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From invariance principles to a class of multifractional fields related to fractional sheets

Céline Lacaux* and Renaud Marty†

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Abstract

In this paper, we study some invariance principles where the limits are Gaussian random fields sharing many properties with multifractional Brownian sheets. In particular, they satisfy the same type of self-similarity and Hölder regularity properties. We also extend the invariance principles mentioned above in a stable setting.

Key-words: Multifractional anisotropic random fields, invariance principle, local self-similarity, sample path properties.

MSC classification (2010): 60F17, 60G22, 60G17, 60G52, 60G60.

1 Introduction

Fractional Brownian motion has been extensively studied because it is a relevant model for many problems where fractal properties occur, which is due in particular to its self-similarity, the stationarity of its increments and its regularity properties. It is also universal in the class of fractional processes because it satisfies the following invariance principle. Consider $X = \{X_n, n \in \mathbb{N}\}$ a centered, stationary and Gaussian sequence and the process

$$S_H^N = \left\{ \frac{1}{N^H} \sum_{n=1}^{[Nt]} X_n, t \in [0, +\infty) \right\}$$

where $H \in (0, 1)$ and $[y]$ is the integer part of the real number y . Then, S_H^N converges in distribution to a fractional Brownian motion of Hurst index H (see Theorem 7.2.11 of [22]) as soon as one of the three following properties is fulfilled:

- If $H \in (1/2, 1)$, there exists $\sigma_H > 0$ such that $\mathbb{E}[X_0 X_n] \sim \sigma_H n^{2H-2}$ as $n \rightarrow \infty$
- If $H \in (0, 1/2)$, there exists $\sigma_H < 0$ such that $\mathbb{E}[X_0 X_n] \sim \sigma_H n^{2H-2}$ as $n \rightarrow \infty$ and $\sum_{n=-\infty}^{\infty} \mathbb{E}[X_0 X_n] = 0$.
- If $H = 1/2$, $\sum_{n=1}^{\infty} |\mathbb{E}[X_0 X_n]| < \infty$ and $\sum_{n=-\infty}^{\infty} \mathbb{E}[X_0 X_n] > 0$.

*Institut Élie Cartan Nancy, UMR 7502, Nancy Université-CNRS-INRIA, BP 70239, F-54506 Vandoeuvre-lès-Nancy, France & MAP 5, CNRS UMR 8145, Université Paris Descartes, 45 rue des Saints-Pères, 75006 Paris, France. Email: Celine.Lacaux@iecn.u-nancy.fr

†Institut Elie Cartan Nancy, UMR 7502, Nancy Université-CNRS-INRIA, BP 70239, F-54506 Vandoeuvre-lès-Nancy Cedex, France. Email: Renaud.Marty@iecn.u-nancy.fr

The most famous generalization of the fractional Brownian motion is the fractional Brownian field $B_H = (B_H(u))_{u \in \mathbb{R}^d}$ (see [17, 13]). This isotropic Gaussian random field B_H , whose increments are stationary, is self-similar of order H , which means that

$$\forall \varepsilon \in [0, +\infty), \left\{ B_H(\varepsilon u), u \in \mathbb{R}^d \right\} \stackrel{\text{dist.}}{=} \varepsilon^H \left\{ B_H(u), u \in \mathbb{R}^d \right\} \quad (1)$$

where $\stackrel{\text{dist.}}{=}$ stands for the equality of the finite-dimensional distributions. However, a drawback of the model B_H is the strong homogeneity of its properties. For instance, the self-similarity property (1) is a global property, which is too restrictive for some applications, and the pointwise Hölder exponent of B_H at any point t equals to its Hurst parameter H . Thus, many generalizations of fractional Brownian fields have been introduced to model phenomena whose Hölder regularity may vary along the trajectories. The most famous of them is the class of multifractional Brownian fields (see [4, 21]). Each of them is defined from a Hurst function h instead of a Hurst index, and has locally but not globally (in general) the same properties as a fractional Brownian motion.

Another drawback of fractional Brownian fields is their isotropy property, which is not suitable for applications in medicine [8] or hydrology [5]. Thus, some fractional anisotropic generalizations of the fractional Brownian motion have also been introduced. In particular, a fractional Brownian sheet (see [12, 1]), that we still denote by B_H , of order $H = (H_1, \dots, H_d) \in (0, 1)^d$ exhibits different scaling properties in the d orthogonal directions, which is summarized in the following self-similarity property:

$$\forall \varepsilon \in [0, +\infty)^d, \left\{ B_H(\varepsilon_1 u_1, \dots, \varepsilon_d u_d), u \in \mathbb{R}^d \right\} \stackrel{\text{dist.}}{=} \prod_{k=1}^d \varepsilon_k^{H_k} \left\{ B_H(u), u \in \mathbb{R}^d \right\}. \quad (2)$$

This property is more general than (1) but is still a global property. Moreover, the rectangular increments of B_H are stationary and its pointwise Hölder regularity may vary with the direction but not along a trajectory (except on the axes). Then, to allow more flexibility, a local version of the property (2) has been introduced in [16], following the way [4] has introduced a local version of (1). As multifractional Brownian fields have been defined, replacing H by a function $t \mapsto h(t) = (h_1(t_1), \dots, h_d(t_d))$, multifractional Brownian sheets have been defined and studied in [16, 11]. Their pointwise Hölder exponent can vary with the point t and with the direction. Observe that multifractional Brownian fields are not in general isotropic but their Hölder regularity do not depend on the directions.

As mentioned previously for fractional processes, a question of interest is the existence of universal multifractional processes or, in other words, invariance principles for multifractional processes. This question in the case of long-range dependence is addressed in [9] where we describe the main result. We consider a centered Gaussian field $X = \{X_n(H), H \in (1/2, 1), n \in \mathbb{N}\}$, a continuous function $h : \mathbb{R} \rightarrow [a, b] \subset (1/2, 1)$ and for every $N \in \mathbb{N}$,

$$S_h^N = \left\{ \sum_{n=1}^{\lfloor Nt \rfloor} \frac{X_n(h(n/N))}{N^{h(n/N)}}, t \in [0, +\infty) \right\}. \quad (3)$$

As N goes to ∞ , under an assumption of long-range dependence, [9] proves that the finite-dimensional distributions of S_h^N converge to those of a multifractional Gaussian process S_h . More precisely, S_h is locally asymptotically self-similar in the sense of [4] and its pointwise

Hölder exponent may vary along the trajectories. This theorem is a generalization of the classical invariance principle ([22]) in a framework of long-range dependence.

In this paper we generalize to different directions the invariance principles we mention above. Our main results deal a multidimensional version of the sequence (3). More precisely, we consider a sequence of the form

$$S_h^N = \left\{ \sum_{n_1=1}^{[Nt_1]} \dots \sum_{n_d=1}^{[Nt_d]} \frac{X_n(h_n^N)}{N^{r_n^N}}, t \in [0, +\infty)^d \right\}$$

where $X = \{X_n(H), n \in \mathbb{N}^d, H \in (0, 1)^d\}$ is a Gaussian or an α -stable ($0 < \alpha < 2$) random field defined by a harmonizable integral representation. For suitable families $(h_n^N)_{n,N}$ and $(r_n^N)_{n,N}$ we prove the convergence in distribution of S_h^N to a multifractional field S_h as N goes to ∞ . Moreover, in the Gaussian case, we study the local self-similarity and the pointwise Hölder regularity properties of the limit S_h . These properties are the same as those satisfied by multifractional Brownian sheets. We also introduce pointwise multi-Hölder exponents which are related to the regularity of the rectangular increments.

The paper is organized as follows. Section 2 is devoted to establish the invariance principle in the Gaussian case. We get limit fields whose we study local self-similarity and regularity properties in Section 3. We extend the invariance principle in a stable setting in Section 4. Some technical lemmas are postponed to the Appendix.

2 Invariance principle

Let $\{W_H(x), x \in \mathbb{R}^d, H \in (0, 1)^d\}$ be a real centered Gaussian field such that for every $H \in (0, 1)^d, K \in (0, 1)^d, \zeta \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^d$,

$$\mathbb{E}[W_H(\zeta)W_K(\eta)] = C(H, K) \prod_{j=1}^d \inf(\zeta_j, \eta_j), \quad (4)$$

with $C : (0, 1)^d \times (0, 1)^d \rightarrow \mathbb{R}$ a symmetric and smooth function. Let us emphasize that $\{W_H(x), x \in \mathbb{R}^d, H \in (0, 1)^d\}$ exists if and only if C is a covariance function. We also remark that, if we fix $H \in (0, 1)^d$, then $\{W_H(x), x \in \mathbb{R}^d\}$ is a Brownian sheet.

Consider the set

$$\mathcal{F} = \left\{ f : \mathbb{R}^d \rightarrow \mathbb{C}, f \in L^2(\mathbb{R}^d), \forall \xi \in \mathbb{R}^d, f(\xi) = \overline{f(-\xi)} \right\}$$

and for each $H \in (0, 1)^d$, let $\widehat{W}_H(dx)$ be the Fourier transform of the real Gaussian random measure associated with W_H (see [22] for example). Then, for each H , for any $f, g \in \mathcal{F}$,

$$\mathbb{E} \left(\int_{\mathbb{R}^d} f(x) \widehat{W}_H(dx) \overline{\int_{\mathbb{R}^d} g(x) \widehat{W}_K(dx)} \right) = C(H, K) \int_{\mathbb{R}^d} f(x) g(x) dx. \quad (5)$$

Let $a : (0, 1)^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ be a measurable and bounded function satisfying

$$\forall H \in (0, 1)^d, a(H, \cdot) \in \mathcal{F}. \quad (6)$$

Then, the random field $X = \{X_n(H), n \in \mathbb{N}^d, H \in (0, 1)^d\}$ such that

$$X_n(H) = \int_{\mathbb{R}^d} a(H, x) \exp(i\langle n, x \rangle) \prod_{j=1}^d \frac{\exp(ix_j) - 1}{|x_j|^{1/2+H_j}} \widehat{W}_H(dx)$$

is well-defined and is a centered real-valued Gaussian field. Using (5), the covariance between $X_n(H)$ and $X_m(K)$ is

$$C(H, K) \int_{\mathbb{R}^d} a(H, x) \overline{a(K, x)} e^{i\langle n-m, x \rangle} \prod_{j=1}^d \frac{|\exp(ix_j) - 1|^2}{|x_j|^{1+H_j+K_j}} dx := r_{H,K}(n-m). \quad (7)$$

Our aim is to consider a field S_h^N defined as a weighted sum of X which converges to a random field sharing many properties with a multifractional Brownian sheet. Before we introduce the weighted sum S_h^N , let us introduce several notations we use throughout the paper.

Notation

- For any $t = (t_1, \dots, t_d) \in [0, +\infty)^d$ and $N \in \mathbb{N}^*$, let

$$\mathcal{D}_t^N = \left\{ k = (k_1, \dots, k_d) \in \mathbb{N}^d / 1 \leq k_j \leq [Nt_j], \forall 1 \leq j \leq d \right\}.$$

- For any integer $1 \leq j \leq d$, let us consider a function $h_j : [0, \infty) \rightarrow (0, 1)$ and set

$$\forall t = (t_1, \dots, t_d) \in [0, +\infty)^d, h(t) = (h_1(t_1), \dots, h_d(t_d)). \quad (8)$$

- In the following, for $k = (k_1, \dots, k_d)$, $h_k^N := (h_{k_1,1}^N, \dots, h_{k_d,d}^N)$ where for $1 \leq j \leq d$, $h_{k,j}^N = h_j(k_j/N)$.
- In order to simplify some notation, for $\alpha = (\alpha_1, \dots, \alpha_d) \in [0, +\infty)^d$, we write $|\alpha| = \sum_{j=1}^d \alpha_j$.

