Asymptotic behavior of a hard thin linear elastic interphase: An energy approach

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The mechanical problem of two elastic bodies separated by a thin elastic film is studied here. The stiffness of the three bodies is assumed to be similar. The asymptotic behavior of the film as its thickness tends to zero is studied using a method based on asymptotic expansions and energy minimization. Several cases of interphase material symmetry are studied (from isotropy to triclinic symmetry). In each case, non-local relations are obtained relating the jumps in the displacements and stress vector fields at order one to these fields at order zero.

1. Introduction

During the mechanical assembly of structures, interphases can have crucial effects. In particular, imperfections in the assembly can lead to structural failure. Although the thickness of interphases is generally very small in comparison with the dimensions of the structure, their mechanical role cannot be neglected and they need to be taken into account in modeling procedures. From the numerical point of view, the thinness of interphases gives rise to problems which are very difficult to solve. In particular, the number of degrees of freedom adopted in studies using a finite element approach can be very large, which affects the convergence and the accuracy of the solution. Interphase modeling therefore has to be performed before solving the problem numerically. One classical technique consists in replacing the thin interphase by an interface of zero thickness, while keeping some important mechanical properties of the interphase. From the geometrical point of view, the interphase is eliminated, although it is accounted for mechanically. The resulting equivalent interface model is simpler to implement in numerical simulations than the original multi-scale problem. This idea was the starting-point of several studies published during the last years (Caillerie, 1980; Ait-Moussa, 1989; Klarbring, 1991; Licht, 1993; Licht and Michaille, 1996, 1997; Ould-Khaoua et al., 1996; Ganghoffer et al., 1997; Geymonat and Krasucki, 1997; Lebon et al., 1997; Zaittouni et al., 2002; Lebon and Rizzoni, 2008; Lebon and Zaittouni, 2010). To model the equivalent interface, asymptotic techniques are necessary, i.e., we take the thickness of the interface to be a small parameter which tends to zero. Interface models usually relates the stress vector to the jump in the displacement (or in the velocity). In most cases, like in soft interface models (Geymonat et al., 1999; Krasucki et al., 2001; Lebon et al., 2004; Lebon and Ronel-Idrissi, 2004; Pelissou and Lebon, 2009; Rekik and Lebon, 2010), this means that not only the thickness of the interface but also its rigidity is small. In the present study on a hard interface model, only the thickness is assumed to be small, and the stiffness of the adherents and the interphase are taken to be similar.

Some studies, focused on adherents and a flat interphase with a comparable level of rigidity (Caillerie, 1980; Abdelmoula et al., 1998; Lebon and Ronel, 2007; Lebon and Rizzoni, 2010), have already established that at the first order ($\varepsilon \to 0$) one obtains a perfect interface model, which prescribes the vanishing of the jumps in the stress and the displacement vectors. At a higher order (the second term in the expansion), an imperfect interface model is obtained, with a transmission condition involving the first order displacement and traction vectors and their derivatives (Abdelmoula et al., 1998; Lebon and Ronel, 2007; Lebon and Rizzoni, 2010). The higher order term, giving rise to an imperfect interface model, can be interpreted as a correction of the leading solution corresponding to the perfect interface model.

All these studies model the interphase as an isotropic, linear elastic material. Even though in many practical cases the adhesive is an isotropic material, typically an epoxy resin, it is possible that the process of producing a thin layer of adhesive causes the material to become anisotropic or layered. In this paper, we extend the results obtained in Abdelmoula et al. (1998), Lebon and Ronel

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(2007), Lebon and Rizzoni (2010) to the case of an anisotropic adhesive.

