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Finite index subgroups of mapping class groups

A. J. Berrick, V. Gebhardt and L. Paris

May 12, 2011

Abstract

Let \( g \geq 3 \) and \( n \geq 0 \), and let \( \mathcal{M}_{g,n} \) be the mapping class group of a surface of genus \( g \) with \( n \) boundary components. We prove that \( \mathcal{M}_{g,n} \) contains a unique subgroup of index \( 2^{g-1}(2^g - 1) \) up to conjugation, a unique subgroup of index \( 2^{g-1}(2^g + 1) \) up to conjugation, and the other proper subgroups of \( \mathcal{M}_{g,n} \) are of index greater than \( 2^{g-1}(2^g + 1) \). In particular, the minimum index for a proper subgroup of \( \mathcal{M}_{g,n} \) is \( 2^{g-1}(2^g - 1) \).


0 Introduction and statement of results

The interaction between mapping class groups and finite groups has long been a topic of interest. The famous Hurwitz bound of 1893 showed that a closed Riemann surface of genus \( g \) has an upper bound of \( 84(g - 1) \) for the order of its finite subgroups, and Kerckhoff showed that the order of finite cyclic subgroups is bounded above by \( 4g + 2 \) \([18], [19]\).

The subject of finite index subgroups of mapping class groups was brought into focus by Grossman’s discovery that the mapping class group \( \mathcal{M}_{g,n} = \pi_0(\text{Homeo}(\Sigma_{g,n})) \) of an oriented surface \( \Sigma_{g,n} \) of genus \( g \) and \( n \) boundary components is residually finite, and thus well-endowed with subgroups of finite index \([16]\). (\( \text{Homeo}(\Sigma_{g,n}) \) denotes the space of those homeomorphisms of \( \Sigma_{g,n} \) that preserve the orientation and are the identity on the boundary.) This prompts the “dual” question:

What is the minimum index \( \text{mi}(\mathcal{M}_{g,n}) \) of a proper subgroup of finite index in \( \mathcal{M}_{g,n} \)?

Results to date have suggested that, like the maximum finite order question, the minimum index question should have an answer that is linear in \( g \). The best previously published bound is \( \text{mi}(\mathcal{M}_{g,n}) > 4g + 4 \) for \( g \geq 3 \) (see \([25]\)). This inequality is used by Aramayona and Souto to prove that, if \( g \geq 6 \) and \( g' \leq 2g - 1 \), then any nontrivial homomorphism \( \mathcal{M}_{g,n} \to \mathcal{M}_{g',n'} \) is induced by an embedding \([1]\). It is also an important ingredient in the proof of Zimmermann \([33]\) that, for \( g = 3 \) and \( 4 \), the minimal nontrivial quotient of \( \mathcal{M}_{g,0} \) is \( \text{Sp}_{2g}(\mathbb{F}_2) \).

The “headline” result of this paper is the following exact, exponential bound.

**Theorem 0.1.** For \( g \geq 3 \) and \( n \geq 0 \),

\[
\text{mi}(\mathcal{M}_{g,n}) = \text{mi}(\text{Sp}_{2g}(\mathbb{Z})) = \text{mi}(\text{Sp}_{2g}(\mathbb{F}_2)) = 2^{g-1}(2^g - 1).
\]

This exponential bound is all the more surprising since in similar questions we get linear (expected) bounds. For instance, Bridson \([6, 7]\) has proved that a mapping class group of a surface

1
of genus $g$ cannot act by semisimple isometries, without a global fixed point, on a CAT(0) space of dimension less than $g$. The exact minimal dimension for such an action is unknown. On the other hand, it has also been shown by Bridson (see [6]) that $M_{g,n}$ has only finitely many irreducible linear representations over any algebraically closed field, up to dimension $(g + 1)$. Later, Funar [15] showed that there is no linear representation with infinite image up to dimension $\sqrt{g} + 1$. However, there is an obvious linear representation of rank $2g$ which comes from the action of $M_{g,n}$ on the homology of $\Sigma_{g,0}$ (the map $\theta_{g,n}$ defined below). It is expected that this representation is minimal in some sense (see [13]).

The nontrivial quotient of $M_{g,n}$ of minimal order is unknown, but obviously its order must be at least $\text{mi}(M_{g,n})$. This quotient is known to be $\text{Sp}_{2g}(F_2)$ if $(g, n) = (3, 0)$ or $(4, 0)$ (see [33]). A consequence of the above theorem is that $M_{g',n'}$ cannot be a quotient of $M_{g,n}$ if $3 \leq g' < g$.

Our proof of this result is constructive, in ways that we now describe. From the surface $\Sigma_{g,n}$ we obtain a closed oriented surface $\hat{\Sigma}_g$ of genus $g$ by gluing a disk along each boundary component.

The embedding $\Sigma_{g,n} \hookrightarrow \hat{\Sigma}_g$ induces a first epimorphism $M_{g,n} \twoheadrightarrow M_{g,0}$. The action of $\text{Homeo}(\hat{\Sigma}_g)$ on $H_1(\hat{\Sigma}_g) = \mathbb{Z}^{2g}$ induces a second epimorphism $M_{g,0} \twoheadrightarrow \text{Sp}_{2g}(\mathbb{Z})$ onto the integral symplectic group, and, passing mod 2, we obtain a third epimorphism $\text{Sp}_{2g}(\mathbb{Z}) \twoheadrightarrow \text{Sp}_{2g}(F_2)$, where $F_2 = \mathbb{Z}/2\mathbb{Z}$. From now on we denote by $\theta_{g,n} : M_{g,n} \twoheadrightarrow \text{Sp}_{2g}(F_2)$ the composition of these three epimorphisms.

The orthogonal groups $O^+_{2g}(F_2)$ and $O^-_{2g}(F_2)$ are subgroups of $\text{Sp}_{2g}(F_2)$. The cardinalities of $\text{Sp}_{2g}(F_2)$, $O^+_{2g}(F_2)$ and $O^-_{2g}(F_2)$ can be found for instance in [30], and from this data it is easily shown that, for $g \geq 2$, the indices of $O^+_{2g}(F_2)$ and $O^-_{2g}(F_2)$ in $\text{Sp}_{2g}(F_2)$ are $N_g^+ = 2^{g-1}(2^g + 1)$ and $N_g^- = 2^{g-1}(2^g - 1)$, respectively. The following, more or less known to experts but seemingly unpublished, is the starting-point for our main result (Theorem 0.3).

**Theorem 0.2.** Let $g \geq 3$.

1. $O^+_{2g}(F_2)$ is the unique subgroup of $\text{Sp}_{2g}(F_2)$ of index $N_g^-$, up to conjugation.
2. $O^-_{2g}(F_2)$ is the unique subgroup of $\text{Sp}_{2g}(F_2)$ of index $N_g^+$, up to conjugation.
3. All the other proper subgroups of $\text{Sp}_{2g}(F_2)$ are of index at least $2N_g^-$. 

We set $O^+_{g,n} = \theta_{g,n}^{-1}(O^+_{2g}(F_2))$ and $O^-_{g,n} = \theta_{g,n}^{-1}(O^-_{2g}(F_2))$. Thus, by the above, $O^-_{g,n}$ is an index $N_g^-$ subgroup of $M_{g,n}$, and $O^+_{g,n}$ is an index $N_g^+$ subgroup of $M_{g,n}$. Here is our main result.

**Theorem 0.3.** Let $g \geq 3$ and $n \geq 0$.

1. $O^-_{g,n}$ is the unique subgroup of $M_{g,n}$ of index $N_g^-$, up to conjugation.
2. $O^+_{g,n}$ is the unique subgroup of $M_{g,n}$ of index $N_g^+$, up to conjugation.
3. If $g = 3$ then all the other proper subgroups of $M_{3,n}$ are of index strictly greater than $N_g^+ = 36$.
4. If $g \geq 4$ then all the other proper subgroups of $M_{g,n}$ are of index at least $5N_{g-1}^- > N_g^+$.
Since \( m \) is an upper bound for the minimum index of a group \( G \) if and only if there is a nontrivial homomorphism from \( G \) to the symmetric group \( \mathfrak{S}_m \) on \( m \) letters, one would like to understand the permutation representations associated to Theorem 0.3.

We denote by \( \phi_{g,n}^+ : M_{g,n} \rightarrow \mathfrak{S}_{N_g^+} \) (resp. \( \phi_{g,n}^- : M_{g,n} \rightarrow \mathfrak{S}_{N_g^-} \)) the permutation representation induced by the action of \( M_{g,n} \) on the right cosets of \( O_{g,n}^+ \) (resp. \( O_{g,n}^- \)). Corresponding to the numerical relations
\[
N_g^+ = 3N_g^{+1} + N_g^{-1}, \quad N_g^- = 3N_g^{-1} + N_g^{+1},
\]
we prove the following.

**Theorem 0.4.**
1. Let \( g \geq 3 \) and \( n \geq 1 \). Then \( \phi_{g,n}^+ : M_{g,n} \rightarrow \mathfrak{S}_{N_g^-} \) is, up to equivalence, the unique extension of the representation \((\phi_{g-1,n}^-)^3 \oplus \phi_{g-1,n}^+ \) from \( M_{g-1,n} \) to \( M_{g,n} \), and \( \phi_{g,n}^+ : M_{g,n} \rightarrow \mathfrak{S}_{N_g^+} \) is, up to equivalence, the unique extension of the representation \( \phi_{g-1,n}^- \oplus (\phi_{g-1,n}^+)^3 \) from \( M_{g-1,n} \) to \( M_{g,n} \).

2. Let \( g \geq 3 \) and \( n \geq 0 \). Let \( b \) be a nonseparating simple closed curve on \( \Sigma_{g,n} \), and let \( T_b \) be the Dehn twist around \( b \). Then the cycle structure of the image of \( T_b \) under \( \phi_{g,n}^- \) is
\[
(1)^{2g-2} (2)^{2g-2(2g-1)-1},
\]
and the cycle structure of the image of \( T_b \) under \( \phi_{g,n}^+ \) is
\[
(1)^{2g-2} (2)^{2g-2(2g-1)+1}.
\]

**Remark.** Implicit in the statement of Theorem 0.4 is the fact that, if \( n \geq 1 \), then \( M_{g-1,n} \) naturally embeds into \( M_{g,n} \). This embedding will be described in Section 2. However, there is no natural embedding of \( M_{g-1,0} \) into \( M_{g,0} \), hence Part (1) of the theorem would make no sense for \( n = 0 \).

We observe that the abelianization of \( M_{g,n} \) is isomorphic to \( \mathbb{Z}/12\mathbb{Z} \) if \( (g,n) = (1,0) \), \( \mathbb{Z}/n\mathbb{Z} \) if \( g = 1 \) and \( n \geq 1 \), and \( \mathbb{Z}/10\mathbb{Z} \) if \( g = 2 \) (see [20]). Hence, the minimum index of \( M_{g,n} \) is 2 if \( g = 1 \) or 2. Note that \( M_{g,n} \) is perfect if \( g \geq 3 \) (see [27, 20]). If \( g = 2 \) we have \( N_g^- = 6 \) and \( N_g^+ = 10 \), and there are six proper subgroups of index at most 10 in \( M_{g,n} \): one of index 2, one of index 5, two of index 6 and two of index 10. The description of these subgroups as well as the proof of this fact are given in Section 3.

The remainder of the paper is divided into two parts. Part I starts with some preliminaries on permutations (Section 1) and presentations of mapping class groups (Section 2). Then we determine the subgroups of \( M_{2,n} \) of index at most 10 = \( N_2^+ \) (Section 3) and the subgroups of \( M_{3,n} \) of index at most 36 = \( N_3^+ \) (Section 4). We prove our theorems by induction on the genus. The starting case, \( g = 3 \), is made in Section 4, and the inductive step is the object of Part II. We first treat the case of a surface with a unique boundary component (Sections 5 to 7) and we extend the result to surfaces with several boundary components in Section 8. Theorem 0.1 is proved in Section 6. This can be read independently from the rest.

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Part I
Preliminaries

1 Useful information on permutations

We define the minimum index $\text{mi}(G)$ of a nontrivial group $G$ to be the element of the ordered set $2 < 3 < 4 < \cdots < \infty$ corresponding to the minimum among the indices of all proper subgroups of $G$ with finite index, and $\infty$ when $G$ has no proper finite index subgroup (such a $G$ has been called counter-finite).

Permutations are important to our investigation because of the well-known relationship between index $m$ subgroups and maps to $S_m$. A homomorphism $\varphi : G \to S_m$ is called transitive if its image acts transitively on $\{1, \ldots, m\}$. If $\varphi : G \to S_m$ is transitive, then $\text{Stab}_\varphi(1) = \{ \gamma \in G \mid \varphi(\gamma)(1) = 1 \}$ is a subgroup of $G$ of index $m$. Conversely, if $H$ is a subgroup of $G$ of index $m$, then there exists a transitive homomorphism $\varphi : G \to S_m$ such that $H = \text{Stab}_\varphi(1)$ (take the action of $G$ on the right cosets of $H$). It follows that $\text{mi}(G)$ is also the smallest $m \geq 2$ such that there exists a nontrivial homomorphism $\varphi : G \to S_m$. Of course, if such a homomorphism does not exist for any $m$, then $\text{mi}(G) = \infty$.

The minimum index has the property that, if $G \twoheadrightarrow H$ is an epimorphism, then $\text{mi}(G) \leq \text{mi}(H)$. (Indeed, the pre-image of an index $m$ subgroup under an epimorphism is an index $m$ subgroup.) From the definition of $\theta_{g,n} : M_{g,n} \twoheadrightarrow \text{Sp}_{2g}(\mathbb{F}_2)$ as the composition of epimorphisms

$$M_{g,n} \twoheadrightarrow M_{g,0} \twoheadrightarrow \text{Sp}_{2g}(\mathbb{Z}) \twoheadrightarrow \text{Sp}_{2g}(\mathbb{F}_2),$$

we therefore have

$$\text{mi}(M_{g,n}) \leq \text{mi}(M_{g,0}) \leq \text{mi}(\text{Sp}_{2g}(\mathbb{Z})) \leq \text{mi}(\text{Sp}_{2g}(\mathbb{F}_2)).$$

Since many generators of mapping class groups commute with each other, we need some preliminary results that discuss aspects of commuting permutations. In the following, $C_k$ denotes the cyclic group of order $k$, and an orbit of cardinality $k$ under the action of a permutation or a permutation group is called a $k$-orbit.

Lemma 1.1. Let $u \in S_m$ have cycle type $(1)^{\ell_1}(2)^{\ell_2} \cdots (m)^{\ell_m}$, and let

$$I(u) = \{ k \in \{1, 2, \ldots, m\} \mid \ell_k > 0 \},$$

so that $\sum_{k \in I(u)} k \ell_k = m$.

