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Brownian penalisations related to excursion lengths, VII

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Abstract. Limiting laws, as $t \to \infty$, for Brownian motion penalised by the longest length of excursions up to t, or up to the last zero before t, or again, up to the first zero after t, are shown to exist, and are characterized.

Résumé. Il est prouvé que les lois limites, lorsque $t \to \infty$, du mouvement brownien pénalisé par la plus grande longueur des excursions jusqu'en t, ou bien jusqu'au dernier zéro avant t, ou encore jusqu'au premier zéro après t, existent. Ces lois limites sont décrites en détail.

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1. Introduction

(a) Let $(\Omega, (X_t, t \geq 0), (\mathcal{F}_t, t \geq 0), (P_x, x \in \mathbb{R}))$ denote the canonical realisation of the Wiener process, i.e. $\Omega = \mathcal{C}([0,\infty), \mathbb{R})$; $(X_t, t \geq 0)$ is the coordinate process on Ω; $(\mathcal{F}_t, t \geq 0)$ denotes its natural filtration, and $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$. $(P_x, x \in \mathbb{R})$ is the family of Wiener measures such that $P_x(X_0 = x) = 1$, for every $x \in \mathbb{R}$. We write simply P for P_0.

(b) Let $(\Gamma_t, t \geq 0)$ denote an $\mathbb{R}_+$-valued, $(\mathcal{F}_t)$ adapted process defined on Ω, which satisfies: $0 < E_x(\Gamma_t) < \infty$ for every $t \geq 0$, and every $x \in \mathbb{R}$. With the help of this process Γ – which we call the penalisation process – we define the family of probabilities $P_x^{(t)}$ by:

$$P_x^{(t)}(\Lambda_t) = \frac{E_x(1_{\Lambda_t} \Gamma_t)}{E_x(\Gamma_t)} \quad (\Lambda_t \in \mathcal{F}_t). \tag{1.1}$$

In several preceding papers ([11–13,15–17], see also [14] for a survey), we have shown that for many penalisation processes $(\Gamma_t, t \geq 0)$, the following holds:

$$\text{(i)} \quad \lim_{t\to\infty} P_x^{(t)}(\Lambda_s) \text{ exists, for every fixed } s \geq 0, \text{ and } \Lambda_s \in \mathcal{F}_s. \tag{1.2}$$

$$\text{(ii)} \quad \text{This limit is of the form } E_x(1_{\Lambda_s} M_s^\Gamma), \text{ where } (M_s^\Gamma, s \geq 0) \text{ is a } (\mathcal{F}_s, P_x) \text{ positive martingale.} \tag{1.3}$$

Here is our main tool to prove (1.2) and (1.3).

Theorem 1.1. *Assume that*:

$$\text{(i)} \quad \frac{E(\Gamma_t | \mathcal{F}_s)}{E(\Gamma_t)} \underset{t\to\infty}{\longrightarrow} M_s \quad a.s.,$$

(ii) $E(M_s) = 1$, *for every* $s \geq 0$.

Then,

(1) $\forall s \geq 0,\ \forall \Lambda_s \in \mathcal{F}_s, \quad \dfrac{E(1_{\Lambda_s}\Gamma_t)}{E(\Gamma_t)} \underset{t\to\infty}{\longrightarrow} E(1_{\Lambda_s}M_s),$

(2) $(M_s, s \geq 0)$ *is a* $(\mathcal{F}_s, P)$ *positive martingale*.

The proof of this Theorem 1.1 is quite elementary. It hinges on Scheffé's lemma (see [9], p. 37, Theorem 21).

(c) Assume that the hypotheses of Theorem 1.1 are satisfied, and define, for $s \geq 0$, and $\Lambda_s \in \mathcal{F}_s$:

$$Q(\Lambda_s) := E(1_{\Lambda_s} M_s). \tag{1.4}$$

Then, (1.4) induces a probability Q on $(\Omega, \mathcal{F}_\infty)$. In [11–13,15–17], we have described precisely the main properties of the canonical process $(X_t, t \geq 0)$ under Q.

(d) The aim of the present paper is to show the existence of the limit $P^{(t)}(\Lambda_s)$, as $t \to \infty$, s being fixed, and to study the canonical process $(X_t, t \geq 0)$ under Q – the Wiener measure P penalized by the process $(\Gamma_t, t \geq 0)$ – when $(\Gamma_t, t \geq 0)$ is defined in terms of lengths of excursions. Let us be more precise and fix notation: For $t \geq 0$, we denote by g_t (resp. d_t) the last zero of $(X_u, u \geq 0)$ before time t, resp.: the first zero after time t:

$$g_t = \sup\{s \leq t; X_s = 0\}, \tag{1.5}$$

$$d_t = \inf\{s > t; X_s = 0\}. \tag{1.6}$$

Hence, $(d_t - g_t)$ is the length of the excursion which straddles t. We also introduce:

$$\Sigma_t \equiv \Sigma_{g_t} = \sup\{d_s - g_s; d_s \leq t\}, \tag{1.7}$$

$$\Sigma_t^* \equiv \Sigma_{d_t} = \sup\{d_s - g_s; g_s \leq t\}. \tag{1.8}$$

Thus, Σ_t is the length of the longest excursion before g_t, whereas Σ_t^* is the length of the longest excursion before d_t. Consequently:

$$\Sigma_t^* = \Sigma_t \vee (d_t - g_t). \tag{1.9}$$

We denote by $(A_t, t \geq 0)$ the so-called age process:

$$A_t := t - g_t \quad \text{and} \quad A_t^* := \sup_{s\leq t} A_s.$$

Hence:

$$A_t^* = \Sigma_t \vee (t - g_t) \quad \text{and} \quad \Sigma_t \leq A_t^* \leq \Sigma_t^*; \qquad \Sigma_t^* = A_t^* \vee (d_t - g_t).$$

The aim of this article is to study the effects on Brownian motion of penalisations induced by the following processes $(\Gamma_t, t \geq 0)$:

(i) $\Gamma_t := 1_{(\Sigma_t \leq x)}$ ($x > 0$, fixed). This is studied in Section 2.

(ii) $\Gamma_t := h(\Sigma_t)$, where h is an "integrable function." This study extends that made in (i), and is developed in Section 3;

(iii) $\Gamma_t := 1_{\{A_t^* \leq x\}}$ ($x > 0$, fixed), and $\Gamma_t = 1_{\{\Sigma_t^* \leq x\}}$ ($x > 0$, fixed). This is studied in Section 4.

However, in this case, we have not been able to obtain a full proof of the existence of the penalised measure; we present a conjecture (4.5) upon which the existence rests.

(e) *Some prerequisites relative to the Brownian meander.*

As Σ_t is $\mathcal{F}_{g_t}$-measurable, it turns out that certain features of the part of the trajectory of our Brownian motion $(X_s, s \geq 0)$ between times g_t and t play some important role throughout the discussion. We now gather a few useful facts about the *Brownian meander*:

$$\left(\tilde{m}_u^{(t)} = \frac{1}{\sqrt{t-g_t}} X_{g_t+u(t-g_t)}, 0 \leq u \leq 1\right),$$

which is a well-defined process, whose law, thanks to the scaling property of Brownian motion does not depend on $t > 0$. (Here, we slightly depart from the classical Brownian terminology, for which it is $m^{(t)} \equiv (|\tilde{m}_u^{(t)}|, u \leq 1)$ which is called the Brownian meander.)

The simple fact, obtained by Brownian scaling, that the law of $\tilde{m}^{(t)}$ does not depend on t, can be further extended as follows.

Proposition 1.2. *Let T be a finite $\{\mathcal{F}_{g_t}\}$ stopping time, such that: $P(X_T = 0) = 0$. Then:*

(i) *the process $(\tilde{m}_u^{(T)}, u \leq 1)$ is independent from $\mathcal{F}_{g_T}$, and its law does not depend on T;*
(ii) *for any Borel function $f : \mathbb{R} \to \mathbb{R}_+$, one has:*

$$E\big[f(X_T)|\mathcal{F}_{g_T}\big] = Kf(A_T), \tag{1.10}$$

where K denotes the Markov kernel defined by:

$$Kf(y) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{|z|}{y} \exp\left\{-\frac{z^2}{2y}\right\} f(z)\,\mathrm{d}z \quad (y \in \mathbb{R}_+).$$

In particular, if $T = T_a^A = \inf\{t\colon\ A_t > a\}$, for $a > 0$, then $X_{T_a^A}$ and $\mathcal{F}_{g_{T_a^A}}$ are independent; hence, $X_{T_a^A}$ and T_a^A are independent, and:

$$P(X_{T_a^A} \in \mathrm{d}z) = \frac{|z|}{2a} \exp\left\{-\frac{z^2}{2a}\right\} \mathrm{d}z. \tag{1.11}$$

(iii) *Let $\Psi_{1,x}(z) = \sqrt{\frac{2}{\pi x}}|z|$ and $\Psi_{2,b}(z) = \Phi(\frac{|z|}{\sqrt{b}})$ for some $x > 0$, and $b > 0$, where*

$$\Phi(y) = \sqrt{\frac{2}{\pi}} \int_y^{\infty} \mathrm{d}u\, \mathrm{e}^{-u^2/2} = P\big(|G| > y\big), \tag{1.12}$$

for G a standard Gaussian variable. Then,

$$K\Psi_{1,x}(y) = \sqrt{\frac{y}{x}}, \qquad K\Psi_{2,b}(y) = 1 - \sqrt{\frac{y}{b+y}}.$$

We note that:

$$K\Psi_{1,x}(y) + K\Psi_{2,x-y}(y) = 1 \quad \textit{for all } y < x. \tag{1.13}$$

Proof.

- Points (i) and (ii) are very classical; they are proven and applied in [1–3,10].
- Point (iii) follows from elementary computations. Indeed:

$$K\Psi_{1,x}(y) = \sqrt{\frac{2}{\pi x}}\left(\frac{1}{2}\right)\int_{-\infty}^{\infty} \frac{z^2}{y} \mathrm{e}^{-z^2/(2y)}\,\mathrm{d}z = \sqrt{\frac{y}{x}}$$

whilst:

$$
\begin{aligned}
K\Psi_{2,b}(y) &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{|z|}{y}\mathrm{e}^{-z^2/(2y)}\Phi\left(\frac{|z|}{\sqrt{b}}\right)\mathrm{d}z \\
&= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\mathrm{d}z\,\frac{z}{y}\mathrm{e}^{-z^2/(2y)}\int_{z/\sqrt{b}}^{\infty}\mathrm{e}^{-u^2/2}\,\mathrm{d}u \\
&= \frac{1}{y}\sqrt{\frac{2}{\pi}}\int_0^{\infty}\mathrm{e}^{-u^2/2}\,\mathrm{d}u\int_0^{u\sqrt{b}} z\mathrm{e}^{-z^2/(2y)}\,\mathrm{d}z \quad \text{(by Fubini)} \\
&= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\mathrm{e}^{-u^2/2}\bigl(1-e^{-\{u^2/(2y)\}b}\bigr)\,\mathrm{d}u \\
&= 1-\sqrt{\frac{y}{b+y}}.
\end{aligned}
$$

□

2. Penalisation induced by $\Gamma_t = 1_{(\Sigma_t \le x)}$

Here, $x > 0$ is fixed.

Theorem 2.1.

(1) *For every* $s \ge 0$, *and* $\Lambda_s \in \mathcal{F}_s$,

$$Q(\Lambda_s) := \lim_{t\to\infty}\frac{E(1_{\Lambda_s}1_{\{\Sigma_t\le x\}})}{E(1_{\{\Sigma_t\le x\}})} \quad \textit{exists.} \tag{2.1}$$

(2) *This limit induces a probability* Q *on* $(\Omega, \mathcal{F}_\infty)$ *such that*:

$$Q(\Lambda_s) := E(1_{\Lambda_s}\widetilde{M}_s 1_{\{\Sigma_s\le x\}}) = E(1_{\Lambda_s}M_s), \tag{2.2}$$

where:

$$M_s := \widetilde{M}_s 1_{(\Sigma_s\le x)}, \tag{2.3}$$

$$\widetilde{M}_s := |X_s|\sqrt{\frac{2}{\pi x}} + \Phi\left(\frac{|X_s|}{\sqrt{x-A_s}}\right)1_{(A_s\le x)} \tag{2.4}$$

with Φ *given by* (1.12). *Moreover*, $(M_s, s\ge 0)$ *is a continuous*, *positive martingale*, *such that* $M_0 = 1$.

(3) *Under* Q, *the process* $(X_t, t\ge 0)$ *satisfies*:

(a) $\Sigma_\infty \le x$ *a.s.*, *and* $\sqrt{\frac{\Sigma_\infty}{x}}$ *is uniformly distributed on* [0,1]; (2.5)

(b) $A^*_\infty = \infty$ *a.s.*; (2.6)

(c) *Let* $g = \sup\{t\colon X_t = 0\}$. *Then*, $Q(0 < g < \infty) = 1$, *and the law of* g *is given by*:

$$Q(g \ge t) = E\left[\sqrt{\frac{A_{t\wedge T_x^A}}{x}}\right], \tag{2.7}$$

where $T_x^A := \inf\{t \ge 0 : A_t = x\}$.

(d) *Let* $0 \le y \le x$, *and denote* $T_y^A := \inf\{t \ge 0\colon A_t = y\}$. *Then*:

(i) *The process* $(A_u, u \le T_y^A)$ *is identically distributed under* P *and* Q;
(ii) $(A_u, u \le T_y^A)$ *and* $X_{T_y^A}$ *are independent under either* P *or* Q;
(iii) *The density of the distribution of* $X_{T_y^A}$ *under* Q *equals*:

$$f^Q_{X_{T_y^A}}(z) = \frac{|z|}{2y} \mathrm{e}^{-z^2/(2y)} \left(|z| \sqrt{\frac{2}{\pi x}} + \Phi\left(\frac{|z|}{\sqrt{x-y}} \right) \right) \tag{2.8}$$

(iv) $Q(g > T_y^A) = 1 - \sqrt{\frac{y}{x}}$;
(v) *The process* $(A_u, u \le T_y^A)$ *and the event* $(g > T_y^A)$ *are independent under* Q.

(4) *Moreover, under* Q:

(a) *The processes* $(X_u, u \le g)$ *and* $(X_{g+u}, u \ge 0)$ *are independent*;
(b) *The process* $(X_{g+u}, u \ge 0)$ *is positive, resp.: negative, with probability* $\frac{1}{2}$, *and* $(|X_{g+u}|, u \ge 0)$ *is a* BES(3) *process*;
(c) *Denoting by* $(L_t, t \ge 0)$ *the local time process of* X *at level* 0, *then*:

- $\sqrt{\frac{2}{\pi x}} L_\infty$ *is exponentially distributed, with mean* 1;
- *Conditionally on* $L_\infty = \ell$, *the process* $(X_u, u \le g)$ *is a Brownian motion* B *stopped at* $\tau_\ell := \inf\{t\colon L_t > \ell\}$, *and conditioned on* $\{\Sigma_{\tau_\ell} \le x\}$.

