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IR-HARQ Schemes with Finite-Length Punctured LDPC Codes over the BEC

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Abstract—The incremental redundancy hybrid ARQ (IR-HARQ) schemes based on finite-length LDPC codes are considered in this paper. The transmission is assumed to take place over time varying binary erasure channels, such as mobile wireless channels at the applications layer for instance. Also, the iterative decoding is supposed. Two following issues are addressed: a) establishment of transmission rules for the m -th transmission after $m - 1$ unsuccessful ones; b) approximation of the throughput and delay performance which would be accurate for finite codelengths, down to several hundreds of bits.

I. INTRODUCTION

Hybrid ARQ transmission schemes combine the conventional ARQ with forward error correction. In applications with fluctuating channel conditions, such as mobile and satellite packet data transmission, a scheme known as *incremental redundancy* hybrid ARQ (IR-HARQ) or *Type II* HARQ achieves higher throughput efficiency by adapting its error correcting code redundancy to varying channel conditions. Because of that, the scheme has been adopted by a number of standards for mobile phone networks. IR-HARQ is considered to be one of the most important features of the CDMA2000 1xEV-DO Release 0 and Revision A systems [1], [2]. A historic overview of HARQ schemes, up to 1998, can be found in [3]. For a survey of more recent developments, we direct the reader to [4] and references therein.

In the third generation wireless standards, the IR-HARQ scheme resides in the physical layer and operates over time varying fading channels. The scheme is based on a turbo code dating back to the IS-95 standard. A possible replacement of this code by an LDPC or a fountain code was considered in [5]. In this paper we are concerned with a performance approximation of IR-HARQ schemes based on LDPC codes of some *finite* codelength n , under *iterative decoding*. For simplicity, it is assumed that the schemes operate over time varying binary erasure channels (BECs). In spite of simplicity of the channel model, it is already of interest as it can be used for the applications layer in environments with time varying or unknown channels.

IR-HARQ schemes operates as follows. At the transmitter, the user data bits are encoded by a low rate code, referred to as the *mother* code. Initially, only a selected number of encoded bits are transmitted, and the decoding is attempted at the receiving end. If the decoding fails, the transmitter is notified, and it sends additional parity bits, possibly under different channel conditions and/or at different power. Decoding

is again attempted at the receiving end, where the new bits are combined with those previously received. The procedure is repeated after each subsequent transmission request until all the parity bits of the mother code are transmitted.

The fundamental decision that the transmitter has to make in the course of this protocol is how many bits to send in transmission m after a failed transmission $m - 1$ to achieve a particular tradeoff between the throughput and the number of transmissions. In this paper we address this question for IR-HARQ schemes based on finite-length, iteratively decoded LDPC codes operating over time varying BECs. Another important question that we investigate is an accurate approximation of the average throughput and the average delay for the same type of scheme. The first issue received a limited attention in the literature, being only considered to some extent for IR-HARQ schemes operating at the physical layer over time varying fading channels in 3GPP and 3GPP2 standards' committees and in [6], [5]. The second issue, on the contrary, was quite intensively investigated, see for instance [7], [5], [8]. However, the obtained results only give bounds, obtained either under maximum-likelihood decoding assumption, either under (more practical) iterative decoding, but based on the bit error probability, which means that the bound is tight only for large codelengths. The approach taken in this paper is based on the *block* error performance under *iterative* decoding, which is the main novelty compared to existing results.

This paper is organized as follows: In Sec. II, we describe the mother code model, our IR-HARQ protocol and multiple transmissions channel model. We also derive transmission rules for two approximations of the channel model, used to simplify the analysis. In Sec. III, we derive expressions for the average throughput and delay of the protocol. In Sec. IV, we evaluate numerically the protocol performance of a (3, 6) LDPC code based IR-HARQ scheme. We then discuss our observations and future work.