1. The centralizer $C_{S_m}(u)$ is isomorphic to

$$\prod_{k \in I(u)} ((C_k)^{\ell_k} \times S_{\ell_k}) = \prod_{k \in I(u)} (C_k \wr S_{\ell_k}).$$
2. If $P \leq C_{\mathfrak{A}_m}(u)$ is nonabelian, then, for some $k \in I(u)$, $\ell_k \geq \text{mi}(P)$.

3. If further $P$ is perfect, then $P$ is isomorphic to a subgroup of

$$\prod_{k \in I(u)} ((C_k)^{\ell_k} \rtimes \mathfrak{A}_{\ell_k}) = \prod_{k \in I(u)} (C_k \rtimes \mathfrak{A}_{\ell_k}).$$

If $I(P) = \{ k \in I(u) \mid P \text{ acts nontrivially on the union of the } k\text{-orbits of } u \}$, then the following numerical constraints hold:

(a) $5 \leq \text{mi}(P) \leq \ell_k$ whenever $k \in I(P)$;

(b) $m \geq \text{mi}(P)(\sum_{k \in I(P)} k)$; and

(c) if $m < 5 \text{mi}(P)$, then either $I(P) = \{ 4 \}$ or $I(P) - \{ 1 \} = \{ 2 \}$ or $\{ 3 \}$.

**Proof.** (1) is left to the reader. Let $P_k$ denote the projection of $P$ to the component $C_k \rtimes \mathfrak{S}_{\ell_k}$ of $C_{\mathfrak{A}_m}(u)$. Since $P$ is contained in $\prod_{k \in I(u)} P_k$ and $P$ is nonabelian, there exists $k \in I(u)$ such that $P_k$ is nonabelian, and thus $\ell_k \geq \text{mi}(P_k) \geq \text{mi}(P)$. Finally, (3) results from the fact that any nontrivial perfect subgroup of $(C_k)^{\ell_k} \rtimes \mathfrak{S}_{\ell_k}$ (whose support is the union of the $k$-orbits of $u$) must have nontrivial perfect image in $\mathfrak{S}_{\ell_k}$. Then $\ell_k \geq 5$ and the image lies in the maximum perfect subgroup $\mathfrak{A}_{\ell_k}$ of $\mathfrak{S}_{\ell_k}$. Then (a), (b) and (c) follow readily. □

For $u \in \mathfrak{S}_m$ we write $\{ 1, \ldots, m \} = F(u) \cup S(u)$ for the partition into the fixed set $F(u)$ and support $S(u)$ of the permutation $u$. Evidently, $u$ restricts to the identity map on $F(u)$, respectively to a bijection on $S(u)$.

**Lemma 1.2.** Let $u, v \in \mathfrak{S}_m$ be such that $uvu = vuv$. Then $|S(u)| \leq 2 |S(u) \cap S(v)|$.

**Proof.** If $u(i) \in F(v)$, then $(vuv)(i) = u(i)$. If also $i \in F(v)$, then

$$u(u(i)) = (vuv)(i) = (vuv)(i) = u(i).$$

Thus,

$$i \in u^{-1}(F(v)) \cap F(v) \implies i \in u^{-1}(F(u));$$

in other words, $F(v) \cap u(F(v)) \subseteq F(u)$. Therefore,

$$S(u) \cap F(v) \cap u(F(v)) = \emptyset;$$

and so $S(u) \cap u(F(v)) \subseteq S(u) \cap S(v)$. However, $u$ maps $S(u) \cap F(v)$ bijectively to $S(u) \cap u(F(v))$. The result is now immediate from the fact that

$$|S(u)| = |S(u) \cap S(v)| + |S(u) \cap F(v)|. \quad \Box$$

**Lemma 1.3.** Let $u, v \in \mathfrak{S}_m$ be such that $uvu = vuv$. If $k \in \{ 2, 3 \}$ and all nontrivial orbits of $u$ and $\langle u, v \rangle$ have length $k$, then $u = v$.

**Proof.** We present the argument for $k = 3$; for $k = 2$ it is similar, but simpler. Since $u$ and $v$ are conjugate, they have the same cycle decomposition type. If there are no nontrivial orbits, then $u = v = 1$. So, let $O$ be an orbit of $u$ of length 3. Then $O$ must also be a nontrivial orbit of $\langle u, v \rangle$. Hence, since $v$ is a product of 3-cycles, $v$ acts on $O$ as either 1, $u$ or $u^2$. In each case, the actions of $u$ and $v$ on $O$ commute. Likewise, on each nontrivial orbit of $v$ the actions of $u$ and $v$ commute. Finally, on $F(u) \cap F(v)$, since $u$ and $v$ both act as the identity, the actions of $u$ and $v$ again commute. Hence, $uv = vu$. From $uvu = vuv$, the result follows. □
Lemma 1.4. Let \( u, v_0, v_1 \in \mathfrak{S}_m \) be such that

(a) \( uv_i u = v_i uv_i \) for \( i = 0, 1 \); and

(b) \( v_0 \) and \( v_1 \) commute.

If \( u \) has order 3 and all nontrivial orbits of \( \langle u, v_0 \rangle \) and \( \langle u, v_1 \rangle \) have length 4, then \( v_0 = v_1 \).

**Proof.** If \( u \) is trivial then the result is immediate from (a). Therefore, we can assume that \( u \) contains the 3-cycle \( (1\ 2\ 3) \) and \( \{2, 3, 4\} \) is a nontrivial orbit of \( v_0 \). Now, in \( \mathfrak{S}_4 \)

\[
(1\ 2\ 3)(4\ 3\ 2)(1\ 2\ 3) = (4\ 3\ 2)(1\ 2\ 3)(4\ 3\ 2) = (1\ 4)(2\ 3)
\]

however

\[
(1\ 2\ 3)(4\ 2\ 3)(1\ 2\ 3) \neq (4\ 2\ 3)(1\ 2\ 3)(4\ 3\ 2).
\]

Thus, \( v_0 \) contains the cycle \( (4\ 3\ 2) \) in its cycle decomposition. Because \( v_1 \) commutes with \( v_0 \),

\[
(1.1) \quad v_1(4\ 3\ 2)v_1^{-1} = (v_1(4)\ v_1(3)\ v_1(2))
\]

is a 3-cycle in the decomposition of \( v_0 \). Now, \( \langle u, v_1 \rangle \) has a 4-orbit of the form \( \{1, 2, 3, x\} \), whence, from the above argument in \( \mathfrak{S}_4 \), \( v_1 \) acts on this orbit as \( (3)(x\ 2\ 1), (2)(x\ 1\ 3) \) or \( (1)(x\ 3\ 2) \). However, in the first case, because \( v_1(3) = 3 \), Equality (1.1) implies that \( v_1(2) = 2 \), contradicting \( v_1(2) = 1 \). Similarly, in the second case, \( v_1(2) = 2 \) combines with (1.1) to imply that \( v_1(3) = 3 \), contradicting \( v_1(1) = 3 \).

This leaves the last case, in which \( v_1(3) = 2 \), which from (1.1) gives \( x = v_1(2) = 4 \). That is, \( (4\ 3\ 2) \) is a 3-cycle in the decomposition of \( v_1 \). It follows that the 3-cycles of \( v_0 \) and \( v_1 \) coincide, whence \( v_0 = v_1 \).

Lemma 1.5. Let \( u, v_0, v_1, v_2 \in \mathfrak{S}_m \) be such that

(a) \( uv_i u = v_i uv_i \) for \( i = 0, 1, 2 \);

(b) \( v_0, v_1 \) and \( v_2 \) commute pairwise; and

(c) whenever \( \{i, j, k\} = \{0, 1, 2\} \), there is an isomorphism \( \gamma_{j,k} \) from \( \langle u, v_i, v_j \rangle \) to \( \langle u, v_i, v_k \rangle \) fixing \( u \) and \( v_i \), and sending \( v_j \) to \( v_k \).

If \( u \) is a product of 4-cycles (possibly with fixed points) and all nontrivial orbits of each \( \langle u, v_i \rangle \) have length 4, then \( v_0 = v_1 = v_2 \).

**Proof.** Since, by (a), \( u \) and \( v_i \) are conjugate, they have the same cycle decomposition type, which must be a product of 4-cycles. If there are no nontrivial orbits, then each \( v_i = 1 \).

Let \( O \) be an orbit of \( u \) of length 4. Then \( O \) must also be a nontrivial orbit of each \( \langle u, v_i \rangle \). From (a) it follows that \( O \) is also a nontrivial orbit of each \( v_i \). Likewise, a nontrivial orbit of any one \( v_i \) must also be a nontrivial orbit of \( u \) and thereby of each \( v_i \). Thus, it suffices to restrict attention to the action on each nontrivial orbit \( O \); in effect, \( \langle u, v_0, v_1, v_2 \rangle \leq \mathfrak{S}_4 \). Using (b), from the fact that in \( \mathfrak{S}_4 \) commuting 4-cycles are either the same or mutually inverse, we must have at least two distinct indices \( i, j \in \{0, 1, 2\} \) such that \( v_i = v_j \). Since by (c) \( \gamma_{j,k} \) both fixes \( v_i \) and sends \( v_j \) to \( v_k \), we conclude that \( v_i = v_j = v_k \).
2 Presentation for the mapping class group

Throughout the paper we denote by $T_b$ the Dehn twist about a simple closed curve $b$. We fix a representation of $\Sigma_{g,1}$ as well as the simple closed curves $a_0, a_1, \ldots, a_{2g+1}$ illustrated in Figure 2.1, and we set $T_i = T_{a_i}$ for all $0 \leq i \leq 2g+1$. The following result is shown in [23].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig21.pdf}
\caption{Generators for $M_{g,1}$.}
\end{figure}

**Theorem 2.1.** (Matsumoto [23]). Let $g \geq 2$.

a) $M_{g,1}$ has a presentation with generators $T_0, T_1, \ldots, T_{2g}$ and relations

\begin{align*}
T_i T_j T_i &= T_j T_i T_j \quad \text{if } a_i \text{ and } a_j \text{ intersect in a single point,} \\
T_i T_j &= T_j T_i \quad \text{if } a_i \cap a_j = \emptyset, \\
(T_2 T_3 T_4 T_0)^{10} &= (T_1 T_2 T_3 T_4 T_0)^6, \\
(T_2 T_3 T_4 T_5 T_6 T_0)^{12} &= (T_1 T_2 T_3 T_4 T_5 T_6 T_0)^9 \quad \text{if } g \geq 3.
\end{align*}

b) $M_{g,0}$ is the quotient of $M_{g,1}$ by the additional relation

$$T_1^{2g-2} = (T_0 T_3 T_4 \cdots T_{2g-1})^{4g-4}.$$

We call $k$ simple closed curves $b_1, \ldots, b_k$ in $\Sigma_{g,n}$ a $k$-chain if their intersection numbers satisfy $i(b_i, b_j) = 1$ if $|i - j| = 1$ and $i(b_i, b_j) = 0$ otherwise. Such a $k$-chain is called nonseparating if the complement of $b_1 \cup \cdots \cup b_k$ in $\Sigma_{g,n}$ is connected. Note that, if $(b_1, \ldots, b_k)$ is a nonseparating $k$-chain, then each $b_i$ is nonseparating, too. The next lemma follows from the rigidity of closed Riemann surfaces (see, for example, [14] Sections 1.3 and 2.3).

**Lemma 2.2.** Let $(b_1, \ldots, b_k)$ and $(b'_1, \ldots, b'_k)$ be two nonseparating $k$-chains in $\Sigma_{g,n}$. Then there exists $\alpha \in M_{g,n}$ such that $\alpha(b_i) = b'_i$ for all $1 \leq i \leq k$. In consequence, this $\alpha$ satisfies $\alpha T_{b_i} \alpha^{-1} = T_{b'_i}$ for all $1 \leq i \leq k$.

For studying the mapping class group $M_{g,n}$ with $n \geq 2$ we use the following convention: $a_0, a_2, \ldots, a_{2g}, a_{2g+1}$ and $b_1, \ldots, b_n$ are the simple closed curves illustrated in Figure 2.2, $T_i = T_{a_i}$ for all $0 \leq i \leq 2g+1$, $i \neq 1$, and $T'_j = T_{b_j}$ for all $1 \leq j \leq n$. In order to unify statements for $n = 1$ and $n > 1$, we make the further convention that, when $n = 1$, $b_1$ in Figure 2.2 coincides with $a_1$ in Figure 2.1. Then for $n = 1$ the element $T'_1$ is simply $T_1$.

The following is well-known. (It can be found for instance in [21, Prop. 2.10 and Thm 3.1].
Proposition 2.3. Let \( g \geq 2 \) and \( n \geq 1 \). Then \( \mathcal{M}_{g,n} \) is generated by \( T_0, T_2, \ldots, T_{2g}, T'_1, \ldots, T'_n \).

Observation. Let \( g \geq 3 \) and \( n \geq 1 \). There is an injective homomorphism \( \mathcal{M}_{g-1,n} \to \mathcal{M}_{g,n} \) which sends \( T'_j \) to \( T'_j \) and \( T_i \) to \( T_i \) for all \( j \in \{1, \ldots, n\} \) and \( i \in \{0, 2, \ldots, 2g - 2\} \). It is easily seen that this homomorphism is induced by some embedding of \( \Sigma_{g-1,n} \) into \( \Sigma_{g,n} \). From now on we will assume \( \mathcal{M}_{g-1,n} \) to be embedded into \( \mathcal{M}_{g,n} \) via this homomorphism. Note that such a homomorphism does not exist for \( n = 0 \).

3 The genus 2 case

In this section we describe all the subgroups of \( \mathcal{M}_{2,n} \) of index at most \( N^+_2 = 10 \) up to conjugation. We will see in particular that the genus \( g = 2 \) case is different from the genus \( g \geq 3 \) case.

The first difference comes from the fact that the abelianization of \( \mathcal{M}_{2,n} \) is nontrivial, while the group \( \mathcal{M}_{g,n} \) is perfect if \( g \geq 3 \) (see [27], [20]). More precisely, the abelianization of \( \mathcal{M}_{2,n} \) is \( \mathbb{Z}/10\mathbb{Z} \) [24]. So, if \( ab : \mathcal{M}_{2,n} \to \mathbb{Z}/10\mathbb{Z} \) denotes the projection of \( \mathcal{M}_{2,n} \) onto its abelianization, then \( ab^{-1}(\mathbb{Z}/5\mathbb{Z}) \) is a subgroup of index 2, \( ab^{-1}(\mathbb{Z}/2\mathbb{Z}) \) is a subgroup of index 5, and \( \text{Ker}(ab) \) is a subgroup of index 10. Note that all these subgroups contain the commutator subgroup \( \mathcal{M}'_{2,n} \). In particular, since \( \text{Sp}_4(\mathbb{F}_2)' \) has index 2 in \( \text{Sp}_4(\mathbb{F}_2) \), neither \( ab^{-1}(\mathbb{Z}/2\mathbb{Z}) \) nor \( \text{Ker}(ab) \) is a pre-image under \( \theta_{2,n} \) of a subgroup of \( \text{Sp}_4(\mathbb{F}_2) \).