In this section, we study the behavior as N tends to $+\infty$ of the random field $S_h^N = \{S_h^N(t), t \in [0, +\infty)^d\}$ defined by

$$S_h^N(t) := \sum_{k \in \mathcal{D}_t^N} \frac{X_k(h_k^N)}{N^{|h_k^N|}} = \sum_{k_1=1}^{[Nt_1]} \sum_{k_2=1}^{[Nt_2]} \dots \sum_{k_d=1}^{[Nt_d]} \frac{X_k(h_k^N)}{N^{\sum_{j=1}^d h_{k,j}^N}}. \quad (9)$$

We assume that the function a satisfies the following technical condition.

Condition (A). There exist two measurable functions $a_0 : (0, 1)^d \rightarrow \mathbb{C}$ and $a_1 : (0, 1)^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ satisfying the following properties.

- For every $(H, x) \in (0, 1)^d \times \mathbb{R}^d$, $a(H, x) = a_0(H) + a_1(H, x)$.
- The functions a_0 and a_1 are of class \mathcal{C}^d with respect to H .
- For every $\kappa = (\kappa_1, \dots, \kappa_d) \in \{0, 1\}^d$ and every compact set \mathcal{K} of $(0, 1)^d$,

$$\lim_{x \rightarrow 0} \sup_{H \in \mathcal{K}} \{|\partial_H^\kappa a_1(H, x)|\} = 0 \quad \text{and} \quad \sup_{(H,x) \in \mathcal{K} \times \mathbb{R}^d} \{|\partial_H^\kappa a_1(H, x)|\} < \infty$$

with $\partial_H^\kappa = \partial_{H_1}^{\kappa_1} \dots \partial_{H_d}^{\kappa_d}$.

These properties imply that for every $\kappa \in \{0, 1\}^d$ and every compact set \mathcal{K} of $(0, 1)^d$,

$$\sup_{(H,x) \in \mathcal{K} \times \mathbb{R}^d} \{|\partial_H^\kappa a(H, x)|\} < \infty.$$

They also imply that a_0 is a real-valued function and that $a_1(H, \cdot) \in \mathcal{F}$ for any $H \in (0, 1)^d$.

Now we state the main result of this paper.

Theorem 1. *Assume that for any $1 \leq j \leq d$, h_j is a C^1 -function and that Condition (A) is fulfilled. Then, the finite-dimensional distributions of S_h^N converge to those of the real-valued centered Gaussian field S_h whose covariance is given, for all s and $t \in [0, +\infty)^d$, by*

$$\mathbb{E}[S_h(t)S_h(s)] = \int_{\mathbb{R}^d} dy \int_{\mathcal{D}_t \times \mathcal{D}_s} C_0(h(\theta), h(\sigma)) e^{i\langle y, \theta - \sigma \rangle} \prod_{j=1}^d |y_j|^{1-h_j(\theta_j)-h_j(\sigma_j)} d\theta d\sigma, \quad (10)$$

with $\mathcal{D}_u = \prod_{j=1}^d [0, u_j]$ and

$$C_0(H, K) = C(H, K) a_0(H) a_0(K). \quad (11)$$

In the special case where $C \equiv 1$,

$$S_h \stackrel{\text{f.d.d.}}{=} \left\{ \int_{\mathbb{R}^d} \widehat{W}(dy) \int_{\mathcal{D}_t} a_0(h(\theta)) e^{i\langle y, \theta \rangle} \prod_{j=1}^d |y_j|^{1/2-h_j(\theta_j)} d\theta, t \in [0, +\infty)^d \right\} \quad (12)$$

with \widehat{W} the Fourier transform of the real Gaussian measure associated with a Brownian sheet.

Observe that $S_h(t) = 0$ if there exists j such that $t_j = 0$. This is also true for multifractional sheets.

Before we prove this theorem, let us relate it with some previous works [9, 22, 23].

Remark 1. *Assume $h \equiv H \in (0, 1)^d$. Then according to Theorem 1, the random field*

$$t \mapsto N^{-\sum_{j=1}^d H_j} \sum_{k_1=1}^{[Nt_1]} \sum_{k_2=1}^{[Nt_2]} \cdots \sum_{k_d=1}^{[Nt_d]} X_k(H)$$

converges to a fractional Brownian sheet B_H of order H . If $d = 1$, this is the classical principle invariance described in [22].

If h is not a constant function, the limit is not a multifractional Brownian sheet. For instance, if $d = 2$ and $C \equiv 1$,

$$S_h(t) = a_0(h_1(t_1)) a_0(h_2(t_2)) B_h(t) - R(t)$$

where B_h is the multifractional Brownian sheet defined by

$$B_h(t) = \int_{\mathbb{R}^2} \widehat{W}(dx) \left(\frac{e^{it_1 x_1} - 1}{ix_1 |x_1|^{h_1(t_1)-1/2}} \right) \left(\frac{e^{it_2 x_2} - 1}{ix_2 |x_2|^{h_2(t_2)-1/2}} \right)$$

and

$$R(t) = \int_{\mathbb{R}^2} \widehat{W}(dx) f_1(t, x) g_2(t, x) + \int_{\mathbb{R}^2} \widehat{W}(dx) f_2(t, x) g_1(t, x) - \int_{\mathbb{R}^2} \widehat{W}(dx) g_1(t, x) g_2(t, x)$$

with

$$f_k(t, x) = a_0(h_k(t_k)) \left(\frac{e^{it_k x_k} - 1}{ix_k |x_k|^{h_k(t_k) - 1/2}} \right)$$

and

$$g_j(x, t) = \int_0^{t_j} h'_j(\theta_j) (a'_0(h_j(\theta_j)) - a_0(h_j(\theta_j)) \ln |x_j|) \left(\frac{e^{i\theta_j x_j} - 1}{ix_j |x_j|^{h_j(\theta_j) - 1/2}} \right) d\theta_j.$$

Remark 2. Assume that $d = 1$ and that Condition (A) is fulfilled. Let $r_{H,K}$ be defined by (7). Then one can prove that for any compact set \mathcal{K} of $(1/2, 1)$,

$$\sup_{H, K \in \mathcal{K}} \left| j^{2-H-K} r_{H,K}(j) - C(H, K) a_0(H) a_0(K) \int_{\mathbb{R}} \exp(ix) |x|^{1-H-K} dx \right| \xrightarrow{j \rightarrow \infty} 0.$$

Hence, the Gaussian process $\{X_H(t), t \in \mathbb{R}, H \in (1/2, 1)\}$ satisfies the assumptions of [9], in which h takes its values in a compact subset of $(1/2, 1)$. Theorem 1 is then a generalization of Theorem 1 of [9], with some restrictions on the form of X and the regularity of h , to the case where h takes its values in $(0, 1)$.

Remark 3. Let $d = 1$. Assume that $a \equiv 1$ and $W_H = W$ does not depend on H . Then under the assumptions of Theorem 1, the finite-dimensional distributions of $\{S_h^N(t), t \in [0, +\infty)\}$ converge to those of the process $\{Y_h(t), t \in [0, +\infty)\}$ defined by

$$Y_h(t) = \int_{\mathbb{R}} \widehat{W}(dy) \int_0^t \frac{\exp(iy\theta)}{|y|^{h(\theta) - 1/2}} d\theta$$

with \widehat{W} the Fourier transform of a real Gaussian measure. This process, called integrated fractional white noise, has first been introduced in [23] as an alternative to multifractional Brownian motion because its magnitude is much less sensitive to the variations of the Hölder exponent. Here we give a new motivation to the introduction of the process Y_h : it is the limit of an invariance principle, so it can be chosen as a universal multifractional Gaussian model. Note that this has already been noticed in [9] in the case of a function h with values in a compact set of $(1/2, 1)$.

Let us now conclude this section by proving Theorem 1.

Proof. Let $t, s \in [0, T]^d$. Then,

$$\begin{aligned} \mathbb{E}[S_h^N(t) S_h^N(s)] &= \sum_{k \in \mathcal{D}_t^N} \sum_{l \in \mathcal{D}_s^N} \frac{1}{N |h_k^N|_N |h_l^N|_N} \mathbb{E}[X_l(h_l^N) X_k(h_k^N)] \\ &= \int_{\mathbb{R}^d} dx \sum_{k \in \mathcal{D}_t^N} \sum_{l \in \mathcal{D}_s^N} \frac{b(\frac{l}{N}, \frac{k}{N}, x) e^{i\langle l-k, x \rangle}}{N |h_k^N|_N |h_l^N|_N} \prod_{j=1}^d \frac{|\exp(ix_j) - 1|^2}{|x_j|^{1+h_{l,j}^N + h_{k,j}^N}} \end{aligned}$$

where $b(\theta, \sigma, x) = C(h(\theta), h(\sigma))a(h(\theta), x)\overline{a(h(\sigma), x)}$. Then, a change of variables leads to

$$\mathbb{E}[S_h^N(t)S_h^N(s)] = \int_{\mathbb{R}^d} \Phi^N(s, t, x)dx$$

where

$$\Phi^N(s, t, x) := \sum_{k \in \mathcal{D}_t^N} \sum_{l \in \mathcal{D}_s^N} b\left(\frac{l}{N}, \frac{k}{N}, \frac{x}{N}\right) \prod_{j=1}^d \frac{|e^{ix_j/N} - 1|^2 e^{i(l_j - k_j)x_j/N}}{|x_j|^{1+h_{l,j}^N+h_{k,j}^N}}.$$

Let us define the function q by

$$q(\xi) = \frac{1 - e^{i\xi}}{\xi}, \quad \xi \in \mathbb{R} \setminus \{0\}.$$

We can write

$$\begin{aligned} \Phi^N(s, t, x) &:= \frac{\overline{q\left(\frac{x_1}{N}\right)}}{N} \sum_{l_1=1}^{[Ns_1]} \frac{e^{il_1x_1/N}}{|x_1|^{h_1(l_1/N)-1/2}} \frac{q\left(\frac{x_1}{N}\right)}{N} \sum_{k_1=1}^{[Nt_1]} \frac{e^{-ik_1x_1/N}}{|x_1|^{h_1(k_1/N)-1/2}} \\ &\quad \dots \frac{\overline{q\left(\frac{x_d}{N}\right)}}{N} \sum_{l_d=1}^{[Ns_d]} \frac{e^{il_dx_d/N}}{|x_d|^{h_d(l_d/N)-1/2}} \frac{q\left(\frac{x_d}{N}\right)}{N} \sum_{k_d=1}^{[Nt_d]} \frac{e^{-ik_dx_d/N}}{|x_d|^{h_d(k_d/N)-1/2}} b\left(\frac{l}{N}, \frac{k}{N}, \frac{x}{N}\right). \end{aligned}$$

Because of the regularity of h , a and C , we can apply $2d$ times Lemma 3 (stated and proved in the Appendix) with $\alpha = 2$. This leads to

$$|\Phi^N(s, t, x)| \leq \max_{\kappa \in \{0,1\}^{2d}} \sup_{(\theta, \sigma) \in [0, T]^{2d}} \left| \partial^\kappa b\left(\theta, \sigma, \frac{x}{N}\right) \right| \prod_{j=1}^d g(x_j)^2$$

where $\partial^\kappa b = \partial_{\theta_1}^{\kappa_1} \dots \partial_{\theta_d}^{\kappa_d} \partial_{\sigma_1}^{\kappa_{d+1}} \dots \partial_{\sigma_d}^{\kappa_{2d}} b$ and

$$g(\xi) = c_{T,h} (1 + |\ln |\xi||) \left(\frac{1}{|\xi|^{H_+-1/2}} \mathbb{I}_{|\xi| < 1} + \frac{1}{|\xi|^{H_-+1/2}} \mathbb{I}_{|\xi| \geq 1} \right) \quad (13)$$

with $H_+ = \max_{1 \leq j \leq d} \max_{[0, T]} h_j$, $H_- = \min_{1 \leq j \leq d} \min_{[0, T]} h_j$ and $c_{T,h}$ a finite positive constant. which only depends on T and the infinite norm of each h'_j on $[0, T]$.

