -\binom{n}{n/2} + \sum_{j=1}^{n}\binom{n}{j}\left(\binom{j-1}{n-j} - \binom{j-1}{n-j-1}\right).
\end{aligned}$$

for n even. According to Lemma 14, we get:

$$\begin{aligned}
\sum_{j=\lceil(n+1)/2\rceil}^{n}\binom{n}{j}\left[\binom{j-1}{n-j-1} - \binom{j-1}{n-j}\right] &= -1, \quad n \text{ odd},\\
\sum_{j=\lceil(n+1)/2\rceil}^{n}\binom{n}{j}\left[\binom{j-1}{n-j-1} - \binom{j-1}{n-j}\right] &= 1 - \binom{n}{n/2}, \quad n \text{ even}.
\end{aligned}$$

Then, we substitute these two last expressions in Eq. (10) to obtain

$$\alpha_n = (-1)^n \frac{(1-2^n)\mathbf{1}_{[n\geq 1]}}{(n-1)!},$$

and thus

$$\sum_{i=0}^{\infty} \alpha_n x^n = -xe^{-x} + 2xe^{-2x}.$$

Proceeding along the same line, β_n is given by

$$\begin{aligned}
\beta_n &= \sum_{j=\lceil \frac{n+1}{2} \rceil}^{n} \left[2 \sum_{i=n-j+1}^{j} \frac{(-1)^{i+j} 2(n-i)(n-j)}{(n-j)!(n-i)!(i+j-n+1)!} \right. \\
&\qquad \left. - \frac{2(n-j)^2}{(n-j)!^2(2j-n+1)!} \right] \\
&= (-1)^n \left(\frac{(-2+2^n)\mathbf{1}_{[n\geq 1]}}{(n-1)!} - \frac{2\mathbf{1}_{[i\geq 2]}}{(i-2)!} \right),
\end{aligned}$$

and again we can simplify the power serie $\sum_{i=0}^{\infty} \beta_n x^n$:

$$\sum_{i=0}^{\infty} \beta_n x^n = 2xe^{-x} - 2(x+x^2)e^{-2x}.$$

Then, substituting α_n and β_n in Eq. (9) yields the result. $\square$

Indeed, Corollary 15 suggests the possibility of finding a simple expression for variance of the Euler's characteristic in higher dimensions. Applying it for $d = 2$, we have a sum in a squared term given by:

$$\left(n + \frac{2(n-i)(n-j)}{1+i+j-n} \right)^2 = n^2 + \frac{4n(n-i)(n-j)}{1+i+j-n} + \left(\frac{2(n-i)(n-j)}{1+i+j-n} \right)^2.$$

With respect to the case in one dimension, no extra knowledge is needed to simplify the first two terms, and, if $x = \lambda(2\epsilon)^2$, they are given respectively by:

$$\left(\frac{a}{2\epsilon}\right)^2 \left((-x+x^2)e^{-x} + 2(x-2x^2)e^{-2x}\right)$$

and

$$\left(\frac{a}{2\epsilon}\right)^2 \left(4(x-x^2)e^{-x} + 4(-x+2x^2)e^{-2x}\right).$$

Unfortunately, we are not able to find a way to express the third term without an infinite serie. This holds for any dimension: we can always find a closed-form for the terms depending on n^d and n^{d-1}, but we cannot go any further.

Theorem 16. *We have $D\chi \leq 2$ and $\|D\chi\|_{L^\infty(\Omega, L^2(\mathbb{T}_a^d))} < \infty$ and*

$$\mathbf{P}(\chi - \bar{\chi} \geq x) \leq \exp\left(-\frac{x}{4} \log\left(1 + \frac{2x}{\mathbf{V}_\lambda[\chi]} \right) \right).$$

Proof. In two dimensions, the Euler characteristic is:

$$\chi = \beta_0 - \beta_1 + \beta_2.$$

Therefore we can bound $D\chi$ by the variation of $\beta_0 - \beta_1$ added to the variation of β_2 when we add a vertex to a simplicial complex.