The second difference comes from the fact that \( \text{Sp}_4(\mathbb{F}_2) = \mathfrak{S}_6 \) [11] contains more than two subgroups of index at most 10, up to conjugation. In addition to \( O^+_5(\mathbb{F}_2) = \mathfrak{S}_5 \) [11] and \( O^+_4(\mathbb{F}_2) \), of indices 6 and 10 respectively, it contains the alternating group \( \mathfrak{A}_6 \) of index 2, and another subgroup of index 6 which can be described as follows. The group \( \text{Sp}_4(\mathbb{F}_2) = \mathfrak{S}_6 \) has a non-inner
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Let \( \alpha \) be an automorphism defined by
\[
\alpha : \begin{cases}
(1 2) \mapsto (1 2)(3 5)(4 6) \\
(2 3) \mapsto (1 3)(2 4)(5 6) \\
(3 4) \mapsto (1 2)(3 6)(4 5) \\
(4 5) \mapsto (1 3)(2 5)(4 6) \\
(5 6) \mapsto (1 2)(3 4)(5 6)
\end{cases}
\]

(Since \( \text{Out}(\mathcal{S}_6) \) has order 2, \( \alpha \) is essentially unique.) It turns out that \( \alpha(O_4^-(\mathbb{F}_2)) \) is a subgroup of \( \text{Sp}_4(\mathbb{F}_2) \) of index 6 which is not conjugate to \( O_4^-(\mathbb{F}_2) \). On the other hand, \( \alpha(O_4^+(\mathbb{F}_2)) \) is conjugate to \( O_4^+(\mathbb{F}_2) \) (so there are four subgroups and not five), and \( ab^{-1}(\mathbb{Z}/5\mathbb{Z}) = \theta_2^{-1}(\mathfrak{A}_6) \).

So, we have the following subgroups of index at most 10 in \( \mathcal{M}_{2,n} \) up to conjugation:

- \( \theta_2^{-1}(\mathfrak{A}_6) = ab^{-1}(\mathbb{Z}/5\mathbb{Z}) \) of index 2,
- \( \text{ab}^{-1}(\mathbb{Z}/2\mathbb{Z}) \) of index 5,
- \( \theta_2^{-1}(O_4^-(\mathbb{F}_2)) \) of index 6,
- \( \theta_2^{-1}(\alpha(O_4^-(\mathbb{F}_2))) \) of index 6,
- \( \text{Ker(ab)} \) of index 10, and
- \( \theta_2^{-1}(O_4^+(\mathbb{F}_2)) \) of index 10.

We show that these are all the proper subgroups of index at most 10 in \( \mathcal{M}_{2,n} \) up to conjugation.

Let \( n \geq 1, m \geq 1 \) and \( w \in \mathcal{S}_m \). If \( w^{10} = 1 \), then there is a permutation representation \( \text{cycl}_w : \mathcal{M}_{2,n} \to \mathcal{S}_m \) which sends \( T_i \) and \( T'_j \) to \( w \) for all \( i \in \{0, 2, 3, 4\} \) and all \( j \in \{1, \ldots, n\} \).

Such a representation is called a cyclic representation of \( \mathcal{M}_{2,n} \). It is transitive if and only if \( w \) is a cycle of length \( m \) and \( m \in \{1, 2, 5, 10\} \).

**Lemma 3.1.** For \( n \geq 1 \), there are exactly three conjugacy classes of non-cyclic transitive permutation representations of \( \mathcal{M}_{2,n} \) of degree at most 10, namely, two conjugacy classes of degree 6, and a unique conjugacy class of degree 10. More specifically, up to conjugacy in \( \mathcal{S}_n \), the two transitive permutation representations \( \mathcal{M}_{2,n} \to \mathcal{S}_n \) are given by

\[
\phi_{2,n} : \begin{cases}
T'_j \mapsto (1 2) & \text{for } 1 \leq j \leq n \\
T_2 \mapsto (2 3) \\
T_3 \mapsto (3 4) \\
T_4 \mapsto (4 5) \\
T_0 \mapsto (5 6)
\end{cases}
\]

\[
\phi_{n}^O : \begin{cases}
T'_j \mapsto (1 2)(3 5)(4 6) & \text{for } 1 \leq j \leq n \\
T_2 \mapsto (1 3)(2 4)(5 6) \\
T_3 \mapsto (1 2)(3 6)(4 5) \\
T_4 \mapsto (1 3)(2 5)(4 6) \\
T_0 \mapsto (1 2)(3 4)(5 6)
\end{cases}
\]
We first show that each \( S \) of the common nontrivial cycles of \( \phi \) are independent of choice of \( j \) and, up to conjugacy in \( S \) Finite index subgroups of mapping class groups

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and, up to conjugacy in \( S_{10} \), the unique permutation representation \( M_{2,n} \to S_{10} \) is given by

\[
\phi_{2,n}^* : \begin{cases} 
T_j' \mapsto (3\ 5)(6\ 8)(9\ 10) & \text{for } 1 \leq j \leq n \\
T_2 \mapsto (2\ 3)(4\ 6)(7\ 9) \\
T_3 \mapsto (1\ 2)(6\ 10)(8\ 9) \\
T_4 \mapsto (2\ 4)(3\ 6)(5\ 8) \\
T_0 \mapsto (4\ 7)(6\ 9)(8\ 10)
\end{cases}
\]

Proof. In the case \( n = 1 \), this result can be easily proved with a direct calculation. Using the presentation of \( M_{2,1} \) from Theorem 2.1, one can use coset enumeration techniques to perform a systematic search for representatives of the conjugacy classes of subgroups of \( M_{2,1} \) of index at most \( K \) for a (small) integer \( K \); see [29]. When \( K = 10 \), this systematic search shows that there are exactly six conjugacy classes of proper subgroups of \( M_{2,1} \) of index at most 10; the columns of the coset table for the each of the constructed subgroups yield the images of the generators \( T_0, \ldots, T_4 \) under the corresponding permutation representation. The computation is very easy and can be performed with any mathematical software such as Magma or GAP.

Now suppose that \( n \geq 2 \). Let \( \varphi : M_{2,n} \to S_m \) be a non-cyclic and transitive representation with \( m \leq 10 \). For \( 1 \leq j \leq n \), we denote by \( M(j) \) the subgroup of \( M_{2,n} \) generated by \( T_j', T_2, T_3, T_4, T_0 \). This group is isomorphic to \( M_{2,1} \) via an isomorphism \( \gamma_j : M_{2,1} \to M(j) \) which sends \( T_1 \) to \( T_j' \) and \( T_1 \) to \( T_i \) for all \( i \in \{2, 3, 4, 0\} \). We denote by \( \varphi_j : M_{2,1} \to S_m \) the composition of \( \gamma_j \) with \( \varphi \).

Observe that

\[(\ast) \quad \varphi_j(T_i) = \varphi(T_i) \quad \text{for all } j \in \{1, \ldots, n\} \text{ and } i \in \{0, 2, 3, 4\}.
\]

By the case \( n = 1 \), for each \( j \in \{1, \ldots, n\} \) there is a decomposition \( \{1, \ldots, m\} = S_j^{(1)} \cup S_j^{(2)} \cup S_j^{(3)} \) as follows. Each \( S_j^{(k)} \) is invariant under the action of \( \varphi_j(M_{2,1}) \); either \( S_j^{(1)} = \emptyset \), or \( \varphi_j \) restricted to \( S_j^{(1)} \) is equivalent to an element of \( \{\phi_{2,1}^-, \phi_1^+, \phi_{2,1}^+\} \); if \( S_j^{(2)} \) is nonempty, then there exists \( w_j \in S_m - \{1\} \) such that \( S(w_j) = S_j^{(2)} \) and the restriction of \( \varphi_j \) to \( S_j^{(2)} \) is cyclic, and \( \varphi_j(M_{2,1}) \) acts trivially on \( S_j^{(3)} \).

Let \( \phi \in \{\phi_{2,1}^-, \phi_1^+, \phi_{2,1}^+\} \) and set \( N = N_\phi = 6 \) if \( \phi = \phi_{2,1}^+ \) or \( \phi_1^+ \), and \( N = N_\phi = 10 \) if \( \phi = \phi_{2,1}^- \).

The following claims are readily verified from the description of \( \phi \), using either GAP or Magma where necessary:

1. \( \phi(T_2) \) and \( \phi(T_3) \) have no common cycle in their decompositions;
2. \( S(\{\phi(T_i) \mid i = 1, 3, 4, 0\}) = \{1, \ldots, N\} \), where, for \( X \subseteq S_N \), \( S(X) \) denotes \( \cup_{w \in X} S(w) \);
3. the (simultaneous) centralizer of \( \{\phi(T_i) \mid i = 1, 3, 4, 0\} \) in \( S_N \) is \( \{1, \phi(T_1)\} \);
4. the support of each cycle in the decomposition of \( \phi(T_1) \) intersects \( S(\{\phi(T_i) \mid i = 0, 2, 3, 4\}) \) nontrivially.

We first show that each \( S_j^{(2)} = \emptyset \). Whenever \( S_j^{(2)} \neq \emptyset \), then by \( (1) \) we get that \( w_j \) is the product of the common nontrivial cycles of \( \varphi_j(T_2) \) and \( \varphi_j(T_3) \). However, by \( (\ast) \), these common cycles are independent of choice of \( j \). It follows that for all \( j \in \{1, \ldots, n\} \), \( S_j^{(2)} \neq \emptyset \), \( w_j = w_1 \) and \( S_j^{(2)} = S(w_j) = S(w_1) = S_1^{(2)} \). However, \( \varphi \) is transitive and non-cyclic. Hence, \( S_j^{(2)} = \emptyset \) for all \( j \in \{1, \ldots, n\} \).
It now follows that each $S_j^{(1)} \neq \emptyset$, and therefore, the restriction of each $\varphi_j$ to $S_j^{(1)}$ is equivalent to an element of $\{ \phi_{2,1}, \phi_{1}^+, \phi_{2,1}^+ \}$. Since we know by (*) that for each $i \neq 1$ the $\varphi_j(T_i)$ agree, it remains to show that all the $\varphi_j(T_1)$ also coincide, and that each $S_j^{(3)}$ is empty.

Without loss of generality we can assume that $S_1^{(1)} = \{1, \ldots, N\}$ and the restriction of $\varphi_1$ to $\{1, \ldots, N\}$ is an element $\phi$ in $\{ \phi_{2,1}, \phi_{1}^+, \phi_{2,1}^+ \}$, where $N = N_0$. Let $j \in \{1, \ldots, n\}$. Since $T'_j$ commutes with $T_1, T_3, T_4, T_0$, the permutation $T_1$ belongs to the centralizer of $\{ \varphi_1(T_1) | i = 1, 3, 4, 0 \}$. Combining (2) and (3) we get that this centralizer is $\{1, \phi(T_1)\} \times S_{m-N}$. On the other hand, by (4), the support of each cycle of $\varphi_j(T_1)$ intersects $S(\{\varphi_j(T_i) | i = 0, 2, 3, 4\})$ nontrivially, and, by (*),

$$S(\{\varphi_j(T_i) | i = 0, 2, 3, 4\}) = S(\{\varphi_1(T_i) | i = 0, 2, 3, 4\}) \subseteq \{1, \ldots, \).

This implies that $\varphi_j(T_1) = \phi(T_1)$ and, therefore, each $\varphi_j(T_1)$ is the same. Then, finally, by transitivity, it must be that $m = N$, and the proof is complete.

**Proposition 3.2.** Let $n \geq 0$. Then $M_{2,n}$ has precisely six proper subgroups of index at most 10 up to conjugation, namely, $ab^{-1}(\mathbb{Z}/5\mathbb{Z}) = \theta_2^{-1}(\mathbb{Z}/3\mathbb{Z})$ of index 2, $ab^{-1}(\mathbb{Z}/2\mathbb{Z})$ of index 5, $\theta_2^{-1}(O_4^-)(\mathbb{F}_2)$ of index 6, $\theta_2^{-1}(\alpha(O_4^-)(\mathbb{F}_2))$ of index 10, $\text{Ker}(ab)$ of index 10, and $\theta_2^{-1}(O_4^+)(\mathbb{F}_2)$ of index 10.

**Proof.** For $n \geq 1$, the claim is a direct consequence of Lemma 3.1 together with the description of the subgroups of $M_{2,n}$ given in the beginning of the section. By Theorem 2.1, $M_{2,0}$ is a quotient of $M_{2,1}$ by one additional relation, so for the case $n = 0$ it is sufficient to check that the representations of $M_{2,1}$ given in Lemma 3.1 satisfy the additional relation of $M_{2,0}$; this is the case, as can easily be verified.

---

### 4 The genus 3 case

In this section we calculate the subgroups of index at most 36 in $M_{2,n}$ up to conjugation. (Note that $N_2^- = 28$ and $N_2^+ = 36$.) We argue in the same manner as for the case of genus 2 surfaces (see Section 3), with direct calculations often made with computers. However, we should point out here that, in this case, the computations are far from being elementary, and we often approach the limit of what can currently be done with computers (especially in the proof of Lemma 4.1). Recall also that the case of surfaces of genus $g = 3$ will be the first step in the induction to prove Theorem 0.3.