Since h is \mathcal{C}^1 , C is smooth and since Condition (A) is fulfilled,

$$\sup_{y \in \mathbb{R}^d} \sup_{(\theta, \sigma) \in [0, T]^{2d}} |\partial^\kappa b(\theta, \sigma, y)| < +\infty.$$

The sequence of functions $\{x \mapsto \Phi^N(s, t, x)\}$ is then uniformly bounded by an $L^1(\mathbb{R}^d)$ -function. In order to use the bounded convergence theorem, it remains to prove the convergence of $\Phi^N(s, t, x)$ for almost every x . We write

$$\Phi^N(s, t, x) = \sum_{n=0}^3 \Phi_n^N(s, t, x)$$

where, for $n \in \{0, 1, 2, 3\}$,

$$\Phi_n^N(s, t, x) = \frac{1}{N^{2d}} \sum_{k \in \mathcal{D}_t^N} \sum_{l \in \mathcal{D}_s^N} b_n\left(\frac{l}{N}, \frac{k}{N}, \frac{x}{N}\right) \prod_{j=1}^d \frac{|q\left(\frac{x_j}{N}\right)|^2 e^{i(l_j - k_j)x_j/N}}{|x_j|^{h_{l,j}^N + h_{k,j}^N - 1}}$$

with

$$\begin{aligned} b_0(\theta, \sigma, y) &= C(h(\theta), h(\sigma))a_0(h(\theta))a_0(h(\sigma)) := C_0(h(\theta), h(\sigma)), \\ b_1(\theta, \sigma, y) &= C(h(\theta), h(\sigma))a_1(h(\theta), y)a_0(h(\sigma)), \\ b_2(\theta, \sigma, y) &= C(h(\theta), h(\sigma))a_0(h(\theta))\overline{a_1(h(\sigma), y)}, \\ \text{and } b_3(\theta, \sigma, y) &= C(h(\theta), h(\sigma))a_1(h(\theta), y)\overline{a_1(h(\sigma), y)}. \end{aligned}$$

For each $n \in \{1, 2, 3\}$, using $2d$ times Lemma 3 as previously, we obtain that

$$|\Phi_n^N(s, t, x)| \leq \max_{\kappa \in \{0,1\}^{2d}} \sup_{(\theta, \sigma) \in [0, T]^{2d}} \left| \partial^\kappa b_n\left(\theta, \sigma, \frac{x}{N}\right) \right| g(x_j)^2$$

with g defined by (13). Thanks to Condition (A), for almost every x ,

$$\lim_{N \rightarrow \infty} \sum_{n=1}^3 \Phi_n^N(s, t, x) = 0.$$

Moreover, by the Riemann sum theorem, for almost every x ,

$$\lim_{N \rightarrow \infty} \Phi_0^N(s, t, x) = \int_{\mathcal{D}_t} d\theta \int_{\mathcal{D}_s} d\sigma C_0(h(\theta), h(\sigma)) \prod_{j=1}^d e^{i(\theta_j - \sigma_j)x_j} |x_j|^{1 - h_j(\theta_j) - h_j(\sigma_j)}.$$

Hence, by the bounded convergence theorem we get

$$\lim_{N \rightarrow +\infty} \mathbb{E}[S_h^N(t)S_h^N(s)] = \int_{\mathbb{R}^d} dy \int_{\mathcal{D}_t \times \mathcal{D}_s} C_0(h(\theta), h(\sigma)) e^{i\langle y, \theta - \sigma \rangle} \prod_{j=1}^d |y_j|^{1 - h_j(\theta_j) - h_j(\sigma_j)} d\theta d\sigma.$$

Since S_h^N is a real-valued centered Gaussian random field, this proves the convergence of the finite-dimensional distributions of S_h^N to those of a real-valued centered Gaussian random field S_h whose covariance is given by (10). Moreover, if $C \equiv 1$, the integral representation (12) of S_h is a direct consequence of the isometry property of \widehat{W} on the space of square integrable functions. \square

3 Sample path properties of the limit field

This section is devoted to the smoothness of the sample paths and the self-similarity properties of the limit field S_h . To this aim, let us first introduce and study some directional increments of S_h .

3.1 Covariance structure of directional increments

As usual, (e_1, \dots, e_d) denotes the canonical basis of \mathbb{R}^d . Let us consider a random field $Y = \{Y(t), t \in \mathbb{R}^d\}$ and $\delta \in \mathbb{R}$. Then for every $k = 1, \dots, d$ and $\delta \in \mathbb{R}$, the directional increment field $\{\Delta_\delta^{(k)} Y(t), t \in \mathbb{R}^d\}$ is defined by

$$\Delta_\delta^{(k)} Y(t) = Y(t + \delta e_k) - Y(t).$$

Then, the rectangular increment of Y at point t in direction (k_1, \dots, k_p) of size $\delta = (\delta_1, \dots, \delta_p)$ is defined by

$$\Delta^{(k_1, \dots, k_p)} Y(t, \delta) = \Delta_{\delta_p}^{(k_p)} \Delta_{\delta_{p-1}}^{(k_{p-1})} \dots \Delta_{\delta_1}^{(k_1)} Y(t).$$

Observe that the increment $\Delta^{(1, \dots, d)} Y(t, s-t)$ is the classical rectangular increment between t and s of the random field Y . For the sake of simplicity, we set

$$\Delta Y(t, s) := \Delta^{(1, \dots, d)} Y(t, s-t) = (-1)^d \sum_{\kappa_1=0}^1 \dots \sum_{\kappa_d=0}^1 (-1)^{\sum_{i=1}^d \kappa_i} Y(t + \kappa \star (s-t))$$

where for a and $b \in \mathbb{R}^k$,

$$a \star b = (a_1 b_1, a_2 b_2 \dots, a_k b_k).$$

All the properties of S_h we study are consequences of the Proposition 1 stated below, which focuses on the behavior of the covariance

$$\mathbb{E}\left(\Delta^{(k_1, \dots, k_p)} S_h(t, \nu \star u) \Delta^{(k_1, \dots, k_p)} S_h(t, \nu \star v)\right) \quad (14)$$

as ν tends to $0_{\mathbb{R}_+^p}$ for any (k_1, \dots, k_p) . In particular, since S_h is a real-valued Gaussian random field, these behaviors are related to some local asymptotic self-similarity properties and to some Hölder smoothness properties, see Sections 3.2 to 3.5.

Proposition 1. *Let $p \in \{1, \dots, d\}$ and $k = (k_1, \dots, k_p) \in \{1, \dots, d\}^p$ with $k_l \neq k_j$ if $l \neq j$. Then, for every $T \in \mathbb{R}_+$,*

$$\lim_{\nu \rightarrow 0_{\mathbb{R}_+^p}} \sup_{\substack{u, v \in [0, T]^p \\ t \in [0, T]^d}} \left| \frac{\mathbb{E}[\Delta^{(k_1, \dots, k_p)} S_h(t, \nu \star u) \Delta^{(k_1, \dots, k_p)} S_h(t, \nu \star v)]}{\nu_1^{2h_{k_1}(t_{k_1})} \dots \nu_p^{2h_{k_p}(t_{k_p})}} - E_{h,1}^k(t, u, v) E_{h,2}^k(t) \right| = 0$$

where

$$E_{h,1}^k(t, u, v) = \int_{\mathbb{R}^p} dy \prod_{j=1}^p \frac{(e^{iy_j u_j} - 1)(e^{-iy_j v_j} - 1)}{|y_j|^{2h_{k_j}(t_{k_j})+1}} \quad (15)$$

and

$$E_{h,2}^k(t) = \int_{\mathbb{R}^{d-p}} dy \int_{\mathcal{D}_t^k \times \mathcal{D}_t^k} d\xi d\eta \prod_{j \in J(k)} \frac{e^{iy_j(\xi_j - \eta_j)}}{|y_j|^{h_j(\xi_j) + h_j(\eta_j) - 1}} C_0\left(h\left(\xi^{k,t}\right), h\left(\eta^{k,t}\right)\right) \quad (16)$$

with $J(k) = \{1, \dots, d\} \setminus \{k_1, \dots, k_p\}$, $\mathcal{D}_t^k = \prod_{j \in J(k)} [0, t_j]$ and

$$\left(\zeta^{k,t}\right)_j = \begin{cases} \zeta_j & \text{if } j \in J(k) \\ t_j & \text{else.} \end{cases}$$

Here, by convention, $E_{h,2}^{(1, \dots, d)}(t) = C_0(h(t), h(t)) = a_0(h(t))^2 C(h(t), h(t))$.

Remark 4. $E_{h,1}^k(t, \cdot, \cdot)$ is the covariance function of the fractional Brownian sheet $B_{(h_{k_1}(t_{k_1}), \dots, h_{k_p}(t_{k_p}))}$ defined by

$$B_{(h_{k_1}(t_{k_1}), \dots, h_{k_p}(t_{k_p}))}(u) = \int_{\mathbb{R}^p} \prod_{j=1}^p \frac{e^{iy_j u_j} - 1}{iy_j |y_j|^{h_{k_j}(t_{k_j}) - 1/2}} \widehat{W}(dy), \quad u \in \mathbb{R}_+^p. \quad (17)$$

Moreover, if we denote $J(k) = \{k_{p+1}, \dots, k_d\}$ with $k_l < k_{l+1}$, then

$$E_{h,2}^k(t) = \text{Var} \tilde{S}(t_{k_{p+1}}, \dots, t_{k_d}) \in \mathbb{R}_+$$

where \tilde{S} satisfies the same kind of invariance principle as S_h (that is, which can be introduced by Theorem 1 replacing d by $d - p$ and with suitable functions C, a and h).

Let us now prove Proposition 1.