Remark 2.2. *Obviously, the probability measure* Q *defined by* (2.1) *and the martingale* (M_s) *depend on the parameter* x. *In this section, since* x *is fixed, there is no ambiguity. A generalisation of Theorem* 2.1 *will be given in Section* 3 *and we shall denote* $Q^{(x)}$ (*resp.* M_s^x) *the p.m.* (*resp. the martingale*) *defined by* (2.1) (*resp.* (2.3)).

Proof of Theorem 2.1.
(1) We begin with the following lemma.

Lemma 2.3.

$$P(\Sigma_t \le x) = P\left(\Sigma_1 \le \frac{x}{t}\right) \underset{t\to\infty}{\sim} \sqrt{\frac{x}{t}} \tag{2.9}$$

(the equality in (2.9) follows from the scaling property).

Proof. We might use the computation in Exercise 4.19 of [10], p. 507, which discusses a result of Knight [7], but we shall proceed from scratch by showing that:
(a) if S_β denotes an independent exponential time, with parameter β, then:

$$\beta \int_0^\infty \mathrm{e}^{-\beta t} P(\Sigma_t \le x)\, \mathrm{d}t = P(\Sigma_{S_\beta} \le x) = \frac{E[\sinh(\sqrt{2\beta}|X_{T_x^A}|)]}{E[\cosh(\sqrt{2\beta} X_{T_x^A})]}. \tag{2.10}$$

Let us admit this result for one moment; then, as $\beta \to 0$, we obtain:

$$P(\Sigma_{S_\beta} \le x) \sim \sqrt{2\beta} E\big[|X_{T_x^A}|\big].$$

Using (1.11), we have

$$E\big[|X_{T_x^A}|\big] = \frac{1}{2} \int_{-\infty}^{\infty} \frac{z^2}{x} \exp\left(-\frac{z^2}{2x}\right) \mathrm{d}z = \sqrt{\frac{\pi x}{2}}.$$

Thus, we have obtained:

$$\int_0^\infty e^{-\beta t} P(\Sigma_t \le x)\,dt \underset{\beta\to 0}{\sim} \sqrt{\frac{\pi x}{\beta}}. \tag{2.11}$$

(b) We now prove formula (2.10): we first note that:

$$\{\Sigma_{S_\beta} \le x\} = \{g_{S_\beta} \le T_x^A\} = \{S_\beta \le d_{T_x^A}\}, \tag{2.12}$$

hence:

$$P(\Sigma_{S_\beta} \le x) = 1 - E\big[e^{-\beta d_{T_x^A}}\big].$$

Let (θ_u) denote the usual family of time-translation operators:

$$X_s \circ \theta_u = X_{s+u} \quad (s, u \ge 0). \tag{2.13}$$

Since: $d_{T_x^A} = T_x^A + T_0 \circ \theta_{T_x^A}$, we obtain:

$$P(\Sigma_{S_\beta} \le x) = 1 - E\big[e^{-\beta T_x^A} E_{X_{T_x^A}}\big[e^{-\beta T_0}\big]\big],$$

$$P(\Sigma_{S_\beta} \le x) = 1 - E\big[e^{-\beta T_x^A} e^{-\sqrt{2\beta}|X_{T_x^A}|}\big].$$

Next, we use the independence of T_x^A and $X_{T_x^A}$, recalled in Proposition 1.2(ii), which yields:

$$P(\Sigma_{S_\beta} \le x) = 1 - E\big[e^{-\beta T_x^A}\big] E\big[e^{-\sqrt{2\beta}|X_{T_x^A}|}\big].$$

Formula (2.10) now follows from Wald's identity:

$$E\big[e^{\sqrt{2\beta} X_{T_x^A}}\big] E\big[e^{-\beta T_x^A}\big] = 1 \tag{2.14}$$

(where we have used again the independence of T_x^A and $X_{T_x^A}$) together with the symmetry of the distribution of $X_{T_x^A}$. □

(2) *We now show relations* (2.1) *and* (2.3).

Let $T_0 = \inf\{s \ge 0; X_s = 0\}$ and $t \ge s \ge 0$. First, we observe:

$$\{\Sigma_t \le x\} = \{\Sigma_s \le x\} \cap \big(\{d_s > t\} \cup \big\{d_s \le t \wedge (s - A_s + x),\, \Sigma_{t-d_s} \circ \theta_{d_s} \le x\big\}\big). \tag{2.15}$$

Hence:

$$P(\Sigma_t \le x | \mathcal{F}_s) = (1) + (2),$$

where

$$(1) = 1_{\{\Sigma_s \le x\}} P\big(\{d_s > t\} | \mathcal{F}_s\big),$$
$$(2) = 1_{\{\Sigma_s \le x\}} P\big(\big\{d_s \le t \wedge (s - A_s + x),\, \Sigma_{t-d_s} \circ \theta_{d_s} \le x\big\} | \mathcal{F}_s\big).$$

We now study the asymptotic behavior of (1), then (2), as $t \to \infty$.

As is well known:

$$P_y(T_0 \ge t) = P_0\left(|G| \le \frac{|y|}{\sqrt{t}}\right) \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi}}|y|\frac{1}{\sqrt{t}}, \tag{2.16}$$

where G denotes a standard $\mathcal{N}(0,1)$ Gaussian variable.

Since $d_s = s + T_0 \circ \theta_s$, the equivalence (2.16) implies:

$$(1) = 1_{\{\Sigma_s \le x\}} P_{X_s}(T_0 > t - s) \underset{t\to\infty}{\sim} \left(\sqrt{\frac{2}{\pi}}|X_s| 1_{\{\Sigma_s \le x\}}\right) \frac{1}{\sqrt{t}}. \tag{2.17}$$

As for the term (2), we apply both the strong Markov property at time d_s and Lemma 2.3:

$$(2) \underset{t\to\infty}{\sim} 1_{\{\Sigma_s \le x\}} \sqrt{\frac{x}{t}} P\big(\{d_s \le s - A_s + x\} | \mathcal{F}_s\big).$$

From the Markov property at time s, together with (1.12) and (2.16) we deduce:

$$(2) \underset{t\to\infty}{\sim} 1_{\{\Sigma_s \le x\}} \sqrt{\frac{x}{t}} 1_{\{A_s \le x\}} \Phi\left(\frac{|X_s|}{\sqrt{x - A_s}}\right).$$

Finally

$$\begin{aligned} P(\Sigma_t \le x | \mathcal{F}_s) &\underset{t\to\infty}{\sim} \frac{1}{\sqrt{t}} \sqrt{x} \left(|X_s| \sqrt{\frac{2}{\pi x}} + \Phi\left(\frac{|X_s|}{\sqrt{x - A_s}}\right) 1_{\{A_s \le x\}}\right) 1_{\{\Sigma_s \le x\}} \\ &\underset{t\to\infty}{\sim} \frac{\sqrt{x}}{\sqrt{t}} \widetilde{M}_s 1_{\{\Sigma_s \le x\}} = \frac{\sqrt{x}}{\sqrt{t}} M_s. \end{aligned}$$

It is clear that (2.9) implies that:

$$\frac{P(\Sigma_t \le x | \mathcal{F}_s)}{P(\Sigma_t \le x)} \underset{t\to\infty}{\sim} \frac{M_s}{E[M_s]} = M_s,$$

since $E[M_s] = 1$ follows from the next point.

(3.a) We begin with the following lemma.

Lemma 2.4.

(1) *Let $f : (y, a) \to f(y, a)$ be a $\mathcal{C}^{2,1}_{(y,a)}$ function, from $\mathbb{R} \times \mathbb{R}_+$ to $\mathbb{R}$. Then, $(f(X_t, A_t), t \ge 0)$ is a $((\mathcal{F}_t), P)$ semi-martingale, which decomposes as*:

$$\begin{aligned} f(X_t, A_t) = f(0,0) &+ \int_0^t \frac{\partial f}{\partial y}(X_s, A_s)\,\mathrm{d}X_s + \int_0^t \left(\frac{1}{2}\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial a}\right)(X_s, A_s)\,\mathrm{d}s \\ &+ \sum_{s \le t}\big(f(0, A_s) - f(0, A_{s-})\big). \end{aligned}$$

In particular, if:

(i) *$f(0, a)$ does not depend on $a \ge 0$*;

(ii) $\dfrac{1}{2}\dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial f}{\partial a} = 0$,

then, $(f(X_t, A_t), t \ge 0)$ is a $((\mathcal{F}_t), P)$ local martingale, with Itô representation:

$$f(X_t, A_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial y}(X_s, A_s)\,\mathrm{d}X_s.$$

(2) *Likewise, let $f : (y, a) \to f(y, a)$ be a $\mathcal{C}^{2,1}_{(y,a)}$ function defined on $\mathbb{R}_+ \times \mathbb{R}_+$.*

Then, $(f(|X_t|, A_t), t \geq 0)$ *is a* $((\mathcal{F}_t), P)$ *semimartingale*, *which decomposes as*:

$$f\big(|X_t|, A_t\big) = f(0,0) + \int_0^t \frac{\partial f}{\partial y}\big(|X_s|, A_s\big)\,\mathrm{sgn}(X_s)\,\mathrm{d}X_s + \frac{\partial f}{\partial y}(0,0)L_t + \int_0^t \left(\frac{1}{2}\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial a}\right)\big(|X_s|, A_s\big)\,\mathrm{d}s + \sum_{s\le t}\big(f(0, A_s) - f(0, A_{s-})\big),$$

where $(L_t, t \geq 0)$ *denotes the local time of* $(X_t, t \geq 0)$ *at level* 0.

In particular, *if* f *satisfies* (i) *and* (ii) *above*, *as well as*:

(iii) $\dfrac{\partial f}{\partial y}(0,0) = 0$,

then $(f(|X_t|, A_t), t \geq 0)$ *is a* $((\mathcal{F}_t), P)$ *local martingale*, *with Itô representation*:

$$f\big(|X_t|, A_t\big) = f(0,0) + \int_0^t \frac{\partial f}{\partial y}\big(|X_s|, A_s\big)\,\mathrm{sgn}(X_s)\,\mathrm{d}X_s.$$

Proof.

(a) Since the process $(A_t, t \geq 0)$ has bounded variation, we may apply Itô's formula to $f(X_t, A_t)$ to obtain:

$$f(X_t, A_t) = f(0,0) + \int_0^t \left(\frac{\partial f}{\partial y}(X_s, A_s)\,\mathrm{d}X_s + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(X_s, A_s)\,\mathrm{d}s\right) + \gamma_t,$$

where

$$\gamma_t = \int_0^t \frac{\partial f}{\partial a}(X_s, A_s)\,\mathrm{d}s + \sum_{s\le t}\big(f(0, A_s) - f(0, A_{s-})\big)$$

since the continuous part (A_t^c) of (A_t) is equal to t, and moreover if $\varDelta A_s \neq 0$, then $X_s = 0$, and $A_s = 0$.

(b) We use similar arguments, together with the Tanaka decomposition:

$$|X_t| = \int_0^t \mathrm{sgn}(X_s)\,\mathrm{d}X_s + L_t, \quad t \geq 0,$$

where $(L_t, t \geq 0)$ denotes the local time of X at 0.

We then use the fact that the support of $\mathrm{d}L_s$ is $\{s\colon X_s = 0\}$, and if s is a zero of X then: $A_s = 0$.

Thus:

$$\int_0^t \frac{\partial f}{\partial y}\big(|X_s|, A_s\big)\,\mathrm{d}L_s = \frac{\partial f}{\partial y}(0,0)L_t.$$

□

(3.b) *We now show that* $(M_s, s \geq 0)$ *is a local martingale*.

We define:

$$T_y^A := \inf\{t \geq 0\colon A_t \geq y\} \quad \text{and} \quad T_y^\varSigma := \inf\{t \geq 0\colon \varSigma_t \geq y\} \quad (y > 0).$$

Clearly, one has:

$$T_x^\varSigma = d_{T_x^A} = T_x^A + T_0 \circ \theta_{T_x^A}.$$

We denote:

$$\begin{aligned}\widehat{M}_s &:= 1_{\{s\le T_x^A\}}\left(|X_s|\sqrt{\frac{2}{\pi x}}+\Phi\left(\frac{|X_s|}{\sqrt{x-A_s}}\right)\right)\\ &\quad + 1_{\{s>T_x^A\}}\sqrt{\frac{2}{\pi x}}(\operatorname{sgn} X_{T_x^A})X_s.\end{aligned} \tag{2.18}$$

Then, clearly:

$$M_s=\widetilde{M}_s 1_{\{\Sigma_s\le x\}}=\widetilde{M}_s 1_{\{s\le T_x^\Sigma\}}=\widehat{M}_{s\wedge T_x^\Sigma}. \tag{2.19}$$

Thus, it remains to prove that $(\widehat{M}_s, s\ge 0)$ is a local martingale, and, for this purpose, it suffices to apply point (2) of Lemma 2.4 to the function:

$$f(y,a)=y\sqrt{\frac{2}{\pi x}}+\Phi\left(\frac{y}{\sqrt{x-a}}\right)\quad (y\ge 0, x>a\ge 0).$$

We note that (i), (ii) and (iii) of Lemma 2.4 hold:

(α) $f(0,a)=\Phi(0)=1$ (hence, (i) is satisfied).

(β) $$\frac{\partial^2 f}{\partial y^2}(y,a)=\sqrt{\frac{2}{\pi}}\frac{y}{(x-a)^{3/2}}\mathrm{e}^{-y^2/(2(x-a))};$$

$$\frac{\partial f}{\partial a}(y,a)=-\frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{y}{(x-a)^{3/2}}\mathrm{e}^{-y^2/(2(x-a))},$$

so that: $\frac{1}{2}\frac{\partial^2 f}{\partial y^2}+\frac{\partial f}{\partial a}=0$ (hence, (ii) is satisfied).

(γ) $$\frac{\partial f}{\partial y}(y,a)=\sqrt{\frac{2}{\pi x}}-\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{x-a}}\mathrm{e}^{-y^2/(2(x-a))},$$

so that: $\frac{\partial f}{\partial y}(0,0)=0$ (hence, (iii) is satisfied).