**Lemma 4.1.** For $n \geq 1$, there are exactly two conjugacy classes of nontrivial transitive permutation representations of $M_{3,n}$ of degree at most 36, namely a unique conjugacy class of degree 28 and a unique conjugacy class of degree 36. More specifically, up to conjugacy in $S_{28}$, the unique permutation representation $M_{3,n} \to S_{28}$ is given by

$$\phi_{3,n} : \begin{cases}
T'_j \mapsto (14\,18)(16\,21)(17\,22)(19\,23)(24\,26)(27\,28) & \text{for } 1 \leq j \leq n \\
T_0 \mapsto (1\,3)(2\,5)(4\,8)(20\,25)(24\,28)(26\,27) \\
T_2 \mapsto (10\,14)(12\,16)(13\,17)(15\,19)(20\,24)(25\,28) \\
T_3 \mapsto (6\,10)(7\,12)(9\,13)(11\,15)(24\,27)(26\,28) \\
T_4 \mapsto (3\,6)(4\,7)(5\,9)(15\,20)(19\,24)(23\,26) \\
T_5 \mapsto (2\,4)(5\,8)(6\,11)(10\,15)(14\,19)(18\,23) \\
T_6 \mapsto (1\,2)(3\,5)(6\,9)(10\,13)(14\,17)(18\,22)
\end{cases}$$
and, up to conjugacy in $\mathfrak{S}_{36}$, the unique permutation representation $M_{3,n} \rightarrow \mathfrak{S}_{36}$ is given by

$$
\phi_{3,n}^{+} : \left\{
\begin{array}{l}
T_{j}^{0} \mapsto (6 \ 9)(10 \ 13)(14 \ 18)(15 \ 19)(20 \ 22)(21 \ 25)(26 \ 28)(27 \ 30)(31 \ 32)(34 \ 35) \\
T_{0} \mapsto (1 \ 2)(11 \ 17)(14 \ 22)(16 \ 24)(18 \ 20)(21 \ 28)(23 \ 29)(25 \ 26)(27 \ 30)(31 \ 32) \\
T_{2} \mapsto (4 \ 6)(7 \ 10)(11 \ 14)(12 \ 15)(16 \ 21)(17 \ 22)(23 \ 27)(24 \ 28)(29 \ 32)(33 \ 35) \\
T_{3} \mapsto (3 \ 4)(5 \ 7)(8 \ 12)(14 \ 20)(18 \ 22)(21 \ 26)(25 \ 28)(27 \ 31)(30 \ 32)(33 \ 36) \\
T_{4} \mapsto (2 \ 3)(7 \ 11)(10 \ 14)(12 \ 16)(13 \ 18)(15 \ 21)(19 \ 25)(29 \ 33)(31 \ 34)(32 \ 35) \\
T_{5} \mapsto (3 \ 5)(4 \ 7)(6 \ 10)(9 \ 13)(16 \ 23)(21 \ 27)(24 \ 29)(25 \ 30)(26 \ 31)(28 \ 32) \\
T_{6} \mapsto (5 \ 8)(7 \ 12)(10 \ 15)(11 \ 16)(13 \ 19)(14 \ 21)(17 \ 24)(18 \ 25)(20 \ 26)(22 \ 28)
\end{array}\right.

$$(1 \leq j \leq n$$)

In particular, $m_{i}(M_{3,n}) = 28$.

**Proof.** In the case $n = 1$, the result is shown by a direct computation. Using the presentation of $M_{3,1}$ from Theorem 2.1, one can use coset enumeration techniques to perform a systematic search for representatives of the conjugacy classes of subgroups of $M_{3,1}$ of index at most $K$ for a (small) integer $K$; see [29]. When $K = 36$, this systematic search shows that there are exactly two conjugacy classes of proper subgroups of $M_{3,1}$ of index at most 36: exactly one conjugacy class of subgroups of index 28 and exactly one conjugacy class of subgroups of index 36. The columns of the coset table for the each of the constructed subgroups yield the images of the generators $T_{0}, \ldots, T_{6}$ under the corresponding permutation representations $\phi_{3,1}^{3}$ of degree 28 and $\phi_{3,1}^{+}$ of degree 36.

We used the implementation of the low index subgroup search provided in MAGMA [5], filling the coset table in column major order. We ran a development version of MAGMA V2.15. The computation took approximately 47.5 hours on a GNU/Linux system with an Intel E8400 64-bit CPU (core: 3 GHz, FSB: 1333 MHz) and a main memory bandwidth of 6.5 GB/s (X38 chipset, dual channel DDR2 RAM, memory bus: 1066 MHz). We remark that the use of column major order is crucial for the running time; tests for indices between 10 and 15 suggest a speed-up by a factor between $10^{3}$ and $10^{4}$ compared to row major order.

Now suppose that $n \geq 2$. Let $\varphi : M_{3,n} \rightarrow \mathfrak{S}_{m}$ be a transitive representation with $m \leq 36$. For $1 \leq j \leq n$, we denote by $M^{(j)}$ the subgroup of $M_{3,n}$ generated by $T_{j}^{0}, T_{2}, \ldots, T_{6}, T_{0}$. This group is isomorphic to $M_{3,1}$ via an isomorphism $\gamma_{j} : M_{3,1} \rightarrow M^{(j)}$ which sends $T_{j}$ to $T_{j}^{0}$ and $T_{i}$ to $T_{i}$ for all $i \in \{2, \ldots, 6, 0\}$. We denote by $\varphi_{j} : M_{3,1} \rightarrow \mathfrak{S}_{m}$ the composition of $\gamma_{j}$ with $\varphi$. Observe that

$$\varphi_{j}(T_{i}) = \varphi(T_{i}) \text{ for all } j \in \{1, \ldots, n\} \text{ and } i \in \{0,2,\ldots,6\}.$$

By the case $n = 1$, for each $j \in \{1, \ldots, n\}$ there is a decomposition $\{1, \ldots, m\} = S_{j}^{(1)} \sqcup S_{j}^{(2)}$ as follows. Each $S_{j}^{(k)}$ is invariant under the action of $\varphi_{j}(M_{3,1})$; the restriction of $\varphi_{j}$ to $S_{j}^{(1)}$ is equivalent to an element of $\{\phi_{3,1}^{-}, \phi_{3,1}^{+}\}$; and $\varphi_{j}(M_{3,1})$ acts trivially on $S_{j}^{(2)}$. (Note here a difference from the $g = 2$ case: because $M_{3,1}$ is a perfect group [27] it has no cyclic representations.)

Let $\phi \in \{\phi_{3,1}^{-}, \phi_{3,1}^{+}\}$ and set $N = N_{\phi} = 28$ if $\phi = \phi_{3,1}^{-}$, and $N = N_{\phi} = 36$ if $\phi = \phi_{3,1}^{+}$. Again, the following claims may be confirmed from the description of $\phi$, using either GAP or MAGMA where necessary:

1. $S(\{\phi(T_{i}) \mid i = 1, 3, 4, 5, 6, 0\}) = \{1, \ldots, N\}$, where, for $X \subseteq \mathfrak{S}_{N}$, $S(X)$ denotes $\bigcup_{w \in X} S(w)$;
2. the unique element of the centralizer of \( \{ \phi(T_i) \mid i = 1, 3, 4, 5, 6, 0 \} \) in \( \mathcal{S}_N \) having the same cycle decomposition type as \( \phi(T_1) \): 

3. the support of each nontrivial cycle in the decomposition of \( \phi(T_1) \) intersects \( S(\{ \phi(T_i) \mid i = 0, 2, 3, 4, 5, 6 \}) \) nontrivially.

Without loss of generality we can assume that \( S_1^{(1)} = \{1, \ldots, N\} \) and the restriction of \( \varphi_1 \) to \( \{1, \ldots, N\} \) is an element \( \phi \in \{ \phi_{5,1}, \phi_{3,1} \} \), where \( N = N_\phi \). Let \( j \in \{1, \ldots, n\} \). Since \( T_j' \) commutes with \( T_1, T_3, T_4, T_5, T_6, T_0 \), the permutation \( \varphi_j(T_1) \) belongs to the centralizer of \( \{ \varphi_1(T_i) \mid i = 1, 3, 4, 5, 6, 0 \} \). By (1), this centralizer is \( Z \times \mathcal{S}_{m-N} \), where \( Z \) is the centralizer of \( \{ \phi(T_i) \mid i = 1, 3, 4, 5, 6, 0 \} \) in \( \mathcal{S}_N \). By (3), the support of each nontrivial cycle of \( \varphi_j(T_1) \) intersects \( S(\{ \varphi_j(T_i) \mid i = 0, 2, 3, 4, 5, 6 \}) \) nontrivially, and, by (*), 

\[
S(\{ \varphi_j(T_i) \mid i = 0, 2, 3, 4, 5, 6 \}) = S(\{ \varphi_1(T_i) \mid i = 0, 2, 3, 4, 5, 6 \}) \subseteq \{1, \ldots, N\}.
\]

Thus, \( \varphi_j(T_1) \in Z \). Now, since \( T_j' \) and \( T_1' \) are conjugate in \( \mathcal{M}_{3,1} \), \( \varphi_j(T_1) \) and \( \varphi_1(T_1) \) share the same cycle decomposition type; hence, by (2), \( \varphi_j(T_1) = \varphi_1(T_1) \). To complete the proof, observe that transitivity forces \( m = N \).

**Proposition 4.2.** Let \( n \geq 0 \). Then \( \mathcal{M}_{3,n} \) has precisely two proper subgroups of index at most 36 up to conjugation, namely, \( \mathcal{O}_{3,n}^- \) of index 28, and \( \mathcal{O}_{3,n}^+ \) of index 36.

**Proof.** For \( n \geq 1 \), the claim is a direct consequence of Lemma 4.1 and the definitions of \( \mathcal{O}_{3,n}^- \) respectively \( \mathcal{O}_{3,n}^+ \). By Theorem 2.1, \( \mathcal{M}_{3,0} \) is a quotient of \( \mathcal{M}_{3,1} \) by one additional relation, so for the case \( n = 0 \) it is sufficient to check that the representations of \( \mathcal{M}_{3,1} \) given in Lemma 4.1 satisfy the additional relation of \( \mathcal{M}_{3,0} \); this is the case, as can easily be verified.

**Part II**

**Induction arguments**

We turn now to the proof of our main result, Theorem 0.3. As pointed out before, we argue by induction on the genus. Recall that the case \( g = 3 \) is proved in Section 4 (see Proposition 4.2). Thus:

- from now on, we suppose that \( g \geq 4 \) plus the inductive hypothesis that Theorem 0.3 holds for a surface of genus \( g - 1 \).

Recall that we have defined \( N_g^- = 2^{g-1}(2^g - 1) \) and \( N_g^+ = 2^{g-1}(2^g + 1) \). Throughout the arguments below, we shall rely on the following numerical relationships for \( g \geq 4 \):

\[
N_g^+ = 3N_{g-1}^+ + N_{g-1}^-, \quad N_g^- = 3N_{g-1}^- + N_{g-1}^+,
\]

\[
4N_{g-1}^- < N_g^- < N_g^+ < 5N_{g-1}^- < 2N_g^-.
\]

(Actually, only the third inequality requires \( g \geq 4 \); the others hold for \( g \geq 2 \).)

Theorem 0.2 will be entirely proved in Section 6. Theorem 0.4 will be proved in Sections 7 and 8.
5 Factorization through symplectic groups

Our goal in this section is to prove the following theorem.

**Theorem 5.1.** For \( g \geq 4 \), let \( \varphi : M_{g,1} \to \mathcal{S}_m \) be a nontrivial homomorphism, with \( m < 5 N_{g-1} \).
Then there is a homomorphism \( \bar{\varphi} : \text{Sp}_{2g}(\mathbb{F}_2) \to \mathcal{S}_m \) such that the following diagram commutes.

\[
\begin{array}{c}
M_{g,1} \\
\theta_{g,1} \downarrow \\
\text{Sp}_{2g}(\mathbb{F}_2) \xrightarrow{\bar{\varphi}} \mathcal{S}_m \\
\end{array}
\]

The proof strategy is as follows. To show that the kernel of \( \varphi \) contains the kernel of \( \theta_{g,1} \), we first prove that the image \( w_i = \varphi(T_i) \) has order 2 in \( \mathcal{S}_m \). For this, we consider the cycle decomposition type \((1)^{i_1}(2)^{i_2} \cdots (m)^{i_m} \) of the permutation \( w_i \). We first exclude the possibility of cycles of length at least 5, then of cycles of length 4, and finally – the most delicate case – we exclude cycles of length 3. Hence, \( w_i \) is reduced to being an involution after all. It then remains to show that the kernel of \( \theta_{g,1} \) is the normal closure of the square of any of our standard generators of \( M_{g,1} \).

5.1 Reduction to involutions

**Proposition 5.2.** Let \( g \geq 4 \), and let \( b \) be a nonseparating simple closed curve in \( \Sigma_{g,1} \). Then the centralizer \( Z_b \) of \( T_b \) in \( M_{g,1} \) contains an index 2 subgroup \( Z^+_b \) with the properties:

\begin{enumerate}
    \item \( Z^+_b \) is perfect;
    \item \( N_{g-1} \leq \text{mi}(Z^+_b) \); and
    \item \( T_b \in Z^+_b \).
\end{enumerate}

**Proof.** It is known that \( Z_b \) is the set of mapping classes that fix the curve \( b \) up to isotopy (see e.g. [26]). We take \( Z^+_b \) to be the subgroup of \( Z_b \) consisting of those classes that also preserve the orientation of \( b \). Evidently, \( Z^+_b \) has index 2 in \( Z_b \) and contains \( T_b \). From [26], \( Z^+_b \) is the image of \( M_{g-1,3} \) in \( M_{g,1} \) under the homomorphism induced by a quotient map \( \Sigma_{g-1,3} \to \Sigma_{g,1} \) identifying two boundary circles with the closed curve \( b \). Since by [17] \( M_{g-1,3} \) is perfect, we have (a). From the epimorphism \( M_{g-1,3} \to Z^+_b \) we know that \( \text{mi}(M_{g-1,3}) \leq \text{mi}(Z^+_b) \), and by induction we have \( \text{mi}(M_{g-1,3}) = N_{g-1} \), thus \( N_{g-1} \leq \text{mi}(Z^+_b) \).

As ever, we consider the simple closed curves \( a_0, a_1, \ldots, a_{2g+1} \) illustrated in Figure 2.1, we denote by \( T_i \) the Dehn twist about \( a_i \), and write \( w_i \) for the image under \( \varphi \) of the Dehn twist \( T_i \) \((i = 0, 1, \ldots, 2g+1) \). In the sequel we repeatedly use the fact that, since \( T_i \) and \( T_j \) are conjugate in the mapping class group, \( w_i = \varphi(T_i) \) and \( w_j = \varphi(T_j) \) are conjugate in the symmetric group, and so share the same cycle decomposition type, say \((1)^{\ell_1}(2)^{\ell_2} \cdots (m)^{\ell_m} \), where \( \sum_{k=1}^m k\ell_k = m \). The fact that \( T_i \in Z^+_b \) implies that, whenever \( \ell_k > 0 \) with \( k > 1 \), \( \varphi(Z^+_a) \) acts nontrivially on the union of the \( k \)-orbits of \( w_i \). Therefore, the above proposition combines with Lemma 1.1 (3) (a), (c) to yield the following.
Corollary 5.3. For $g \geq 4$, let $\varphi : \mathcal{M}_{g,1} \to \mathfrak{S}_m$ be a nontrivial homomorphism, with $m < 5N_{g-1}^-$. Then there exists $k \in \{2, 3, 4\}$ such that every $w_i$ has the same cycle decomposition type $(1)^{\ell_1}(k)^{\ell_k}$ with $\ell_k \geq N_{g-1}^-$. □

Lemma 5.4. For $g \geq 3$, let $\varphi : \mathcal{M}_{g,1} \to \mathfrak{S}_m$ be a group homomorphism. If $S(w_{2g-2}) \cap S(w_{2g}) = \emptyset$, then $\varphi$ is trivial.