Proof. Let $u, v \in [0, T]^p$, $t \in [0, T]^d$ and $\nu \in (0, 1]^p$. Because of the symmetry properties of (10), we can assume without loss of generality that $k = (1, \dots, p)$. Then, we study

$$I_\nu(t, u, v) = \frac{\mathbb{E}[\Delta^{(1, \dots, p)} S_h(t, \nu \star u) \Delta^{(1, \dots, p)} S_h(t, \nu \star v)]}{\nu_1^{2h_1(t_1)} \dots \nu_p^{2h_p(t_p)}}.$$

By definition of $\Delta^{(1, \dots, p)} S_h$ and by (10), we have:

$$I_\nu(t, u, v) = \prod_{j=1}^p \nu_j^{-2h_j(t_j)} \int_{\mathbb{R}^d} dx \int_{\mathcal{D}_{t, \nu \star u} \times \mathcal{D}_{t, \nu \star v}} d\theta d\sigma \prod_{j=1}^d \frac{e^{i(\theta_j - \sigma_j)x_j}}{|x_j|^{h_j(\theta_j) + h_j(\sigma_j) - 1}} C_0(h(\theta), h(\sigma))$$

where for all $\delta \in \mathbb{R}_+^p$, $\mathcal{D}_{t, \delta} = \prod_{j=1}^p [t_j, t_j + \delta_j] \times \prod_{j=p+1}^d [0, t_j]$. Some substitutions lead to

$$I_\nu(t, u, v) = \int_{\mathbb{R}^d} dy \int_{\mathcal{D}_{t, u} \times \mathcal{D}_{t, v}} d\xi d\eta \Phi_{1, t, \nu}(y, \xi, \eta) \Phi_2(y, \xi, \eta) C_0(h(\xi(t^p, \nu)), h(\eta(t^p, \nu)))$$

where for all $w \in \mathbb{R}_+^d$, $w^p = (w_1, \dots, w_p)$,

$$\Phi_{1, t, \nu}(y, \xi, \eta) = \prod_{j=1}^p \frac{e^{i(\xi_j - \eta_j)y_j} \nu_j^{h_j(\nu_j \xi_j + t_j) + h_j(\nu_j \eta_j + t_j) - 2h_j(t_j)}}{|y_j|^{h_j(\nu_j \xi_j + t_j) + h_j(\nu_j \eta_j + t_j) - 1}},$$

$$\Phi_2(y, \xi, \eta) = \prod_{j=p+1}^d \frac{e^{i(\xi_j - \eta_j)y_j}}{|y_j|^{h_j(\xi_j) + h_j(\eta_j) - 1}}$$

and

$$\forall \vartheta \in \mathbb{R}_+^p, (\zeta(\vartheta, \nu))_j = \begin{cases} \nu_j \zeta_j + \vartheta_j & \text{if } 1 \leq j \leq p \\ \zeta_j & \text{if } p+1 \leq j \leq d. \end{cases}$$

Note that $\Phi_{1, t, \nu}(y, \xi, \eta)$ (respectively $\Phi_2(y, \xi, \eta)$) does not depend on $(y_j, \xi_j, \eta_j, t_j)_{p+1 \leq j \leq d}$ (respectively $(y_j, \xi_j, \eta_j, t_j)_{1 \leq j \leq p}$).

Let us fix $y \in \mathbb{R}^d$ with $y_j \neq 0$ for every $j \in \{1, \dots, d\}$. First let us consider $\mathfrak{C}_y \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^p, \mathbb{C})$ the function defined by

$$\mathfrak{C}_y(\vartheta, \varsigma) = \int_{\mathcal{D}_t^{(1, \dots, p)} \times \mathcal{D}_t^{(1, \dots, p)}} d\xi_{p+1} \dots d\xi_d d\eta_{p+1} \dots d\eta_d \Phi_2(y, \xi, \eta) C_0(h(\xi(\vartheta, 0)), h(\eta(\varsigma, 0)))$$

with $\mathcal{D}_t^{(1, \dots, p)} = \prod_{j=p+1}^d [0, t_j]$.

Let $j \in \{1, \dots, p\}$. We define the operator $I_{j,1}^\rho$ from $\mathcal{C}([0, 2T]^j \times [0, 2T]^{j-1}, \mathbb{C})$ to $\mathcal{C}([0, 2T]^{j-1} \times [0, 2T]^{j-1}, \mathbb{C})$ by

$$I_{j,1}^\rho \varphi(\vartheta, \varsigma) = \begin{cases} \int_0^{u_j} d\theta \frac{\rho^{h_j(\rho\theta+t_j)-h_j(t_j)} e^{iy_j\theta}}{|y_j|^{h_j(\rho\theta+t_j)-1/2}} \varphi(\vartheta, t_j + \rho\theta, \varsigma) & \text{if } \rho \in (0, 1] \\ \int_0^{u_j} d\theta \frac{e^{iy_j\theta}}{|y_j|^{h_j(t_j)-1/2}} \varphi(\vartheta, t_j, \varsigma) = \frac{e^{iy_j u_j} - 1}{iy_j |y_j|^{h_j(t_j)-1/2}} \varphi(\vartheta, t_j, \varsigma) & \text{if } \rho = 0 \end{cases}$$

for any $\vartheta \in [0, 2T]^{j-1}$, $\varsigma \in [0, 2T]^{j-1}$ and $\varphi \in \mathcal{C}([0, 2T]^j \times [0, 2T]^{j-1}, \mathbb{C})$. For the sake of simplicity, we do not emphasize the dependence of the operator $I_{j,1}^\rho$ on (y, u) .

We also define the operator $I_{j,2}^\rho$ from $\mathcal{C}([0, 2T]^j \times [0, 2T]^j, \mathbb{C})$ to $\mathcal{C}([0, 2T]^j \times [0, 2T]^{j-1}, \mathbb{C})$ by

$$I_{j,2}^\rho \varphi(\vartheta, \varsigma) = \begin{cases} \int_0^{v_j} d\sigma \frac{\rho^{h_j(\rho\sigma+t_j)-h_j(t_j)} e^{-iy_j\sigma}}{|y_j|^{h_j(\rho\sigma+t_j)-1/2}} \varphi(\vartheta, \varsigma, t_j + \rho\sigma) & \text{if } \rho \in (0, 1] \\ \int_0^{v_j} d\sigma \frac{e^{-iy_j\sigma}}{|y_j|^{h_j(t_j)-1/2}} \varphi(\vartheta, \varsigma, t_j) = \frac{1 - e^{-iy_j v_j}}{iy_j |y_j|^{h_j(t_j)-1/2}} \varphi(\vartheta, \varsigma, t_j) & \text{if } \rho = 0 \end{cases}$$

for any $\vartheta \in [0, 2T]^j$, $\varsigma \in [0, 2T]^{j-1}$ and $\varphi \in \mathcal{C}([0, 2T]^j \times [0, 2T]^j, \mathbb{C})$. As for $I_{j,1}^\rho$ we do not emphasize the dependence of the operator $I_{j,2}^\rho$ on (y, v) .

We also consider the operator $I_j^\rho = I_{j,1}^\rho I_{j,2}^\rho$. Then the Fubini Theorem leads to

$$I_\nu(t, u, v) = \int_{\mathbb{R}^d} dy I_1^{\nu_1} \dots I_p^{\nu_p} \mathfrak{E}_y.$$

By the Fubini Theorem, we also have

$$\int_{\mathbb{R}^d} dy I_1^0 \dots I_p^0 \mathfrak{E}_y = E_{h,1}^k(t, u, v) E_{h,2}^k(t)$$

with $E_{h,1}^k(t, u, v)$ defined by (15) and $E_{h,2}^k(t)$ by (16). Therefore

$$\begin{aligned} \left| I_\nu(t, u, v) - E_{h,1}^k(t, u, v) E_{h,2}^k(t) \right| &= \left| \int_{\mathbb{R}^d} dy (I_1^{\nu_1} \dots I_p^{\nu_p} - I_1^0 \dots I_p^0) \mathfrak{E}_y \right| \\ &\leq \sum_{j=1}^p \left(\int_{\mathbb{R}^d} dy \left| I_1^{\nu_1} \dots I_{j-1}^{\nu_{j-1}} I_j^{y, u, \nu_{j-1}} (I_{j,1}^{\nu_j} - I_{j,1}^0) I_{j,2}^0 I_{j+1}^0 \dots I_p^0 \mathfrak{E}_x \right| \right. \\ &\quad \left. + \int_{\mathbb{R}^d} dy \left| I_1^{\nu_1} \dots I_{j-1}^{\nu_{j-1}} I_{j,1}^{\nu_j} (I_{j,2}^{\nu_j} - I_{j,2}^0) I_{j+1}^0 \dots I_p^0 \mathfrak{E}_x \right| \right) \end{aligned}$$

Thanks to Lemma 5 we have the following properties for $I_{j,1}^\rho$ and $I_{j,2}^\rho$. Let $y \in \mathbb{R}_+^d$ with $y_j \neq 0$ and $\rho \in (0, 1]$.

(i) Let $\varphi \in \mathcal{C}([0, 2T]^j \times [0, 2T]^{j-1}, \mathbb{C})$. Then, for every $\vartheta, \varsigma \in [0, 2T]^{j-1}$.

$$\left| (I_{j,1}^\rho - I_{j,1}^0) \varphi(\vartheta, \varsigma) \right| \leq c_1 \rho |\ln(\rho)| g(y_j) \max \left(\max_{w \in [0, 2T]} |\varphi(\vartheta, w, \varsigma)|, \max_{w \in [0, 2T]} |\partial_w \varphi(\vartheta, w, \varsigma)| \right)$$

where $c_1 = c_1(T, h)$ only depends on T and h and

$$g(r) = \left(\frac{1}{|r|^{1/2+H_-}} \mathbf{1}_{|r| \geq 1} + \frac{1}{|r|^{H_+-1/2}} \mathbf{1}_{|r| < 1} \right) (1 + |\ln |r||) \quad (18)$$

with $H_+ = \max_{1 \leq k \leq d} \max_{[0, 2T]} h_k$ and $H_- = \min_{1 \leq k \leq d} \min_{[0, 2T]} h_k$. Moreover,

$$\left| I_{j,1}^\rho \varphi(\vartheta, \varsigma) \right| \leq c_1 g(y_j) \max \left(\max_{w \in [0, 2T]} |\varphi(\vartheta, w, \varsigma)|, \max_{w \in [0, 2T]} |\partial_w \varphi(\vartheta, w, \varsigma)| \right)$$

Let us remark that this last inequality also holds for $\rho = 0$ choosing $c_1 \geq 1$.

(ii) Let $\varphi \in \mathcal{C}([0, 2T]^j \times [0, 2T]^j, \mathbb{C})$. Then for every $\vartheta \in [0, 2T]^j$ and $\varsigma \in [0, 2T]^{j-1}$,

$$\left| (I_{j,2}^\rho - I_{j,2}^0) \varphi(\vartheta, \varsigma) \right| \leq c_1 \rho |\ln(\rho)| g(y_j) \max \left(\max_{w \in [0, 2T]} |\varphi(\vartheta, \varsigma, w)|, \max_{w \in [0, 2T]} |\partial_w \varphi(\vartheta, \varsigma, w)| \right)$$

and

$$\left| I_{j,2}^\rho \varphi(\vartheta, \varsigma) \right| \leq c_1 g(y_j) \max \left(\max_{w \in [0, 2T]} |\varphi(\vartheta, \varsigma, w)|, \max_{w \in [0, 2T]} |\partial_w \varphi(\vartheta, \varsigma, w)| \right).$$

Let us remark that this last inequality also holds for $\rho = 0$.