Finally, one has:

$$M_s=1+\int_0^{s\wedge T_x^\Sigma}\left\{\sqrt{\frac{2}{\pi x}}+\frac{1}{\sqrt{(x-A_u)_+}}\Phi'\left(\frac{|X_u|}{\sqrt{(x-A_u)_+}}\right)\right\}\operatorname{sgn}(X_u)\,\mathrm{d}X_u, \tag{2.20}$$

with the natural convention:

$$\Phi\left(\frac{y}{z}\right)=0\quad\text{and}\quad\frac{1}{z}\Phi'\left(\frac{y}{z}\right)=0,\quad\text{if } z=0 \text{ and } y>0. \tag{2.21}$$

(3.c) *We now show that $(M_s, s\ge 0)$ is a true martingale.*

Since T_x^Σ is a finite stopping time, the identity (2.19) and Doob's optional stopping theorem imply that $(M_s, s\ge 0)$ is a martingale as soon as $(\widehat{M}_s, s\ge 0)$ is a martingale.

Due to (2.21), we have:

$$\widehat{M}_s=|X_s|\sqrt{\frac{2}{\pi s}}+\Phi\left(\frac{|X_s|}{\sqrt{(x-A_s)_+}}\right). \tag{2.22}$$

From the preceding (3.b), since $(\widehat{M}_s, s \geq 0)$ is a positive local martingale, in order to prove that it is a true martingale, it suffices to show: $E[\widehat{M}_s] = 1$. One has: $E(\widehat{M}_s) = (1) + (2)$, with:

$$(1) = E\left[1_{\{s \leq T_x^A\}}\left(|X_s|\sqrt{\frac{2}{\pi x}} + \Phi\left(\frac{|X_s|}{\sqrt{x - A_s}}\right)\right)\right],$$

$$(2) = \sqrt{\frac{2}{\pi x}} E\big[1_{\{s > T_x^A\}}(\operatorname{sgn} X_{T_x^A}) X_s\big].$$

Since T_x^A is a $(\mathcal{F}_{g_s})$ stopping time, $\{s \leq T_x^A\}$ belongs to $\mathcal{F}_{g_s}$. Using now Proposition 1.2, we get:

$$\begin{aligned}
(1) &= E\left[1_{\{s \leq T_x^A\}} E\left[|X_s|\sqrt{\frac{2}{\pi x}} + \Phi\left(\frac{|X_s|}{\sqrt{x - A_s}}\right)\Big|\mathcal{F}_{g_s}\right]\right] \\
&= E\big[1_{\{s \leq T_x^A\}}\big(K\Psi_{1,x}(A_s) + K\Psi_{2,x-A_s}(A_s)\big)\big] \\
&= E\left[1_{\{s \leq T_x^A\}}\left(\sqrt{\frac{A_s}{x}} + 1 - \sqrt{\frac{A_s}{x}}\right)\right] = P\big(s \leq T_x^A\big).
\end{aligned}$$

As for (2), we use again Proposition 1.2:

$$\begin{aligned}
(2) &= \sqrt{\frac{2}{\pi x}} E\big[1_{\{s > T_x^A\}}(\operatorname{sgn} X_{T_x^A})(X_{T_x^A} + X_s - X_{T_x^A})\big] \\
&= \sqrt{\frac{2}{\pi x}} E\big[1_{\{s > T_x^A\}}|X_{T_x^A}|\big] \\
&= \sqrt{\frac{2}{\pi x}} P\big(s > T_x^A\big) E\big[|X_{T_x^A}|\big] \\
&= P\big(s > T_x^A\big) K\Psi_{1,x}(x) = P\big(s > T_x^A\big).
\end{aligned}$$

Finally:

$$E(\widehat{M}_s) = P\big(s \leq T_x^A\big) + P\big(s > T_x^A\big) = 1. \tag{2.23}$$

This ends the proof of point (2) in Theorem 2.1.

(4) *We now show that, under Q, $\sqrt{\frac{\Sigma_\infty}{x}}$ is uniformly distributed on* $[0, 1]$.

(4.a) We have:

$$Q(\Sigma_s \leq x) = \lim_{t \to \infty} \frac{P(\Sigma_s \leq x, \Sigma_t \leq x)}{P(\Sigma_t \leq x)} = \frac{P(\Sigma_t \leq x)}{P(\Sigma_t \leq x)} = 1.$$

Thus:

$$Q(\Sigma_\infty \leq x) = 1. \tag{2.24}$$

(4.b) *Proof of point* (3.a) (of Theorem 2.1).

Let $y \in [0, x]$, and $s > 0$. According to (2.2) and Doob's optional stopping theorem, we have:

$$Q(\Sigma_s > y) = Q\big(T_y^\Sigma < s\big) = E[1_{\{T_y^\Sigma < s\}} M_s] = E[1_{\{T_y^\Sigma < s\}} M_{T_y^\Sigma}].$$

Consequently:

$$Q(\Sigma_\infty > y) = Q\big(T_y^\Sigma < \infty\big) = E[M_{T_y^\Sigma}]. \tag{2.25}$$

However:

$$M_{T_y^\Sigma} = \begin{cases} 1, & \text{if the length of the 1st excursion of length} \geq y \text{ is smaller than } x, \\ 0, & \text{otherwise.} \end{cases} \tag{2.26}$$

To compute $E(M_{T_y^\Sigma})$, we use the excursions theory for Brownian motion (see, for instance, [10], Chapter XII). Let $\mathbf{e} = (e_s, s > 0)$ be the excursion process related to (X_t) under P. We introduce, for any excursion ε, its duration $\zeta(\varepsilon)$, and let

$$U = \{\varepsilon; \zeta(\varepsilon) \geq y\}, \qquad \Gamma = \{\varepsilon; y \leq \zeta(\varepsilon) \leq x\} \quad (y < x) \tag{2.27}$$

and N_t^U the number of excursions starting before time t, and belonging to the set U:

$$N_t^U = \sum_{s \leq t} 1_{\{e_s \in U\}}.$$

We now consider the $(\mathcal{F}_{\tau_l})$ stopping time S^y:

$$S^y = \inf\{l > 0; \zeta(e_l) \geq y\} = \inf\{t > 0; N_t^U > 0\},$$

where $\tau_l = \inf\{t \geq 0; L_t > l\}$.

From [10], Lemma 1.13, Chapter XII and (2.25), one has:

$$E[M_{T_y^\Sigma}] = P\big(\zeta(e_{S^y}) \in U\big) = \frac{\mathbf{n}(\Gamma)}{\mathbf{n}(U)},$$

with $\mathbf{n}$ denoting Itô's excursion measure.

It is easy to compute $\mathbf{n}(\Gamma)$ and $\mathbf{n}(U)$:

$$\mathbf{n}(\Gamma) = \frac{1}{\sqrt{2\pi}} \int_y^x \frac{\mathrm{d}r}{\sqrt{r^3}} = \frac{2}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{y}} - \frac{1}{\sqrt{x}} \right), \qquad \mathbf{n}(U) = \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{y}}.$$

Consequently:

$$Q(\Sigma_\infty > y) = 1 - \sqrt{\frac{y}{x}}. \tag{2.28}$$

This ends the proof of point (3.a) in Theorem 2.1.

(4.c) *We determine the density function of L_∞ under Q.*

Let $a > 0$. We have:

$$Q(L_\infty > a) = Q(\tau_a < \infty) = \lim_{t \to \infty} Q(\tau_a < t),$$

where $\tau_a = \inf\{s; L_s \geq a\}$.

According to (2.2), (2.3) and the optimal stopping theorem, we get:

$$Q(\tau_a < t) = E[1_{\{\tau_a < t\}} M_t] = E[1_{\{\tau_a < t\}} M_{\tau_a}] = P(\tau_a < t, \Sigma_{\tau_a} \leq x).$$

Consequently, using notation introduced in (4.b) above, we obtain:

$$Q(L_\infty > a) = P(\Sigma_{\tau_a} \leq x) = \exp\big(-a\mathbf{n}(\zeta \geq x)\big).$$

Since:

$$\mathbf{n}(\zeta \geq x) = \int_x^\infty \frac{\mathrm{d}r}{\sqrt{2\pi r^3}} = \sqrt{\frac{2}{\pi x}},$$

we get:

$$Q(L_\infty > a) = \exp\left\{-a\sqrt{\frac{2}{\pi x}}\right\} \quad (a > 0). \tag{2.29}$$

This proves the first part of point (4.c) in Theorem 2.1.

(5) *We show that, for any $y < x$, $E[M_{T_y^A}] = 1$.*

According to identity (2.3), we have:

$$M_{T_y^A} = |X_{T_y^A}|\sqrt{\frac{2}{\pi x}} + \Phi\left(\frac{|X_{T_y^A}|}{\sqrt{x-y}}\right). \tag{2.30}$$

From Proposition 1.2, we deduce:

$$E[M_{T_y^A}] = K\Psi_{1,x}(y) + K\Psi_{2,x-y}(y) = \sqrt{\frac{y}{x}} + 1 - \sqrt{\frac{y}{x}} = 1. \tag{2.31}$$

We note that the preceding computation yields another proof of (2.28).

(6) *We now show: $Q(A_\infty^* = \infty) = 1$.*

Let $0 < \eta < 1$. We will prove that $Q(A_\infty^* = \infty) \geq \sqrt{1-\eta}$, which will yield the result.

Similarly to the proof of point (4.b) one has: $Q(T_y^A < \infty) = E[M_{T_y^A}]$, for any $y \in [0, x[$. Identity (2.31) implies that $Q(T_y^A < \infty) = 1$.

From point (3.a) of Theorem 2.1:

$$Q(\Sigma_\infty < x - x\eta) = \sqrt{1-\eta}.$$

Consequently:

$$Q\big(\{\Sigma_\infty < x - x\eta\} \cap \{T_y^A < \infty\}\big) = \sqrt{1-\eta}, \quad \text{where } x - x\eta < y < x.$$

Let us observe that on the set $\{\Sigma_\infty < x(1-\eta)\} \cap \{T_y^A < \infty\}$, X_t does not vanish after time T_y^A. In particular, one has: $A_\infty^* = \infty$. Consequently $Q(A_\infty^* = \infty) \geq \sqrt{1-\eta}$.

We also observe that $\Sigma_\infty < \infty$ and $A_\infty^* < \infty$, Q a.s. imply that:

$$Q(g < \infty) = 1, \tag{2.32}$$

where

$$g = \sup\{t\colon\ X_t = 0\}. \tag{2.33}$$

(7) *We then prove simultaneously that the process $(A_u, u \leq T_y^A)$ has the same distribution under P and Q, and that it is independent from the r.v. $X_{T_y^A}$ under Q (and under P), and we compute the density of $X_{T_y^A}$ under Q.*

(7.a) From the definition (2.2) of Q, for every positive functional F, and every Borel function $h : \mathbb{R} \to \mathbb{R}_+$, we have:

$$E_Q\big[F\big(A_u, u \leq T_y^A\big)h(X_{T_y^A})\big] = E\big[F\big(A_u, u \leq T_y^A\big)h(X_{T_y^A})M_{T_y^A}\big]. \tag{2.34}$$

Note that $M_{T_y^A}$ is a function of $X_{T_y^A}$, and recall that from Proposition 1.2, $X_{T_y^A}$ and $(A_u, u \leq T_y^A)$ are P-independent.

Thus:

$$E_Q\big[F\big(A_u, u \leq T_y^A\big)h(X_{T_y^A})\big] = E\big[F\big(A_u, u \leq T_y^A\big)\big]E\big[h(X_{T_y^A})M_{T_y^A}\big]. \tag{2.35}$$

(7.b) Now, taking $h \equiv 1$, and using (2.31), we obtain:

$$E_Q\big[F\big(A_u, u \leq T_y^A\big)\big] = E\big[F\big(A_u, u \leq T_y^A\big)\big]$$

thus proving the first point announced in (7).

(7.c) We may now write (2.35) as follows:

$$\begin{aligned} E_Q\big[F\big(A_u, u \le T_y^A\big)h(X_{T_y^A})\big] &= E_Q\big[F\big(A_u, u \le T_y^A\big)\big]E\big[h(X_{T_y^A})M_{T_y^A}\big] \\ &= E_Q\big[F\big(A_u, u \le T_y^A\big)\big]E_Q\big[h(X_{T_y^A})\big], \end{aligned}$$

thus proving the independence of $(A_u, u \le T_y^A)$ and $X_{T_y^A}$ under Q.

(7.d) *We now compute the density of* $X_{T_y^A}$ *under* Q.

One has:

$$E_Q\big[h(X_{T_y^A})\big] = E\big[h(X_{T_y^A})M_{T_y^A}\big]. \tag{2.36}$$

Hence, from formulae (2.30) and (1.11) we get:

$$Q(X_{T_y^A} \in \mathrm{d}z) = \frac{|z|}{2y}\mathrm{e}^{-z^2/(2y)}\left(|z|\sqrt{\frac{2}{\pi x}} + \Phi\left(\frac{|z|}{\sqrt{x-y}}\right)\right)\mathrm{d}z. \tag{2.37}$$

(8) *We show the independence under* Q *of* $(A_u, u \le T_y^A)$ *and of the set* $\{g > T_y^A\}$, *and we compute* $Q(g > T_y^A)$.

For every positive functional F, one has:

$$E_Q\big[1_{\{g>T_y^A\}}F\big(A_u, u \le T_y^A\big)\big] = E_Q\big[F\big(A_u, u \le T_y^A\big)Q\big(g > T_y^A|\mathcal{F}_{T_y^A}\big)\big].$$

We now admit for an instant the result of Lemma 2.5, which will be proved below:

$$Q(g > t|\mathcal{F}_t) = \Phi\left(\frac{|X_t|}{\sqrt{(x-A_t)_+}}\right)\frac{1}{M_t}1_{\{A_t^* \le x\}}. \tag{2.38}$$

Replacing t by the $(\mathcal{F}_t)$-stopping time T_y^A, we get:

$$Q\big(g > T_y^A|\mathcal{F}_{T_y^A}\big) = \Phi\left(\frac{|X_{T_y^A}|}{\sqrt{x-y}}\right)\frac{1}{M_{T_y^A}}. \tag{2.39}$$

Since the right-hand side of (2.39) only depends on $X_{T_y^A}$, we deduce from the above point (7) that:

$$E_Q\big[1_{(g>T_y^A)}F\big(A_u, u \le T_y^A\big)\big] = Q\big(g > T_y^A\big)E_Q\big[F\big(A_u, u \le T_y^A\big)\big].$$

Hence the desired independence property.