Proof. First note that, since conjugate permutations have bijective supports, by Lemma 2.2 the cardinality $r = |S(w_i)|$ is independent of choice of $i$, and for all 3-chains $(a_i, a_j, a_k)$ yields the same cardinalities $s = |S(w_i) \cap S(w_j)|$ and $t = |S(w_i) \cap S(w_k)|$. By assumption, the 3-chain $(a_{2g-2}, a_{2g-1}, a_{2g})$ gives $t = 0$, which implies that

$$(S(w_{2g-2}) \cap S(w_{2g})) \cup (S(w_{2g-1}) \cap S(w_{2g})) \subseteq S(w_{2g-1}),$$

and so $r \geq 2s$. On the other hand, Lemma 1.2 asserts that $r \leq 2s$. Thus, for any 3-chain $(a_i, a_j, a_k)$ we have

$$S(w_j) = (S(w_i) \cap S(w_j)) \cup (S(w_j) \cap S(w_k)).$$

Turning now to the 3-chains $(a_0, a_4, a_3)$ and $(a_0, a_4, a_5)$ and $(a_3, a_4, a_5)$, we observe that, since $t = 0$, the subsets $S(w_0) \cap S(w_4)$, $S(w_3) \cap S(w_4)$ and $S(w_4) \cap S(w_5)$ of $S(w_4)$ are pairwise disjoint, but each of cardinality half that of $S(w_4)$. For avoidance of contradiction, it must be that $r = s = 0$; whence, since by Theorem 2.1 the image of $\varphi$ is generated by the $w_i$, $\varphi$ is trivial. □

Proposition 5.5. For $g \geq 4$, let $\varphi : \mathcal{M}_{g,1} \to \mathfrak{S}_m$ with $m < 5N_{g-1}^-$. Then for $i = 0, 1, \ldots, 2g+1$ there are no 4-cycles in the cycle decomposition of $w_i$.

Proof. Again write $\ell_k$ for the number of $k$-cycles in the decomposition of $w_{2g}$, and suppose that $\ell_4 > 0$. Then Corollary 5.3 forces the cycle decomposition type of every $w_j$ to be $(1)^{\ell_1}(4)^{4}$ where $\ell_4 \geq N_{g-1}^-$. The image $\varphi(\mathcal{M}_{g,1})$ contains the subgroup $H = \langle w_{2g}, w_{2g+1} \rangle$, which, since $w_{2g}$ is a product of 4-cycles, is a homomorphic image (via $x_i \mapsto w_{2g+i}$) of the group

$$\hat{H} = \langle x_0, x_1 \mid x_0^4 = x_1^4 = 1, x_0x_1x_0 = x_1x_0x_1 \rangle$$

of order 96 (denoted $(-2, 3|4)$ in [10, p.74]).

From now on $\mathcal{M}_{g-1,1}$ is considered as the subgroup of $\mathcal{M}_{g,1}$ generated by $T_0, T_1, \ldots, T_{2g-2}$. We write $\hat{\Omega}_k$ for the union of the orbits $\Omega_{k,i}$ ($i = 1, \ldots, h_k$) of cardinality $k$ of $H$. Thus, $|\hat{\Omega}_k| = kh_k$ and $k$ divides 96. From the fact that the generators $w_{2g}$ and $w_{2g+1}$ of $H$ contain no nontrivial cycles of length less than 4 it follows that $h_2 = h_3 = 0$. Since $\varphi(\mathcal{M}_{g-1,1})$ lies in the centralizer of $H$, it acts, for any $k$, both on $\hat{\Omega}_k$ and on the set $\{\Omega_{k,i} \mid i = 1, \ldots, h_k\}$. The former corresponds to a homomorphism $\psi_k : \mathcal{M}_{g-1,1} \to \mathfrak{S}_{kh_k}$ and the latter to a homomorphism $\nu_k : \mathcal{M}_{g-1,1} \to \mathfrak{S}_{kh_k}$.

Claim. If $\nu_k$ is trivial, then so is $\psi_k$.

To prove this, since $\psi_1 = \psi_2$ we may assume that $k \geq 4$. Assuming that $\nu_k$ is trivial, we observe that each $k$-orbit $\Omega_{k,i}$ of $H$ must be invariant under the action of $\varphi(\mathcal{M}_{g-1,1})$. Now partition $\Omega_{k,i}$ by means of the orbits of $w_{2g}$, as

$$\Omega_{k,i} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_p \sqcup F,$$
where each orbit $P_j$ of $w_{2g}$ has length 4, and $F$ consists of the fixed points of $w_{2g}$. Then $\varphi(M_{g-1,1})$ acts on $P_1 \sqcup \cdots \sqcup P_p$, on $\{P_1, \ldots, P_p\}$, and on $F$. The action on the set $\{P_1, \ldots, P_p\}$ must be trivial because

$$p \leq \frac{k}{4} \leq \frac{96}{4} < 2^{g-2}(2^{g-1} - 1) = N_{g-1}^{-1} = \text{mi}(M_{g-1,1})$$

(by induction). This leaves $\varphi(M_{g-1,1})$ acting on each 4-orbit $P_j$, where the action must again be trivial, since certainly $4 < \text{mi}(M_{g-1,1})$. Hence, the action of $\varphi(M_{g-1,1})$ on $\Omega_{k,i}$ stabilises $\Omega_{k,i} \cap S(w_{2g})$ pointwise, and likewise $\Omega_{k,i} \cap S(w_{2g+1})$. However, since $H$ is generated by $w_{2g}$ and $w_{2g+1}$, the union of $\Omega_{k,i} \cap S(w_{2g})$ and $\Omega_{k,i} \cap S(w_{2g+1})$ is equal to $\Omega_{k,i}$. Thus, the action of $\varphi(M_{g-1,1})$ is trivial on each $\Omega_{k,i}$, and therefore also on their union $\hat{\Omega}_k$ as required.

In consequence, if $\varphi(M_{g-1,1})$ acts nontrivially on some $\hat{\Omega}_k$, then $h_k \geq \text{mi}(M_{g-1,1})$. Since

$$kh_k = |\hat{\Omega}_k| \leq m < 5N_{g-1}^{-1} = 5\text{mi}(M_{g-1,1}),$$

the only possibilities are $k = 1$ or $k = 4$. From Lemma 5.4, $M_{g-1,1}$ must act nontrivially on some $\hat{\Omega}_k$ with $k > 1$, and therefore $k = 4$ indeed occurs.

The inequality $h_4 \geq N_{g-1}^{-1}$ implies that $h_1 \leq m - 4h_4 < N_{g-1}^{-1}$, thus $M_{g-1,1}$ acts trivially on $\hat{\Omega}_1$. It follows that

$$S(\varphi(M_{g-1,1})) \subseteq \hat{\Omega}_1 \subseteq S(H),$$

where, for a subgroup $G$ of $S_m$, $S(G)$ denotes its support. Since $S(\varphi(M_{g-1,1}))$ contains $S((w_1, w_2))$, which by Lemma 2.2 has the same cardinality as $S(H)$, the above inclusions are indeed equalities.

So, $w_{2g}$ is a product of 4-cycles and all nontrivial orbits of $H = \langle w_{2g}, w_{2g+1} \rangle$ have length 4. Applying again Lemma 2.2, we get that $w_4$ is a product of 4-cycles and, for $i \in \{0, 3, 5\}$, all nontrivial orbits of $\langle w_4, w_i \rangle$ have length 4. By Lemma 1.5 we deduce that $w_0 = w_3 = w_5$. But, since $w_3 = w_5$, we also have

$$w_3 = (w_5w_6)w_3(w_5w_6)^{-1} = (w_5w_6)w_5(w_5w_6)^{-1} = w_6.$$

Thus $w_5 = w_6$, and therefore $w_0 = w_1 = \cdots = w_{2g}$. It follows that the image of $\varphi$ is cyclic, and hence, because $M_{g,1}$ is perfect, $\varphi$ is trivial – a contradiction. \hfill \Box

**Proposition 5.6.** For $g \geq 4$, let $\varphi : M_{g,1} \rightarrow S_m$ with $m < 5N_{g-1}^{-1}$. Then for $0 \leq i \leq 2g + 1$ there are no 3-cycles in the cycle decomposition of $w_i$.

**Proof.** Suppose to the contrary that $\ell_3 > 0$. In particular, $\varphi$ cannot be trivial. Then Corollary 5.3 forces the cycle decomposition type of each $w_i$ to be $(1)^{\ell_1}(3)^{\ell_3}$ where $\ell_3 \geq N_{g-1}^{-1}$. The image $\varphi(M_{g,1})$ contains the subgroup $H = \langle w_{2g}, w_{2g+1} \rangle$, which, since $w_{2g}$ is a product of 3-cycles, is a homomorphic image (via $x_i \mapsto w_{2g+i}$) of the group

$$H = \langle x_0, x_1 \mid x_0^3 = x_1^3 = 1, x_0x_1x_0 = x_1x_0x_1 \rangle$$

of order 24. Again, we consider $M_{g-1,1}$ as the subgroup of $M_{g,1}$ generated by $T_0, T_1, \ldots, T_{2g-2}$. Moreover, we write $\hat{\Omega}_k$ for the union of the orbits $\Omega_{k,i}$ ($i = 1, \ldots, h_k$) of cardinality $k$ of $H$. Thus, $|\hat{\Omega}_k| = kh_k$ and $k$ divides 24. From the fact that the generators $w_{2g}$ and $w_{2g+1}$ of $H$ contain no nontrivial cycles of length less than 3 it follows that $h_2 = 0$.
Since $\varphi(\mathcal{M}_{g-1,1})$ lies in the centralizer of $H$, it acts, for any $k$, both on $\hat{\Omega}_k$ and on the set $\{\Omega_{k,i} \mid i = 1, \ldots, h_k\}$. The former corresponds to a homomorphism $\psi_k : \mathcal{M}_{g-1,1} \to \mathfrak{S}_{kh_k}$ and the latter to a homomorphism $\nu_k : \mathcal{M}_{g-1,1} \to \mathfrak{S}_{h_k}$.

**Claim 1.** If $\nu_k$ is trivial, then so is $\psi_k$.

Indeed, suppose that $\nu_k$ is trivial. Then each $k$-orbit $\Omega_{k,i}$ of $H$ must be invariant under the action of $\varphi(\mathcal{M}_{g-1,1})$. But

$$|\Omega_{k,i}| = k \leq 24 < 2^{g-2}(2^{g-1} - 1) = \text{mi}(\mathcal{M}_{g-1,1}),$$

thus $\varphi(\mathcal{M}_{g-1,1})$ acts trivially on each $\Omega_{k,i}$, that is, $\mathcal{M}_{g-1,1}$ acts trivially on $\hat{\Omega}_k$.

**Claim 2.** $\psi_k$ is trivial if $k \geq 5$. Moreover, either $\psi_3$ or $\psi_4$ is nontrivial, but not both.

Indeed, if $\psi_k$ is nontrivial, then $\nu_k$ is nontrivial by Claim 1, thus

$$\frac{m}{k} \geq h_k \geq \text{mi}(\mathcal{M}_{g-1,1}) = N_{g-1},$$

therefore $k \leq 4$ as $m < 5N_{g-1}$. If both $\psi_3$ and $\psi_4$ are nontrivial, then

$$m \geq 3h_3 + 4h_4 \geq 7N_{g-1},$$

which also contradicts $m < 5N_{g-1}$. Finally, by Lemma 5.4 one of the $\psi_k$ with $k \geq 3$ must be nontrivial, so either $\psi_3$ or $\psi_4$ is nontrivial.

**Claim 3.** $\psi_3$ is nontrivial.

Suppose otherwise that $\psi_3$ is trivial. Then, by Claim 2, $\psi_4$ is nontrivial, thus $h_4 \geq N_{g-1}$. As $h_1 + 4h_4 \leq m < 5N_{g-1}$, it follows that $h_1 < N_{g-1}$, thus $\mathcal{M}_{g-1,1}$ acts trivially on $\hat{\Omega}_1$. So,

$$S(\varphi(\mathcal{M}_{g-1,1})) \subseteq \hat{\Omega}_1 \subseteq S(H).$$

Since $S(\varphi(\mathcal{M}_{g-1,1}))$ contains $S(\langle w_1, w_2 \rangle)$, which has the same cardinality as $S(H)$, the above inclusions are indeed equalities.

So, all nontrivial orbits of $H$ have length 4. Applying Lemma 2.2, we get that $w_2$ is a product of disjoint 3-cycles and, for $i \in \{1, 3\}$, all nontrivial orbits of $\langle w_2, w_i \rangle$ have length 4. By Lemma 1.4 it follows that $w_1 = w_3$. Hence, we also have

$$w_1 = (w_3w_4)w_1(w_3w_4)^{-1} = (w_3w_4)w_3(w_3w_4)^{-1} = w_4,$$

thus $w_3 = w_4$, therefore $w_0 = w_1 = \cdots = w_{2g}$. This implies that the image of $\varphi$ is cyclic, and hence, because $\mathcal{M}_{g,1}$ is perfect, $\varphi$ is trivial – a contradiction.

**Claim 4.** $\psi_1$ is nontrivial.

Suppose instead that $\psi_1$ is trivial. Then we have the inclusions

$$S(\varphi(\mathcal{M}_{g-1,1})) \subseteq \hat{\Omega}_3 \subseteq S(H)$$

which imply as in the previous claim that $S(H) = \hat{\Omega}_3$. So, all nontrivial orbits of $w_{2g}$ and $\langle w_{2g}, w_{2g+1} \rangle$ have length 3; thus, by Lemma 1.3, $w_{2g} = w_{2g+1}$. It follows that the image of $\varphi$ is cyclic, and hence, because $\mathcal{M}_{g,1}$ is perfect, $\varphi$ is trivial – a contradiction.
The following claim is a direct consequence of the braid relation between \(w_{2g}\) and \(w_{2g+1}\) (see the proof of Lemma 1.3).

**Claim 5.** \(\Omega_{3,1}, \ldots, \Omega_{3,h_3}\) are precisely the common 3-orbits of \(w_{2g}\) and \(w_{2g+1}\).

We turn now to conclude the proof of Proposition 5.6 with counting arguments. If \((b_1, \ldots, b_p)\) is a \(p\)-chain of nonseparating closed curves, we denote by \(c_p(b_1, \ldots, b_p)\) the number of common 3-orbits of \(\varphi(T_{b_1}), \ldots, \varphi(T_{b_p})\). We do not assume that the whole chain is nonseparating, but we suppose that each pair \((b_i, b_{i+1})\) is nonseparating, so that \((T_{b_i}, T_{b_{i+1}})\) is conjugate to \((T_{2g}, T_{2g+1})\) (see Lemma 2.2). By Claims 1, 3 and 5 we have

\[
c_2(b_1, b_2) = c_2(a_{2g}, a_{2g+1}) = h_3 \geq N_{g-1}^-,
\]

and by Claim 4 we have \(h_1 \geq N_{g-1}^-\). Recall also that \(\ell_3\) is the number of 3-orbits of \(w_{2g}\).