Since $\partial_w I_{j,k}^\rho \varphi = I_{j,k}^\rho \partial_w \varphi$ ($k \in \{1, 2\}$), by applying (i) and (ii), we obtain:

$$\begin{aligned} & \left| I_\nu(t, u, v) - E_{h,1}^k(t, u, v) E_{h,2}^k(t) \right| \\ & \leq 2c_1^p \sum_{j=1}^p \nu_j |\ln \nu_j| \int_{\mathbb{R}^d} dy \prod_{j=1}^p g(y_j)^2 \max_{\varepsilon \in \{0,1\}^{2p}} \sup_{\vartheta, \varsigma \in [0, 2T]^p} \left| \partial_{\vartheta_1}^{\varepsilon_1} \cdots \partial_{\vartheta_p}^{\varepsilon_p} \partial_{\varsigma_1}^{\varepsilon_{p+1}} \cdots \partial_{\varsigma_p}^{\varepsilon_{2p}} \mathbf{c}_x \right| \end{aligned}$$

Moreover, applying Lemma 4, we obtain:

$$\begin{aligned} & \max_{\varepsilon \in \{0,1\}^{2p}} \sup_{\vartheta, \varsigma \in [0, 2T]^p} \left| \partial_{\vartheta_1}^{\varepsilon_1} \cdots \partial_{\vartheta_p}^{\varepsilon_p} \partial_{\varsigma_1}^{\varepsilon_{p+1}} \cdots \partial_{\varsigma_p}^{\varepsilon_{2p}} \mathbf{c}_x \right| \\ & \leq c_2 \prod_{j=p+1}^d g(x_k)^2 \max_{\varepsilon \in \{0,1\}^{2d}} \sup_{\vartheta, \varsigma \in [0, 2T]^d} \left| \partial_{\vartheta_1}^{\varepsilon_1} \cdots \partial_{\vartheta_d}^{\varepsilon_d} \partial_{\varsigma_1}^{\varepsilon_{d+1}} \cdots \partial_{\varsigma_d}^{\varepsilon_{2d}} \tilde{C}_0 \right| \end{aligned}$$

where $\tilde{C}_0(\vartheta, \varsigma) = C_0(h(\vartheta), h(\varsigma))$ and where the finite positive constant c_2 only depends on (T, h) . Therefore,

$$\left| I_\nu(t, u, v) - E_{h,1}^k(t, u, v) E_{h,2}^k(t) \right| \leq c_3 \sum_{j=1}^p \nu_j |\ln \nu_j|$$

with $c_3 = 2c_1^p c_2 \|g\|_{L^2(\mathbb{R})}^{2d} \max_{\varepsilon \in \{0,1\}^{2d}} \sup_{\vartheta, \varsigma \in [0, 2T]^d} \left| \partial_{\vartheta_1}^{\varepsilon_1} \cdots \partial_{\vartheta_d}^{\varepsilon_d} \partial_{\varsigma_1}^{\varepsilon_{d+1}} \cdots \partial_{\varsigma_d}^{\varepsilon_{2d}} \tilde{C}_0 \right|$. One can easily check that $c_3 < \infty$, which concludes the proof. \square

3.2 Local asymptotic self-similarity properties

In general, a multifractional Brownian sheet satisfies neither a global self-similarity property (1) or (2) nor a local asymptotically self-similarity property in the sense of [4]. However, restricted on a line directed by e_k , it is locally self-similar. Moreover, it also satisfies a local version, introduced in [16], of the self-similarity property (2). This local property, which involves its rectangular increments, takes account of its local anisotropic behavior. The next proposition proves, in particular, that all these self-similarity properties also hold for the limit field S_h . Since S_h is a centered real-valued Gaussian random field, it is a direct consequence of Proposition 1.

Proposition 2. *Let $p \in \{1, \dots, d\}$ and $k = (k_1, \dots, k_p) \in \{1, \dots, d\}^p$ with $k_l \neq k_j$ if $l \neq j$. Then, for any $t \in [0, +\infty)^d$,*

$$\lim_{\varepsilon \rightarrow 0^+_{\mathbb{R}^p}} \left\{ \left(\prod_{j=1}^p \varepsilon_j^{-h_{k_j}(t_{k_j})} \right) \Delta^{(k_1, \dots, k_p)} S_h(t, \varepsilon \star u), u \in [0, +\infty)^p \right\} \stackrel{\text{dist.}}{=} \{ \mathcal{T}_{h,k,t}(u), u \in [0, +\infty)^p \}$$

where

$$\mathcal{T}_{h,k,t}(u) = \left(E_{h,2}^k(t) \right)^{1/2} B_{(h_{k_1}(t_{k_1}), \dots, h_{k_p}(t_{k_p}))}(u)$$

with $E_{h,2}^k(t)$ defined by (16) and $B_{(h_{k_1}(t_{k_1}), \dots, h_{k_p}(t_{k_p}))}$ the fractional Brownian sheet defined by (17).

Observe that the limit field $\mathcal{T}_{h,k,t}$ is non-degenerate if and only if $E_{h,2}^k(t) \neq 0$. In the following remark, we assume that $\mathcal{T}_{h,k,t}$ is a non-degenerate random field.

Remark 5. *Choosing $p = 1$ and $k \in \{1, \dots, d\}$, Proposition 2 can be written as*

$$\lim_{\varepsilon \rightarrow 0^+} \left\{ \varepsilon^{-h_k(t_k)} (S_h(t + \varepsilon u e_k) - S_h(t)), u \in [0, +\infty) \right\} \stackrel{\text{dist.}}{=} \{ \mathcal{T}_{h,k,t}(u), u \in [0, +\infty) \}, \quad (19)$$

where the tangent process $\mathcal{T}_{h,k,t}$ is a fractional Brownian motion of order $h_k(t_k)$. In other words, S_h is locally asymptotically self-similar with respect to k^{th} -component at point t with tangent field a fractional Brownian motion.

3.3 Modulus of continuity and Hölder exponents

Since S_h is a centered real-valued Gaussian random field, its sample path properties are related to the behavior of its variogram. Hence, we start by giving an upper bound this variogram, that is we give a Kolmogorov-type estimate for the increments of S_h which takes account of the anisotropic behavior of S_h .

Lemma 1. *Let $T \in [0, +\infty)$. There exists two finite positive constants $c > 0$ and $\eta > 0$ such that, for every t and s in $[0, T]^d$ with $\|t - s\| \leq \eta$,*

$$\mathbb{E}[(S_h(t) - S_h(s))^2] \leq c \left(\sum_{j=1}^d |t_j - s_j|^{h_j(\min(t_j, s_j))} \right)^2. \quad (20)$$

Proof. Let $s, t \in [0, T]^d$. Observe that we can assume without loss of generality that $t_j \neq s_j$ for all $1 \leq j \leq d$. Otherwise $S_h(t) = S_h(s)$ and (20) holds for any $c \in (0, +\infty)$. Let us now consider the sequence $(u^{(j)})_{0 \leq j \leq d}$ of $[0, T]^d$ defined by

$$u(0) = t \quad \text{and} \quad u(j+1) = u(j) + (s_{j+1} - t_{j+1})e_{j+1}.$$

Observe that $u(d) = s$ and for any $0 \leq j \leq d-1$, $u(j+1) - u(j) = (s_{j+1} - t_{j+1})e_{j+1}$. Then using the Minkowski inequality, we have

$$\mathbb{E} \left[(S_h(t) - S_h(s))^2 \right] \leq \left(\sum_{j=1}^d \mathbb{E} \left[\{S_h(u(j)) - S_h(u(j-1))\}^2 \right]^{1/2} \right)^2.$$

Let $1 \leq j \leq d$. By Proposition 1 (applied with $p = 1$ and $k = j$), there exists r_j such that for all $\nu \in (0, r_j)$,

$$\sup_{v \in [0, T]^d} \left| \nu^{-2h_j(v_j)} \mathbb{E} \left[(S_h(v) - S_h(v + \nu e_j))^2 \right] - E_{h,1}^j(v, e_j, e_j) E_{h,2}^j(v) \right| \leq 1.$$

with $E_{h,1}^j$ and $E_{h,2}^j$ defined by (15) and (16). Since for any $1 \leq j \leq d$, $v \mapsto E_{h,1}^j(v, e_j, e_j)$ and $v \mapsto E_{h,2}^j(v)$ are continuous on the compact set $[0, T]^d$, there exists a finite positive constant c such that for all $1 \leq j \leq d$, for all $\nu \in (0, \min_{1 \leq k \leq d} r_k)$ and for all $v \in [0, T]^d$,

$$\mathbb{E} \left[(S_h(v) - S_h(v + \nu e_j))^2 \right] \leq c \nu^{2h_j(v_j)}.$$

Applying this inequality with $v = u(j-1)$ and $\nu = s_j - t_j$ (respectively $v = u(j)$ and $\nu = t_j - s_j$) if $s_j > t_j$ (respectively if $s_j < t_j$), one obtains that there exists $\eta \in (0, 1)$ such that

$$\mathbb{E} \left[(S_h(u(j)) - S_h(u(j-1)))^2 \right] \leq c |t_j - s_j|^{2h_j(\min(t_j, s_j))}$$

for $s, t \in [0, T]^d$ with $\|t - s\| \leq \eta$. This leads to the conclusion. \square

Observe that, by Lemma 1, for any compact set K of \mathbb{R}^d , there exists a finite positive constant $c = c(K)$ such that

$$\forall s, t \in K, \mathbb{E} \left[(S_h(t) - S_h(s))^2 \right] \leq c \rho_K(s, t)^2 \quad (21)$$

with ρ_K the metric defined on \mathbb{R}^d by

$$\rho_K(s, t) := \sum_{j=1}^d |s_j - t_j|^{\min_K h_j}, \quad \forall s, t \in \mathbb{R}^d. \quad (22)$$

Henceforth S_h admits a continuous version on \mathbb{R}^d , which we still denote by S_h .

To conclude this section, we are interested in the modulus of continuity of the continuous Gaussian random field S_h . Many authors have studied moduli of continuity in a Gaussian setting, e.g. see [14, 20, 4, 2, 24, 19]. In particular, [2] establishes a sharp modulus of continuity for fractional Brownian sheets using a wavelet expansion. Here, we cannot follow the approach of [2] since it is based on an integral representation. Nevertheless, the upper bound we give is sufficient to obtain the Hölder regularity properties of S_h .

Proposition 3. For any non-empty compact set K of \mathbb{R}^d and any $\eta > 0$,

$$\lim_{\delta \rightarrow 0} \sup_{\substack{s, t \in K \\ 0 < \|t-s\| \leq \delta}} \frac{|S_h(t) - S_h(s)|}{\rho_K(s, t) |\log \rho_K(s, t)|^{1/2+\eta}} = 0 \quad \text{almost surely}$$

where the metric ρ_K is defined by (22).

Proof. Let K be a non-empty compact set of \mathbb{R}^d and E be the diagonal matrix given by

$$E = \text{diag}\left(\frac{1}{\min_K h_1}, \dots, \frac{1}{\min_K h_d}\right).$$

Since the eigenvalues of E are positive, according to Chapter 6 of [18], there exists an unit sphere S_E (for a suitable norm) such that we can write any $\xi \in \mathbb{R}^d \setminus \{0\}$ uniquely as

$$\xi = \tau_E(t)^E \ell_E(t)$$

with $\tau_E(t) > 0$ and $\ell_E(t) \in S_E$. By convention $\tau_E(0) = 0$. Then, for any $t, s \in \mathbb{R}^d$ such that $t \neq s$,

$$\rho_K(t, s) = \rho_K(0, t-s) = \tau_E(t-s) \rho_K(0, \ell_E(t-s))$$

by definition of E and ρ_K . Therefore by continuity and positiveness of $u \mapsto \rho_K(0, u)$ on the compact set S_E , there exists two finite positive constants c_1 and c_2 such that

$$\forall s, t \in \mathbb{R}^d, c_1 \tau_E(t-s) \leq \rho_K(t, s) \leq c_2 \tau_E(t-s). \quad (23)$$

The upper bound in (23) and Lemma 1 prove that the centered Gaussian random field S_h satisfies the assumption of Proposition 5.3 of [6] (with $\beta = 0$). Both this proposition and (23) lead to the conclusion. \square

The following corollary is a direct consequence of the previous proposition.

Corollary 1. Let $K = \prod_{j=1}^d [a_j, b_j]$ be a non-empty compact set of \mathbb{R}^d . Let $H_j = \min_{[a_j, b_j]} h_j$ for $1 \leq j \leq d$ and let $H = \min_{1 \leq j \leq d} H_j$.

1. Then for any $\varepsilon \in (0, H)$, S_h is Hölderian on K of order $H - \varepsilon$.
2. Let $t \in K$ and $K_j = \{t + re_j \in K / r \in \mathbb{R}\}$. Then, for any $\varepsilon \in (0, H_j)$, S_h is Hölderian on K_j of order $H_j - \varepsilon$.