II. IR-HARQ SCHEMES

A. The Mother Code

The mother code is taken at random from an irregular LDPC code ensemble of length n , parametrized by degree distributions $\lambda(x)$ and $\rho(x)$ (see [9] for more details about LDPC codes). Note that each code in the ensemble corresponds to a different Tanner graph with n variable and k check nodes, in which fraction $i\lambda_i/(\sum_i i\lambda_i)$ of the variable nodes have degree i , $1 \leq i \leq n$, and fraction $j\rho_j/(\sum_j j\rho_j)$ of the check

nodes have degree j , $1 \leq j \leq k$. The performance of iterative decoding of such ensembles is well understood when n is sufficiently large and when they are used for a transmission over a channel with some fixed erasure rate ϵ . Namely, as long as $\epsilon < \epsilon^*$ where

$$\epsilon^* = \min_{x \in (0,1)} \frac{x}{\lambda(1 - \rho(1 - x))},$$

the iterative message passing algorithm leads to vanishing bit-erasure probability as the number of iterations grows.

B. The Multiple-Transmissions Model and Protocol

We consider an IR HARQ scheme with at most M transmissions. We assume that the channel erasure probability during a single transmission does not change, and is equal to ϵ_j during transmission j , $1 \leq j \leq M$. Prior to transmission j , the transmitter knows the channels of the previous transmissions, that is, $\epsilon_1, \dots, \epsilon_{j-1}$, and may or may not know ϵ_j .

If transmission $m - 1$ fails, for $1 \leq m < M$, the transmitter needs to decide how many and which bits to send in transmission m , and at which power. For a given power, higher number of transmitted bits improves the probability of successful decoding, and therefore decreases the expected number of transmissions. On the other hand, higher number of transmitted bits reduces the expected throughput. Our first objective is to determine the transmission rules based on the knowledge of the used LDPC ensemble.

We first have to understand the performance of iterative decoding of our code ensembles after each transmission. To that end, we adopt a probabilistic model in which the transmitter assigns a bit corresponding to a variable node of degree i to transmission j with probability q_{ij} ¹. Clearly, the transmitter has also the constraint (known as *rate compatible puncturing*) to assign to transmission j only the bits which have not been assigned to any of the previous transmissions. The IR-HARQ protocol outlined below implements our model while conforming to rate compatible puncturing; it is based on the one proposed in [6]. The values of q_{ij} are determined prior to transmission j , which can be done by using one of the approaches we propose in Sec. II-D.

IR-HARQ Protocol

Let $\mathcal{V}_i$ be the set the variable nodes with degree i .

Before the IR HARQ protocol starts

- 1) For each variable node v of degree i , generate a number θ_v independently and uniformly at random over $[0, 1)$.
- 2) Determine q_{ij} (see Sec. II-D).
- 3) Compute p_{i1} as $p_{i1} = 1 - q_{i1}$.
- 4) For each node v s.t. $\theta_v \geq p_{i1}$, assign bit corresponding to v to transmission 1.

¹Note that in this case the concentration of code performances around the average performance may be assumed, and it becomes sufficient to describe the performance of almost any code from the ensemble. If the code bits were punctured with probability q regardless of their degree, the concentration property would be lost [10].

If transmission $j - 1$ fails for $2 \leq j < M - 1$

- 1) Determine q_{ij} (see Sec. II-D).
- 2) Compute p_{ij} as $p_{ij} = p_{i(j-1)} - q_{ij}$.
- 3) For each node v s.t. $p_{ij} \leq \theta_v < p_{i(j-1)}$, assign the bit corresponding to v to transmission j .

If transmission $M - 1$ fails

transmit all remaining bits.

The values of q_{ij} can be determined by one of two methods derived in Sec. II-D. These methods are based on two following approximations of our multiple transmissions channel model for each IR transmission, the so called equivalent erasure channel and the equivalent punctured code model:

1) *Equivalent Erasure Channel Approximation*: Under our model, after m transmissions, a bit corresponding to a variable node of degree i will not reach the destination if it is assigned to one of the first m transmissions and then erased by the channel (which happens with the probability $\sum_{j=1}^m q_{ij}\epsilon_j$) or if it is simply not assigned to one of the first m transmission (which happens with probability $1 - \sum_{j=1}^m q_{ij}$). Equivalently, after m transmissions, a bit corresponding to a variable node of degree i is erased independently and uniformly at random with probability ν_{im} given by

$$\nu_{im} = (1 - \sum_{j=1}^m q_{ij}) + \sum_{j=1}^m q_{ij}\epsilon_j = 1 - \sum_{j=1}^m q_{ij}(1 - \epsilon_j).$$

Consequently, the expected number of erased bits after the first m transmissions is given by

$$\nu_m = \sum_i \lambda_i \nu_{im}. \quad (1)$$

Therefore, upon transmission m , we can approximate our model as that of the mother code being sent over a BEC with erasure rate ν_m .