The equilibrium problem involved in the interphase/adherents system is presented in Section 2. The mathematical methods used so far for this purpose have often been matched asymptotic expansions (Eckhaus, 1979; Sanchez-Hubert and Sanchez-Palencia, 1992). In this paper, an energy approach is also used. The main assumption adopted, which is introduced in Section 3, is the existence of expansions in series of the displacements and stress vector fields in terms of the small parameter describing the thickness. The second assumption is that we can obtain the fields which are stationary points of the energy of the system by finding the stationary points of the energies obtained at each level in the expansion. In the second part of Section 3, the minimization is performed at orders -1, 0, 1 and 2. Two types of relations are obtained: either an interface relation or an equilibrium relation. In particular, at orders -1 and 1, we obtain conditions on the displacement fields at order zero and order one, respectively, determining the jumps at the interface. At orders 0 and 2, we obtain the equilibrium equations for the adherents written in terms of the displacement fields at order zero and order one, respectively. The former are balance equations for the zero order stress and displacement vector fields associated with a perfect interface law, and the latter are balance equations for the first order stress and displacement vector fields associated with an imperfect interface law, involving tangential derivatives and first order terms. We also find that some (natural) boundary condition arising at order 2 (Eq. (54)) are not verified by the classical asymptotic expansion assumed here. We interpret this as an indication of a phenomenon of boundary layer, whose analysis is beyond the scope of this paper.

In Section 4 and in the Appendices, several cases of anisotropy are analyzed. In the case of isotropy, we obtain the same results as those presented in Lebon and Rizzoni (2010). In the case of orthotropic symmetry, that of transverse isotropy, in the case where a symmetry axis is running perpendicular and parallel to the interface, and in the case of monoclinic and triclinic materials, we obtain the forms of the coefficients involved in the imperfect interface relations.

2. Statement of the problem

Let S be an open bounded subset of R^2 with a smooth boundary and let us take a thin interphase B^ε with cross-section S and a constant small thickness $\varepsilon \ll 1$. The interphase lies between two bodies $\Omega^\varepsilon_\pm \subset R^3$, as shown in Fig. 1. Let $S^\varepsilon_\pm$ denote the flat interfaces between the interphase and the two bodies and let $\Omega^\varepsilon = \Omega^\varepsilon_\pm \cup S^\varepsilon_\pm \cup B^\varepsilon$ denote the composite comprising the interphase and the two bodies. We take an orthogonal frame (O,x_1,x_2,x_3) with its origin at the center of the interphase midplane and with x_3-axis running perpendicular to the interfaces $S^\varepsilon_\pm$. The adhesion between the bodies and the interphase is assumed to be perfect. Let $u^\varepsilon : \Omega^\varepsilon \mapsto R^3$ be a displacement field defined in Ω^ε. The continuity conditions across the surfaces $S^\varepsilon_\pm$ are

$$[u^\varepsilon]_\pm = 0 \quad \text{on } S^\varepsilon_\pm, \tag{1}$$

where

$$[u^\varepsilon]_\pm := u^\varepsilon\left(x_1, x_2, \left(\pm\frac{\varepsilon}{2}\right)^+\right) - u^\varepsilon\left(x_1, x_2, \left(\pm\frac{\varepsilon}{2}\right)^-\right), \tag{2}$$

gives the jumps in the displacement across $S^\varepsilon_\pm$. In (2), $u^\varepsilon(x_1, x_2, (\frac{\varepsilon}{2})^+)$ (resp. $u^\varepsilon(x_1, x_2, (\frac{\varepsilon}{2})^-)$) indicates the limit of $u^\varepsilon(x_1,x_2,x_3)$ as x_3 tends to $(\frac{\varepsilon}{2})$, $x_3 \geqslant (\frac{\varepsilon}{2})$ (resp. $x_3 \leqslant \frac{\varepsilon}{2}$).

The interphase and the two bodies are assumed to be homogeneous and linear elastic. We take b_{ijkl} to denote the components of the elasticity tensor b at the interphase and $a_{\pm ijkl}$ to denote the components of the elasticity tensors $a_\pm$ of the two bodies. Let e be the strain tensor

$$e(u^\varepsilon) = \frac{1}{2}\left(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T\right). \tag{3}$$

In a general anisotropic context, linear elasticity gives the Cauchy stress tensor σ^ε as follows:

$$\sigma^\varepsilon = b(e) \quad \text{in } B^\varepsilon, \tag{4}$$

$$\sigma^\varepsilon = a_\pm(e) \quad \text{in } \Omega^\varepsilon_\pm. \tag{5}$$