3.4 Multi-Hölder continuity property

The notion of double Hölder continuity has been introduced in [10]. This property, which is defined in the framework of functions of two variables, is related to rectangular increments and can be readily extended to functions of d variables as follows.

Definition 1. Let $U \subset \mathbb{R}^d$. A function $\varphi : U \rightarrow \mathbb{R}$ is said to be multi-Hölder continuous on U of indexes $(\alpha_1, \dots, \alpha_d) \in (0, \infty)^d$ if there exists a finite constant $c > 0$ such that

$$\forall s, t \in U, |\Delta\varphi(t, s)| \leq c \prod_{j=1}^d |s_j - t_j|^{\alpha_j}$$

where $\Delta\varphi(t, s)$ is the rectangular increment of φ between s and t (see Section 3.1).

D. Feyel and A. De La Pradelle have also introduced in [10] a framework to study double Hölder continuity property for a two-variable random field Y . More precisely, they prove a Kolmogorov-type lemma (Lemma 17 page 283 of [10]) by means of Liouville spaces. By following the same lines as their proof, one obtains the lemma below.

Lemma 2. *Let $Y = (Y(t), t \in [0, \infty)^d)$ be a continuous real-valued random field and let $K \subset [0, \infty)^d$ be a non-empty compact set. Assume that there exist $c \in (0, \infty)$, $q \in (0, \infty)$, $\eta \in (0, 1]$ and a vector of indexes $(\beta_1, \dots, \beta_d) \in (0, \infty)^d$ such that, for every $(s, t) \in K \times K$ satisfying $\|t - s\| \leq \eta$,*

$$\mathbb{E}[(\Delta Y(s, t))^q] \leq c \prod_{j=1}^d |t_j - s_j|^{1+\beta_j}. \quad (24)$$

Then, for every $(\alpha_1, \dots, \alpha_d) \in \prod_{j=1}^d [0, \beta_j/q)$, almost surely, Y is multi-Hölder on K of indexes $(\alpha_1, \dots, \alpha_d)$.

We can now establish a multi-Hölder continuity property for S_h .

Proposition 4. *Let $K = \prod_{j=1}^d [a_j, b_j]$ be a non-empty compact set of $[0, \infty)^d$. Then, almost surely, the random field S_h is multi-Hölder on K of indexes $(\alpha_1, \dots, \alpha_d)$ for every $(\alpha_1, \dots, \alpha_d) \in \prod_{j=1}^d [0, \min_{[a_j, b_j]} h_j)$.*

Proof. Let $s, t \in K$. We can check that

$$|\Delta S_h(t, s)| = |\Delta S_h(\min(s, t), \max(s, t))|$$

where $\min(s, t) = (\min(t_1, s_1), \dots, \min(t_d, s_d))$ and $\max(s, t) = (\max(t_1, s_1), \dots, \max(t_d, s_d))$. Hence, without loss of generality, we assume that $t_j \geq s_j$ for each $j = 1, \dots, d$. Because of Lemma 1, there exists $\eta \in (0, 1]$ so that for every $\nu \in (0, \infty)^d$ with $\|\nu\| \leq \eta$ and every $v \in K$

$$\begin{aligned} \mathbb{E}[(\Delta S_h(v, v + \nu))^2] &\leq (1 + E_{h,1}^{(1,\dots,d)}(v, \vec{1}, \vec{1}) E_{h,2}^{(1,\dots,d)}(v)) \prod_{j=1}^d \nu_j^{2h_j(v_j)} \\ &\leq \sup_{u \in K} (1 + E_{h,1}^{(1,\dots,d)}(u, \vec{1}, \vec{1}) E_{h,2}^{(1,\dots,d)}(u)) \prod_{j=1}^d \nu_j^{2 \min_{[a_j, b_j]} h_j} \end{aligned}$$

where $\vec{1} = (1, 1, \dots, 1) \in \mathbb{R}^d$. Observe that $C = \sup_{u \in K} (1 + E_{h,1}^{(1,\dots,d)}(u, \vec{1}, \vec{1}) E_{h,2}^{(1,\dots,d)}(u)) < \infty$. Then, choosing $v = t$ and $\nu = t - s$ and since S_h is Gaussian, $Y = S_h$ satisfies (24) on K with $\beta_j = \min_{[a_j, b_j]} h_j - 1/q$. The conclusion follows from Lemma 2. \square

3.5 Hölder exponents

Let us first focus on the pointwise Hölder exponent

$$H_{S_h, k}(t) := \sup \left\{ H > 0 : \lim_{r \rightarrow 0} \frac{S_h(t + r e_k) - S_h(t)}{|r|^H} \right\}$$

of S_h at point t on the direction e_k and on the global pointwise Hölder exponent

$$H_{S_h}(t) := \sup \left\{ H > 0 : \lim_{s \rightarrow 0} \frac{S_h(t + s) - S_h(t)}{\|s\|^H} \right\}$$

of S_h at point t . We briefly explain how to derive them from the previous sections. On one hand, Corollary 1 gives a lower bound of these exponents, which is improved by applying Proposition 4 when t has at least one null coordinate. On the other hand, the upper bound is derived from Proposition 2 in which the limit field has to be non-degenerate.

Proposition 5. *Let $1 \leq k \leq d$, $t = (t_1, \dots, t_d) \in \mathbb{R}_+^d$ and $N_t = \{1 \leq l \leq d; t_l = 0\}$.*

1. (a) *If $N_t \setminus \{k\} \neq \emptyset$, $H_{S_h, k}(t) = +\infty$ almost surely.*
 (b) *If $N_t \setminus \{k\} = \emptyset$ and $E_{h,2}^k(t) \neq 0$, then $H_{S_h, k}(t) = h_k(t_k)$ almost surely.*
2. (a) *Assume that $h_k(t_k) = \min_{1 \leq j \leq d} h_j(t_j)$. If $E_{h,2}^k(t) \neq 0$, then almost surely*

$$H_{S_h}(t) = h_k(t_k) = \min_{1 \leq j \leq d} h_j(t_j).$$

- (b) *Assume that $N_t = \{k_1, \dots, k_p\}$ with $1 \leq p \leq d$. If $E_{h,2}^{(k_1, \dots, k_p)}(t) \neq 0$, then almost surely*

$$H_{S_h}(t) = \sum_{j=1}^p h_{k_j}(0) = \sum_{l \in N_t} h_l(t_l).$$

Let us now comment the assumption of the previous proposition.

Remark 6. *The condition*

$$E_{h,2}^k(t) \neq 0. \tag{25}$$

is equivalent to assuming that the tangent field $\mathcal{T}_{h,k,t}$ is non-degenerate. Then, in view of (16), k has to be chosen (as done in the previous proposition) such that $t_j \neq 0$ for $j \notin \{k_1, \dots, k_p\}$. Moreover, (25) also implies that there exist $\theta, \sigma \in \mathcal{D}_t$, such that

$$a_0(h(\theta))a_0(h(\sigma))C(h(\theta), h(\sigma)) \neq 0. \tag{26}$$

It would be interesting to exhibit necessary and sufficient conditions on t and the functions C , a and h such that condition (25) holds. However, we have chosen not to focus on this question. We can easily check that $E_{h,2}^k(t) \neq 0$ in many cases, as for instance when (26) holds for any $\theta, \sigma \in \mathcal{D}_t$ or when for t small enough if (26) holds for $\theta = \sigma = 0$.

Before we prove the previous proposition, observe that the pointwise Hölder exponents may vary with the position and that the directional pointwise Hölder exponent may also depend on the direction as in the framework of multifractional Brownian sheets (see [16, 11]). Let us also mention that our proof holds for multifractional or fractional Brownian sheets. The statements of Theorem 6.11 of [16] and Proposition 13 of [11] have to be slightly rectified in the case where $N_t \neq \emptyset$.

Proof. 1. If $N_t \setminus \{k\} \neq \emptyset$, then $S_h(t + re_k) = S_h(t) = 0$ for any r and $H_{S_h, k}(t) = +\infty$. Assume now that $N_t \setminus \{k\} = \emptyset$ and $E_{h,2}^k(t) \neq 0$. By proposition 2

$$\frac{S_h(t + \varepsilon e_k) - S_h(t)}{\varepsilon^{h_k(t_k)}} \xrightarrow[\varepsilon \rightarrow 0_+]{\text{dist.}} \mathcal{T}_{h,k,t}(1) \neq 0 \quad \text{almost surely}$$

since $\mathcal{T}_{h,k,t}(1) = \sqrt{E_{h,2}^{k,1}(t)} B_{h_k(t_k)}(1)$ with $B_{h_k(t_k)}$ the fractional Brownian motion defined by (17). This implies that $H_{S_h, k}(t) \leq h_k(t_k)$ almost surely (see for example the proof of Proposition 2.3 of [3]). We conclude the proof of assertion 1. by applying Corollary 1.

2. By definition, $H_{S_h}(t) \leq \min_{1 \leq j \leq d} H_{S_h,j}(t)$. Then if $h_k(t_k) = \min_{1 \leq j \leq d} h_j(t_j)$ and $E_{h,2}^k(t) \neq 0$, assertion 2.(a) is a direct consequence of assertion 1. and of Corollary 1.

Let us now assume that $N_t = \{k_1, \dots, k_p\}$ with $1 \leq p \leq d$.

• Let us prove, using the multi-Hölder continuity property, that $H_{S_h}(t) \geq \sum_{j=1}^p h_{k_j}(0)$. To this aim, observe that for any s ,

$$S_h(t+s) = \Delta S_h(0, t+s)$$

since S_h vanishes on the axes. Then let $T \in (0, 1)$. Applying Proposition 4, one can find a finite positive random variable $c(t_{p+1}, \dots, t_d)$ such that

$$\forall s \in [0, T]^d, |S_h(t+s)| = |\Delta S_h(0, t+s)| \leq c \prod_{j=1}^p |s_j|^{\min_{[0,T]} h_{k_j} - \varepsilon},$$

which proves that $H_{S_h}(t) \geq \sum_{j=1}^p \min_{[0,T]} h_{k_j} - 2p\varepsilon$ for ε small enough. Therefore, letting ε and T tends to 0, by continuity of each h_j , we obtain that

$$H_{S_h}(t) \geq \sum_{j=1}^p h_{k_j}(0)$$

• Since S_h vanishes on the axes and $t_{k_j} = 0$ for $1 \leq j \leq p$, we have:

$$\forall \delta \in \mathbb{R}_+^p, \Delta^{(k_1, \dots, k_p)} S_h(t, \delta) = S_h\left(t + \sum_{j=0}^p \delta_j e_{k_j}\right)$$

Then, since $E_{h,2}^{(k_1, \dots, k_p)}(t) \neq 0$, Proposition 2 implies that

$$\frac{S_h(t + \varepsilon u)}{\varepsilon^{\sum_{j=1}^p h_{k_j}(t_{k_j})}} = \frac{\Delta^{(k_1, \dots, k_p)} S_h\left(t, \varepsilon \vec{1}_{\mathbb{R}^p}\right)}{\varepsilon^{\sum_{j=1}^p h_{k_j}(t_{k_j})}} \xrightarrow[\varepsilon \rightarrow 0_+]{\text{dist.}} \mathcal{T}_{h, (k_1, \dots, k_p), t}\left(\vec{1}_{\mathbb{R}^p}\right) \neq 0 \quad \text{almost surely.}$$

where we have set $u = \sum_{j=0}^p e_{k_j}$. Therefore, $H_{S_h}(t) \leq \sum_{j=1}^p h_{k_j}(t_{k_j})$ almost surely, which concludes the proof. \square

One can also consider the local Hölder exponent at point t , which is related to the Hölder property fulfilled around t . In our framework and under the assumptions of Proposition 5, one can prove that it is equal to $\min_{1 \leq j \leq d} h_j(t)$. Observe that the pointwise and the local Hölder exponent at point t are different if $t = 0$. This is also true for multifractional or fractional sheets.