Also, from (2.39), (2.34) and Proposition 1.2, we deduce:

$$Q\big(g > T_y^A\big) = E_Q\left[\Phi\left(\frac{|X_{T_y^A}|}{\sqrt{x-y}}\right)\frac{1}{M_{T_y^A}}\right] = E\left[\Phi\left(\frac{|X_{T_y^A}|}{\sqrt{x-y}}\right)\right] = 1 - \sqrt{\frac{y}{x}}.$$

(9) To end the proof of Theorem 2.1, we shall now use the technique of progressive enlargement of filtration (see [6]). We have proven (see point (6) above) that $Q(g < \infty) = 1$, where g is defined by (2.33). We denote by $(\mathcal{G}_t, t \ge 0)$ the smallest filtration which contains $(\mathcal{F}_t, t \ge 0)$, and which makes g a $(\mathcal{G}_t, t \ge 0)$ stopping time. In order to use the technique of enlargement of filtration, we need the following lemma.

Lemma 2.5. (i) *For any* $t > 0$, *we have*:

$$Z_t := Q(g > t|\mathcal{F}_t) = \Phi\left(\frac{|X_t|}{\sqrt{x-A_t}}\right)\frac{1}{M_t}1_{\{A_t^* \le x\}}. \tag{2.40}$$

(ii) *Let* (k_s) *be a predictable and non-negative process*; *then*

$$E_Q[k_g] = \sqrt{\frac{2}{\pi x}} E\left[\int_0^\infty k_s 1_{\{s \le T_x^A\}} \, \mathrm{d}L_s\right]. \tag{2.41}$$

Proof.

(i) For every $\Lambda_t \in \mathcal{F}_t$, one has:

$$Q\big(\Lambda_t \cap \{g > t\}\big) = Q\big(\Lambda_t \cap \{d_t < \infty\}\big) = E[1_{\Lambda_t} M_{d_t}] = P\big(\Lambda_t \cap \{\Sigma_{d_t} \le x, A_{d_t} \le x\}\big).$$

Observe that:

$$\{\Sigma_{d_t} \le x, A_{d_t} \le x\} = \big\{A_t^* \le x, d_t - t \le x - A_t\big\} = \big\{A_t^* \le x, T_0 \circ \theta_t \le x - A_t\big\}.$$

Applying the Markov property at time t and identity (2.16) leads to:

$$\begin{aligned} Q\big(\Lambda_t \cap (g > t)\big) &= E\left[1_{\Lambda_t} 1_{\{A_t^* \le x\}} \Phi\left(\frac{|X_t|}{\sqrt{x - A_t}}\right)\right] \\ &= E_Q\left[1_{\Lambda_t} 1_{\{A_t^* \le x\}} \Phi\left(\frac{|X_t|}{\sqrt{x - A_t}}\right) \frac{1}{M_t}\right]. \end{aligned}$$

(ii) Replacing t in (2.40) by any $(\mathcal{F}_t)$ finite stopping time T, taking the expectation, and using (2.3), we get:

$$\begin{aligned} Q(g \le T) = E_Q\big[1_{[0,T]}(g)\big] &= 1 - E\left[1_{\{A_T^* \le x\}} \Phi\left(\frac{|X_T|}{\sqrt{x - A_T}}\right)\right] \\ &= \sqrt{\frac{2}{\pi x}} E\big[|X_{T \wedge T_x^A}|\big] = \sqrt{\frac{2}{\pi x}} E\left[\int_0^\infty 1_{[0,T]}(s) 1_{\{s < T_x^A\}} \, \mathrm{d}L_s\right]. \end{aligned}$$

Point (ii) now follows by an application of the monotone class theorem. □

We now indicate how to make use of Lemma 2.5.

Our study in (3.c) and in particular (2.20) imply that:

$$M_t = \mathcal{E}(J)_t := \exp\left\{\int_0^t J_s \, \mathrm{d}X_s - \frac{1}{2}\int_0^t J_s^2 \, \mathrm{d}s\right\}, \quad t < T_x^\Sigma, \tag{2.42}$$

with

$$J_s = \left(\sqrt{\frac{2}{\pi x}} + \frac{1}{\sqrt{x - A_s}} \Phi'\left(\frac{|X_s|}{\sqrt{x - A_s}}\right)\right) \frac{\operatorname{sgn} X_s}{M_s}, \quad s < T_x^\Sigma. \tag{2.43}$$

From Girsanov's theorem, there exists a $((\mathcal{F}_t, t \ge 0), Q)$ Brownian motion $(\beta_t, t \ge 0)$ such that:

$$X_t = \beta_t + \int_0^t \left(\sqrt{\frac{2}{\pi x}} + \frac{1}{\sqrt{(x - A_s)_+}} \Phi'\left(\frac{|X_s|}{\sqrt{(x - A_s)_+}}\right)\right) \frac{\operatorname{sgn} X_s}{M_s} \, \mathrm{d}s \tag{2.44}$$

and the enlargement formulae imply that there exists a $((\mathcal{G}_t)_{t \ge 0}, Q)$ Brownian motion $(\widetilde{\beta}_t, t \ge 0)$ such that:

$$\beta_t = \widetilde{\beta}_t + \int_0^{t \wedge g} \frac{\mathrm{d}\langle Z, \beta\rangle_u}{Z_u} - \int_{t \wedge g}^t \frac{\mathrm{d}\langle Z, \beta\rangle_u}{1 - Z_u}. \tag{2.45}$$

Once we shall have computed explicitly $\mathrm{d}\langle Z, \beta\rangle_u$, these formulae (2.44) and (2.45) will help us to describe the process $(X_t, t \ge 0)$ under Q (see points (11) and (12) below).

(10) *We compute the law of g under Q.*

Note that from our convention (2.21) we have:

$$\Phi\left(\frac{|X_t|}{\sqrt{(x-A_t)_+}}\right)=\Phi\left(\frac{|X_t|}{\sqrt{x-A_t}}\right)1_{\{A_t<x\}}. \tag{2.46}$$

Since $\{A_t^*<x\}=\{A_{g_t}^*<x, t-g_t<x\}$, from (2.40), we have:

$$\begin{aligned}Q(g\ge t)&=E\left[1_{\{A_t^*<x\}}\Phi\left(\frac{|X_t|}{\sqrt{x-A_t}}\right)\right]\\&=E\left[1_{\{A_t^*<x\}}E\left\{\Phi\left(\frac{|X_t|}{\sqrt{x-A_t}}\right)\Big|\mathcal{F}_{g_t}\right\}\right].\end{aligned}$$

Using Proposition 1.2 we get:

$$Q(g\ge t)=E\left[1_{\{A_t^*<x\}}K\Psi_{2,x-A_t}(A_t)\right]=E\left[1_{\{A_t^*<x\}}\left(1-\sqrt{\frac{A_t}{x}}\right)\right].$$

This yields formula (2.7).

(11) *We now show that, under Q,* $(|X_{g+u}|, u\ge 0)$ *is a three-dimensional Bessel process, which is independent from* $(X_u, u\le g)$.

From formulae (2.44) and (2.45), we have, for $t\ge g$:

$$\begin{aligned}X_t&=\widetilde{\beta}_t-\int_g^t\frac{\mathrm{d}\langle Z,\beta\rangle_u}{1-Z_u}\\&\quad+\int_0^t\left(\sqrt{\frac{2}{\pi x}}+\frac{1}{\sqrt{(x-A_u)_+}}\Phi'\left(\frac{|X_u|}{\sqrt{(x-A_u)_+}}\right)\right)\frac{\operatorname{sgn}X_u}{M_u}\,\mathrm{d}u.\end{aligned} \tag{2.47}$$

However, due to Itô's formula, (2.20), (2.3) and Lemma 2.5, the martingale part $\widetilde{Z}$ of Z, satisfies:

$$\begin{aligned}\mathrm{d}\widetilde{Z}_t&=-\frac{1}{M_t^2}\Phi\left(\frac{|X_t|}{\sqrt{(x-A_t)_+}}\right)\\&\quad\times\left\{\sqrt{\frac{2}{\pi x}}+\frac{1}{\sqrt{(x-A_t)_+}}\Phi'\left(\frac{|X_t|}{\sqrt{(x-A_t)_+}}\right)\right\}(\operatorname{sgn}X_t)\,\mathrm{d}X_t\\&\quad+\frac{1}{M_t}\frac{1}{\sqrt{(x-A_t)_+}}\Phi'\left(\frac{|X_t|}{\sqrt{(x-A_t)_+}}\right)(\operatorname{sgn}X_t)\,\mathrm{d}X_t\\&=\frac{1}{M_t^2}\sqrt{\frac{2}{\pi x}}(\operatorname{sgn}X_t)\left\{\frac{|X_t|}{\sqrt{(x-A_t)_+}}\Phi'\left(\frac{|X_t|}{\sqrt{(x-A_t)_+}}\right)\right.\\&\quad\left.-\Phi\left(\frac{|X_t|}{\sqrt{(x-A_t)_+}}\right)\right\}\mathrm{d}X_t.\end{aligned} \tag{2.48}$$

Hence, with the help of (2.44):

$$\begin{aligned}\mathrm{d}\langle Z,\beta\rangle_t&=\frac{1}{M_t^2}\sqrt{\frac{2}{\pi x}}(\operatorname{sgn}X_t)\left\{\frac{|X_t|}{\sqrt{(x-A_t)_+}}\Phi'\left(\frac{|X_t|}{\sqrt{(x-A_t)_+}}\right)\right.\\&\quad\left.-\Phi\left(\frac{|X_t|}{\sqrt{(x-A_t)_+}}\right)\right\}\mathrm{d}t.\end{aligned} \tag{2.49}$$

On the other hand, from Lemma 2.5, (2.3) and (2.4), we have:

$$\frac{1}{1-Z_t} = \frac{M_t}{M_t - \Phi(|X_t|/\sqrt{(x-A_t)_+})} = \sqrt{\frac{\pi x}{2}}\frac{M_t}{|X_t|}. \tag{2.50}$$

Since $X_g = 0$, plugging now (2.49) and (2.50) into (2.47) leads, after simplification:

$$\begin{aligned} X_{t+g} &= \widetilde{\beta}_{t+g} - \widetilde{\beta}_g + \int_0^t \frac{\operatorname{sgn} X_{s+g}}{M_{s+g}}\left(\sqrt{\frac{2}{\pi x}} + \frac{1}{|X_{s+g}|}\Phi\left(\frac{|X_{s+g}|}{\sqrt{(x-A_{s+g})_+}}\right)\right)\mathrm{d}s \\ &= \widetilde{\beta}_{t+g} - \widetilde{\beta}_g + \int_0^t \frac{\operatorname{sgn} X_{s+g}}{|X_{s+g}|}\,\mathrm{d}s \quad \big(\text{from (2.3) and (2.4)}\big). \end{aligned} \tag{2.51}$$

It now remains to note that $\operatorname{sgn}(X_{s+g})$ is constant under Q, and that:

$$Q\big(\operatorname{sgn}(X_{s+g}) = 1, \forall s > 0\big) = Q\big(\operatorname{sgn}(X_{s+g}) = -1, \forall s > 0\big) = \frac{1}{2}.$$

We also note that the independence of $(X_u, u \le g)$ and $(X_{t+g}, t \ge 0)$ follows classically from the fact that the modification of equation (2.51) written for $|X_{t+g}|$, $t \ge 0$, admits only one strong solution.

(12) *We now describe the law of $(X_u, u \le g)$ under Q.*

From (2.41), we deduce that for any $\mathbb{R}_+$-valued predictable process $(F(X_u, u \le s), s \ge 0)$, and any Borel function $h \ge 0$:

$$\begin{aligned} &E_Q\big[F(X_u, u \le g)h(L_g)\big] \\ &\quad = \sqrt{\frac{2}{\pi x}} E\left[\int_0^\infty F(X_u, u \le s)h(L_s)1_{\{A_s^* \le x\}}\,\mathrm{d}L_s\right] \\ &\quad = \sqrt{\frac{2}{\pi x}} E\left[\int_0^\infty F(X_u, u \le \tau_\ell)h(\ell)1_{\{A_{\tau_\ell}^* \le x\}}\,\mathrm{d}\ell\right] \\ &\qquad \text{with } \tau_\ell = \inf\{s \ge 0\colon L_s > \ell\}. \end{aligned} \tag{2.52}$$

Taking $F = 1$ in (2.52), we deduce that the density function of $L_g = L_\infty$ under Q is

$$\sqrt{\frac{2}{\pi x}} P\big(A_{\tau_\ell}^* \le x\big).$$

Consequently:

$$E_Q\big[F(X_u, u \le g)h(L_g)\big] = \int_0^\infty E\big[F(X_u, u \le \tau_\ell)|\big(A_{\tau_\ell}^* \le x\big)\big]h(\ell)Q(L_\infty \in \mathrm{d}\ell).$$

Thus, the law under Q of $(X_u, u \le g)$, conditioned on $\{L_\infty = \ell\}$, is that of Brownian motion stopped at τ_ℓ, and conditioned by the event $\{A_{\tau_\ell}^* \le x\} = \{\Sigma_{\tau_\ell} \le x\}$.

This ends up the proof of Theorem 2.1. □

Remark 2.6. *The technique of enlargement of filtration, applied before g (see (2.44) and (2.45), with $t \le g$), yields:*

$$\begin{aligned} X_t &= \widetilde{\beta}_t + \int_0^{g\wedge t} \frac{\mathrm{d}\langle Z, \beta\rangle_u}{Z_u} \\ &\quad + \int_0^{g\wedge t}\left\{\sqrt{\frac{2}{\pi x}} + \frac{1}{\sqrt{(x-A_u)_+}}\Phi'\left(\frac{|X_u|}{\sqrt{(x-A_u)_+}}\right)\right\}\frac{\operatorname{sgn} X_u}{M_u}\,\mathrm{d}u. \end{aligned}$$

Hence with (2.49) *and* (2.3)–(2.4), *we obtain*:

$$X_t = \widetilde{\beta}_t + \int_0^{g\wedge t} \frac{1}{\sqrt{(x-A_u)_+}} \frac{\Phi'}{\Phi}\left(\frac{|X_u|}{\sqrt{(x-A_u)_+}}\right) \operatorname{sgn} X_u \, \mathrm{d}u. \tag{2.53}$$

We note that, in this equation, the drift term tends to $-\infty$ (*resp*: $+\infty$) *when* $u \to T_x^A$, *with* $X_u > 0$ (*resp*: $X_u < 0$).

(2.53) *shows that* $((X_t, A_t), t \le g)$ *is Markov. We would like to point out that the theory of enlargement of filtrations is very helpful here since it permits to determine the law of* $((X_t, A_t), t \le g)$ – *as expressed in Theorem* 2.1, *point* (4.c) – *although it seems rather difficult to do it only from* (2.53).