A body force density f is applied to Ω^ε and a surface force density g to $\Gamma_g \subset \partial\Omega^\varepsilon$. On $\Gamma_u = \partial\Omega^\varepsilon \backslash \Gamma_g \backslash (\partial\Omega^\varepsilon \cap \partial B^\varepsilon)$, we prescribe the homogeneous boundary conditions

$$u^\varepsilon = 0 \quad \text{on } \Gamma_u. \tag{6}$$

We also make the following assumptions:

$$(\mathrm{H1}) \begin{cases} a_\pm,\ b \in L^\infty(\Omega), \\ a_{\pm ijkl} = a_{\pm klij} = a_{\pm jilk} = a_{\pm ijlk}, \\ b_{ijkl} = b_{klij} = b_{jilk} = b_{ijlk}, \\ \exists \eta_\pm,\ \eta > 0 : a_\pm(e)\cdot(e) \geqslant \eta_\pm |e|^2, \\ b(e)\cdot(e) \geqslant \eta_\pm |e|^2, \quad \forall e : e = e^T, \end{cases}$$

$$(\mathrm{H2})\ \exists \varepsilon_0 : B_\varepsilon \cap (\Gamma_g \cup \ \mathrm{supp}(f)) = \emptyset, \quad \forall \varepsilon < \varepsilon_0,$$

$$(\mathrm{H3})\ f \in (L^2(\Omega))^3,\ g \in (L^2(\Gamma_g))^3.$$

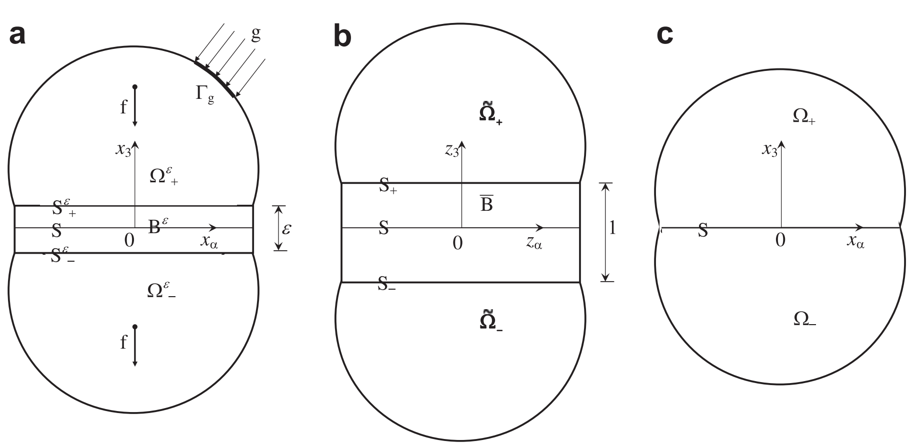

Fig. 1. (a) Initial configuration with a thin interphase placed between two bodies; (b) rescaled configuration with the two bodies separated by an interphase of unit thickness; (c) limit configuration, where the interphase is replaced by an interface.

Assumption (H1) deals with the usual symmetry properties and positive definiteness hypothesis about the elasticity tensors. Assumption (H2) means that Γ_g is located far from the interphase. In (H3), the fields of the external forces are endowed with sufficient regularity to ensure the existence of equilibrium configurations (see below).

The composite body equilibrium configurations are the minimizers of the total energy

$$E^{\varepsilon}(u) = \int_{\Omega^{\varepsilon}_{\pm}} \left(\frac{1}{2} a_{\pm}(e(u)) \cdot e(u) - f \cdot u\right) dx - \int_{\Gamma_g} g \cdot u \, ds_x + \int_{B^{\varepsilon}} \frac{1}{2} b(e(u)) \cdot e(u) \, dx, \tag{7}$$

in the space of kinematically admissible displacements

$$V^{\varepsilon} = \left\{ u \in H(\Omega^{\varepsilon}; R^3) : u = 0 \quad \text{on } \Gamma_u \right\}, \tag{8}$$

where $H(\Omega^{\varepsilon}; R^3)$ is the space of the vector-valued functions on the set Ω^{ε}, which are continuous and differentiable as many times as necessary. The assumptions (H1), (H2) and (H3) ensure the existence of a unique minimizer u^{ε} in V^{ε} (Ciarlet, 1988, Theorem 6.3-2).