If we add a vertex on the torus, either the vertex is isolated or not. In the first case, it forms a new connected component incrementing β_0 by 1, and the number of holes that is β_1 is the same. Otherwise, as there is no new connected component,

β_0 is the same, but the new vertex can at most fill a hole incrementing β_1 by 1. Therefore, the variation of $\beta_0 - \beta_1$ is at most 1.

Now, let us look at the variation of β_2 when we add a vertex to a simplicial complex. According to Proposition 3 is at most 1, showing that $D\chi \leq 2$. Then, we use Eq. (4) to complete the proof. $\square$

5. Nth order moments

For this section, without loss of generality, using Proposition 9, we can choose $k = 1/2\epsilon$, so $\lambda_\tau = \lambda(2\epsilon)^d$, $\epsilon_\tau = 1/2$ and $ak = a/2\epsilon$.

We are interested in the central moment, so we introduce the following notation for the centralized number of $(k-1)$-simplices: $\tilde{N}_k = N_k - \bar{N}_k$.

Finally, let us denote that $\binom{i}{j} = 0$ as soon as $i \leq 0$ or $j \leq 0$ or $i - j \leq 0$ for i and j integers.

Definition 7. *We extend the Definition 6 used in the second order moments calculations.*

Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be three simplices with common vertices. For $L \in \mathcal{P}(\{1,2,3\})$, let us denote m_L the number of vertices belonging exactly to the list L of simplices.

Then $M = m_{123} + m_{12} + m_{13} + m_{23} + m_1 + m_2 + m_3$ is the total number of vertices and $\mathcal{J}_3$ represents the integral on these three simplices:

$$\mathcal{J}_3 = \int_{\Delta^{p_1}} \int_{\Delta^{p_2}} \int_{\Delta^{p_3}} h_{p_1} h_{p_2} h_{p_3} \ dx_1 \dots \ dx_M.$$

with p_i being the number of vertices of simplex $\mathcal{C}_i$ for $i = 1, \cdots, 3$, for instance $p_1 = m_{123} + m_{12} + m_{13} + m_1$, and $x_1, \cdots, x_M$ being the M vertices.

Definition 8. *We denote $\mathcal{J}_3(i,j,s,t)$ the integral defined above such that*

- $m_{123} = 2t - i - j + s \vee 0$
- $m_{12} = i + j - s - t \vee 0$
- $m_{13} = i - t \vee 0$
- $m_{23} = j - t \vee 0$
- $m_1 = k - i \vee 0$
- $m_2 = k - j \vee 0$
- $m_3 = k - s \vee 0.$

Theorem 17. *The third moment of the number of $(k-1)$-simplices is given by:*

$$\mathbf{E}_\lambda \left[\tilde{N_k}^3 \right] = \sum_{i,j,s,t} \lambda^{3k-i-j} t! \binom{k}{i} \binom{k}{j} \binom{k}{s} \binom{i}{t} \binom{j}{t} \binom{t}{i+j-s-t} \mathcal{J}_3(i,j,s,t),$$

with $s \geq |i - j|$.

Proof. The chaos decomposition of the number of $(k-1)$-simplices is as shown in lemma 10:

$$\tilde{N_k} = I_1(f_1) + \cdots + I_k(f_k) = \sum_{i=1}^{k} I_i(f_i),$$

with

$$f_i(x_1, \cdots, x_i) = \binom{k}{i} \int h(x_1, \cdots, x_k) \lambda^{k-i} \ \mathrm{d}x_k \dots \ \mathrm{d}x_{i+1},$$

And

$$I_i(f_i) = \int f_i(\ \mathrm{d}\omega(x_1) - \ \mathrm{d}\lambda(x_1)) \dots (\ \mathrm{d}\omega(x_i) - \ \mathrm{d}\lambda(x_i)).$$

Then, we define $g_{i,j,i+j-s} = \sum_{t=\lceil \frac{i+j-s}{2} \rceil}^{i+j-s\wedge i\wedge j} t!\binom{i}{t}\binom{j}{t}\binom{t}{i+j-s-t} f_i \circ_t^{u-t} f_j$ and using the chaos expansion (cf Proposition 6):

$$\begin{aligned}
\tilde{N_k}^3 &= (I_1(f_1)+\cdots+I_k(f_k))^3 \\
&= \left(\sum_{i=1}^{k}\sum_{j=1}^{k} I_i(f_i)I_j(f_j)\right)(I_1(f_1)+\cdots+I_k(f_k)) \\
&= \sum_{i,j=1}^{k}\sum_{s=|i-j|}^{i+j} I_s(g_{i,j,i+j-s})(I_1(f_1)+\cdots+I_k(f_k)) \\
&= \sum_{i,j,l=1}^{k}\sum_{s=|i-j|}^{i+j} I_s(g_{i,j,i+j-s})I_l(f_l).
\end{aligned}$$

