Adding new Representations of Mathematical Objects to Aplusix

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Abstract: In the framework of the ReMath European project, we initiated two new representations for algebraic expressions in Aplusix: a Tree & Natural representation mixing tree representations and natural representations of algebraic expressions, and a graphical representations to show curves and sets of solutions. Our work has been driven by two approaches: (1) a traditional approach focusing on the definition of new objects with their functionalities, beginning with a specification phase and continuing with a development phase; (2) an experiment-driven approach taking into account future cross-experiments. Solutions arisen from the cross-fertilization of these two approaches are presented in the paper.
Keywords: Algebraic, Natural, Tree, Graphical representations; Display, Editing; Experiment, Cross-experiment, Scenario; TEL.

Introduction

In December 2005, the IST4-26751 European project called ReMath has begun. ReMath stands for: Representing Mathematics with Digital Media [ReMath]. Its first objective is to enrich state-of-the-art dynamic digital artefacts for doing mathematics with new representations of mathematical objects. Its second objective is to work on scenarios for the use of these artefacts. Its third objective is to carry out empirical research involving cross-experiments in realistic educational contexts.
Our team, working for long on the dynamic digital artefact APLUSIX [NIC 03, CHA 04], decided to participate to the ReMath project and found in the first objective of ReMath the opportunity to introduce in our TEL (Technology Enhanced Learning) environment new representations of algebraic expressions which were already in our mind for years: a tree representation and a graphical representation.
But, independently of the nature of the new representations targeted, the framework chosen by the ReMath project is interesting and worth an explanation. Most of the members of the ReMath were already grouped in the European Research Team Telma [Telma] of the European network of excellence Kaleidoscope [Kaleidoscope]. During years 2005 and 2006, the Telma team worked out on the definition of guidelines for doing experiments [CEG 05], conducted and exemplified its work with an innovative *cross-experiment* scenario: each partner organised a real experiment with a TEL environment developed by another partner and produced very interesting report on it. The approach was very fruitful, for the definition of the guidelines and for the feedback received by the teams developing the TEL environments. Benefits were partly due to the fact that researchers doing the experiments were not the researchers who developed the TEL environment: each one was working with his own theoretical framework (theory of didactical situations, anthropologic approach, ACT theory, socio-constructivism, cognitive theory … [ART 05]), and, as a consequence, the cross-experiments confront theoretical frameworks. On one side, it reveals the actual differences between those theoretical frameworks, and on the other side, it shows the ability of each TEL environment to be used in other contexts than those foreseen by the developers.

Thus, in parallel with a reflection on the definition of the new representations, we have thought about scenarios of cross-experiments of the new representations. As long as the new representations were not available, the definition of scenarios of cross-experiments was theoretically oriented, but the work of specifying the new representations has been influenced by the organisation of the near future cross-experiments. It is particularly true for our new tree representations whose progresses have been partly driven by questions and solutions raised by our partner and by us about the foreseen scenarios of use. We can think about Experiment Driven Design, and compare that approach to the traditional approach, where the design is driven by the definition of the objects and the functionalities it must offer. It is not just 'use cases', *à la* UML.

New representations

Beforehand, just few words about Aplusix and the natural representation of algebraic expressions used until now.

Aplusix have been though as a microworld and an exerciser for doing algebra. Students freely write algebraic expressions and work on it: at each step of their reasoning processes, students define a new expression which should be equivalent to the expression defined at the previous step and which should progress toward a solution. Aplusix gives two fundamental feedbacks, first it verifies the semantic equivalence between successive steps and, second, it can verify that the form of the final expression corresponds to a solution.

Till the beginning, we used at the interface of our system only the natural representation of algebraic expressions that everyone writes on paper and blackboard, see figure 1. An important part of our work has been also dedicated to the definition of the way students will edit natural representations. Results of our work have been written in [NIC 04].

For the definition of that *first* version of the TEL environment Aplusix, a traditional approach has been followed: (1). a specification phase, (2). a development phase, (3). an experiment phase, and back to (1) when adding new functionalities.

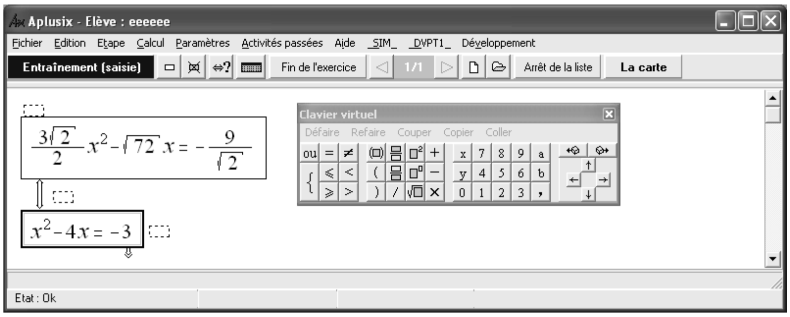

Figure 1: Natural representation of algebraic expression in Aplusix.