Note that the supports of the 3-orbits of \(\varphi(T_i)\) there are at most \(|X|/3\) that are not 3-orbits of \(\varphi(T_{i-1})\). We therefore obtain in turn

\[
c_3(b_1, b_2, b_3) \geq h_3 - \frac{|X|}{3}, \quad c_4(b_1, b_2, b_3, b_4) \geq h_3 - \frac{2|X|}{3},
\]

\[
c_5(b_1, b_2, b_3, b_4, b_5) \geq h_3 - \frac{3|X|}{3} = h_3 - |X|.
\]

Using again the assumption that \(m < 5N_{g-1}^-\), we have

\[|X| = m - h_1 - 3h_3 < 4N_{g-1}^- - 3h_3 \leq h_3.\]

Hence

\[c_5(b_1, b_2, b_3, b_4, b_5) \geq h_3 - |X| > 0.\]

So, there is a common 3-cycle in the decomposition of every generator \(w_0, \ldots, w_4\) of \(\varphi(M_{2,1})\). By restricting attention to the support of this 3-cycle, we deduce that \(\varphi\) induces a nontrivial homomorphism from \(M_{2,1}\) to \(S_3\) whose image is generated by elements of order 3; in other words, from \(M_{2,1}\) onto the cyclic group of order 3. This, however, contradicts Proposition 3.2. 

\[\square\]

### 5.2 Normal generation of \(\text{Ker} \theta_{g,1}\) by the square of a Dehn twist

Thanks to Corollary 5.3 and Propositions 5.5 and 5.6, we know (under the inductive hypothesis that Theorem 0.3 holds for a surface of genus \(g-1\)) that, if \(g \geq 4\) and \(\varphi : M_{g,1} \to S_m\) is a permutation representation with \(m < 5N_{g-1}^-\), then \(w_i = \varphi(T_i)\) is an involution for all \(i \in \{0, 1, \ldots, 2g+1\}\). So, in order to prove Theorem 5.1, still under the inductive hypothesis, it suffices to show the following.

**Theorem 5.7.** Let \(g \geq 2\). Then the kernel of \(\theta_{g,1}\) is the smallest normal subgroup of \(M_{g,1}\) containing \(T_i^2\).

This theorem is known to experts but, as far as we know, is nowhere in the literature. So, we include a proof but skip some details.
Proof. We say that a normal subgroup $H$ of a group $G$ is normally generated by a subset $S \subseteq H$ if $H$ is the smallest normal subgroup of $G$ that contains $S$. For $i, j \in \{1, \ldots, g\}$ we denote by $e_{i,j}$ the $g \times g$ matrix whose entries are all zero except the $(i,j)$-th entry which is equal to 1. Then the kernel of $\mu_g : \text{Sp}_{2g}(\mathbb{Z}) \to \text{Sp}_{2g}(\mathbb{F}_2)$ is normally generated by $\left( I_g \quad 2e_{1,1} \\ 0 \quad I_g \right)$, where $I_g$ denotes the identity matrix. This fact can be deduced from [3] (see also [31]) and its proof is left to the reader.

We denote by $\hat{\theta}_g : \mathcal{M}_{g,1} \to \text{Sp}_{2g}(\mathbb{Z})$ the natural epimorphism, so that $\theta_{g,1} = \mu_g \circ \hat{\theta}_g$. Consider the simple closed curves $c_2, \ldots, c_g, c'_2, \ldots, c'_g, d_2, \ldots, d_g$ illustrated in Figure 5.1. By [27] and [4] (see also [28]), $\ker \hat{\theta}_g$ is normally generated by $\{T_{ci}T_{ci}^{-1}, T_{di} \mid 2 \leq i \leq g\}$. On the other hand, a direct calculation shows that $\hat{\theta}_g(T_i^2) = \left( I_g \quad 2e_{1,1} \\ 0 \quad I_g \right)$. By the above, it follows that $\ker \theta_{g,1}$ is normally generated by $\{T_{ci}T_{ci}^{-1}, T_{di} \mid 2 \leq i \leq g\} \cup \{T_i^2\}$.

Let $H$ be the normal subgroup of $\mathcal{M}_{g,1}$ normally generated by $T_1^2$, and let $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_{g,1}/H$ be the quotient map. The following relations hold in $\mathcal{M}_{g,1}/H$:

$$
\begin{align*}
\pi(T_i)^2 &= 1 \quad &\text{if } 1 \leq i \leq 2g, \\
\pi(T_i)\pi(T_j) &= \pi(T_j)\pi(T_i) \quad &\text{if } |i-j| \geq 2, \\
\pi(T_i)\pi(T_{i+1})\pi(T_i) &= \pi(T_{i+1})\pi(T_i)\pi(T_{i+1}) \quad &\text{if } 1 \leq i \leq 2g - 1.
\end{align*}
$$

Thus, there exists a homomorphism $q : \mathcal{G}_{2g+1} \to \mathcal{M}_{g,1}/H$ that sends $s_i = (i, i+1)$ to $\pi(T_i)$ for all $1 \leq i \leq 2g$. On the other hand, the following relations hold in $\mathcal{M}_{g,1}$ (see [21], for instance):

$$
\begin{align*}
T_{di} &= (T_1T_2 \cdots T_{2i-2})^{4i-2}, \\
T_{ci}T_{ci}' &= (T_1T_2 \cdots T_{2i-1})^{2i},
\end{align*}
$$

whenever $i \in \{2, \ldots, g\}$. Moreover, since $s_1s_2 \cdots s_j$ is a cycle of length $j+1$,

$$(s_1s_2 \cdots s_j)^{j+1} = 1, \quad \text{for } 1 \leq j \leq 2g. $$

It follows that

$$
\begin{align*}
\pi(T_{di}) &= q((s_1s_2 \cdots s_{2i-2})^{4i-2}) = q(1) = 1, \\
\pi(T_{ci}T_{ci}^{-1}) &= \pi(T_{ci}T_{ci}') = q((s_1s_2 \cdots s_{2i-1})^{2i}) = q(1) = 1
\end{align*}
$$

for all $i \in \{2, \ldots, g\}$. Thus, $H = \ker \theta_{g,1}$. 

![Figure 5.1. Some curves in Σg,1.](image-url)
Theorem 6.1. For \( g \geq 4 \), let \( \varphi : \mathcal{M}_{g,1} \to \mathcal{G}_m \) be a nontrivial homomorphism, with \( m < 5N_{g-1} \). As pointed out before, thanks to Corollary 5.3 and Propositions 5.5 and 5.6, \( w_i = \varphi(T_i) \) is an involution for all \( i \in \{0,1, \ldots, 2g+1\} \). By Theorem 5.7 we conclude that there exists a homomorphism \( \bar{\varphi} : \text{Sp}_{2g}(\mathbb{F}_2) \to \mathcal{G}_m \) such that \( \varphi = \bar{\varphi} \circ \theta_{g,1} \).

6 Large subgroups of \( \text{Sp}_{2g}(\mathbb{F}_2) \)

The aim of this section is to prove Theorem 0.2, which, together with Theorem 5.1, proves Theorem 0.3 for the case of a surface with \( n = 1 \) boundary component. The extension to surfaces with several boundary components will be the object of Section 8.

The maximal subgroups of the finite classical groups have been described using the classification of finite simple groups. (Below, \( 2^n \) denotes an elementary abelian group of rank \( n \), \( A.B \) an extension of a group \( A \) by a group \( B \), and \( r \) a cyclic group of order \( r \).) Their orders are also known (see for example [30]).

Theorem 6.1. ([22, Theorem 4.1 and §3]; [2, §1]) If \( H \) is a maximal subgroup of \( \text{Sp}_{2g}(\mathbb{F}_2) \) for \( g \geq 1 \), then one of the following holds.

1. \( H = O^+_{2g}(\mathbb{F}_2) \), of order \( 2^{g^2-g+1}(2^g+1)\prod_{i=1}^{g-1}(2^{2i}-1) \).

2. \( H = \text{Sp}_{2g}(\mathbb{F}_2) \), of order \( 2^{g^2-g+1}(2^g-1)\prod_{i=1}^{g-1}(2^{2i}-1) \).

3. \( |H| \leq 2^{6g} \).

4. \( H \leq \mathcal{S}_{2g+2} \), of order at most \( (2g+2)! \).

5. \( H = \text{Sp}_{2k}(\mathbb{F}_2) \cdot r \), where \( r > 1 \) is a prime divisor of \( g \) and \( kr = g \). Here, \( H \) has order \( r \cdot 2^{g^2} \prod_{i=1}^{k}(2^{2ri}-1) \).

6. \( H = \text{Sp}_{2k}(\mathbb{F}_2) \cdot \mathcal{S}_r \), where \( r > 1 \) is a divisor of \( g \) and \( kr = g \). Here, \( H \) has order \( r! \cdot 2^{g^2} \prod_{i=1}^{k}(2^{2i}-1)^r \).

7. \( H \) is the stabilizer of a totally isotropic subspace of \( \mathbb{F}_2^{2g} \) under the natural action of \( \text{Sp}_{2g}(\mathbb{F}_2) \).
   That is, \( H \) is
   \[
   (2^{\frac{k(k+1)}{2}} \cdot 2^{2k(g-k)}) \times (\text{Sp}_{2g-2k}(\mathbb{F}_2) \times \text{GL}_k(\mathbb{F}_2))
   \]
   for some integer \( k \in \{1, \ldots, g\} \), and has order
   \[
   2^{g^2} \cdot \prod_{i=1}^{g-k}(2^{2i}-1) \cdot \prod_{i=1}^{k-1}(2^{2i+1}-1) \cdot \prod_{i=1}^{k}(2^{2i}-1) \cdot \prod_{i=1}^{k}(2^{2i+1}-1) .
   \]

8. \( H \) is the stabilizer of a nonsingular subspace of \( \mathbb{F}_2^{2g} \) under the natural action of \( \text{Sp}_{2g}(\mathbb{F}_2) \).
   That is, \( H \) is \( \text{Sp}_{2k}(\mathbb{F}_2) \times \text{Sp}_{2g-2k}(\mathbb{F}_2) \) for some \( k \in \{1, \ldots, g-1\} \), and has order
   \[
   2^{g^2+2k^2-2gk} \cdot \prod_{i=1}^{g-k}(2^{2i}-1) \cdot \prod_{i=1}^{k}(2^{2i}-1) \cdot \prod_{i=1}^{k}(2^{2i+1}-1) .
   \]
Moreover, the maximal subgroups falling within case (1) lie in a single conjugacy class, as do the subgroups falling within case (2).

Proof. The first part is shown in [22, Theorem 4.1 and §3]. Their orders may be found in [30, pp. 19, 70, 141]. For the last two cases, the description of $H$ follows from [32, p. 63], for example. Cases 2 and 1 correspond to the Aschbacher class $C_9$ [2, §1], that is, contain the subgroups of $\text{Sp}_{2g}(\mathbb{F}_2)$ stabilizing some quadratic form polarizing to the symplectic bilinear form defining $\text{Sp}_{2g}(\mathbb{F}_2)$; Case 2 corresponds to forms of Witt index $g$, Case 1 to forms of Witt index $g - 1$. By [2, Theorems $B\Delta$ and $BO$], the action of $\text{Sp}_{2g}(\mathbb{F}_2)$ is transitive on stabilizers of forms of the same similarity type, showing that the groups falling within Case 2, respectively the groups falling within Case 1, form a single conjugacy class.

Corollary 6.2. Let $g \geq 3$ and let $H$ be a subgroup of $\text{Sp}_{2g}(\mathbb{F}_2)$ of index (strictly) less than $2N_g^-$. Then either $H \cong O_{2g}^-(\mathbb{F}_2)$ or $H \cong O_{2g}^+(\mathbb{F}_2)$.

Remark. $SO_{2g}^-\mathbb{F}_2)$ is an index 2 subgroup of $O_{2g}^-(\mathbb{F}_2)$ and, therefore, an index $2N_g^-$ subgroup of $\text{Sp}_{2g}(\mathbb{F}_2)$.

Proof. Recall again that $N_g^- = [\text{Sp}_{2g}(\mathbb{F}_2) : O_{2g}^-(\mathbb{F}_2)]$ and note that the result holds for proper subgroups of $O_{2g}^-(\mathbb{F}_2)$ and $O_{2g}^+(\mathbb{F}_2)$ because $|O_{2g}^+(\mathbb{F}_2)| < |O_{2g}^-(\mathbb{F}_2)|$. Thus, we need consider only maximal subgroups $H$. For applications, the assertion is presented in terms of the index; however, the data relate to the order, so one needs to check that of the maximal subgroups listed in Theorem 6.1, in all except for the first two cases the order of $H$ is less than

$$\frac{|O_{2g}^-(\mathbb{F}_2)|}{2} = 2^{g^2-g}(2^g + 1) \prod_{i=1}^{g-1} (2^{2i} - 1).$$

For $g = 3$ the result holds by [8]. (Alternatively, we can compute the conjugacy classes of maximal subgroups of $\text{Sp}_6(\mathbb{F}_2)$ using MAGMA [5]: their orders are 1512, 4320, 4608, 10752, 12096, 23040, 40320, 51840. The maximal subgroups of order 40320 and 51840 can be checked to be isomorphic to $O_6^+(\mathbb{F}_2)$ respectively $O_6^-(\mathbb{F}_2)$, so the claim holds.)

So, assume that $g \geq 4$. In general, it is a routine matter to use the formulae of Theorem 6.1 to check that $H$ has order less than $|O_{2g}^-(\mathbb{F}_2)|/2$. To indicate the flavor of the verification, we discuss the two most delicate cases, namely (6) and (8).