When taking the expectation of $\tilde{N_k}$, we use the isometry formula in Eq. (1). Denoting $u = i+j-s$, we obtain:

$$\begin{aligned}
\mathbf{E}_\lambda\left[\tilde{N_k}^3\right] &= \mathbf{E}_\lambda\left[\sum_{i,j=1}^{k}\sum_{s=|i-j|\vee 1}^{i+j\wedge k} I_s(g_{i,j,u})I_s(f_s)\right] \\
&= \sum_{i,j=1}^{k}\sum_{s=|i-j|\vee 1}^{i+j\wedge k}\int g_{i,j,u} f_s \lambda^s \,\mathrm{d}x_1\ldots\,\mathrm{d}x_s \\
&= \sum_{i,j=1}^{k}\sum_{s=|i-j|\vee 1}^{i+j\wedge k}\sum_{t=\lceil\frac{u}{2}\rceil}^{u\wedge i\wedge j}\lambda^s t!\binom{i}{t}\binom{j}{t}\binom{t}{u-t}\int(f_i\circ_t^{u-t} f_j)f_s \,\mathrm{d}x_1\ldots\,\mathrm{d}x_s.
\end{aligned}$$

Then we recognize the integral defined in Definition 8:

$$\mathbf{E}_\lambda\left[\tilde{N_k}^3\right] = \sum_{i,j,=1}^{k}\sum_{s=|i-j|\vee 1}^{i+j\wedge k}\sum_{t=\lceil\frac{u}{2}\rceil}^{u\wedge i\wedge j}\lambda^{3k-i-j} t!\binom{k}{i}\binom{k}{j}\binom{k}{s}\binom{i}{t}\binom{j}{t}\binom{t}{u-t} \mathcal{J}_3(i,j,s,t).$$

Finally, relaxing the boundaries on the sums conclude the proof. □

Definition 9. *Let $\mathcal{C}_1,\cdots,\mathcal{C}_n$ be n simplices with common vertices. For $L \in \mathcal{P}(\{1,\cdots,n\})$, let us denote m_L the number of vertices belonging exactly to the list L of simplices.*

Then $M = \sum_{L\in\mathcal{P}(\{1,\cdots,n\})} m_L$ is the total number of vertices and $\mathcal{J}_n$ represents the integral on these n simplices:

$$\mathcal{J}_n = \int_{\Delta^{p_1}}\cdots\int_{\Delta^{p_n}} h_{p_1}\ldots h_{p_n}\; dx_1\ldots\; dx_M.$$

with p_i being the number of vertices of simplex $\mathcal{C}_i$ for $i=1,\cdots,n$, and $x_1,\cdots,x_M$ being the M vertices.

Theorem 18. *The expression of the nth power of the number of $(k-1)$-simplices is given by:*

$$\tilde{N}_k^n = \sum_{i_1,\cdots,i_n}\sum_{s_1,\cdots s_{n-2}}\sum_{t_1,\cdots t_{n-2}}\left(\prod_{j=1}^{n-2} t_j!\binom{m_{j,1}}{t_j}\binom{m_{j,2}}{t_j}\binom{t_j}{u_j-t_j}\right) I_a(\circ_{j\in A} f_{i_j})I_b(\circ_{j\in\bar{A}} f_{i_j}). \tag{11}$$

With for $j\in\{1,\cdots,n-2\}$:

- $1 \le i_1,\cdots,i_n \le k$,

- $s_j \geq |m_{j,1} - m_{j,2}|$,
- $m_{j,1} = i_{2j-1}$ *if* $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ *and* $s_{2(j-\lfloor \frac{n}{2} \rfloor)-1}$ *otherwise,*
- $m_{j,2} = i_{2j}$ *if* $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ *and* $s_{2(j-\lfloor \frac{n}{2} \rfloor)}$ *otherwise,*
- $u_j = m_{j,1} + m_{j,2} - s_j$,
- $A \subset \{1, \cdots, n\}$,
- *If* n *is even, then* $a = s_{n-3}$ *and* $b = s_{n-2}$,
- *If* n *is odd, then* $a = s_{n-2}$ *and* $b = i_n$.