So, after discussions with our 7 partners from ReMath, we envisaged to initiate two new representations in Aplusix: a Tree & Natural representation mixing tree representations and natural representations of algebraic expressions, and a graphical representations to show curves corresponding to algebraic expressions and sets of solutions for equations, inequations, and systems of equations.

Tree & Natural representation

Motivations for the introduction for the Tree & Natural Representation were numerous.

From an epistemological point of view, trees are natural representations of algebraic expressions.

From a didactical point of view, the introduction of trees would permit an interesting work for students who will have to change of register [DUV 06]. We believe that the learning of the tree representation and the understanding of the mapping between the natural representation and the tree representation will help students understand the algebraic expressions.

From the computer science point of view, trees are fundamental objects on top of which most of composed data structures are defined. As a matter of fact, the internal objects used inside Aplusix to represent algebraic expressions, and their visual features are trees.

The first questions raised during the specification phases were about the kind of tree we would like to have.

Would it be the internal trees used by Aplusix for the algebraic expressions? Would it be special trees with algebraic specific characteristics or abstract tree without any special link with algebra? What will be the link between the tree representation and the already existing natural representation of algebraic expression? Would it be only a new way to display algebraic expressions, or would it add new ways to work on it (to edit it)? Would every algebraic expression have a tree representation, even an ill-formed one? Would there be ill-formed trees? Would every tree correspond to an algebraic expression? Would trees will be an object on his own?
Few questions were closer to mathematics: what would be the arity of the "-" operator in the tree? Would parentheses be represented in the tree (trees do not need parentheses)?
Many answers were possible, but we had to find a coherent set of answers for all theses questions, , and take into account the framework of the ReMath project (the cross-experiment aspect).

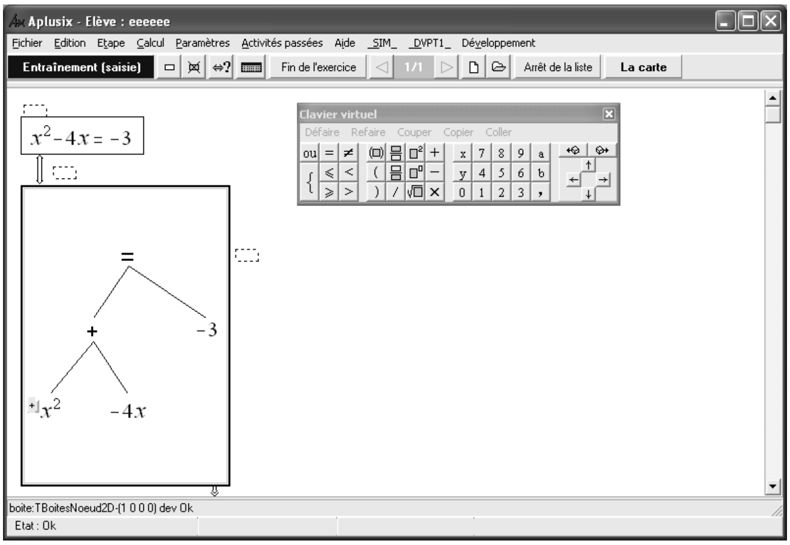

Figure 2: Tree & Natural representation of algebraic expression in Aplusix. When the mouse cursor is over a non-elementary leaf, a "+' button appears allowing to expand it in its tree form.

Reflections about editing and manipulations which should be available and discussions with our colleagues about experiments help us to converge toward a coherent -we hope- solution.
Trees will be authentic objects of our microworld, corresponding to abstract trees in computer science.
When trees will represent algebraic expressions, there could be a mixed representation of tree and natural representations: a leaf of the tree will be natural representations, and could be expanded as a tree by a click on a button; a subpart of the tree could be collapsed in its natural representation by a click on a button.
The natural editing of leaves (which are natural representations) will be mixed with a tree editing of the tree part (adding/deleting a father or a son to a node, changing a node). Two modes will be implemented, one mode without scaffolding (called *free mode*) where the student will be able to build any sort of ill-formed representation; another mode with scaffolding (called *controlled mode*) where Aplusix will oblige the student to write correct algebraic operators in the internal nodes of the tree and to respect the arity (number of arguments) of the operators.
Two new sorts of exercise will be added for defining scenarios of use: they will consist of asking the student to build the tree representation of an expression given in the natural representation and the inverse. For both, trees will only be available in a fully expanded representation. The mix Tree and Natural representation will not be available for these sorts of exercises but will be available in other contexts for exploration of the structure of the expression.

Half a page for describing all of our choices is not enough. The specifications we gave in the ReMath project for our Tree and Natural representation were about 20 pages long, in particular, selection, cursors and the "-" operator were described.
At the end of December 2006, a first prototype of Aplusix with trees was ready. Screenshot shown in Figure 2 have been obtained from it. The main functionalities are already available (display, insert, delete, drag&drop, the two new sorts of exercise). The months before experimentations (foreseen in late 2007) will be used by us for fine-tuning the prototype and by our colleagues to familiarize with it and conceive their experiments.