Observe that pointwise and local exponents of S_h do not directly involve all the values of $h_j(t_j)$ with $t_j \in [0, +\infty)$ and $1 \leq j \leq d$. However, the knowledge of all these values is required to obtain pointwise multi-Hölder exponents which are introduced below. This proves that all the range of h is involved in the regularity of the sample paths.

Definition 2. Let us consider φ a real-valued function defined on a neighborhood U of t . The pointwise multi-Hölder exponent of φ at point $t \in \mathbb{R}_+^d$ is the only multi-indexes (H_1, \dots, H_d) such that

(i) for any $\alpha \in \prod_{j=1}^d (0, H_j)$, $\lim_{s \rightarrow t} \frac{\Delta\varphi(t,s)}{\prod_{j=1}^d |t_j - s_j|^{\alpha_j}} = 0$

(ii) and for any $\alpha \in \mathbb{R}_+^d \setminus \prod_{j=1}^d (0, H_j]$, $\frac{\Delta\varphi(t,s)}{\prod_{j=1}^d |t_j - s_j|^{\alpha_j}}$ does not converge when $s \rightarrow t$.

In our framework and also in the framework of multifractional sheets, this exponent is given by the function h .

Proposition 6. Assume $a_0(h(t))C(h(t), h(t)) \neq 0$. Then the pointwise multi-Hölder exponent of S_h at point t is almost surely equal to $h(t)$.

Proof. Since $\Delta S_h(t, t + \varepsilon) = \Delta^{(1, \dots, d)} S_h(t, \varepsilon)$, by Proposition 2,

$$\lim_{\varepsilon \rightarrow 0_{\mathbb{R}^d}^+} \frac{\Delta S_h(t, t + \varepsilon)}{\prod_{j=1}^d \varepsilon_j^{h_j(t_j)}} \stackrel{\text{dist.}}{=} |a_0(h(t))| \sqrt{C(h(t), h(t))} B_{h(t)}(1, \dots, 1)$$

where $B_{h(t)}$ is the fractional Brownian sheet defined by (17). Then, since $B_{h(t)}(1, \dots, 1) \neq 0$ almost surely, for any $\alpha \in \mathbb{R}_+^d \setminus \prod_{j=1}^d (0, h_j(t_j)]$, there exists a sequence $(\varepsilon^{(n)})_{n \in \mathbb{N}}$ which converges to $0_{\mathbb{R}^d}^+$ and such that

$$\frac{|\Delta S_h(t, t + \varepsilon^{(n)})|}{\prod_{j=1}^d (\varepsilon_j^{(n)})^{\alpha_j}} \rightarrow \infty$$

almost surely. Therefore property (ii) of the definition 2 holds with $H_j = h_j(t_j)$, $1 \leq j \leq d$. Then, one concludes applying Proposition 4. \square

4 Extension to stable fields

Let $0 < \alpha \leq 2$ and $W_\alpha(d\xi)$ be a complex isotropic α -stable random measure on \mathbb{R}^d with Lebesgue control measure (see [22] p.281). If $\alpha = 2$, $W_\alpha(d\xi)$ is a complex isotropic Gaussian random measure. We recall that, for any function $g : \mathbb{R}^d \rightarrow \mathbb{C}$ in $L^\alpha(dx)$,

$$W_\alpha(g) = \int_{\mathbb{R}^d} g(x) W_\alpha(dx)$$

is well-defined and is an isotropic α -stable random variable. Moreover, if for any function $g : \mathbb{R}^d \rightarrow \mathbb{C}$ in $L^\alpha(dx)$, we have:

$$\forall z \in \mathbb{C}, \mathbb{E} \left(\exp \left[i \Re \left(\bar{z} \int_{\mathbb{R}^d} g(\xi) W_\alpha(d\xi) \right) \right] \right) = \exp \left(-c_\alpha^\alpha |z|^\alpha \int_{\mathbb{R}^d} |g(x)|^\alpha dx \right) \quad (27)$$

with $c_\alpha = \left(\frac{1}{2\pi} \int_0^\pi |\cos(x)|^\alpha dx \right)^{1/\alpha}$.

Let $a : (0, 1)^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ be a measurable and bounded function. Then, the random field $X_\alpha = \{X_{n,\alpha}(H), n \in \mathbb{N}^d, H \in (0, 1)^d\}$ such that

$$X_{n,\alpha}(H) = \Re \left(\int_{\mathbb{R}^d} a(H, x) \exp(i \langle n, x \rangle) \prod_{j=1}^d \frac{\exp(ix_j) - 1}{|x_j|^{1/\alpha + H_j}} W_\alpha(dx) \right)$$

is well-defined and is a real-valued symmetric α -stable random field. Observe that if a satisfies (6) and if $\alpha = 2$, then the field X_α has the same distribution as the field defined in Section 3 with $C \equiv 1/2$.

For any integer $1 \leq j \leq d$, let us consider a function $h_j : [0, \infty) \rightarrow (0, 1)$. Then, let us define $h : [0, \infty)^d \rightarrow (0, 1)^d$ by (8). We now consider

$$\forall t \in [0, +\infty)^d, S_{h,\alpha}^N(t) := \sum_{k \in \mathcal{D}_t^N} \frac{X_{k,\alpha}(h_k^N)}{N|h_k^N|}$$

where \mathcal{D}_t^N and h_k^N are defined in Section 2. The next theorem is the analogous of Theorem 1 for $S_{h,\alpha}^N$.

Theorem 2. *Assume that the function a satisfies Condition (A). Then, the finite dimensional distributions of $S_{h,\alpha}^N$ converge to those of $S_{h,\alpha}$ defined by*

$$\forall t \in [0, +\infty)^d, S_{h,\alpha}(t) = \Re \left(\int_{\mathbb{R}^d} W_\alpha(dx) \int_{\mathcal{D}_t} a_0(h(\theta)) e^{i\langle x, \theta \rangle} \prod_{j=1}^d |x_j|^{1-1/\alpha-h_j(\theta_j)} d\theta \right),$$

where $\mathcal{D}_t = \prod_{j=1}^d [0, t_j]$.

Observe that if $h \equiv H$, the limit field $S_{h,\alpha}$ is a real harmonizable fractional stable field of order H .

Proof. Let $m, M \in \mathbb{N} \setminus \{0\}$ and $T > 0$. For any $l = 1 \dots m$, let $t^{(l)} = (t_1^{(l)}, \dots, t_d^{(l)}) \in [0, T]^d$ and $\lambda_l \in \mathbb{R}$. In order to simplify the notation, we set $\mathcal{D}^{N,l} := \mathcal{D}_{t^{(l)}}^N$. Then, using the linearity of the integral with respect to W_α and (27), we obtain that

$$\mathbb{E} \left(\exp \left(i \sum_{l=1}^m \lambda_l S_{h,\alpha}^N(t^{(l)}) \right) \right) = \exp(-c_\alpha^\alpha I_\alpha(N)) \quad (28)$$

with

$$I_\alpha(N) := \int_{\mathbb{R}^d} \left| \sum_{l=1}^m \lambda_l \sum_{k \in \mathcal{D}^{N,l}} a(h_k^N, x) \exp(i\langle k, x \rangle) \prod_{j=1}^d \frac{\exp(ix_j) - 1}{N^{h_j(k_j/N)} |x_j|^{1/\alpha+h_j(k_j/N)}} \right|^\alpha dx.$$

The change of variables $y = Nx$ leads to $I_\alpha(N) = \int_{\mathbb{R}^d} |\Phi_\alpha^N(y)|^\alpha dy$ with

$$\Phi_\alpha^N(y) := \sum_{l=1}^m \lambda_l \sum_{k \in \mathcal{D}^{N,l}} a(h_k^N, \frac{y}{N}) e^{i\langle k, y/N \rangle} \prod_{j=1}^d \frac{e^{iy_j/N} - 1}{|y_j|^{1/\alpha+h_j(k_j/N)}}.$$

As in the proof of Theorem 1, we apply d times Lemma 3 to each function $\Phi_{\alpha,l}^N$ ($1 \leq l \leq m$) defined by

$$\Phi_{\alpha,l}^N(y) := \sum_{k \in \mathcal{D}^{N,l}} b \left(\frac{k}{N}, \frac{y}{N} \right) \prod_{j=1}^d e^{ik_j y_j / N} \frac{e^{iy_j/N} - 1}{|y_j|^{1/\alpha+h_j(k_j/N)}}$$

with $b(\theta, x) = a(h(\theta), x)$. This leads to

$$|\Phi_\alpha^N(y)| \leq \max_{\kappa \in \{0,1\}^d} \sup_{\theta \in [0,T]^d} \left| \delta_\theta^\kappa b\left(\theta, \frac{y}{N}\right) \right| \sum_{l=1}^m |\lambda_l| \prod_{j=1}^d \tilde{g}_\alpha(y_j)$$

where

$$\tilde{g}_\alpha(\xi) = c_{T,h} (1 + |\ln |\xi||) \left(\frac{1}{|\xi|^{H_+ - 1 + 1/\alpha}} \mathbb{I}_{|\xi| < 1} + \frac{1}{|\xi|^{H_- + 1/\alpha}} \mathbb{I}_{|\xi| \geq 1} \right)$$

with $H_+ = \max_{1 \leq j \leq d} \max_{[0,T]} h_j$, $H_- = \min_{1 \leq j \leq d} \min_{[0,T]} h_j$ and $c_{T,h}$ a finite positive constant which only depends on T and of each infinite norm of h'_j on $[0, T]$. Moreover, using the smoothness of h and Condition (A), we see that

$$\sup_{\theta \in [0,T]^d} \left| \delta_\theta^\kappa b\left(\theta, \frac{y}{N}\right) \right| < +\infty.$$

The sequence of functions $\{x \mapsto |\Phi_\alpha^N(s, t, x)|^\alpha\}_N$ is then uniformly bounded by an $L^1(\mathbb{R}^d)$ -function. Then, in order to obtain the convergence $I_\alpha(N)$, it remains to prove the convergence of each $\Phi_{\alpha,l}^N(y)$ for almost every y . To this aim, we write:

$$\Phi_{\alpha,l}^N(y) = \Phi_{\alpha,l,0}^N(y) + \Phi_{\alpha,l,1}^N(y)$$

where

$$\Phi_{\alpha,l,0}^N(y) = \sum_{k \in \mathcal{D}^{N,l}} a_0\left(h\left(\frac{k}{N}\right)\right) \prod_{j=1}^d e^{ik_j y_j / N} \frac{e^{iy_j / N} - 1}{|y_j|^{1/\alpha + h_j(k_j/N)}}$$

and

$$\Phi_{\alpha,l,1}^N(y) = \sum_{k \in \mathcal{D}^{N,l}} a_1\left(h\left(\frac{k}{N}\right), \frac{y}{N}\right) \prod_{j=1}^d e^{ik_j y_j / N} \frac{e^{iy_j / N} - 1}{|y_j|^{1/\alpha + h_j(k_j/N)}}.$$

As in the proof of Theorem 1, Lemma 3 and Condition (A) lead to $\lim_{N \rightarrow +\infty} \Phi_{\alpha,l,1}^N(y) = 0$. Then, using Riemann sum,

$$\lim_{N \rightarrow +\infty} \Phi_{\alpha,l}^N(y) = \lim_{N \rightarrow +\infty} \Phi_{\alpha,l,0}^N(y) = \int_{\mathcal{D}_t^{(l)}} a_0(h(\theta)) \prod_{j=1}^d \frac{iy_j e^{iy_j \theta_j}}{|y_j|^{1/\alpha + h_j(\theta_j)}} d\theta.$$

Hence, by the bounded convergence theorem we get the convergence of $I_\alpha(N)$, which leads (combined with (28)) to

$$\lim_{N \rightarrow \infty} \mathbb{E} \left(\exp \left(i \sum_{l=1}^m \lambda_l S_{h,\alpha}^N(t^{(l)}) \right) \right) = \exp \left(-c_\alpha^\alpha \int_{\mathbb{R}^d} \left| \sum_{l=1}^m \lambda_l \int_{\mathcal{D}_t^{(l)}} a_0(h(\theta)) \prod_{j=1}^d \frac{e^{iy_j \theta_j}}{|y_j|^{1/\alpha + h_j(\theta_j) - 1}} d\theta \right|^\alpha dy \right).$$

and concludes the proof. \square

Let us mention that one can prove an analogous of Theorem 2 for the stable limit field $S_{h,\alpha}$ studying the characteristic function of each quantity of the form

$$\sum_{l=1}^m \lambda_l \Delta^{(k_1, \dots, k_p)} S_{h,\alpha}(t, \varepsilon \star u_l).$$

However, the arguments used in the Gaussian framework cannot be applied to study the regularity of the limit field $S_{h,\alpha}$. Since this field is α -stable, such study can certainly be done using a series representation (see [15, 6, 7] for example) but this is out of the scope of this paper.