Proof. The decomposition of the centralized number of $(k-1)$-simplices is:

$$\tilde{N_k} = I_1(f_1) + \cdots + I_k(f_k) = \sum_{i=1}^{k} I_i(f_i).$$

Now, we raise $\tilde{N_k}$ to the nth power:

$$\tilde{N_k}^n = \left(\sum_{i=1}^{k} I_i(f_i) \right)^n .$$

First, we consider the case where n is even, we can group the factors by 2:

$$\tilde{N}_k^n = \left(\sum_{i_1=1}^{k} I_{i_1}(f_{i_1}) \sum_{i_2=1}^{k} I_{i_2}(f_{i_2}) \right) \cdots \left(\sum_{i_{n-1}=1}^{k} I_{i_{n-1}}(f_{i_{n-1}}) \sum_{i_n=1}^{k} I_{i_n}(f_{i_n}) \right).$$

We then use the chaos expansion of Proposition 6:

$$\begin{aligned} I_i(f_i)I_j(f_j) &= \sum_{s=0}^{2(i\wedge j)} I_{i+j-s} \left(\sum_{s \leq 2t \leq 2(s \wedge i \wedge j)} t! \binom{i}{t} \binom{j}{t} \binom{t}{s-t} f_i \circ_t^{s-t} f_j \right) \\ &= \sum_{s=|i-j|}^{i+j} I_s \left(\sum_{i+j-s \leq 2t \leq 2(i+j-s) \wedge i \wedge j)} t! \binom{i}{t} \binom{j}{t} \binom{t}{i+j-s-t} f_i \circ_t^{i+j-s-t} f_j \right). \end{aligned}$$

Let us denote $g_s = t!\binom{i}{t}\binom{j}{t}\binom{t}{i+j-s-t} f_i \circ_t^{i+j-s-t} f_j$, so we can re-write, relaxing the boundaries on the sums:

$$I_i(f_i)I_j(f_j) = \sum_{s \geq |i-j|} \sum_t I_s(g_s).$$

Thus, we have:

$$\tilde{N}_k^n \quad = \quad \sum_{i_1,i_2=1}^{k} \sum_{s_1 \geq |i_1-i_2|} \sum_{t_1} I_{s_1}(g_{s_1}) \cdots \sum_{i_{n-1},i_n=1}^{k} \sum_{s_{n/2} \geq |i_{n-1}-i_n|} \sum_{t_{n/2}} I_{s_{n/2}}(g_{s_{n/2}}).$$

We go on grouping terms by 2 until we only have a product of 2 chaos left: First we made $n/2$ chaos expansions, leading to $n/2$ sums with indexes s_j, $j = 1, \cdots, n/2$. To reduce the number of chaos to 2, we have to make other chaos expansions. For $j \geq \frac{n}{2} + 1$, the sum indexed by s_j represents the expansion of the chaos indexed $s_{2(j-\frac{n}{2})-1}$ and $s_{2(j-\frac{n}{2})-1}$. We have 2 chaos remaining when $j = 2(j - \frac{n}{2}) + 2$, i.e. when $j = n - 2$.

Moreover, there are as much sums indexed with t_j as with s_j, that is $n-2$. Thus we can write:

$$\tilde{N}_k^n = \sum_{i_1,\cdots,i_n=1}^{k} \sum_{s_1,\cdots s_{n-2}} \sum_{t_1,\cdots t_{n-2}} I_{s_{n-3}}(\phi_{s_{n-3}}) I_{s_{n-2}}(\phi_{s_{n-2}}),$$

With $s_j \geq |m_{j,1} - m_{j,2}|$ for $j \in \{1, \cdots, n-2\}$ if we denote:

- $m_{j,1} = i_{2j-1}$ if $1 \leq j \leq \frac{n}{2}$ and $s_{2(j-\frac{n}{2})-1}$ otherwise,

- $m_{j,2} = i_{2j}$ if $1 \leq j \leq \frac{n}{2}$ and $s_{2(j-\frac{n}{2})}$ otherwise.