Graphical representations
The work on graphical representations has been scheduled after the work on tree representations because the researchers who will drive the experiments in the ReMath project are more interested by the tree representations which are more novel. The graphical representations of algebraic expressions will be only for display (no editing is imagined). Our objective is *only* to produce graphical representations to show curves corresponding to algebraic expressions and sets of solutions for equations, inequations, and systems of equations.

Nevertheless, motivations for the introduction for the graphical representations were numerous.
First, teachers were asking for it (graphical representations of expressions are present in most curricula) and seem not to understand why we did not have implemented them yet.
The necessity of combining symbolic and graphical representations in mathematics and in the learning of mathematics is well known. Our main idea for Aplusix consists of developing graphical representations in a way that favour the understanding of the notion of algebraic equivalence (equivalent expressions have identical graphical representation).

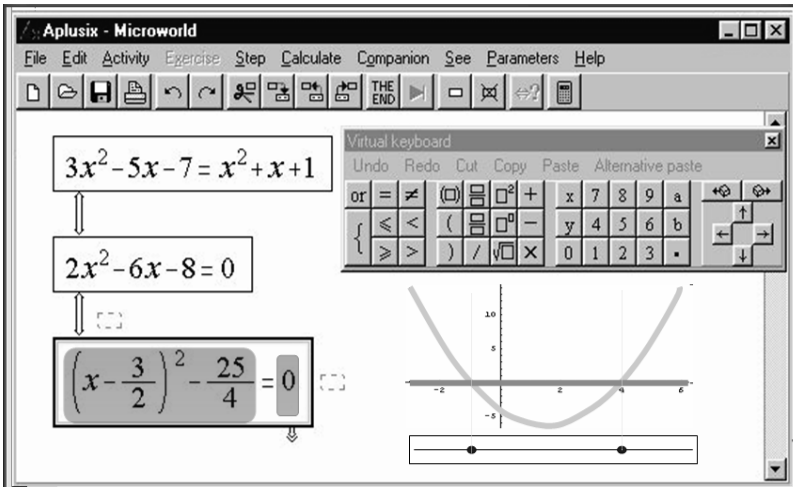

Figure 3: Graphical representation of an equation in Aplusix. The figure in two dimensions is an intermediate representation which represents the functions of both sides of the equal sign. The figure in one dimension is the final representation (the set of solutions). This is not a real screenshot of Aplusix as graphical representations are not yet implemented.

Main questions concerning graphical representations were about the way to represent the solution of equations (solution of $2x^2–8=0$ are x=2 or x=–2, should they be only represented by two points on a line?). Other questions concern the representation of *identical* objects (should $2x^2–8$ and 2(x–2)(x+2) be represented twice or once, and how could they be represented two times as they correspond to the same object?)

We decided to show: polynomial and rational expressions of one variable with curves in the 2D-space; equations of one unknown of the form f(x)=g(x) with curves of f(x) and g(x) in the 2D-space, and solutions in an independent 1D-space (see figure 3); inequations of one unknown of the form f(x)<g(x) with the curves of f(x) and g(x) in the 2D-space, solutions in a 1D-space; linear systems of equations of 2 unknowns with lines in the 2D-space and the intersection point (in the usual case where the system has one).

Specifications given in the ReMath project were about 10 pages long. In particular, discussions about representations of identical objects were developed.

Conclusion

What have to be considered when someone wants to add a new representation of a mathematical object? First, it must be decided whether the representation will be an object on its own, which could be edited, manipulated, or just a new way of displaying the mathematical object. But this is not enough. Representing algebraic expressions is not just displaying it, not even just displaying and editing it. Of course adding new representations of mathematical objects must include a work of specification of the display and the editing of the object but it should also contain a work on scenarios involving the new representations (even if students are free to use them the way they want). This work on possible uses of representations does help to find and choose between possible solutions about the representations and help to make a coherent set of solutions to all the questions related to the representations. Thinking about experiments, or even best, working with colleagues from other laboratories (with different cultures) and foreseeing cross-experiments is a great challenge which improves the design process.

During the first year of the ReMath project, we tried to achieve that goal. It led us to the definition of two new representations, especially of the Natural and Tree representation of an algebraic expression. If tree representations have yet been uses in mathematics, this representation, mixing natural and tree representations, seems to us totally new and very promising. The design benefits from a reflection on future experiments which orients us toward a richer object where its construction is at the centre of the design. We hope this new representation, and the new graphical representation, will reinforced the global objectives of our algebraic microworld: working freely on algebraic expressions having two fundamental feedbacks, one about equivalence between different steps of a reasoning process, the other about when the end of an exercise is reached.

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