3. Penalisation with a function of Σ_t

The aim of this section is to extend the results of the preceding section, by replacing the penalisation functional $1_{\{\Sigma_t \le x\}}$ by a functional of the form $h(\Sigma_t)$. We shall use the following notation: let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function which is almost surely differentiable and such that:

$$\int_0^\infty \psi(x)\,\mathrm{d}x = 1.$$

In particular, ψ is a probability density on $\mathbb{R}_+$. We also introduce:

$$h(x) := 2\sqrt{x}\,\psi(x) \quad (x \ge 0), \tag{3.1}$$

$$\Psi_1(x) := \int_0^x \psi(y)\,\mathrm{d}y \quad (x \ge 0), \tag{3.2}$$

$$h_1(x) := 2x\psi(x) + 1 - \Psi_1(x) \quad (x \ge 0). \tag{3.3}$$

We assume that:

$$x\psi(x) \underset{x\to 0}{\longrightarrow} 0 \tag{3.4}$$

as well as:

$$x\psi(x) \underset{x\to \infty}{\longrightarrow} 0 \tag{3.5}$$

and

$$\int_0^\infty x\left|\psi'(x)\right|\mathrm{d}x < \infty. \tag{3.6}$$

We shall now take $h(\Sigma_t)$ as our penalisation functional. Note that Theorem 2.1 corresponds to the following choice of ψ:

$$\psi(u) = \frac{1}{2\sqrt{ux}} 1_{[0,x]}(u) \quad (x \text{ fixed}). \tag{3.7}$$

Remark 3.1. *We note that the definition and assumptions* (3.1) *to* (3.6) *imply that*:

$$\int_0^\infty \sqrt{x}\left|h'(x)\right|\mathrm{d}x < \infty, \tag{3.8}$$

$$h_1(x) = -\int_x^\infty \sqrt{y}\,h'(y)\,\mathrm{d}y \tag{3.9}$$

and

$$h_1(0) = 1. \tag{3.10}$$

Indeed:

$$-\int_x^\infty \sqrt{y}h'(y)\,\mathrm{d}y = -\big[2y\psi(y)\big]_x^\infty + \int_x^\infty \psi(y)\,\mathrm{d}y$$
$$= 2x\psi(x) + 1 - \Psi_1(x).$$

Theorem 3.2.

1. *For any* $s \ge 0$, *and every* $\Lambda_s \in \mathcal{F}_s$:

$$\lim_{t\to\infty} \frac{E[1_{\Lambda_s} h(\Sigma_t)]}{E[h(\Sigma_t)]} \quad \textit{exists}. \tag{3.11}$$

2. *This limit is equal to*:

$$Q^\psi(\Lambda_s) := E\big[1_{\Lambda_s} M_s^\psi\big] \tag{3.12}$$

with:

$$M_s^\psi = \sqrt{\frac{2}{\pi}} h(\Sigma_s)|X_s| + h_1(\Sigma_s)\Phi\left(\frac{|X_s|}{\sqrt{(\Sigma_s - A_s)_+}}\right)$$
$$+ \sqrt{\frac{2}{\pi}} \int_0^{|X_s|/\sqrt{(\Sigma_s - A_s)_+}} h_1\left(A_s + \frac{X_s^2}{v^2}\right) \mathrm{e}^{-v^2/2}\,\mathrm{d}v; \tag{3.13}$$

the function Φ *has been defined by* (1.12) *and the convention* (2.21) *holds.*

Moreover, $(M_s^\psi, s \ge 0)$ *is a continuous positive martingale such that* $M_0^\psi = 1$.

3. *The formula* (3.12) *induces a probability on* $(\Omega, \mathcal{F}_\infty)$.
4. *Under* Q^ψ, *the canonical process* $(X_t, t \ge 0)$ *satisfies*:

(a) Σ_∞ *is finite a.s., and admits* ψ *as its density.*
(b) A_∞^* *is a.s. infinite.*
(c) *Let* $g := \sup\{s\colon X_s = 0\}$. *Then*: $Q^\psi(0 < g < \infty) = 1$, *and for every* $t \ge 0$:

$$Q^\psi(g > t) = -\int_0^\infty h'(x) E\Big[\sqrt{A_{t\wedge T_x^A}}\Big]\,\mathrm{d}x. \tag{3.14}$$

(d) *The processes* $(X_t, t \le g)$ *and* $(X_{g+t}, t \ge 0)$ *are independent.*
(e) *With probability* $1/2$, $(X_{g+t}, t \ge 0)$ *is positive* (*resp. negative*), *and* $(|X_{g+t}|, t \ge 0)$ *is the* 3-*dimensional Bessel process starting from* 0.
(f) *Conditionally on* $L_\infty (= L_g) = \ell$, *and* $\Sigma_{\tau_\ell} \le x$, *the law of* $(X_u, u \le g)$ *is that of Brownian motion* B *considered until* τ_ℓ, *i.e.*: $(B_u, u \le \tau_\ell)$, *and conditioned on* $\Sigma_{\tau_\ell} \le x$.

Remark 3.3. *It seems*, *when first looking at the definition of* M_s^ψ (*see* (3.13)) *that there might be some ambiguity concerning the definition of* M_s^ψ *at the times* s, *which are ends of excursions*; *but, a careful inspection shows that there is no ambiguity.*

Remark 3.4. (1) *The martingale* $(M_s^\psi, s \ge 0)$ *defined by* (3.13) *is equal to*:

$$M_s^\psi = -\int_0^\infty \sqrt{x}h'(x)M_s^x\,\mathrm{d}x, \tag{3.15}$$

where $(M_s^x, s \geq 0)$ *is the martingale defined in Theorem* 2.1 *by* (2.3) (*see also Remark* 2.2).

Indeed, *we have*:

$$-\int_0^\infty \sqrt{x} h'(x) M_s^x \,\mathrm{d}x = (1) + (2), \tag{3.16}$$

with:

$$(1) = -\sqrt{\frac{2}{\pi}} |X_s| \int_{\Sigma_s}^\infty h'(x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}} h(\Sigma_s)|X_s| \quad \bigl(\textit{from } (3.5)\bigr) \tag{3.17}$$

and:

$$\begin{aligned}
(2) &= -\int_{\Sigma_s}^\infty 1_{\{A_s < x\}} \Phi\left(\frac{|X_s|}{\sqrt{x - A_s}}\right) \sqrt{x} h'(x)\,\mathrm{d}x \\
&= -\sqrt{\frac{2}{\pi}} \int_{\Sigma_s \vee A_s}^\infty \mathrm{d}x \sqrt{x} h'(x) \int_{|X_s|/\sqrt{x - A_s}}^\infty \mathrm{e}^{-v^2/2}\,\mathrm{d}v \\
&= -\sqrt{\frac{2}{\pi}} \int_0^\infty \mathrm{e}^{-v^2/2}\,\mathrm{d}v \int_{\Sigma_s \vee (A_s + X_s^2/v^2)} \sqrt{x} h'(x)\,\mathrm{d}x \quad (\textit{by Fubini}) \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty \mathrm{e}^{-v^2/2}\,\mathrm{d}v\, h_1\left(\Sigma_s \vee \left\{A_s + \frac{X_s^2}{v^2}\right\}\right)\mathrm{d}v \quad \bigl(\textit{from } (3.9)\bigr) \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty h_1(\Sigma_s) 1_{\{v \geq |X_s|/\sqrt{(\Sigma_s - A_s)_+}\}} \mathrm{e}^{-v^2/2}\,\mathrm{d}v \\
&\quad + \sqrt{\frac{2}{\pi}} \int_0^\infty h_1\left(A_s + \frac{X_s^2}{v^2}\right) 1_{\{v < |X_s|/\sqrt{(\Sigma_s - A_s)_+}\}} \mathrm{e}^{-v^2/2}\,\mathrm{d}v \\
&= h_1(\Sigma_s)\Phi\left(\frac{|X_s|}{\sqrt{(\Sigma_s - A_s)_+}}\right) \\
&\quad + \sqrt{\frac{2}{\pi}} \int_0^{|X_s|/\sqrt{(\Sigma_s - A_s)_+}} h_1\left(A_s + \frac{X_s^2}{v^2}\right)\mathrm{e}^{-v^2/2}\,\mathrm{d}v.
\end{aligned} \tag{3.18}$$

Thus, *the validity of relation* (3.15) *now follows immediately from the conjunction of* (3.17) *and* (3.18).

(2) *From* (3.15), *we get*:

$$Q^\psi(\Lambda) = -\int Q^{(x)}(\Lambda)\sqrt{x} h'(x)\,\mathrm{d}x \quad (\Lambda \in \mathcal{F}_\infty) \tag{3.19}$$

with $Q^{(x)}(\Lambda)$ *the probability measure defined via* (2.2) (*see Remark* 2.2). *The reader may have been surprised to find that*, *in the disintegration formula* (3.15) *of the martingale* M^ψ, *the representing measure* $(-\sqrt{x} h'(x)\,\mathrm{d}x)$ *is not in general positive*; *nonetheless*, *it has total integral equal to* 1, *i.e.*:

$$-\int_0^\infty \sqrt{x} h'(x)\,\mathrm{d}x = 1. \tag{3.20}$$

We shall discuss the positivity of $Q^{(x)}(\Lambda)$ *and the martingale* M^ψ *in Theorem* 3.5 *below.*

Proof of Theorem 3.2. (1) To prove this theorem, it suffices, thanks to Remark 3.4, to prove point (1) of the theorem (which we shall do in the second step in this proof). From now, we use the representation (3.19) to compute

$Q^\psi(\Sigma_\infty > a)$:

$$Q^\psi(\Sigma_\infty > a) = -\int_0^\infty \sqrt{x}h'(x)Q^{(x)}(\Sigma_\infty > a)\,\mathrm{d}x$$

$$= -\int_0^\infty \sqrt{x}h'(x)Q^{(x)}\left(\sqrt{\frac{\Sigma_\infty}{x}} > \sqrt{\frac{a}{x}}\right)\mathrm{d}x$$

$$= -\int_0^\infty \sqrt{x}h'(x)1_{\{a<x\}}\left(1-\sqrt{\frac{a}{x}}\right)\mathrm{d}x \tag{3.21}$$

(from point (3.a) in Theorem 2.1)

$$= -\int_a^\infty h'(x)\big(\sqrt{x}-\sqrt{a}\big)\,\mathrm{d}x. \tag{3.22}$$

Hence, Σ_∞ is a finite r.v. under Q^ψ with density function:

$$-\frac{1}{2\sqrt{a}}\int_a^\infty h'(x)\,\mathrm{d}x = \frac{1}{2\sqrt{a}}h(a) = \psi(a), \tag{3.23}$$

from (3.5) and (3.1). □

(2) *We now show point (1) in Theorem 3.2.*

(a) We first estimate $E[h(\Sigma_t)]$. We have:

$$E\big[h(\Sigma_t)\big] = -E\left[\int_{\Sigma_t}^\infty h'(u)\,\mathrm{d}u\right] \quad \big(\text{since } h(\infty)=0\big)$$

$$= -\int_0^\infty h'(u)P(\Sigma_t < u)\,\mathrm{d}u$$

$$= -\int_0^\infty h'(u)P(t\Sigma_1 < u)\,\mathrm{d}u$$

$$= -\frac{1}{\sqrt{t}}\int_0^\infty h'(u)\rho\left(\frac{u}{t}\right)\sqrt{u}\,\mathrm{d}u, \tag{3.24}$$

where $\rho(y) = \dfrac{P(\Sigma_1 < y)}{\sqrt{y}}$, $y > 0$, and we have used the scaling property:

$$\Sigma_t \overset{(d)}{=} t\Sigma_1 \quad (\text{under } P).$$

From (2.9), $\lim_{y\to 0}\rho(y) = 1$ and ρ is bounded on $]0,\infty[$. The dominated convergence theorem implies that:

$$E\big[h(\Sigma_t)\big] \underset{t\to\infty}{\sim} -\frac{1}{\sqrt{t}}\int_0^\infty h'(u)\sqrt{u}\,\mathrm{d}u = \frac{h_1(0)}{\sqrt{t}} = \frac{1}{\sqrt{t}} \quad \big(\text{from (3.10)}\big). \tag{3.25}$$

(b) We now write:

$$E\big[1_{\Lambda_s}h(\Sigma_t)\big] = E\big[1_{\Lambda_s}E\big(h(\Sigma_t)|\mathcal{F}_s\big)\big].$$

In order to estimate:

$$\Delta := E\big[h(\Sigma_t)|\mathcal{F}_s\big] \quad (s<t) \tag{3.26}$$

two cases need to be studied:

(b.i): $d_s = s + T_0\circ\theta_s > t$ and (b.ii): $s + T_0\circ\theta_s < t$.

We then write: $\Delta = \Delta_1 + \Delta_2$ with:

$$\Delta_1 := E\big[h(\Sigma_t)1_{\{s+T_0\circ\theta_s>t\}}|\mathcal{F}_s\big], \qquad \Delta_2 := E\big[h(\Sigma_t)1_{\{s+T_0\circ\theta_s<t\}}|\mathcal{F}_s\big],$$

which we study separately.

(b.i) On the set: $\{s+T_0\circ\theta_s>t\}$, one has: $\Sigma_s=\Sigma_t$; hence relation (2.16) implies:

$$\Delta_1 = h(\Sigma_s)P_{|X_s|}(T_0>t-s) \underset{t\to\infty}{\sim} h(\Sigma_s)\sqrt{\frac{2}{\pi}}\frac{|X_s|}{\sqrt{t}}. \tag{3.27}$$

(b.ii) On $\{s+T_0\circ\theta_s<t\}=\{d_s<t\}$, the r.v. Σ_t may be decomposed as follows: $\Sigma_t=\widetilde{\Sigma}\vee\Sigma'$, with $\widetilde{\Sigma}:=\Sigma_s\vee(A_s+d_s-s)$ and $\Sigma':=\Sigma_{t-d_s}\circ\theta_{d_s}$.