Then, denoting $u_j = m_{j,1} + m_{j,2} - s_j$ and A the subset of $\{1, \cdots, n\}$ such that if $j \in A$ then the chaos i_j is expanded in the chaos s_{n-3}, we have:

$$I_{s_{n-3}}(\phi_{s_{n-3}}) I_{s_{n-2}}(\phi_{s_{n-2}}) = \left(\prod_{j=1}^{n-2} t_j! \binom{m_{j,1}}{t_j} \binom{m_{j,2}}{t_j} \binom{t_j}{u_j - t_j} \right) I_{s_{n-3}}(\circ_{j \in A} f_{i_j}) I_{s_{n-2}}(\circ_{j \in \bar{A}} f_{i_j}).$$

The notation $\circ_{j \in A} f_{i_j}$ represents the product defined in Eq. (3) of the functions f_{i_j} for $j \in A$, but whom variables depend on all the $i_1, \cdots, i_n, s_1, \cdots, s_{n-2}$, and $t_1, \cdots, t_{n-2}$.

Now, if n is odd, we consider $n-1$ which is even, therefore we have:

$$\begin{aligned} \tilde{N}_k^n &= \sum_{i_1, \cdots, i_{n-1}=1}^{k} \sum_{s_1, \cdots, s_{n-3}} \sum_{t_1, \cdots, t_{n-3}} I_{s_{n-4}}(\phi_{s_{n-4}}) I_{s_{n-3}}(\phi_{s_{n-3}}) \sum_{i_n=1}^{k} I_{i_n}(f_{i_n}) \\ &= \sum_{i_1, \cdots, i_n=1}^{k} \sum_{s_1, \cdots, s_{n-2}} \sum_{t_1, \cdots, t_{n-2}} I_{s_{n-2}}(\phi_{s_{n-2}}) I_{i_n}(f_{i_n}), \end{aligned}$$

with $s_j \geq |m_{j,1} - m_{j,2}|$ for $j \in \{1, \cdots, n-2\}$ using the same notations for $n-1$ instead of n:

- $m_{j,1} = i_{2j-1}$ if $1 \leq j \leq \frac{n-1}{2}$ and $s_{2(j-\frac{n-1}{2})-1}$ otherwise,
- $m_{j,2} = i_{2j}$ if $1 \leq j \leq \frac{n-1}{2}$ and $s_{2(j-\frac{n-1}{2})}$ otherwise.

And with $u_j = m_{j,1} + m_{j,2} - s_j$,

$$I_{s_{n-2}}(\phi_{s_{n-2}}) = \left(\prod_{j=1}^{n-2} t_j! \binom{m_{j,1}}{t_j} \binom{m_{j,2}}{t_j} \binom{t_j}{u_j - t_j} \right) I_{s_{n-2}}(\circ_{j \in \{1, \cdots, n-1\}} f_{i_j}),$$

concluding the proof. □

Theorem 19. *The expression of the nth moment of the number of $(k-1)$-simplices is given by:*

$$\mathbf{E}_\lambda \left[\tilde{N_k}^n \right] = \sum_{i_1, \cdots, i_n} \sum_{s_1, \cdots, s_{n-3}} \sum_{t_1, \cdots, t_{n-2}} \lambda^{nk+c} \left(\prod_{j=1}^{n} \lambda^{-i_j} \binom{k}{i_j} \right)$$
$$\left(\prod_{j=1}^{n-2} t_j! \binom{m_{j,1}}{t_j} \binom{m_{j,2}}{t_j} \binom{t_j}{u_j - t_j} \right) \mathcal{J}_n(i_1, \cdots, i_n, s_1, \cdots, s_{n-3}, t_1, \cdots, t_{n-2}).$$

With for $j \in \{1, \cdots, n-2\}$:

- *if $j \leq n-3$, $s_j \geq |m_{j,1} - m_{j,2}|$,*
- *$m_{j,1} = i_{2j-1}$ if $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ and $s_{2(j-\lfloor \frac{n}{2} \rfloor)-1}$ otherwise,*
- *$m_{j,2} = i_{2j}$ if $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ and $s_{2(j-\lfloor \frac{n}{2} \rfloor)}$ otherwise,*
- *$m_{j,3} = s_j$ if $1 \leq j \leq n-3$ and s_{n-3} otherwise,*
- *$u_j = m_{j,1} + m_{j,2} - m_{j,3}$,*
- *If n is even, then $c = s_{n-3}$ and*
  $s_{n-3} \geq |m_{n-2,1} - m_{n-2,2}| \vee |m_{n-3,1} - m_{n-3,2}|$,
- *If n is odd, then $c = i_n$ and $i_n \geq |m_{n-2,1} - m_{n-2,2}|$.*

Proof. The expression of the nth power of the number of $(k-1)$-simplices is given in Eq. (11):

$$\tilde{N}_k^n = \sum_{i_1,\cdots,i_n=1}^{k} \sum_{s_1,\cdots,s_{n-2}} \sum_{t_1,\cdots,t_{n-2}} \left(\prod_{j=1}^{n-2} t_j! \binom{m_{j,1}}{t_j} \binom{m_{j,2}}{t_j} \binom{t_j}{u_j - t_j} \right)$$
$$I_a(\circ_{j\in A} f_{i_j}) I_b(\circ_{j\in \bar{A}} f_{i_j}).$$

If n is even, we have:

$$\tilde{N}_k^n = \sum_{i_1,\cdots,i_n=1}^{k} \sum_{s_1,\cdots,s_{n-2}} \sum_{t_1,\cdots,t_{n-2}} \left(\prod_{j=1}^{n-2} t_j! \binom{m_{j,1}}{t_j} \binom{m_{j,2}}{t_j} \binom{t_j}{u_j - t_j} \right)$$
$$I_{s_{n-3}}(\circ_{j\in A} f_{i_j}) I_{s_{n-2}}(\circ_{j\in \bar{A}} f_{i_j}).$$

So let us focus on the only part of the equation which is likely to change when we take the expected value, that we will denote:

$$K = \sum_{s_{n-3}} \sum_{s_{n-2}} I_{s_{n-3}}(\circ_{j\in A} f_{i_j}) I_{s_{n-2}}(\circ_{j\in \bar{A}} f_{i_j}).$$

We then use the property of Eq. (1) and recognize the integral from Definition 9:

$$\begin{aligned}
\mathbf{E}_\lambda[K] &= \sum_{s_{n-3}} \left(\prod_{j=1}^{n} \lambda^{k-i_j} \binom{k}{i_j} \right) \lambda^{s_{n-3}} \mathcal{J}_n(i_1,\cdots,i_n,s_1,\cdots,s_{n-3},t_1,\cdots,t_{n-2}) \\
&= \sum_{s_{n-3}} \lambda^{nk+s_{n-3}} \left(\prod_{j=1}^{n} \lambda^{-i_j} \binom{k}{i_j} \right) \mathcal{J}_n(i_1,\cdots,i_n,s_1,\cdots,s_{n-3},t_1,\cdots,t_{n-2}),
\end{aligned}$$

with $s_{n-3} \geq |m_{n-2,1} - m_{n-2,2}| \vee |m_{n-3,1} - m_{n-3,2}|$.

Then for n odd we directly write:

$$\begin{aligned}
K' &= \sum_{i_n} \sum_{s_{n-2}} I_{i_n}(\circ_{j\in I} f_{i_j}) I_{s_{n-2}}(\circ_{j\in \bar{I}} f_{i_j}), \\
\mathbf{E}_\lambda[K'] &= \sum_{i_n} \lambda^{nk+i_n} \left(\prod_{j=1}^{n} \lambda^{-i_j} \binom{k}{i_j} \right) \mathcal{J}_n(i_1,\cdots,i_n,s_1,\cdots,s_{n-3},t_1,\cdots,t_{n-2}),
\end{aligned}$$

with $i_n \in \{|m_{n-2,1} - m_{n-2,2}| \vee 1, k\}$.

The binomials with the i_j allow us to relax the boundaries on the sums on i_j, concluding the proof. □

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