Case (6). First observe that $r! < 2^{(r^2 - 2r + 3)/2}$, and, for $r$ in the range $[2, g]$, the function $(r^2 - 2r + 3)/2 + \frac{g}{2}$ achieves its maximum value of $(g^2 + 3)/2$ at the endpoints. Thus, on putting $j = ir$, we have that $|H| = r! \cdot 2^{(r^2 - 2r + 3)/2} \prod_{i=1}^{k} (2^{2i} - 1)^r$ is bounded above by

$$2^{(g^2 + 3)/2} \prod_{r|j, j \leq g} (2^{2j} - 1) < 2^{(g^2 + 2g + 3)/2} (2^g + 1) \prod_{r|j, j \leq g-1} (2^{2j} - 1).$$

Since $2^{(g^2 + 2g + 3)/2} < 2^{g^2 - g(2^2 - 1)}$, and $(2^2 - 1) \prod_{r|j, j \leq g-1} (2^{2j} - 1) < \prod_{i=1}^{g-1} (2^{2i} - 1)$, we obtain $|H| < \frac{|O_{2g}^-\mathbb{F}_2)|}{2}$, as required.
Case (8). Since, for $k$ in the range $[1, g-1]$, the function $2k(k-g)$ achieves its maximum of $-2(g-1)$ at the endpoints, we have $2^{g^2+2k^2-2g} < 2^{g^2-g}$. Meanwhile, using the symmetry of the product below to assume that $k \leq \frac{g}{2}$, and observing that $i < i + g - k = j$ say,

$$\left( \prod_{i=1}^{g-k} (2^{2i} - 1) \right) \cdot \left( \prod_{i=1}^{k} (2^{2i} - 1) \right)$$

$$= (2^{2k} - 1) \left( \prod_{i=1}^{g-k} (2^{2i} - 1) \right) \cdot \left( \prod_{i=1}^{k-1} (2^{2i} - 1) \right)$$

$$< (2^g + 1) \left( \prod_{i=1}^{g-k} (2^{2i} - 1) \right) \cdot \left( \prod_{j=g-k+1}^{g-1} (2^j - 1) \right).$$

When combined with the exponential inequality above, this again yields $|H| < |O^-_{2g}(\mathbb{F}_2)|/2$. □

**Proof of Theorem 0.2.** As stated before, we have $|\text{Sp}_{2g}(\mathbb{F}_2) : O^-_{2g}(\mathbb{F}_2)| = N_g^- = 2^{g-1}(2^g - 1)$ and $|\text{Sp}_{2g}(\mathbb{F}_2) : O^+_{2g}(\mathbb{F}_2)| = N_g^+ = 2^{g-1}(2^g + 1)$ by [30, pp. 70 and 141]. Finally, Corollary 6.2 ensures that the conjugacy classes of $O^+_{2g}(\mathbb{F}_2)$ and $O^-_{2g}(\mathbb{F}_2)$ are the only conjugacy classes of subgroups of $\text{Sp}_{2g}(\mathbb{F}_2)$ of index less than $2N_g^-$. □

Now, thanks to Theorem 5.1 and Theorem 0.2, we can prove our main theorem for the case of a surface with $n = 1$ boundary component. However, we have to keep the inductive hypothesis (that Theorem 0.3 holds for a surface of genus $g-1$), as we have used the equality $\text{mi}(\mathcal{M}_{g-1,3}) = N_{g-1}^-$ in the proof of Proposition 5.2.

**Proposition 6.3.** Let $g \geq 4$.

1. $O^-_{g,1}$ is the unique subgroup of $\mathcal{M}_{g,1}$ of index $N_g^- = 2^{g-1}(2^g - 1)$, up to conjugation.

2. $O^+_{g,1}$ is the unique subgroup of $\mathcal{M}_{g,1}$ of index $N_g^+ = 2^{g-1}(2^g + 1)$, up to conjugation.

3. All the other proper subgroups of $\mathcal{M}_{g,1}$ are of index at least $5N_g^- > N_g^+$.

**Proof.** Let $H$ be a subgroup of $\mathcal{M}_{g,1}$ of index $m < 5N_g^-$. By Theorem 5.1 there is a subgroup $\tilde{H}$ of $\text{Sp}_{2g}(\mathbb{F}_2)$ of index $m$ such that $H = \tilde{H}/\text{ker}(\gamma)$. Since $5N_g^- < 2N_g^-$, by Theorem 0.2, we must necessarily have either $\tilde{H} = O^+_{2g}(\mathbb{F}_2)$ or $\tilde{H} = O^-_{2g}(\mathbb{F}_2)$ up to conjugation, thus either $H = O^+_{g,1}$ or $H = O^-_{g,1}$ up to conjugation. □

### 7 Small symplectic representations of $\mathcal{M}_{g,n}$

The aim of this section is to prove Proposition 7.2, which is the same as Theorem 0.4, except that we do not state that the decompositions in (1) are unique. The uniqueness will follow from Theorem 0.3 proved in Section 8.

The decompositions of $O^\pm_{g,n}$ in (1) reflect analogous properties of the corresponding representations of $\text{Sp}_{2g}(\mathbb{F}_2)$, which in turn arise in a geometric way; while this result may be more or less known to experts, we could not locate a reference and hence establish it in Lemma 7.1.
We choose a basis \( \{ e_1, \ldots, e_{2g} \} \) of \( V = \mathbb{F}_2^{2g} \) such that \( (e_i, e_{2g+1-i}) \) for \( i = 1, \ldots, g \) are the symplectic pairs for the action of \( \text{Sp}_{2g}(\mathbb{F}_2) \); we write \( i \) for \( 2g + 1 - i \) to shorten notation. For a vector \( x \in V \), we denote the components of \( x \) with respect to the basis \( \{ e_1, \ldots, e_{2g} \} \) by \( x_1, \ldots, x_{2g} \). We consider \( \text{Sp}_{2g-2}(\mathbb{F}_2) \) as a subgroup of \( \text{Sp}_{2g}(\mathbb{F}_2) \) via the the standard embedding defined by

\[
\omega \mapsto \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \omega & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\]

with respect to the above basis.

For \( g \geq 2 \) and \( \epsilon \in \{ \pm \} \), let \( \bar{\varphi}_\epsilon^g : \text{Sp}_{2g}(\mathbb{F}_2) \to S_N \) denote the permutation representation induced by the action of \( \text{Sp}_{2g}(\mathbb{F}_2) \) on the right cosets of \( O_\epsilon^g(\mathbb{F}_2) \) (cf. Corollary 6.2).

**Lemma 7.1.** Let \( g \geq 3 \). Then,

\[
\bar{\varphi}_\epsilon^+|_{\text{Sp}_{2g-2}(\mathbb{F}_2)} \cong (\bar{\varphi}_{g-1}^-)^3 \oplus \bar{\varphi}_{g-1}^- \quad \text{and} \quad \bar{\varphi}_\epsilon^-|_{\text{Sp}_{2g-2}(\mathbb{F}_2)} \cong (\bar{\varphi}_{g-1}^-)^3 \oplus \bar{\varphi}_g^+.
\]

**Proof.** It is well-known (see, for example, [12] or [30, ch. 11]) that \( \bar{\varphi}_\epsilon^+ \) and \( \bar{\varphi}_\epsilon^- \) arise from the action of \( \text{Sp}_{2g}(\mathbb{F}_2) \) on the two orbits of quadratic forms polarizing to the symplectic form \( \langle - , - \rangle \) preserved by \( \text{Sp}_{2g}(\mathbb{F}_2) \). The two orbits consist of the quadratic forms of Witt index \( g \) (type +) respectively Witt index \( g - 1 \) (type -).

Let \( Q \) denote the set of quadratic forms polarizing to \( \langle - , - \rangle \). The elements of \( Q \) are of the form

\[
Q_\epsilon(x) = \sum_{i=1}^g x_ix_i + (x, b)^2 = \sum_{i=1}^g x_ix_i + (x, b) \quad \text{for } b \in V.
\]

The symplectic group \( \text{Sp}_{2g}(\mathbb{F}_2) \) acts on \( Q \) via \( (\omega \cdot Q)(x) := Q(\omega x) \) for \( \omega \in \text{Sp}_{2g}(\mathbb{F}_2) \) and \( Q \in Q \). In particular, \( (\omega \cdot Q_\epsilon)(x) = Q_{\epsilon'}(x) \) for some \( b' \in V \). It is easy to see that this defines another action of \( \text{Sp}_{2g}(\mathbb{F}_2) \) on the set \( V \) given by \( \omega \cdot b := b'' \). (Note that this action does not respect the structure of \( V \) as a vector space. In particular, it is different from the natural action given by matrix multiplication.) It is easy to verify that the restriction of this action to \( \text{Sp}_{2g-2}(\mathbb{F}_2) < \text{Sp}_{2g}(\mathbb{F}_2) \) can be canonically identified with the analogous action of \( \text{Sp}_{2g-2}(\mathbb{F}_2) \) on quadratic forms on \( \mathbb{F}_2^{2g-2} \).

Now consider the two orbits \( Q^+ \) and \( Q^- \) of quadratic forms of type + respectively type - under the action of \( \text{Sp}_{2g}(\mathbb{F}_2) \). We will show that \( Q' (\epsilon \in \{ \pm \}) \) splits up into four orbits under the action of \( \text{Sp}_{2g-2}(\mathbb{F}_2) \), one for each of the \( 2^2 \) possible values of \( (b_1, b_{2g}) \), and that three of these orbits have the type \( \epsilon \) while the remaining orbit has the type \( -\epsilon \).

Consider first the orbit \( Q^+ \) of quadratic forms of type +, that is, of Witt index \( g \), and define

\[
b_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.
\]
The quadratic forms $Q_b(x)$ for $i = 0, 1, 2, 3$ have Witt index $g$, that is, are elements of $Q^+$. Moreover, $Q^+ = \bigsqcup_{i=1}^{3} Q_{b_i}^{Sp_{2g-2}(\mathbb{F}_2)}$. It is easy to see that the restriction of $Q_b$ to $\mathbb{F}_2^{2g-2}$ is of type $+ (\text{Witt index } g-1)$ for $i = 0, 1, 2$ and of type $- (\text{Witt index } g-2)$ for $i = 3$. Hence $\overline{\phi_g} |_{Sp_{2g-2}(\mathbb{F}_2)} \cong (\overline{\phi_g})^3 \oplus \overline{\phi_g}$ as claimed.

The argument for the orbit $Q^-$ of quadratic forms of type $-$ is analogous. $\blacksquare$

**Proposition 7.2.** (1) Let $g \geq 3$ and $n \geq 1$. Then $\phi_{g,n} : \mathcal{M}_{g,n} \rightarrow \mathcal{S}_{N_g}$ is equivalent to an extension of the representation $(\phi_{g-1,n})^3 \oplus \phi_{g-1,n}^+$ from $\mathcal{M}_{g-1,n}$ to $\mathcal{M}_{g,n}$, and $\phi_{g,n}^+ : \mathcal{M}_{g,n} \rightarrow \mathcal{S}_{N_g}$ is equivalent to an extension of the representation $\phi_{g-1,n}^+ \oplus (\phi_{g-1,n})^3$ from $\mathcal{M}_{g-1,n}$ to $\mathcal{M}_{g,n}$.

(2) Let $g \geq 3$ and $n \geq 0$. Let $b$ be a nonseparating simple closed curve on $\Sigma_{g,n}$, and let $T_b$ be the Dehn twist around $b$. Then the cycle structure of the image of $T_b$ under $\phi_{g,n}$ is

$$(1)^{2^{2g-2}} (2)^{2g-2}(2g-1-1),$$

and the cycle structure of the image of $T_b$ under $\phi_{g,n}^+$ is

$$(1)^{2^{2g-2}} (2)^{2g-2}(2g-1+1).$$

**Proof.** We first consider Part (1). For a simple closed curve $c$ on $\Sigma_{g,1}$, we denote by $[c]$ the class of $c$ in $H_1(\Sigma_{g,1}, \mathbb{Z})$. We consider the curves $u_1, \ldots, u_g, v_1, \ldots, v_g$ indicated in Figure 7.1, and choose $[u_g], [u_{g-1}], \ldots, [u_1], [v_1], [v_2], \ldots, [v_g]$ as basis elements for $H_1(\Sigma_{g,1}, \mathbb{Z})$. With respect to this ordering, the bilinear form $\langle -, - \rangle$ yielding the algebraic intersection number is given by the matrix

$$M = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{bmatrix}.$$  

![Figure 7.1. Basis for $H_1(\Sigma_{g,1}, \mathbb{Z})$.](image)

The image of the Dehn twist $T_b$ about a simple closed curve $b$ under the epimorphism $\theta_{g,1} : \mathcal{M}_{g,1} \rightarrow \text{Sp}_{2g}(\mathbb{F}_2)$ is the map

$$[a] \mapsto [a] + (a, b)[b].$$

Since $\mathcal{M}_{g-1,1} < \mathcal{M}_{g,1}$ is generated by $T_0, T_1, \ldots, T_{2g-2}$ and since $a_0, a_1, \ldots, a_{2g-2}$ intersect neither $u_g$ nor $v_g$, it is clear from the above that $\theta_{g,1}(\mathcal{M}_{g-1,1}) = \text{Sp}_{2g-2}(\mathbb{F}_2) < \text{Sp}_{2g}(\mathbb{F}_2)$, and the restriction of $\theta_{g,1}$ to $\mathcal{M}_{g-1,1}$ coincides with $\theta_{g-1,1} : \mathcal{M}_{g-1,1} \rightarrow \text{Sp}_{2g-2}(\mathbb{F}_2) < \text{Sp}_{2g}(\mathbb{F}_2)$. 

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Let $n \geq 1$. Gluing disks along all boundary components of $\Sigma_{g,n}$ but one, we obtain an embedding $\Sigma_{g,n} \hookrightarrow \Sigma_{g,1}$ that induces a surjective homomorphism $\mu_{g,n} : \mathcal{M}_{g,n} \to \mathcal{M}_{g,1}$ (see [26]). More precisely, the epimorphism $\mu_{g,n}$ is defined sending $T_j$ to $T_1$ for all $j \in \{1, \ldots, n\}$, and $T_i$ to $T_i$ for all $i \in \{0, 2, \ldots, 2g\}$. Observe that $\theta_{g,n} = \theta_{g,1} \circ \mu_{g,n}$ and the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{M}_{g-1,n} & \to & \mathcal{M}_{g,n} \\
\mu_{g-1,n} & \downarrow & \mu_{g,n} \\
\mathcal{M}_{g-1,1} & \to & \mathcal{M}_{g,1}
\end{array}
$$

where $\mathcal{M}_{g-1,n} \to \mathcal{M}_{g,n}$ and $\mathcal{M}_{g-1,1} \to \mathcal{M}_{g,1}$ are the natural embeddings described in Section 2. By the above, it follows that $\theta_{g,n}(\mathcal{M}_{g-1,n}) = \text{Sp}_{2g-2}(\mathbb{F}_2) < \text{Sp}_{2g}(\mathbb{F}_2)$, and the restriction of $\theta_{g,n}$ to $\mathcal{M}_{g-1,n}$ coincides with $\theta_{g-1,n} : \mathcal{M}_{g-1,n} \to \text{Sp}_{2g-2}(\mathbb{F}_2) < \text{Sp}_{2g}(\mathbb{F}_2)$.