We first take the conditional expectation with respect to $\mathcal{F}_{d_s}$:

$$E\big[h(\Sigma_t)1_{\{d_s<t\}}|\mathcal{F}_{d_s}\big] = H(\widetilde{\Sigma},t-d_s)1_{\{d_s<t\}} \tag{3.28}$$

with

$$H(u,v)=E\big[h(u\vee\Sigma_v)\big], \quad u,v\geq 0. \tag{3.29}$$

Let $\widetilde{h}(y):=h(u\vee y)$, $y\geq 0$ with $u>0$ fixed. Then

$$\widetilde{h_1}(x) := -\int_x^\infty \sqrt{y}(\widetilde{h})'(y)\,\mathrm{d}y = -\int_{x\vee u}^\infty \sqrt{y}h'(y)\,\mathrm{d}y.$$

Using (3.25) we get, with the help of (3.9):

$$H(u,v) \underset{v\to\infty}{\sim} \frac{1}{\sqrt{v}}\widetilde{h_1}(0) = -\frac{1}{\sqrt{v}}\int_u^\infty \sqrt{y}h'(y)\,\mathrm{d}y = \frac{h_1(u)}{\sqrt{v}}. \tag{3.30}$$

Then we have successively:

$$\begin{aligned}
&E\big[h(\Sigma_t)1_{\{d_s<t\}}|\mathcal{F}_{d_s}\big] \underset{t\to\infty}{\sim} \frac{h_1(\widetilde{\Sigma})}{\sqrt{t}}1_{\{d_s<t\}},\\
&\Delta_2 \underset{t\to\infty}{\sim} \frac{1}{\sqrt{t}}E\big[h_1\big(\Sigma_s\vee(A_s+d_s-s)\big)1_{\{d_s<t\}}|\mathcal{F}_s\big]\\
&\quad = \frac{1}{\sqrt{t}}E\big[h_1\big(\Sigma_s\vee(A_s+T_0\circ\theta_s)\big)1_{\{s+T_0\circ\theta_s<t\}}|\mathcal{F}_s\big].
\end{aligned}$$

According to the well-known identity:

$$P_y(T_0\in\mathrm{d}u) = \frac{y}{\sqrt{2\pi u^3}}\exp\left\{-\frac{y^2}{2u}\right\}1_{\{u>0\}}\,\mathrm{d}u, \tag{3.31}$$

we deduce:

$$\Delta_2 \underset{t\to\infty}{\sim} \frac{1}{\sqrt{t}}\int_0^\infty h_1\big(\Sigma_s\vee(A_s+u)\big)\frac{|X_s|}{\sqrt{2\pi u^3}}\mathrm{e}^{-X_s^2/(2u)}\,\mathrm{d}u. \tag{3.32}$$

We then write the integral on the RHS, as:

$$\begin{aligned}
&\int_0^{(\Sigma_s-A_s)_+} h_1(\Sigma_s)\frac{|X_s|}{\sqrt{2\pi u^3}}\mathrm{e}^{-X_s^2/(2u)}\,\mathrm{d}u + \int_{(\Sigma_s-A_s)_+}^\infty h_1(A_s+u)\frac{|X_s|}{\sqrt{2\pi u^3}}\mathrm{e}^{-X_s^2/(2u)}\,\mathrm{d}u\\
&\quad = h_1(\Sigma_s)\Phi\left(\frac{|X_s|}{\sqrt{(\Sigma_s-A_s)_+}}\right) + \sqrt{\frac{2}{\pi}}\int_0^{|X_s|/\sqrt{(\Sigma_s-A_s)_+}} h_1\left(A_s+\frac{X_s^2}{v^2}\right)\mathrm{e}^{-v^2/2}\,\mathrm{d}v
\end{aligned} \tag{3.33}$$

(after the change of variables $\frac{X_s^2}{u} = v^2$).

Thus, we finally obtain:

$$\frac{E[h(\Sigma_t)|\mathcal{F}_s]}{E[h(\Sigma_t)]} = \frac{\Delta_1 + \Delta_2}{E[h(\Sigma_t)]} \underset{t\to\infty}{\sim} M_s^{\psi},$$

from (3.13), (3.27), (3.32), (3.33) and (3.25).

(3) *End of the proof of Theorem* 3.2.

We observe that identity (3.15) of Remark 3.4 and (3.12) imply item (3) of Theorem 3.2. We claim that point (4) of Theorem 3.2 may be directly deduced from this property and Theorem 2.1. Indeed as an illustration we prove that $(X_t, t \le g)$ and $(X_{g+t}, t \ge 0)$ are independent under Q^{ψ}.

Let F_1 and F_2 denote two positive functionals. Then:

$$\begin{aligned} &E_{Q^{\psi}}\big[F_1(X_t, t \le g)F_2(X_{g+t}, t \ge 0)\big] \\ &\quad = -\int_0^{\infty} \sqrt{x}h'(x)\,\mathrm{d}x\, E_{Q^{(x)}}\big[F_1(X_t, t \le g)F_2(X_{g+t}, t \ge 0)\big] \\ &\quad = -\int_0^{\infty} \sqrt{x}h'(x)\,\mathrm{d}x\, E_{Q^{(x)}}\big[F_1(X_t, t \le g)\big]E_{Q^{(x)}}\big[F_2(X_{g+t}, t \ge 0)\big] \end{aligned}$$

from Theorem 2.1, (4.a).

But, from Theorem 2.1, (4.b), the law of $(X_{g+t}, t \ge 0)$, under $Q^{(x)}$, does not depend on x, hence it is equal to its law under Q^{ψ} since $-\sqrt{x}h'(x)\,\mathrm{d}x$ is a finite measure on $[0, \infty)$, whose integral is equal to 1.

Consequently, we deduce from the preceding identity that:

$$\begin{aligned} &E_{Q^{\psi}}\big[F_1(X_t, t \le g)F_2(X_{g+t}, t \ge 0)\big] \\ &\quad = E_{Q^{\psi}}\big[F_1(X_t, t \le g)\big]E_{Q^{\psi}}\big[F_2(X_{g+t}, t \ge 0)\big]. \end{aligned}$$

We now take care of the drawback of positivity of Q^{ψ} and M^{ψ} by "changing the parametrisation." Indeed, as in the proof of Theorem 3.5 below, it is more convenient to write the penalisation process $h(\Sigma_t)$ as $h_0(\sqrt{\Sigma_t})$. Note that in the context of Theorem 2.1, which is a particular case of Theorem 3.2, the distribution of $\sqrt{\Sigma_\infty}$ is simpler than that of Σ_∞ since it is equal to a uniform distribution.

This leads us to present more naturally point (3) of Theorem 3.2.

Theorem 3.5. *The hypotheses and notation are those found in Theorem* 3.2. *Consider, for any probability density* ψ *on* $\mathbb{R}_+$, *the disintegration of* Q^{ψ} *with respect to the random variable* Σ_∞, *which admits the density* ψ:

$$Q^{\psi}(\Lambda) = \int_0^{\infty} Q^{\psi}(\Lambda|\Sigma_\infty = y)\psi(y)\,\mathrm{d}y \qquad (\Lambda \in \mathcal{F}_\infty). \tag{3.34}$$

Then: $\mathrm{d}y$ *a.e.*, $Q^{\psi}(\Lambda|\Sigma_\infty = y)$ *does not depend on* ψ.

Thus, if one defines, for $y > 0$,

$$\widehat{Q}^{(y)}(\Lambda) := Q^{\psi_0}(\Lambda|\Sigma_\infty = y) \tag{3.35}$$

for some probability density $\psi_0(y) > 0$ *everywhere, one obtains*:

$$Q^{\psi}(\Lambda) = \int_0^{\infty} \widehat{Q}^{(y)}(\Lambda)\psi(y)\,\mathrm{d}y \quad (\Lambda \in \mathcal{F}_\infty), \tag{3.36}$$

$$\widehat{Q}^{(y)}(\Sigma_\infty = y) = 1 \quad \forall y \ge 0, \tag{3.37}$$

and, furthermore:

$$Q^{(x)} = \frac{1}{\sqrt{x}} \int_0^{\infty} \frac{\mathrm{d}y}{2\sqrt{y}} \widehat{Q}^{(y)}. \tag{3.38}$$

Proof. (1) In order to take into account the penalisation by $\sqrt{\Sigma_t}$, we set:

$$h(x) = h_0\big(\sqrt{x}\big), \quad x \geq 0. \tag{3.39}$$

Consequently the penalisation process $h(\Sigma_t)$ equals $h_0(\sqrt{\Sigma_t})$.

It is clear that our assumptions and notation related to h and ψ may be interpreted in terms of h_0 in the following way:

$$\psi(x) = \frac{1}{2\sqrt{x}} h_0\big(\sqrt{x}\big), \quad x \geq 0, \tag{3.40}$$

$$\int_0^\infty h_0(x)\,\mathrm{d}x = 1, \tag{3.41}$$

$$x h_0(x) \to 0, \quad \text{as } x \to 0 \text{ or } x \to \infty, \tag{3.42}$$

$$\int_0^\infty y\big|h_0'(y)\big|\,\mathrm{d}y < \infty. \tag{3.43}$$

Consequently:

$$\begin{aligned} Q^{\psi}(\Lambda) &= -\int_0^\infty \sqrt{x} h'(x) Q^{(x)}(\Lambda)\,\mathrm{d}x = -\frac{1}{2}\int_0^\infty h_0'\big(\sqrt{x}\big) Q^{(x)}(\Lambda)\,\mathrm{d}x \\ &= -\int_0^\infty h_0'(y) y Q^{(y^2)}(\Lambda)\,\mathrm{d}y, \end{aligned} \tag{3.44}$$

for any $\Lambda \in \mathcal{F}_\infty$.

Let $k : \mathbb{R}_+ \to \mathbb{R}_+$, with compact support and of C^∞ class; applying (3.44) with $h_0 = k/c$ and $c := \int_0^\infty k(y)\,\mathrm{d}y$, we get:

$$-\int_0^\infty k'(y) y Q^{(y^2)}(\Lambda)\,\mathrm{d}y = \left(\int_0^\infty k(y)\,\mathrm{d}y\right) Q^{\psi}(\Lambda), \tag{3.45}$$

$$\int_0^\infty k(y)\,\mathrm{d}\big(y Q^{(y^2)}(\Lambda)\big) = \left(\int_0^\infty k(y)\,\mathrm{d}y\right) Q^{\psi}(\Lambda), \tag{3.46}$$

where $\mathrm{d}(y Q^{(y^2)}(\Lambda))$ denotes the differential in the distribution sense of $y Q^{(y^2)}(\Lambda)$.

The relation (3.46) implies that $\mathrm{d}(y Q^{(y^2)}(\Lambda))$ is a non-negative measure which is absolutely continuous with respect to the Lebesgue measure. We set:

$$\widehat{Q}^{(y^2)}(\Lambda) := \frac{\mathrm{d}}{\mathrm{d}y}\big(y Q^{(y^2)}(\Lambda)\big). \tag{3.47}$$

We observe that, in this definition, $\widehat{Q}^{(y^2)}(\Lambda)$ is only defined a.e. in y. But, it follows from [4] that the quasi-kernel $(\Lambda, y) \to \widehat{Q}^{(y^2)}(\Lambda)$ may be "regularized" as a kernel, so that the definition (3.47) holds for every Λ, a.e.

(2) Coming back to (3.44) and using (3.47) and (3.40) we obtain:

$$\begin{aligned} Q^{\psi}(\Lambda) &= \int_0^\infty h_0(y)\widehat{Q}^{(y^2)}(\Lambda)\,\mathrm{d}y = 2\int_0^\infty y\psi\big(y^2\big)\widehat{Q}^{(y^2)}(\Lambda)\,\mathrm{d}y \\ &= \int_0^\infty \psi(z)\widehat{Q}^{(z)}(\Lambda)\,\mathrm{d}z. \end{aligned} \tag{3.48}$$

Let $x>0$ fixed and $\psi_x(y) := \frac{1}{2\sqrt{xy}} 1_{[0,x]}(y)$. Then

$$\begin{aligned} Q^{\psi_x}(\Lambda) &= -\int_0^\infty \sqrt{y} h_x'(y) Q^{(y)}(\Lambda)\,\mathrm{d}y \\ &= -\int_0^\infty \sqrt{y}\frac{\mathrm{d}}{\mathrm{d}y}\big(2\sqrt{y}\psi_x(y)\big) Q^{(y)}(\Lambda)\,\mathrm{d}y \\ &= Q^{(x)}(\Lambda). \end{aligned}$$

Using (3.48) we obtain:

$$\sqrt{x} Q^{(x)}(\Lambda) = \int_0^x \widehat{Q}^{(y)}(\Lambda)\frac{\mathrm{d}y}{2\sqrt{y}}, \quad x>0. \tag{3.49}$$

(3) *Let us prove that:* $\widehat{Q}^{(x)}(\Sigma_\infty = x) = 1$.

Recall that, from item (3.a) of Theorem 2.1, we have:

$$Q^{(x)}(\Sigma_\infty > a) = 1 - \sqrt{\frac{a}{x}}, \quad \forall a \in]0,x]. \tag{3.50}$$

Hence, for any $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, Borel, we have:

$$\sqrt{x} Q^{(x)}\big[\varphi(\Sigma_\infty)\big] = \int_0^x \frac{\mathrm{d}y}{2\sqrt{y}}\varphi(y).$$

Comparing with (3.49), we deduce: $\widehat{Q}^{(y)}[\varphi(\Sigma_\infty)] = \varphi(y)$, $\mathrm{d}y$ a.e. and it follows that $\widehat{Q}^{(y)}(\Sigma_\infty = y) = 1$, $\mathrm{d}y$ a.e.

(4) We may now conclude. Indeed, for ψ a probability density, one has

$$Q^\psi(\Lambda) = \int_0^\infty Q^\psi(\Lambda | \Sigma_\infty = y)\psi(y)\,\mathrm{d}y \tag{3.51}$$

and, from (3.48):

$$Q^\psi(\Lambda) = \int_0^\infty \widehat{Q}^{(y)}(\Lambda)\psi(y)\,\mathrm{d}y. \tag{3.52}$$

Since $\widehat{Q}^{(x)}$ charges only $\{\Sigma_\infty = x\}$, we deduce from (3.51) and (3.52) that:

$$Q^\psi(\Lambda | \Sigma_\infty = y) = \widehat{Q}^{(y)}(\Lambda). \tag{3.53}$$

This ends the proof of Theorem 3.5. □

4. Penalisation by $1_{\{A_t^* \le x\}}$

Let $x>0$ be fixed. The aim of this section is to study the penalisation of Wiener measure with the functional $\Gamma_t = 1_{(A_t^* \le x)}$.

In fact, because of the non-availability of a Tauberian argument in this case, we need to make a conjecture (see (4.5)), which, if valid, implies the existence of the penalised probability (see Theorem 4.3).