3. Asymptotic analysis

In this section, the asymptotic expansion method is used to obtain the interface conditions giving the effect of a thin interphase on the mechanical behavior of the composite Ω^{ε}. In order to reformulate the equilibrium problem in an interphase domain independent of ε, we introduce the change of variables

$$(z_1, z_2, z_3) = \bar{p}(x_1, x_2, x_3) := (x_1, x_2, x_3 \varepsilon^{-1}), \quad (x_1, x_2, x_3) \in B^{\varepsilon}, \tag{9}$$

$$(z_1, z_2, z_3) = \tilde{p}(x_1, x_2, x_3) := \left(x_1, x_2, x_3 \pm \frac{\varepsilon}{2} \mp \frac{1}{2}\right), \quad (x_1, x_2, x_3) \in \Omega^{\varepsilon}_{\pm}. \tag{10}$$

In particular, B^{ε} is rescaled by a factor ε^{-1} along the interphase thickness and the bodies $\Omega^{\varepsilon}{}_{\pm}$ are shifted by $\pm 1/2(1-\varepsilon)$ in the same direction, as shown in Fig. 1b. In the new coordinate system, the interphase occupies the domain

$$\overline{B} = \left\{ (z_1, z_2, z_3) \in R^3 : (z_1, z_2) \in S, |z_3| < \frac{1}{2} \right\}, \tag{11}$$

and the two bodies occupy the domains $\tilde{\Omega}_{\pm} = \Omega^{\varepsilon}_{\pm} \pm 1/2(1-\varepsilon) i_3$, where i_3 denotes the unit vector along the z_3-axis. Let $S_{\pm} = \left\{ (z_1, z_2, z_3) \in R^3 : (z_1, z_2) \in S, z_3 = \pm\frac{1}{2} \right\}$ denote the interfaces between the interphase and the two bodies after rescaling, and let $\Omega = \tilde{\Omega}_+ \cup \tilde{\Omega}_- \cup \overline{B} \cup S_+ \cup S_-$ denote the configuration of the composite body after the change of variables (Fig. 1b). Lastly, let $\tilde{\Gamma}_u$ and $\tilde{\Gamma}_g$ denote the shifts of Γ_u and Γ_g, respectively.

Let

$$\tilde{u}^{\varepsilon}_{\pm}(z_1, z_2, z_3) := (u^{\varepsilon} \circ \tilde{p}^{-1})(z_1, z_2, z_3), \quad (z_1, z_2, z_3) \in \tilde{\Omega}_{\pm}, \tag{12}$$

be the displacement from configuration Ω of the bodies adjacent to the rescaled interphase, and let

$$\bar{u}^{\varepsilon}(z_1, z_2, z_3) := (u^{\varepsilon} \circ \bar{p}^{-1})(z_1, z_2, z_3), \quad (z_1, z_2, z_3) \in \overline{B}, \tag{13}$$

be the displacement from configuration Ω in the rescaled interphase. In view of the continuity condition (1), we have

$$\tilde{u}^{\varepsilon}_{\pm}\left(z_1, z_2, \pm\frac{1}{2}\right) = \bar{u}^{\varepsilon}\left(z_1, z_2, \pm\frac{1}{2}\right), \quad (z_1, z_2) \in S. \tag{14}$$

Note also that in view of the change of variables, we can write

$$u^{\varepsilon}\left(x_1, x_2, \left(\pm\frac{\varepsilon}{2}\right)^{\mp}\right) = \tilde{u}^{\varepsilon}_{\pm}\left(z_1, z_2, \left(\pm\frac{1}{2}\right)^{\mp}\right), \quad (x_1, x_2), (z_1, z_2) \in S, \tag{15}$$

$$u^{\varepsilon}\left(x_1, x_2, \left(\pm\frac{\varepsilon}{2}\right)^{\pm}\right) = \bar{u}^{\varepsilon}\left(z_1, z_2, \left(\pm\frac{1}{2}\right)^{\pm}\right), \quad (x_1, x_2), (z_1, z_2) \in S. \tag{16}$$

Let $\tilde{f} := f \circ \bar{p}^{-1}$ and $\tilde{g} := g \circ \bar{p}^{-1}$ denote the rescaled external forces. We also rephrase assumption (H2) as follows:

$$(\mathrm{H2}')\, \overline{B} \cap (\tilde{\Gamma}_g \cup \operatorname{supp}(\tilde{f})) = \emptyset. \tag{17}$$

We make no further rescaling assumptions about the unknown displacements, the loads or the elastic properties of the bodies.