We have

$$
\overline{\phi}_g^+|_{\text{Sp}_{2g-2}(\mathbb{F}_2)} \cong (\overline{\phi}_{g-1})^3 + \overline{\phi}_{g-1}^- \quad \text{and} \quad \overline{\phi}_g^-|_{\text{Sp}_{2g-2}(\mathbb{F}_2)} \cong (\overline{\phi}_{g-1})^3 + \overline{\phi}_{g-1}^+
$$

from Lemma 7.1. By composing with $\theta_{g,n}$ we conclude that

$$
\phi_{g,n}|_{\mathcal{M}_{g-1,n}} \cong (\phi_{g-1,n})^3 + \phi_{g-1,n}^- \quad \text{and} \quad \phi_{g,n}|_{\mathcal{M}_{g-1,n}} \cong (\phi_{g-1,n})^3 + \phi_{g-1,n}^+.
$$

Part (2) for $n \geq 1$ follows by induction on $g$, using the cycle structures $(2)^6 (1)^{16}$ of $\phi^+_3(T_b)$ and $(2)^{10} (1)^{16}$ of $\phi^+_3(T_b)$ for $g = 3$ (see Lemma 4.1), and the equivalence $\phi_{g,n}|_{\mathcal{M}_{g-1,n}} \cong (\phi_{g-1,n})^3 + \phi_{g-1,n}^-$ proved above for $g \geq 4$.

Gluing a disk along the boundary component of $\Sigma_{g,1}$ we obtain an embedding $\Sigma_{g,1} \hookrightarrow \Sigma_{g,0}$ that induces a surjective homomorphism $\nu : \mathcal{M}_{g,1} \to \mathcal{M}_{g,0}$ (see e.g. [26]). Observe that $\theta_{g,1} = \theta_{g,0} \circ \nu$. Thus, if $b$ is a nonseparating simple closed curve in $\Sigma_{g,1}$, then $\theta_{g,1}(T_b) = \theta_{g,0}(T_b)$; therefore $\phi_{g,1}(T_b) = \phi_{g,0}(T_b)$. This proves Part (2) for $n = 0$.

8 Surfaces with multiple boundary components

We start the section with a geometrical interpretation of the decomposition given in Proposition 7.2.

Take $\epsilon \in \{\pm\}$ and consider the representation $\phi^\epsilon_{g,1} : \mathcal{M}_{g,1} \to \mathfrak{S}_{N_g}$. Let $w_i$ denote the image of $T_i$ under $\phi^\epsilon_{g,1}$ for all $i \in \{0, 1, \ldots, 2g + 1\}$. Set $H = \langle w_{2g}, w_{2g+1} \rangle$. Since $w_1$ has order 2, there is an epimorphism from the group

$$
\tilde{H} = \langle x_0, x_1 \mid x_0^2 = x_1^2 = 1, x_0x_1x_0 = x_1x_0x_1 \rangle = \mathfrak{S}_3
$$

to $H$ which sends $x_i$ to $w_{2g+i}$ for $i \in \{0, 1\}$. In particular, the order of $H$ divides 6. For $k \geq 1$, let $\hat{\Omega}_k$ denote the union of the $k$-orbits $\Omega_{k,i}$ of $H$, $1 \leq i \leq h_k$. Since the image $\phi^\epsilon_{g,1} |_{\mathcal{M}_{g-1,1}} = \langle w_0, \ldots, w_{2g-2} \rangle$ belongs to the centralizer of $H$, it acts on the set of orbits $\{\Omega_{k,1}, \ldots, \Omega_{k,h_k}\}$ as well as on the whole set $\hat{\Omega}_k$. The first action induces a homomorphism $\nu_k : \mathcal{M}_{g-1,1} \to \mathfrak{S}_{h_k}$ and the second a homomorphism $\psi_k : \mathcal{M}_{g-1,1} \to \mathfrak{S}_{kh_k}$. The following lemma describes these representations.
Lemma 8.1. Let $g \geq 3$. With the above notation, the following hold.

1. $h_3 = N_{g-1}^e$, $h_1 = N_{g-1}^{-e}$, and $h_k = 0$ for all $k \notin \{1,3\}$.
2. $\nu_3 \cong \phi_{g-1,1}^e$, $\psi_3 \cong (\phi_{g-1,1}^e)^3$, and $\nu_1 = \psi_1 \cong \phi_{g-1,1}^{-e}$.
3. $\{1, \ldots, N_{g}^{e}\} = S(\phi_{g,1}(\mathcal{M}_{g,1}))$, where, for $G < \mathfrak{S}_{N_{g}^{e}}$, $S(G)$ denotes the support of $G$, and $\mathcal{M}_{g,1}$ is the subgroup of $\mathcal{M}_{g,1}$ generated by $T_0, T_1, \ldots, T_5$.

Proof. We prove by induction on $g$ that $\langle w_1, w_2 \rangle$ has $N_{g-1}^e$ 3-orbits, has no other nontrivial orbit, and has $N_{g-1}^{-e}$ fixed letters. The case $g = 3$ follows from Lemma 4.1, and the inductive step can easily be derived from the formula $\phi_{g,1}^e|_{\mathcal{M}_{g-1,1}} \cong (\phi_{g-1,1}^e)^3 \oplus \phi_{g-1,1}^{-e}$ of Proposition 7.2. Thanks to Lemma 2.2 this proves Part (1).

Let $\Omega_{3,i}$ be a 3-orbit of $H$. We may write $\Omega_{3,i} = \{a_i, b_i, c_i\}$ so that the restriction of $w_{2g}$ to $\Omega_{3,i}$ is the transposition $(a_i, b_i)$ and the restriction of $w_{2g+1}$ is $(b_i, c_i)$. Set $\hat{\Omega}_3^1 = \hat{\Omega}_3 - S(w_{2g+1}) = \{a_i; 1 \leq i \leq N_{g-1}^e\}$, $\hat{\Omega}_3^3 = \hat{\Omega}_3 - S(w_{2g}) = \{c_i; 1 \leq i \leq N_{g-1}^{-e}\}$, and $\hat{\Omega}_3^2 = \hat{\Omega}_3 - (\hat{\Omega}_3^1 \cup \hat{\Omega}_3^3) = \{b_i; 1 \leq i \leq N_{g-1}^e\}$. Then $\hat{\Omega}_3^2$ is invariant under the action of $\mathcal{M}_{g-1,1}$ for $\ell = 1, 2, 3$, and the action of $\mathcal{M}_{g-1,1}$ on $\hat{\Omega}_3^1$ is equivalent to $\nu_3$. So,

$$\phi_{g,1}^e|_{\mathcal{M}_{g-1,1}} \cong \psi_3 \oplus \psi_1 \cong (\nu_3)^3 \oplus \nu_1.$$ 

For the case $g = 3$, one can easily check the equivalences $\nu_3 \cong \phi_{2,1}^e$ and $\nu_1 \cong \phi_{2,1}^{-e}$ using Lemma 3.1 and Lemma 4.1, so Part (2) holds in this case and we can assume $g \geq 4$. Recall that $h_3 = N_{g-1}^e$ and $h_1 = N_{g-1}^{-e}$. By Lemma 4.1 (if $g = 4$), respectively Proposition 6.3 (if $g > 4$), $\nu_3$ and $\nu_1$ are either trivial or of the form $v_3 = \phi_{g-1,1}^\mu \oplus 1_g$, respectively $v_1 = \phi_{g-1,1}^{\mu'} \oplus 1_g$. Now, the formula of Proposition 7.2 implies that $\phi_{g,1}^e|_{\mathcal{M}_{g-1,1}}$ has no fixed letter, and thus both $\nu_3$ and $\nu_1$ must be nontrivial and we must have $q = q' = 0$, $\mu = \epsilon$, and $\mu' = -\epsilon$. So Part (2) holds.

Part (3) follows by induction on $g$ using Lemma 4.1 for the case $g = 3$ and the equivalence $\phi_{g,1}^e|_{\mathcal{M}_{g-1,1}} \cong (\phi_{g-1,1}^e)^3 \oplus \phi_{g-1,1}^{-e}$ for the inductive step.

Now, we finish the proof of Theorem 0.3 with the following.

Proposition 8.2. Let $g \geq 4$ and $n \geq 0$.

1. $\mathcal{O}_{g,n}^-$ is the unique subgroup of $\mathcal{M}_{g,n}$ of index $N_{g}^- = 2^{g-1}(2^g - 1)$, up to conjugation.
2. $\mathcal{O}_{g,n}^+$ is the unique subgroup of $\mathcal{M}_{g,n}$ of index $N_{g}^+ = 2^{g-1}(2^g + 1)$, up to conjugation.
3. All the other subgroups of $\mathcal{M}_{g,n}$ are of index at least $5N_{g-1}^- > N_{g}^+$.

Proof. Recall that we are under the inductive hypothesis stated at the beginning of Part II, that is, Theorem 0.3 holds for a surface of genus $g - 1$.

The case $n = 1$ is proved in Proposition 6.3, and the case $n = 0$ follows from Proposition 6.3, Theorem 0.2, and the existence of the epimorphisms $\mathcal{M}_{g,1} \twoheadrightarrow \mathcal{M}_{g,0}$, $\mathcal{M}_{g,0} \twoheadrightarrow \Sp_{2g}(\mathbb{F}_2)$ described in Section 0. So, we may assume $n \geq 2$. 

Let \( \varphi : M_{g,n} \to S_m \) be a nontrivial transitive homomorphism with \( m < 5N_{g-1}^- \). As ever, for \( i \in \{0, 2, \ldots, 2g + 1\} \) and \( j \in \{1, \ldots, n\} \), we denote by \( a_i \) and \( b_j \) the simple closed curves illustrated in Figure 2.2, we denote by \( T_i \) the Dehn twist about \( a_i \), by \( T'_j \) the Dehn twist about \( b_j \), and we set \( w_i = \varphi(T_i) \) and \( w'_j = \varphi(T'_j) \). For \( j \in \{1, \ldots, n\} \), we denote by \( \Sigma^{(j)} \) a tubular neighborhood of \( b_j \cup a_0 \cup (\gamma_{2g+1}^{(j)}) \). Observe that \( \Sigma^{(j)} \) is a subsurface of \( \Sigma_{g,n} \) of genus \( g \) with a unique boundary component, and the inclusion \( \Sigma^{(j)} \hookrightarrow \Sigma_{g,n} \) induces an injective homomorphism \( \psi : M_{g,1} \to M_{g,n} \) which sends \( T_j \) to \( T'_j \) and \( T_i \) to \( T_i \) for all \( i \in \{0, 2, \ldots, 2g\} \). Set \( \varphi_j = \varphi \circ \gamma_j : M_{g,1} \to S_m \). By Proposition 6.3, \( \varphi_j \) is of the form \( \varphi_j = \psi_j \otimes 1_n \), where \( \psi_j \) is conjugate to an element of \( \{\varphi_{g,1}, \varphi_{g,2}\} \) and \( 1_n : M_{g,1} \to S_n \) is the trivial representation.

Let \( \Sigma' \) be a tubular neighborhood of \( \cup_{i=0}^\infty a_{2g-1} \). Then \( \Sigma' \) is a subsurface of \( \Sigma_{g,n} \) of genus 3 with a unique boundary component, and is included in \( \Sigma^{(j)} \) for all \( j \in \{1, \ldots, n\} \). Moreover, the inclusion \( \Sigma' \hookrightarrow \Sigma_{g,n} \) induces an embedding \( \gamma' : M_{3,1} \to M_{g,n} \). Set \( \varphi' = \varphi \circ \gamma' : M_{3,1} \to S_m \). We have \( S(\varphi_j(M_{g,1})) = S(\varphi'(M_{3,1})) \) by Lemma 8.1 (3) for all \( j \) and \( \bigcup_{j=1}^n S(\varphi_j(M_{g,1})) = S(\varphi(M_{g,n})) \), thus

\[
S(\varphi_j(M_{g,n})) = S(\varphi_j(M_{g,1})) = S(\varphi'(M_{3,1}))
\]

Since \( \varphi \) is transitive, it follows that there exists \( \epsilon \in \{\pm\} \) such that \( m = N_\epsilon^g \) and \( \varphi_j \) is conjugate to \( \varphi_{g,1}^\epsilon \) for all \( j \in \{1, \ldots, n\} \).

Set \( H = \langle w_{2g}, w_{2g+1} \rangle \); note that \( H \subseteq \bigcap_j \gamma_j(M_{g,1}) \). For \( k \geq 1 \) we denote by \( \tilde{\Omega}_k \) the union of the \( k \)-orbits \( \tilde{\Omega}_{k,i} \) of \( H \), \( 1 \leq i \leq h_k \). On the other hand, we assume that \( M_{g,1} \) is the subgroup of \( M_{g,n} \) generated by \( T_1, \ldots, T_n, T_0, T_2, \ldots, T_{2g-2} \). Since the image \( \varphi(M_{g-1,n}) \) lies in the centralizer of \( H \), it acts on the set of orbits \( \{\tilde{\Omega}_{k,1}, \ldots, \tilde{\Omega}_{k,h_k}\} \) as well as on the whole set \( \tilde{\Omega}_k \). The first action induces a homomorphism \( \nu_k : M_{g-1,n} \to S_{h_k} \), and the second induces a homomorphism \( \psi_k : M_{g-1,n} \to S_{kh_k} \). Applying Lemma 8.1 (1) to \( \varphi_j(M_{g,1}) \) for any \( j \), we get that \( h_3 = N_{g-1}^\epsilon, h_1 = N_{g-1}^\epsilon, \) and \( h_k = 0 \) if \( k \notin \{1, 3\} \).

As in the proof of Lemma 8.1, we set \( \tilde{\Omega}_3^3 = \tilde{\Omega}_3 - S(w_{2g+1}), \tilde{\Omega}_3^3 = \tilde{\Omega}_3 - S(w_{2g}), \) and \( \tilde{\Omega}_3^2 = \tilde{\Omega}_3 - (\tilde{\Omega}_3^1 \cup \tilde{\Omega}_3^3) \). Then \( \tilde{\Omega}_3^2 \) is invariant under the action of \( M_{g-1,n} \) for \( \ell = 1, 2, 3 \), and the action of \( M_{g-1,n} \) on \( \tilde{\Omega}_3^2 \) is equivalent to \( \nu_3 \). Hence,

\[
\psi_{|M_{g-1,n}} \cong \psi_3 \oplus \psi_1 \cong (\nu_3)^3 \oplus \nu_1.
\]

Since \( h_3, h_1 \leq N_{g-1}^+ \), by induction we have \( \nu_3(T'_1) = \nu_3(T'_1) \) and \( \nu_1(T'_j) = \nu_1(T'_j) \), thus \( \varphi(T'_j) = \varphi(T'_j) \) for all \( j \in \{1, \ldots, n\} \). Since \( \varphi_1 \) is conjugate to \( \varphi_{g,1}^+ \), we conclude that \( \varphi \) is conjugate to \( \varphi_{g,1}^+ \).

**Proof of Theorem 0.4.** This follows from Proposition 7.2 and Theorem 0.3, proved above. \( \Box \)

**References**


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