2) *Equivalent Punctured Code Approximation*: By using the above stated IR-HARQ protocol, the bits that the transmitter chooses to sent up to the m -th transmission, can be equivalently seen as obtained by implementing a puncturing device which punctures a bit corresponding to a variable node of degree i with probability p_{im} , where $p_{im} = 1 - \sum_{j=1}^m q_{ij}$, or as shown within the protocol above

$$p_{i1} = 1 - q_{i1}, \text{ and } p_{ij} = p_{i(j-1)} - q_{ij} \text{ for } 1 < j \leq M. \quad (2)$$

Given that a bit corresponding to a variable node of degree i is assigned to one of the first m transmissions, it gets erased with probability $(\sum_{j=1}^m q_{ij}\epsilon_j)/(\sum_{k=1}^m q_{ik})$. So, we can model our IR-HARQ protocol through transmission m as transmission of the punctured mother code over a BEC with erasure rate

$$\epsilon_m = \sum_i \lambda_i \frac{\sum_{j=1}^m q_{ij}\epsilon_j}{\sum_{k=1}^m q_{ik}}, \quad (3)$$

where the considered bit is punctured with probability p_{im} .

C. Performance at Transmission m

Lemma 2.1: If an LDPC code based IR-HARQ scheme assigns a bit corresponding to a variable node of degree i to transmission j with probability q_{ij} , then in the process of iterative decoding after transmission m is complete, the average probability x_ℓ that the message from a variable node to a check node in the ℓ th iteration is the erasure is given by

$$x_\ell^m = \sum_i \lambda_i \nu_{im} [1 - \rho(1 - x_{\ell-1}^m)]^{i-1}$$

Proof: In the ℓ th iteration, a check-to-variable node message sent by a degree i check node is the erasure message if any of the $(i-1)$ incoming messages is an erasure, which on the average occurs with probability $1 - (1 - x_{\ell-1}^m)^{i-1}$. Thus the expected value of the erasure probability of a check-to-variable message in the ℓ th iteration is equal to $1 - \rho(1 - x_{\ell-1}^m)$. Now consider a variable-to-check message in the ℓ th iteration sent by a variable node of degree i . It will be an erasure if the bit corresponding to the node was not received (which happens with probability ν_{im}) and each of the $(i-1)$ incoming check-to-variable messages are erasures. Thus, it is an erasure with probability $\nu_{im} [1 - \rho(1 - x_{\ell-1}^m)]^{i-1}$, and on the average, $x_\ell^m = \sum_i \lambda_i \nu_{im} [1 - \rho(1 - x_{\ell-1}^m)]^{i-1}$. ■

D. Transmission Rules

Suppose that $m-1$ transmissions have taken place and were unsuccessful. We now need to decide which fraction q_{im} of bits corresponding to the variable nodes of degree i to assign to transmission m in order to have a successful decoding with high probability. We propose two following rules on which a decision can be made.

1) *Rules Based on the Density Evolution for the Multiple Channel Transmissions and the Equivalent Erasure Channel Approximation:* Based on Lemma 2.1, the iterative algorithm leads to vanishing bit-erasure probability as the number of iterations grows if we can choose $q_{im}(1 - \epsilon_m)$ so that

$$x - \sum_i \lambda_i \nu_{im} [1 - \rho(1 - x)]^{i-1} < 0 \text{ for all } x \in (0, 1).$$

To simplify the computations that would have to be done by the transmitter at each iteration, we will use the equivalent erasure channel approximation (described in Sec. II-B2), and analyze the decoding after the first m transmissions as decoding of the mother code after a single transmission over the erasure channel with fixed error rate ν_m given by (1). For this model, we need to ensure that

$$\nu_m < \epsilon^*,$$

that is, choose q_{im} s.t. the following condition holds:

$$q_{im}(1 - \epsilon_m) > 1 - \epsilon^* - \sum_{j=1}^{m-1} q_{ij}(1 - \epsilon_j) \quad (4)$$

Note that this condition specifies only the minimum value that the product $q_{im}(1 - \epsilon_m)$ should take in order to guarantee vanishing bit-erasure probability as the number of iterations

grows. The redundancy increment q_{im} has direct influence on throughput, but we need to know the actual probability of successful decoding to be able to study the delay of our scheme. We will address this problem by using the equivalent punctured code approximation of Sec. II-B2.