5 Technical lemmas

Lemma 3. *Let $\phi : [0, T] \rightarrow \mathbb{C}$ and $h : [0, T] \rightarrow (0, 1)$ be two continuously differentiable functions and $\alpha \in (0, 2]$. Let $H_+ = \max_{[0, T]} h$ and $H_- = \min_{[0, T]} h$. Then, for every $x \in \mathbb{R} \setminus \{0\}$, for every $N \in \mathbb{N} \setminus \{0\}$ and every $t \in [0, T]$,*

$$\left| \frac{1 - e^{-ix/N}}{ix} \sum_{k=1}^{[Nt]} \phi\left(\frac{k}{N}\right) e^{ikx/N} |x|^{1-1/\alpha-h(k/N)} \right| \leq c_{T,h} \max(\|\phi\|_\infty, \|\phi'\|_\infty) g_\alpha(x)$$

where

$$g_\alpha(x) = (1 + |\ln|x||) \left(\frac{1}{|x|^{H_+-1+1/\alpha}} \mathbb{I}_{|x|<1} + \frac{1}{|x|^{H_-+1/\alpha}} \mathbb{I}_{|x|\geq 1} \right) \quad (29)$$

and $c_{T,h}$ is a finite positive constant which only depends on T and on $\max_{[0, T]} |h'|$.

Proof. Let $x \in \mathbb{R} \setminus \{0\}$, $N \in \mathbb{N} \setminus \{0\}$ and $t \in [0, T]$. Let us now set

$$g_N(t, x) = \frac{1 - e^{-ix/N}}{ix} \sum_{k=1}^{[Nt]} \phi\left(\frac{k}{N}\right) e^{ikx/N} |x|^{1-1/\alpha-h(k/N)} = \sum_{k=1}^{[Nt]} \phi\left(\frac{k}{N}\right) \frac{e^{ikx/N} - e^{i(k-1)x/N}}{ix|x|^{h(k/N)-1+1/\alpha}}.$$

Then,

$$\begin{aligned} g_N(t, x) &= \phi\left(\frac{[Nt]}{N}\right) \frac{e^{i[Nt]x/N} - 1}{ix|x|^{h([Nt]/N)-1+1/\alpha}} \\ &\quad - \sum_{k=1}^{[Nt]} \frac{e^{i(k-1)x/N} - 1}{ix|x|^{1/\alpha-1}} \phi\left(\frac{k}{N}\right) \left(\frac{1}{|x|^{h(k/N)}} - \frac{1}{|x|^{h((k-1)/N)}} \right) \\ &\quad - \sum_{k=1}^{[Nt]} \frac{e^{i(k-1)x/N} - 1}{ix|x|^{h((k-1)/N)-1+1/\alpha}} \left(\phi\left(\frac{k}{N}\right) - \phi\left(\frac{k-1}{N}\right) \right). \end{aligned} \quad (30)$$

Let us first remark that

$$\left| \phi\left(\frac{[Nt]}{N}\right) \frac{e^{i[Nt]x/N} - 1}{ix|x|^{h([Nt]/N)-1+1/\alpha}} \right| \leq \max(2, T) \|\phi\|_\infty g_\alpha(x) \quad (31)$$

with g_α defined by (29). Moreover, since h is continuously differentiable, by the mean value theorem,

$$\begin{aligned} & \left| \sum_{k=1}^{[Nt]} \frac{e^{i(k-1)x/N} - 1}{ix|x|^{-1+1/\alpha}} \phi\left(\frac{k}{N}\right) \left(\frac{1}{|x|^{h(k/N)}} - \frac{1}{|x|^{h((k-1)/N)}} \right) \right| \\ & \leq \|\phi\|_\infty \|h'\|_\infty \left(\mathbb{I}_{|x|<1} \sum_{k=1}^{[Nt]} \frac{(k-1)|\ln|x||}{N^2|x|^{H_+-1+1/\alpha}} + \mathbb{I}_{|x|\geq 1} \frac{2[Nt]|\ln|x||}{N|x|^{H_-+1/\alpha}} \right), \end{aligned}$$

which implies that

$$\left| \sum_{k=1}^{[Nt]} \frac{e^{i(k-1)x/N} - 1}{ix|x|^{-1+1/\alpha}} \phi\left(\frac{k}{N}\right) \left(\frac{1}{|x|^{h(k/N)}} - \frac{1}{|x|^{h((k-1)/N)}} \right) \right| \leq \max(T^2, 2T) \|h'\|_\infty \|\phi\|_\infty g_\alpha(x). \quad (32)$$

Applying the mean value theorem to the continuously differentiable function ϕ , we also obtain

$$\left| \sum_{k=1}^{[Nt]} \frac{e^{i(k-1)x/N} - 1}{ix|x|^{h((k-1)/N)-1+1/\alpha}} \left(\phi\left(\frac{k}{N}\right) - \phi\left(\frac{k-1}{N}\right) \right) \right| \max(T^2, 2T) \|\phi'\|_\infty g_\alpha(x). \quad (33)$$

Equations (30), (31), (32) and (33) lead to the conclusion. \square

Lemma 4. *Let $\phi : [0, T] \rightarrow \mathbb{C}$ and $h : [0, T] \rightarrow (0, 1)$ be two continuously differentiable functions. Let $\alpha \in (0, 2]$. Then, for every $x \in \mathbb{R} \setminus \{0\}$ and every $t \in [0, T]$,*

$$\left| \int_0^t \phi(\theta) e^{i\theta x} |x|^{1-1/\alpha-h(\theta)} d\theta \right| \leq c_{T,h} \max(\|\phi\|_\infty, \|\phi'\|_\infty) g_\alpha(x)$$

with g_α defined by (29) and $c_{T,h}$ a finite positive constant which only depends on T and on $\max_{[0,T]} |h'|$.

Proof. This is a direct consequence of Lemma 3 in which we make N go to infinity. \square

Lemma 5. *Let $\phi : [0, 2T] \rightarrow \mathbb{C}$ and $h : [0, 2T] \rightarrow (0, 1)$ be two continuously differentiable functions. Let $H_+ = \max_{[0,2T]} h$ and $H_- = \min_{[0,2T]} h$. Then, for all $u \in [0, T]$, $y \in \mathbb{R} \setminus \{0\}$, $t \in [0, T]$ and $\nu \in (0, 1]$*

$$\left| \int_0^u \frac{e^{iy\theta} |\nu|^{h(\nu\theta+t)-h(t)}}{|y|^{h(\nu\theta+t)-1/2}} \phi(\nu\theta+t) d\theta - \phi(t) \frac{e^{iyu} - 1}{iy|y|^{h(t)-1/2}} \right| \leq c_{T,h} \nu |\ln(\nu)| \max(\|\phi\|_\infty, \|\phi'\|_\infty) g(y) \quad (34)$$

and

$$\left| \int_0^u \frac{e^{iy\theta} |\nu|^{h(\nu\theta+t)-h(t)}}{|y|^{h(\nu\theta+t)-1/2}} \phi(\nu\theta+t) d\theta \right| \leq c_{T,h} g(y)$$

with $g = g_2$ defined by (29) and $c_{T,h}$ is a finite positive constant which only depends on T and on $\max_{[0,2T]} |h'|$.

Proof. Let $u \in [0, T]$, $y \in \mathbb{R} \setminus \{0\}$, $t \in [0, T]$ and $\nu \in (0, 1]$. Let us consider

$$G_1(t, y, u, \nu) = \int_0^u \frac{e^{iy\theta} \nu^{h(\nu\theta+t)-h(t)}}{|y|^{h(\nu\theta+t)-1/2}} \phi(\nu\theta + t) d\theta.$$

An integration by parts leads to

$$G_1(t, y, u, \nu) = \phi(t) \frac{e^{iyu} - 1}{iy|y|^{h(t)-1/2}} - G_2(t, y, u, \nu)$$

with

$$G_2(t, y, u, \nu) = \nu^{1-h(t)} |y|^{1/2} \int_0^u \frac{e^{iy\theta} - e^{iyu}}{iy} \left| \frac{\nu}{y} \right|^{h(\nu\theta+t)} \left(\phi'(\nu\theta + t) + h'(\nu\theta + t) \ln \left| \frac{\nu}{y} \right| \phi(\nu\theta + t) \right) d\theta.$$

Then,

$$|G_2(t, y, u, \nu)| \leq \max(1, \|h'\|_\infty) \max(\|\phi\|_\infty, \|\phi'\|_\infty) \nu |\ln \nu| \int_0^u \frac{|e^{iy(\theta-u)} - 1|}{|y|^{1/2+h(\nu\theta+t)}} \nu^{h(\nu\theta+t)-h(t)} d\theta.$$

Let us remark that for every $\theta \in [0, u]$,

$$\nu^{h(\nu\theta+t)-h(t)} \leq \exp(\nu |\ln \nu| T \|h'\|_\infty) \leq \exp(T \|h'\|_\infty)$$

Moreover,

$$\int_0^u \frac{|e^{iy(\theta-u)} - 1|}{|y|^{1/2+h(\nu\theta+t)}} d\theta \leq \max(2T, T^2) g_2(y)$$

with g_2 defined by (29). Therefore,

$$|G_2(t, y, u, \nu)| \leq A(h, T) \max(\|\phi\|_\infty, \|\phi'\|_\infty) \nu |\ln \nu| g_2(y)$$

with $A(h, T) = \exp(T \|h'\|_\infty) \max(1, \|h'\|_\infty) \max(T^2, 2T)$. This is (34). Since

$$\left| \phi(t) \frac{e^{iyu} - 1}{iy|y|^{h(t)-1/2}} \right| \leq \|\phi\|_\infty g_2(y),$$

we have:

$$|G_1(t, y, u, \nu)| \leq A(h, T) \left(\sup_{\rho \in (0,1]} \rho |\ln \rho| + 1 \right) \max(\|\phi\|_\infty, \|\phi'\|_\infty) g_2(y),$$

which concludes the proof. \square

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