We recall that:

$$A_t^* = \sup_{s \le t} A_s = \Sigma_t \vee (t - g_t) = \Sigma_t \vee A_t. \tag{4.1}$$

To prepare for the statement of our conjecture, let, for $\lambda \geq 0$:

$$\theta(\lambda) = \lambda \mathrm{e}^{-\lambda} \int_0^1 \frac{\mathrm{e}^{\lambda z}\,\mathrm{d}z}{\sqrt{z}}. \tag{4.2}$$

From [8], p. 266, formula (9.11.1), the function θ defined above may be expressed in terms of a confluent hypergeometric function, namely:

$$\theta(\lambda) = 2\lambda \mathrm{e}^{-\lambda} \Phi\left(\frac{1}{2}, \frac{3}{2}, \lambda\right). \tag{4.3}$$

We begin with the following lemma.

Lemma 4.1. *The function $1 - \theta(\lambda)$ admits a first strictly positive zero λ_0.*

Proof. Setting $z = 1 - \frac{u}{\lambda}$ in (4.2) and integrating by parts, we get:

$$\theta(\lambda) = 2\lambda\left(1 - \int_0^\lambda \sqrt{1 - \frac{u}{\lambda}}\,\mathrm{e}^{-u}\,\mathrm{d}u\right).$$

From the asymptotic expansion:

$$\sqrt{1 - \frac{u}{\lambda}} = 1 - \frac{u}{2\lambda} - \frac{u^2}{8\lambda^2} + \mathrm{o}\left(\frac{1}{\lambda^2}\right) \quad (u \text{ fixed}),$$

it is easy to deduce:

$$\theta(\lambda) = 1 + \frac{1}{2\lambda} + \mathrm{o}\left(\frac{1}{\lambda}\right) \quad (\lambda \to \infty).$$

As $1 - \theta(0) = 1$, the lemma is proven. □

Before we state the conjecture, we note that, from the scaling property of Brownian motion:

$$T_y^A \overset{\text{(law)}}{=} y T_1^A, \qquad A_t^* \overset{\text{(law)}}{=} t A_1^*, \qquad T_1^A \overset{\text{(law)}}{=} \frac{1}{A_1^*}. \tag{4.4}$$

Conjecture. *There exists a constant C, such that*:

$$P\left(T_1^A \geq \frac{t}{x}\right) = P\left(A_t^* \leq x\right) \underset{t \to \infty}{\sim} C \mathrm{e}^{-\lambda_0 t/x}, \tag{4.5}$$

where λ_0 denotes the first positive zero of $1 - \theta(\cdot)$, as defined in Lemma 4.1.

Remark 4.2. *We note that Theorem 3.1 of Hu–Shi [5] which contains – among other results – an asymptotic estimate of $P(A_1^* \leq x)$ as $x \to 0$ is in agreement with our conjecture. However, Hu–Shi use a refinement of the Tauberian theorem (their Theorem 3.2) which, in all rigor, does not imply the equivalence (4.5) above.*

We are now in a position to state the following theorem.

Theorem 4.3. *Let $x > 0$ be fixed, and $\lambda_0 > 0$ be defined as in Lemma 4.1.*

(1) *Assuming the validity of the conjecture (4.5),*

(a) *for every* $s > 0$, *and* $\Lambda_s \in \mathcal{F}_s$, *the limit*:

$$Q^*(\Lambda_s) := \lim_{t\to\infty} \frac{E[1_{\Lambda_s} 1_{(A_t^* \le x)}]}{P(A_t^* \le x)} \quad \textit{exists}. \tag{4.6}$$

(b) *This limit is equal to*:

$$Q^*(\Lambda_s) := E\big[1_{\Lambda_s} M_s^*\big], \tag{4.7}$$

where:

$$M_s^* = \mathrm{e}^{\lambda_0 s/x} 1_{(A_s^* \le x)} \int_0^{x-A_s} \frac{|X_s|}{\sqrt{2\pi u^3}} \mathrm{e}^{-\{X_s^2/(2u)\}+\lambda_0 u/x} \,\mathrm{d}u. \tag{4.8}$$

(2) (a) $(M_s^*, s \ge 0)$ *is a positive continuous martingale such that* $M_0^* = 1$.
(b) Q^* *induces a probability measure on* $(\Omega, \mathcal{F}_\infty)$, *such that*, *under* Q^*, *the process* $(X_t, t \ge 0)$ *satisfies*:

$$A_\infty^* = x \quad \textit{a.s.} \qquad \textit{and} \qquad T_x^A = \inf\{t\colon A_t = x\} = \infty \quad \textit{a.s.} \tag{4.9}$$

We begin with a preliminary result.

Lemma 4.4. *For every* $\lambda \ge 0$ *and* $y > 0$:

$$\lambda \int_0^\infty \mathrm{e}^{-\lambda t} P\big(A_t^* \le y\big) \,\mathrm{d}t = \frac{\mathrm{e}^{\lambda y} \lambda \sqrt{y} \int_0^y \frac{\mathrm{e}^{-\lambda v}\,\mathrm{d}v}{\sqrt{v}}}{1 + \mathrm{e}^{\lambda y} \lambda \sqrt{y} \int_0^y \frac{\mathrm{e}^{-\lambda v}\,\mathrm{d}v}{\sqrt{v}}}. \tag{4.10}$$

Equivalently, *the complement to* 1 *of the above expression equals*:

$$E\big[\mathrm{e}^{-\lambda T_y^A}\big] = \frac{1}{1 + \mathrm{e}^{\lambda y} \lambda \sqrt{y} \int_0^y \mathrm{e}^{-\lambda v} \frac{\mathrm{d}v}{\sqrt{v}}} = \frac{1}{1 - \theta(-\lambda y)}. \tag{4.11}$$

Proof. We compute $E[\mathrm{e}^{-\lambda T_y^A}]$, with the help of the independence property of $X_{T_y^A}$ and T_y^A. Namely according to Wald's identity (2.14) we have:

$$E\big[\mathrm{e}^{\lambda X_{T_y^A}}\big] E\big[\mathrm{e}^{-\{\lambda^2/2\} T_y^A}\big] = 1. \tag{4.12}$$

As for the calculation of $E[\mathrm{e}^{\lambda X_{T_y^A}}]$ we use Proposition 1.2, and:

$$\begin{aligned}
E\big[\mathrm{e}^{\lambda X_{T_y^A}}\big] &= \frac{1}{2} \int_{-\infty}^\infty \frac{|z|}{y} \mathrm{e}^{-\{z^2/(2y)\}+\lambda z} \,\mathrm{d}z = \frac{1}{2} \int_{-\infty}^\infty |z| \mathrm{e}^{-\{z^2/2\}-\lambda z\sqrt{y}} \,\mathrm{d}z \\
&= \frac{\mathrm{e}^{\{\lambda^2/2\}y}}{2} \int_{-\infty}^\infty |z| \mathrm{e}^{-\{1/2\}(z+\lambda\sqrt{y})^2} \,\mathrm{d}z = \frac{\mathrm{e}^{\lambda^2 y/2}}{2} \int_{-\infty}^\infty \big|z - \lambda\sqrt{y}\big| \mathrm{e}^{-z^2/2} \,\mathrm{d}z \\
&= \frac{\mathrm{e}^{\{\lambda^2/2\}y}}{2} \bigg[\int_{-\infty}^{\lambda\sqrt{y}} (\lambda\sqrt{y} - z) \mathrm{e}^{-z^2/2} \,\mathrm{d}z + \int_{\lambda\sqrt{y}}^\infty (z - \lambda\sqrt{y}) \mathrm{e}^{-z^2/2} \,\mathrm{d}z\bigg] \\
&= \frac{\mathrm{e}^{\{\lambda^2/2\}y}}{2} \bigg[\lambda\sqrt{y} \bigg\{\int_{-\infty}^{\lambda\sqrt{y}} \mathrm{e}^{-z^2/2} \,\mathrm{d}z - \int_{\lambda\sqrt{y}}^\infty \mathrm{e}^{-z^2/2} \,\mathrm{d}z\bigg\} + 2\mathrm{e}^{-\lambda^2 y/2}\bigg] \\
&= 1 + \lambda\sqrt{y} \mathrm{e}^{\{\lambda^2/2\}y} \int_0^{\lambda\sqrt{y}} \mathrm{e}^{-z^2/2} \,\mathrm{d}z.
\end{aligned} \tag{4.13}$$

Hence, from (4.12) and (4.13), and after changing λ in $\sqrt{2\lambda}$, we get:

$$E\big[\mathrm{e}^{-\lambda T_y^A}\big] = \frac{1}{1+\mathrm{e}^{\lambda y}\sqrt{2\lambda y}\int_0^{\sqrt{2\lambda y}}\mathrm{e}^{-z^2/2}\,\mathrm{d}z} \tag{4.14}$$

$$= \frac{1}{1+\mathrm{e}^{\lambda y}\lambda\sqrt{y}\int_0^{y}\mathrm{e}^{-\lambda v}\frac{\mathrm{d}v}{\sqrt{v}}} \tag{4.15}$$

after making the change of variables: $z^2 = 2\lambda v$. Now, Lemma 4.4 follows from (4.15). □

It follows from the scaling property of Brownian motion (and this is confirmed by (4.15)) that:

$$T_y^A \overset{\text{(law)}}{=} yT_1^A \tag{4.16}$$

and

$$A_t^* \overset{\text{(law)}}{=} tA_1^*. \tag{4.17}$$

Proof of Theorem 4.3. (1) *We prove point* (1).

For every $s+x\le t$, one has:

$$1_{\{A_t^*\le x\}} = 1_{\{A_s^*\le x\}}1_{\{A_s+d_s-s\le x, d_s<t\}}1_{\{A_{t-d_s}^*\circ\theta_{d_s}\le x\}}. \tag{4.18}$$

Hence, from conjecture (4.5), after conditioning on $\mathcal{F}_{d_s}$, we get:

$$E[1_{\{A_t^*\le x\}}|\mathcal{F}_s] \underset{t\to\infty}{\sim} C1_{\{A_s^*\le x\}}E\big[1_{\{A_s+d_s-s\le x,d_s<t\}}\mathrm{e}^{-\lambda_0/x(t-d_s)}|\mathcal{F}_s\big]. \tag{4.19}$$

Recall that $d_s = s + T_0\circ\theta_s$, and:

$$P(T_z\in \mathrm{d}u) = \frac{|z|}{\sqrt{2\pi u^3}}\mathrm{e}^{-z^2/(2u)}\,\mathrm{d}u, \tag{4.20}$$

where T_z denotes the first hitting time of z by Brownian motion.

Therefore relation (4.19) implies:

$$E[1_{\{A_t^*\le x\}}|\mathcal{F}_s] \underset{t\to\infty}{\sim} C\mathrm{e}^{-\lambda_0 t/x}\left(1_{\{A_s^*\le x\}}\mathrm{e}^{\lambda_0 s/x}\int_0^{x-A_s}\frac{|X_s|}{\sqrt{2\pi u^3}}\mathrm{e}^{-\{X_s^2/(2u)\}+\lambda_0 u/x}\,\mathrm{d}u\right). \tag{4.21}$$

It is clear that

$$\lim_{t\to\infty}\frac{E[1_{\{A_t^*\le x\}}|\mathcal{F}_s]}{P(A_t^*\le x)} = M_s^* \tag{4.22}$$

follows from (4.21) and conjecture (4.5) (recall that M_s^* has been defined by (4.8)).

It will be shown below (see (2)(e)) that $E[M_s^*]=1$. Therefore point (1) of Theorem 4.3 is a direct consequence of Theorem 1.1.

(2) *We now show that* $(M_s^*, s\ge 0)$*, as defined in* (4.8) *is a martingale.* Note that the results stated in point (2) of Theorem 4.3 hold without assuming (4.5).

(a) We first remark that we may write M_s^*, with the help of the change of variables: $\frac{X_s^2}{u}=v^2$, as:

$$M_s^* = 1_{\{A_s^*\le x\}}\mathrm{e}^{\lambda_0 s/x}\sqrt{\frac{2}{\pi}}\int_{|X_s|/\sqrt{x-A_s}}^{\infty}\mathrm{e}^{-v^2/2+\{\lambda_0 X_s^2/(xv^2)\}}\,\mathrm{d}v \tag{4.23}$$

without assuming the validity of the conjecture.

We also note that $1_{\{A_s^*\le x\}} = 1_{\{T_x^A \ge s\}}$, and that, for $s < T_x^A$: $M_s^* > 0$.

Thus, $(M_s^*, s \ge 0)$ is stopped at its first zero. Moreover, we have:

$$0 \le M_s^* \le \sqrt{\frac{2}{\pi}} \mathrm{e}^{\lambda_0 s/x} 1_{(A_s^*\le x)} \int_{|X_s|/\sqrt{x-A_s}}^{\infty} \exp\left(-\frac{v^2}{2} + \frac{\lambda_0 X_s^2(x-A_s)}{x X_s^2}\right) \mathrm{d}v$$
$$\le \mathrm{e}^{\lambda_0 s/x}\mathrm{e}^{\lambda_0}. \tag{4.24}$$

Thus, to show that $(M_s^*, s \ge 0)$ is a martingale (under P), it suffices to see that it is a local martingale.

Note that the relation (4.23) implies that $M_0^* = 1$ a.s.

(b) Define:

$$f(y,a) = \int_{y/\sqrt{x-a}}^{\infty} \mathrm{e}^{-v^2/2+\{\lambda_0/(xv^2)\}y^2}\,\mathrm{d}v \quad (y \ge 0, 0 < a \le x) \tag{4.25}$$

so that:

$$M_s^* = \mathrm{e}^{\lambda_0 s/x} 1_{(A_s^*\le x)} \sqrt{\frac{2}{\pi}} f\big(|X_s|, A_s\big). \tag{4.26}$$

To prove that $(M_s^*, s \ge 0)$ is a local martingale, we shall apply Lemma 2.4 or more precisely, a slight variant of Lemma 2.4, due to the presence of the factor $\mathrm{e}^{\lambda_0 s/x}$.

In fact, it suffices to show that the function f considered as a function of (y,a) satisfies:

(i) $f(0,a)$ does not depend on a $(a > 0)$; in fact, from (4.25): $f(0,a) = \sqrt{\pi/2}$;

(ii) $\dfrac{\partial f}{\partial a} + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial y^2} + \dfrac{\lambda_0}{x} f = 0$;

(iii) $\dfrac{\partial f}{\partial y}(0,0) = 0$.

(c) We show (iii).