2) *Rules Based on the Equivalent Punctured Code Scaling Approximation:* We first derive the block erasure rate $P_B^{(m)}$ at transmission m . Such estimation is particularly interesting for short-length codes, down to several hundreds-one thousand bits. For punctured LDPC codes of length n , we know from [10] how to estimate P_B quite accurately.

Recall from Sec. II-B2 that we model our IR-HARQ protocol through transmission m as transmission of the punctured mother code over a BEC with erasure rate ϵ_m given by (3), where the bit corresponding to a variable node of degree i is punctured with probability p_{im} . Let us denote this block erasure rate at transmission m by $P_B^{(m)}$. We are going to show that the constraint of keeping $P_B^{(m)}$ below a certain value provides us a requirement on p_{im} , and by (2), on q_{im} . We use the following scaling approximation of [10] for $P_B^{(m)}$:

$$P_B^{(m)} \simeq Q \left(\frac{\sqrt{n}(\epsilon_m^* - \epsilon_m - \beta_m n^{-2/3})}{\alpha_m} \right), \quad (5)$$

where ϵ_m^* is the asymptotic iterative threshold, i.e., the (unique) solution of the density evolution equation

$$x = \sum_i \lambda_i \nu_{im} [1 - \rho(1 - x)]^{i-1}, \quad (6)$$

and α_m and β_m are parameters that explicitly depend on the mother LDPC code ensemble (i.e., λ and ρ), puncturing rates p_{im} , and the threshold ϵ_m^* . Expressions for α_m (scaling parameter) and β_m (shift parameter) are given in Appendix A and come from the results derived in [10].

If we ask for $P_B^{(m)} < \gamma$, where γ is a small positive constant, we have to have

$$\frac{\epsilon_m^* - \epsilon_m - \beta_m n^{-2/3}}{\alpha_m} < \frac{Q^{-1}(\gamma)}{\sqrt{n}}. \quad (7)$$

For simplicity, let us consider a particular case of punctured regular codes of type (x^{c-1}, x^{d-1}, p_m) . Then α_m and β_m can be simply expressed through the scaling and shift parameters of the mother ensemble and the punctured probability p_m (see (11)); moreover, $\epsilon_m^* = p_m + (1 - p_m)\epsilon^*$. Hence (7) becomes

$$(1 - p_m)[(1 - p_m)\epsilon^* + p_m] - \sum_{j=1}^m q_j \epsilon_j < \frac{\alpha Q^{-1}(\gamma)}{\sqrt{n}} + \beta n^{-2/3}.$$

Given that $1 - p_m = \sum_{j=1}^m q_j$, we finally obtain that

$$q_m (1 - \epsilon_m - q_m(1 - \epsilon^*)) < \frac{\alpha Q^{-1}(\gamma)}{\sqrt{n}} + \beta n^{-2/3} - Q_{m-1}, \quad (8)$$

where $Q_{m-1} = \sum_{j=1}^{m-1} q_j [1 - \epsilon_j + (\sum_{j=1}^{m-1} q_j)(1 - \epsilon^*)]$.

Note that here the concentration of the block erasure rate of a code from the punctured ensemble around the average is assumed, while the literature only provides the concentration theorem for the bit erasure rate. However, for ϵ close to ϵ^* ,

the number of erased bits after a decoding failure is linear in the codelength, and such a concentration may be assumed.

III. AVERAGE THROUGHPUT AND DELAY

The standard measure of ARQ protocol efficiency is *throughput*, defined as the average number of user data bits accepted at the receiving end in the time required for transmission of a single bit. We next derive the expected throughput our IR-HARQ scheme for regular codes. The results for irregular codes is obtained by replacing q_m below by $\sum_i \lambda_i q_{im}$.

Let R be the rate of the code, ω_m be the probability that it takes m transmissions for the decoding to be successful. The probability that one of the $m \leq M$ transmissions is successful is $\sum_{m=1}^M \omega_m$. Because our protocol is limited to M transmissions, if none of these transmissions is successful, the throughput is equal to 0. When one of the $m \leq M$ transmissions is successful, the number of user data bits communicated to the receiver is Rn . The number of the encoded bits sent to the receiver through the m th transmission is $n \sum_{j=1}^m q_j$. So, the average throughput η is given by

$$\eta = \frac{R \sum_{m=1}^M \omega_m}{\sum_{m=1}^M \omega_m \left(\sum_{j=1}^m q_j \right)}$$

where $\omega_m = P_B^{(m-1)} - P_B^{(m)}$ and $P_B^{(m)}$ is given by (5).

Note that the above expression for throughput becomes equal to its counterpart in [7] when $q_m = 1/M$ and when $P_B^{(m)}$ is replaced by the probability that the asymptotic bit erasure rate P_b at the transmission m goes to 0. It was proposed in [7] to compute P_b through density evolution, i.e., $P(m) \approx \text{Prob}[P_b^{(m)} \rightarrow 0]$. Clearly, this quantity is a lower bound on the failure probability at transmission m , and hence, gives an upper bound on the average throughput and a lower bound on the average delay. Unfortunately, the bounds were shown to be too optimistic for finite-length codes for small and medium codelengths. This issue can be overcome by using the approximation for $P_B^{(m)}$ which is more accurate.

The second important parameter is the *delay*, which is the average number of bits needed to transmit in order to receive the useful information (user data bits). Similarly, the average delay τ of the IR-HARQ scheme can be shown to be equal to

$$\tau = \frac{n \sum_{m=1}^M \omega_m \left(\sum_{j=1}^m q_j \right)}{\sum_{m=1}^M \omega_m}.$$

IV. EXAMPLES AND DISCUSSION

A. A Scheme Based on the (3, 6) Mother Ensemble

Let us check how tight our predictions are. Fig. 1 compares values of η versus the average channel erasure probability, obtained numerically and using our approximation. We used

a (3, 6)-regular ensemble of length 1024. Note that for (3, 6) LDPC codes, $\epsilon^* = 0.4293$, $\alpha = 0.5603$ and $\beta = 0.616$ [11]. M been fixed to 5 (all 512 information bits are allocated to the first transmission and each following transmission contains 128 redundancy bits). We can see that the predicted curve for η and the simulated one are very close to each other.

Fig. 1. Average throughput η versus equivalent channel erasure probability for (3, 6) LDPC codes of length 1024. $M = 5$. Dashed line - numerical result, straight line - our estimation.

Let us use our approximation to study the effect of the codelength n and of the number of retransmissions M on η and τ . In Fig. 2, the average throughput versus average channel erasure probability is represented for different M and n . The taken ensemble is the (3, 6)-regular one, chosen codelengths - 1024 and 8192. At each retransmission we send the M -th part of the codeword, $M = 5; 15; 100$. The capacity limit for the BEC is $1 - \epsilon$. One can see that the curve obtained for larger n is sharper, owing to the faster decrease of the block erasure rate in the waterfall region. Next, for increasing M , the curves approach the maximum throughput performance η_{max} that can be obtained with given code ensemble; it is given by

$$\eta_{max}(\epsilon) = \frac{R}{1 - \epsilon^*} (1 - \epsilon)$$

for $\epsilon \in [0, \epsilon^*]$ and 0 otherwise.

Fig. 2. Throughput versus channel erasure probability for 5, 15 and 100 retransmissions using (3, 6)-regular LDPC codes. The dotted line represents the capacity limit; straight lines correspond to $n = 1024$; dashed lines - to $n = 8192$.

It is worth to note that approaching to η_{max} by increasing M comes at the price of increasing the average delay (see

Fig. 3 for illustration). In fact, $\tau = RM/\eta$, and $\tau_{max} = RM/\eta_{max} = M(1 - \epsilon^*)/(1 - \epsilon)$.

Fig. 3. Average delay (in slots) versus channel erasure probability for (3,6) LDPC codes of length 1024. The straight line corresponds to $M = 5$, dashed - to $M = 15$, dotted - to $M = 50$.