This follows from:

$$\frac{\partial f}{\partial y}(0,0) = -\frac{1}{\sqrt{x}}\mathrm{e}^{\lambda_0} + \int_{y/\sqrt{x}}^{\infty} \frac{2\lambda_0 y}{xv^2} \mathrm{e}^{-v^2/2+\{\lambda_0/(xv^2)\}}\,\mathrm{d}v\bigg|_{y=0}$$
$$= -\frac{1}{\sqrt{x}}\mathrm{e}^{\lambda_0} + \frac{\lambda_0}{\sqrt{x}}\int_0^1 \frac{\mathrm{d}u}{\sqrt{u}} \mathrm{e}^{-\{y^2/(2xu)\}+\lambda_0 u}\bigg|_{y=0}$$
$$\left(\text{after making the change of variables } \frac{y^2}{2v^2} = u\right)$$
$$= -\frac{\mathrm{e}^{\lambda_0}}{\sqrt{x}}\big(1-\theta(\lambda_0)\big) = 0,$$

from the definition of λ_0.

(d) We show point (ii) above. We start computing the two first derivatives $\frac{\partial f}{\partial y}(y,a)$ and $\frac{\partial f}{\partial a}(y,a)$.

From (4.25), we have:

$$\frac{\partial f}{\partial a}(y,a) = -\frac{y}{2(x-a)^{3/2}} \mathrm{e}^{-\{y^2/(2(x-a))\}+\{\lambda_0/x\}(x-a)} \tag{4.27}$$

and

$$\frac{\partial f}{\partial y}(y,a) = -\frac{1}{\sqrt{x-a}} \mathrm{e}^{-\{y^2/(2(x-a))\}+\{\lambda_0/x\}(x-a)} + \int_{y/\sqrt{x-a}}^{\infty} \frac{2\lambda_0 y}{xv^2} \mathrm{e}^{-v^2/2+\{\lambda_0 y^2/(xv^2)\}}\,\mathrm{d}v \tag{4.28}$$

so that, after making the change of variables $\frac{y^2}{xv^2} = u$:

$$\frac{\partial f}{\partial y}(y,a) = -\frac{1}{\sqrt{x-a}} \mathrm{e}^{-\{y^2/(2(x-a))\}+\{\lambda_0/x\}(x-a)} + \frac{\lambda_0}{\sqrt{x}} \int_0^{1-a/x} \frac{1}{\sqrt{u}} \mathrm{e}^{-\{y^2/(2ux)\}+\lambda_0 u} \,\mathrm{d}u. \tag{4.29}$$

Consequently:

$$\frac{\partial^2 f}{\partial y^2}(y,a) = \frac{y}{(x-a)^{3/2}} \mathrm{e}^{-\{y^2/(2(x-a))\}+\{\lambda_0/x\}(x-a)} - \frac{\lambda_0 y}{x^{3/2}} \int_0^{1-a/x} \mathrm{e}^{-\{y^2/(2ux)\}+\lambda_0 u} \frac{\mathrm{d}u}{u^{3/2}}. \tag{4.30}$$

On the other hand, we write f, after making the change of variables: $\frac{y^2}{xv^2} = u$ in (4.25), as:

$$f(y,a) = \frac{y}{2\sqrt{x}} \int_0^{1-a/x} \mathrm{e}^{-\{y^2/(2ux)\}+\lambda_0 u} \frac{\mathrm{d}u}{u^{3/2}}. \tag{4.31}$$

We then gather (4.27), (4.29), (4.30) and (4.31) to deduce (ii).

Moreover:

$$M_s^* = 1 + \sqrt{\frac{2}{\pi}} \int_0^{s \wedge T_x^A} \mathrm{e}^{\lambda_0 u/x} \frac{\partial f}{\partial y}\big(|X_u|, A_u\big) \operatorname{sgn}(X_u) \,\mathrm{d}X_u. \tag{4.32}$$

Hence, (M_s^*) is a local martingale, which, as we have seen earlier, implies it is a martingale; in particular, $E[M_s^*] = 1$.

(3) *We now prove point* (2)(b) *of Theorem* 4.3.

(a) It is clear that, Q a.s.: $A_\infty^* \le x$, since, for every $s \ge 0$:

$$Q^*\big(A_s^* \le x\big) = \lim_{t\to\infty} \frac{P(\{A_s^* \le x\} \cap \{A_t^* \le x\})}{P(A_t^* \le x)} = 1. \tag{4.33}$$

On the other hand, we get, for every $y < x$:

$$Q^*\big(A_t^* > y\big) = Q^*\big(T_y^A < t\big) = E\big[1_{\{T_y^A < t\}} M_t^*\big] = E\big[1_{\{T_y^A < t\}} M_{T_y^A}\big].$$

Taking the limit $t \to \infty$, we get:

$$Q^*\big(A_\infty^* > y\big) = E\big[M_{T_y^A}^*\big]. \tag{4.34}$$

However, from (4.32):

$$M_{T_y^A \wedge s}^* = 1 + \sqrt{\frac{2}{\pi}} \int_0^{T_y^A \wedge s} \mathrm{e}^{\lambda_0 u/x} \frac{\partial f}{\partial y}\big(|X_u|, A_u\big) \operatorname{sgn}(X_u) \,\mathrm{d}X_u, \quad s \ge 0,\ 0 < y < x. \tag{4.35}$$

Next, we deduce from the form of $\frac{\partial f}{\partial y}$ given by (4.29) the existence of a constant k such that:

$$\left| \frac{\partial f}{\partial y}\big(|X_u|, A_u\big) \right| \le k, \quad 0 \le u \le T_y^A. \tag{4.36}$$

Identity (4.11) implies that, for $\lambda > 0$, small enough:

$$E\big[\mathrm{e}^{\lambda T_y^A}\big] = \frac{1}{1 - \lambda y \mathrm{e}^{-\lambda y} \int_0^1 \mathrm{e}^{\lambda y v} \frac{\mathrm{d}v}{\sqrt{v}}} = \frac{1}{1 - \theta(\lambda y)}. \tag{4.37}$$

From Lemma 4.1 we deduce:

$$E\big[\mathrm{e}^{\lambda T_y^A}\big] < \infty \quad \forall \lambda < \frac{\lambda_0}{y}. \tag{4.38}$$

Combining Burkholder–Davis–Gundy inequality, (4.35), (4.36) and the fact that $E[\mathrm{e}^{\lambda_0 T_y^A/x}] < \infty$, shows that $(M^*_{s\wedge T_y^A})$ belongs to H^1 (i.e., $E[\sup_{s\geq 0}|M^*_{s\wedge T_y^A}|] < \infty$). Hence: $E[M^*_{T_y^A}] = 1$ and finally, from (4.34): $Q^*(A^*_\infty \geq y) = 1$ for any $0 \leq y < x$. Using moreover (4.33) we can conclude that $A^*_\infty = x$, Q^* a.s.

(b) We show that: $T_x^A = \infty$, Q^* a.s.

Since $M_t^* = 0$, for any $t \geq T_x^A$, then, for any $y < x$, one has:

$$Q^*\big(T_y^A > t\big) = E\big[1_{(T_y^A > t)} M_t^*\big] \underset{y\uparrow x}{\longrightarrow} E\big[1_{(T_x^A \geq t)} M_t^*\big] = E\big[M_t^*\big] = 1.$$

Hence,

$$\lim_{y\uparrow x} Q^*\big(T_y^A \leq t\big) = 0.$$

This implies:

$$Q^*\big(T_x^A \leq t\big) = \lim_{y\uparrow x} Q^*\big(T_y^A \leq t\big) = 0 \quad \forall t \geq 0$$

which proves that $T_x^A = \infty$, Q^* a.s.

□

Remark 4.5. *Let us come back to* (4.10). *By an analytic continuation argument*, *relation* (4.10) *may be extended as follows*:

$$\int_0^\infty \mathrm{e}^{\lambda t} P\big(A_t^* \leq x\big)\,\mathrm{d}t = \frac{\mathrm{e}^{-\lambda x}\sqrt{x}\int_0^x \frac{\mathrm{e}^{\lambda v}\,\mathrm{d}v}{\sqrt{v}}}{1 - \lambda\sqrt{x}\mathrm{e}^{-\lambda x}\int_0^x \frac{\mathrm{e}^{\lambda v}\,\mathrm{d}v}{\sqrt{v}}}$$
$$= \frac{\mathrm{e}^{-\lambda x}\sqrt{x}\int_0^x \frac{\mathrm{e}^{\lambda v}\,\mathrm{d}v}{\sqrt{v}}}{1 - \theta(\lambda x)}, \tag{4.39}$$

for any $\lambda < \lambda_0$. Since the first positive zero λ_0 of $1 - \theta$ is simple, *a formal application of classical results about the Mellin–Fourier transformation yield to* (4.5). *However we have not been able to justify this approach.*

Remark 4.6. *Let $\varphi(y) := P(A_1^* < \frac{1}{y}) = P(T_1^A > y)$ $(y > 0)$. Since $A_t^* \leq t$, we have*:

$$P\big(A_t^* < y\big) = P\big(tA_1^* < y\big) = \varphi\left(\frac{t}{y}\right),$$
$$\varphi(a) = 1 \quad \text{if } a < 1,$$
$$\varphi(a) \underset{a\to\infty}{\longrightarrow} 0.$$

The aim of the present remark is to show that φ satisfies the following equation:

$$\varphi(a) = -\int_0^1 \varphi'(a+v)\frac{\mathrm{d}v}{\sqrt{1-v}}, \quad a \geq 1. \tag{4.40}$$

Since $(-\varphi')$ is the density function of T_1^A, (4.40) *may equivalently be presented as*:

$$P\big(T_1^A > a\big) = E\left[\frac{1}{\sqrt{a+1-T_1^A}} 1_{\{a < T_1^A < a+1\}}\right]. \tag{4.41}$$

Indeed, to prove (4.40), *we rewrite* (4.10) *in the form*:

$$\left(\frac{\mathrm{e}^{-\lambda x}}{\lambda} + \sqrt{x}\int_0^x \frac{\mathrm{e}^{-\lambda v}\,\mathrm{d}v}{\sqrt{v}}\right)\left(\int_0^\infty \mathrm{e}^{-\lambda t}\varphi\left(\frac{t}{x}\right)\mathrm{d}t\right) = \frac{1}{\lambda}\left(\sqrt{x}\int_0^x \frac{\mathrm{e}^{-\lambda v}\mathrm{d}v}{\sqrt{v}}\right). \tag{4.42}$$

We denote:

$$\mu_x(s) = \sqrt{x}1_{[0,x]}(s)\frac{1}{\sqrt{s}} + 1_{[x,\infty]}(s) \tag{4.43}$$

so that (4.42) *becomes*, *for every* $t \geq 0$:

$$\left(\varphi\left(\frac{\cdot}{x}\right) * \mu_x\right)(t) = \sqrt{x}\int_0^t 1_{[0,x]}(s)\frac{\mathrm{d}s}{\sqrt{s}} = 2\sqrt{x}\sqrt{x\wedge t}. \tag{4.44}$$

In particular, *when* $t \geq 2x$ *we have*:

$$2x = \int_0^t \varphi\left(\frac{t-s}{x}\right)\mu_x(s)\,\mathrm{d}s.$$

Since $\varphi(\frac{t-s}{x}) = \mu_x(s) = 1$ *for* $s \geq t-x$, *then the previous relation can be reduced to*:

$$x = \int_0^{t-x} \varphi\left(\frac{t-s}{x}\right)\mu_x(s)\,\mathrm{d}s. \tag{4.45}$$

Hence, *differentiating with respect to* t, *one gets*:

$$\varphi(1)\mu_x(t-x) + \frac{1}{x}\int_0^{t-x} \varphi'\left(\frac{t-s}{x}\right)\mu_x(s)\,\mathrm{d}s = 0,$$

which implies, *since* $\varphi(1) = 1$, *and* $\mu_x(t-x) = 1$ *for* $t \geq 2x$:

$$0 = 1 + \frac{1}{\sqrt{x}}\int_0^x \varphi'\left(\frac{t-s}{x}\right)\frac{\mathrm{d}s}{\sqrt{s}} + \frac{1}{x}\int_x^{t-x} \varphi'\left(\frac{t-s}{x}\right)\mathrm{d}s.$$

However:

$$\frac{1}{x}\int_x^{t-x} \varphi'\left(\frac{t-s}{x}\right)\mathrm{d}s = \varphi\left(\frac{t-x}{x}\right) - \varphi(1) = \varphi\left(\frac{t-x}{x}\right) - 1.$$

Hence:

$$\varphi\left(\frac{t-x}{x}\right) = -\frac{1}{\sqrt{x}}\int_0^x \varphi'\left(\frac{t-s}{x}\right)\frac{\mathrm{d}s}{\sqrt{s}} = -\int_0^1 \varphi'\left(\frac{t}{x} - v\right)\frac{\mathrm{d}v}{\sqrt{v}}.$$

Closely related computations are found in [7].

Remark 4.7. *With the help of Girsanov's theorem*, *together with* (4.7) *and* (4.32), *there exists a* Q^**-Brownian motion* $(\beta_t, t \geq 0)$ *such that*:

$$X_t = \beta_t + \int_0^t \frac{\partial f/\partial y}{f}\big(|X_s|, A_s\big)\,\mathrm{sgn}(X_s)\,\mathrm{d}s. \tag{4.46}$$

From (4.25), *we deduce that for* $y > 0$:

$$f(y,a) \underset{a\uparrow x}{\sim} \frac{\sqrt{x-a}}{y}\mathrm{e}^{-y^2/(2(x-a))}, \qquad \frac{\partial f}{\partial y}(y,a) \underset{a\uparrow x}{\sim} -\frac{1}{\sqrt{x-a}}\mathrm{e}^{-y^2/(2(x-a))}.$$

Consequently:

$$\lim_{a\uparrow x}\frac{\partial f/\partial y}{f}(y,a)=-\infty \quad \textit{and}$$

$$\frac{\partial f/\partial y}{f}\big(|X_s|,A_s\big)\operatorname{sgn}(X_s)$$

goes to $-\infty$, *resp.*: $+\infty$, *as* A_s *approaches* x, *and* $X_s>0$, *resp.*: $X_s<0$. *In other terms*, *when the age of* X *approaches* x, *the process* X *is strongly pushed towards* 0.

This explains intuitively why, *under* Q^*, $A^*_\infty=x$ *and* $T^A_x=\infty$, *a.s.*

Remark 4.8. *The rates of decay of* $P(A^*_t\le x)$ *and* $P(\Sigma_t\le x)$ *as* $t\to\infty$ *are radically different*. *The first one is exponential* (*see* (4.5)), *whereas the second one is polynomial* (*see Lemma* 2.3). *We already observed such a difference when we studied* (*see* [13]) *the penalisations of Wiener measure by*, *on one hand*, $1_{\{X^*_t\le x\}}$ (*exponential decay*) *and*, *on the other hand*, $1_{\{S_t<x\}}$ (*polynomial decay*), *where we denote* $X^*_t=\sup_{s\le t}|X_s|$ *and* $S_t=\sup_{s\le t}X_s$.

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