B. Discussion

In this paper we have considered IR-HARQ schemes based on finite-length LDPC codes and have defined the transmission rules as well as a very accurate estimation of its average throughput and delay. The aforementioned estimation is based on the average block erasure performance of punctured codes obtained through the scaling approximation [10]. The transmission was considered to take place over the BEC, but our results can be extended to other symmetric memoryless channels. Moreover, they can also be combined with [12], giving an approximation of the failure probability of data packets at the application layer. A natural way of extending the obtained result is the optimization of parameters of the mother code and of the IR-HARQ scheme by obtaining the best tradeoff between η and τ . Further, the IR-HARQ scheme based on the optimized ensemble may be compared with schemes based on other codes, such as Raptor codes for instance. This point will be presented in our future work.

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APPENDIX

To give the expression for the scaling parameter α_m and the shift parameter β_m , some more definitions of parameters of punctured LDPC codes are needed.

Definition 1: Let the variable node degree distribution from the node perspective be $\Lambda(x) = \sum_{i \in V} \Lambda_i x^{i-1}$, for which Λ_i represent the fraction of variable nodes of degree i , $i \in V$.

Definition 2: For given polynomial $p_m(x)$, let $\lambda_p(x) = \sum_{i \in V} p_i \lambda_i x^{i-1}$ and $\bar{\lambda}_p(x) = \sum_{i \in V} (1 - p_i) \lambda_i x^{i-1}$. Note that that $\lambda(x) = \lambda_p(x) + \bar{\lambda}_p(x)$. Similarly, let $\bar{\Lambda}_p(x) = \sum_{i \in V} (1 - p_i) \Lambda_i x^{i-1}$.

Also, let us introduce the following notation:

$$y(x) = 1 - \rho(\bar{x}), \quad \pi(y) = \epsilon \bar{\lambda}_p(y) + \lambda_p(y), \\ \xi(x) = (\pi'(y))^2 (\bar{y}) (\rho'(1) - \rho'(\bar{x})), \quad \mu(x) = \pi'(y) \rho'(\bar{x}),$$

where $\bar{x} = 1 - x$ and $\bar{y} = 1 - y$. Assume x^* satisfies the equation (6) and $y^* = y(x^*)$, then the expression for α_m and β_m derived in [10] are:

$$\alpha_m = \sqrt{\frac{\xi(x^*)}{\Lambda'(1)}} \left(\frac{1}{\bar{\lambda}_p(y^*)} - \frac{2\bar{\lambda}_p'(y^*)\rho'(1-x^*)(1-\mu(x^*))}{\bar{\lambda}_p(y^*)^2 \cdot \mu(x^*)} \right) \quad (9)$$

$$\beta_m = \left(\frac{b}{\bar{\Lambda}_p'(y^*)x^*\rho'(1-x^*) \cdot \sqrt{\bar{\lambda}_p(y^*)\mu'(x^*)}} \right)^{2/3} \quad (10)$$

where

$$b = x^*\rho'(1-x^*) \frac{\lambda'(y^*)y^* - x^*\rho'(1-x^*)}{\lambda(y^*)y^*} \\ + \frac{(x^*\rho'(1-x^*))^2}{\pi(y^*)} \left(\pi''(y^*) + \frac{\pi'(y^*)}{y^*} - \frac{\pi(y^*)'^2}{\pi(y^*)} \right) \\ \left(\frac{x^*(1-\eta_m^*)\rho'(\bar{x}^*)}{y^*} \right)^2 \cdot \frac{\sum_l l p_{ml}(1-p_{ml})\lambda_{l+1}(y^*)^{l-1}}{\pi(y^*)}$$

As we can see, α_m and β_m only depend on η_m^* , x^* and y^* and on polynomials λ , ρ and p_m . Moreover, they give a very accurate approximation as it is shown in [10].

Interestingly, for regular LDPC codes with parameters $(x^{c-1}, x^{d-1}, p_m)^2$, α_m and β_m can be expressed through the scaling α and shift β parameters of the unpunctured ensemble (i.e. $p_m(x) = 0$):

$$\alpha_m = \frac{\alpha}{1 - p_m}, \quad \beta_m = \frac{\beta}{1 - p_m}. \quad (11)$$

²for this particular case, p_m is a positive constant, $0 \leq p_m \leq 1$