With these assumptions, the rescaled energy takes the form

$$\mathcal{E}^{\varepsilon}(\tilde{u}^{\varepsilon}_{\pm}, \bar{u}^{\varepsilon}) := \int_{\tilde{\Omega}_{\pm}} \left(\frac{1}{2} a_{\pm}\left(e(\tilde{u}^{\varepsilon}_{\pm})\right) \cdot e(\tilde{u}^{\varepsilon}_{\pm}) - \tilde{f} \cdot \tilde{u}^{\varepsilon}_{\pm}\right) dz - \int_{\tilde{\Gamma}_g} \tilde{g} \cdot \tilde{u}^{\varepsilon}_{\pm} \, ds_z + \int_{\overline{B}} \frac{1}{2} \Big(\varepsilon^{-1} K^{33}\left(\bar{u}^{\varepsilon}_{,3}\right) \cdot \bar{u}^{\varepsilon}_{,3} + 2K^{\alpha 3}(\bar{u}^{\varepsilon}_{,\alpha}) \cdot \bar{u}^{\varepsilon}_{,3} + \varepsilon K^{\alpha\beta}\left(\bar{u}^{\varepsilon}_{,\alpha}\right) \cdot \bar{u}^{\varepsilon}_{,\beta}\Big) dz, \tag{18}$$

where a comma is used to denote partial differentiation and K^{jl}, j, $l = 1,2,3$, are the matrices whose components are defined by the relations

$$K^{jl}_{ki} := b_{ijkl}. \tag{19}$$

In view of the symmetry properties of the elasticity tensor b, the matrices K^{jl} have the property that $K^{jl} = (K^{lj})^T$, j, $l = 1,2,3$.

The rescaled equilibrium problem $\widetilde{P}^{\varepsilon}$ can be formulated as follows: find the pair $(\tilde{u}^{\varepsilon}_{\pm}, \bar{u}^{\varepsilon})$ minimizing the energy (18) in the set of displacements

$$V = \left\{ (\tilde{u}_{\pm}, \bar{u}) \in H(\Omega_{\pm}; R^3) \times H(\overline{B}; R^3) : \tilde{u}_{\pm} = \bar{u} \ \text{ on } S_{\pm}, \tilde{u}_{\pm} = 0 \ \text{ on } \tilde{\Gamma}_u \right\}. \tag{20}$$

Since we are looking for the behavior of the minimizer of (7) when the interphase thickness ε is small, we assume that the minimizing displacements can be expressed as the sum of the series

$$\tilde{u}^{\varepsilon}_{\pm} = \tilde{u}^0_{\pm} + \varepsilon \tilde{u}^1_{\pm} + \varepsilon^2 \tilde{u}^2_{\pm} + o(\varepsilon^2), \tag{21}$$

$$\bar{u}^{\varepsilon} = \bar{u}^0 + \varepsilon \bar{u}^1 + \varepsilon^2 \bar{u}^2 + o(\varepsilon^2), \tag{22}$$

where the displacement vectors u^1, u^2 are independent of ε. Substituting this expansion into (12) and (13) and inserting the result into (18), we obtain

$$\mathcal{E}^{\varepsilon}(\tilde{u}_{\pm}, \bar{u}) = \frac{1}{\varepsilon} \mathcal{E}^{-1}(\bar{u}^0) + \mathcal{E}^0\left(\tilde{u}^0_{\pm}, \bar{u}^0, \bar{u}^1\right) + \varepsilon \mathcal{E}^1\left(\tilde{u}^0_{\pm}, \tilde{u}^1_{\pm}, \bar{u}^0, \bar{u}^1, \bar{u}^2\right) + \varepsilon^2 \mathcal{E}^2\left(\tilde{u}^0_{\pm}, \tilde{u}^1_{\pm}, \tilde{u}^2_{\pm}, \bar{u}^0, \bar{u}^1, \bar{u}^2, \bar{u}^3\right) + o(\varepsilon^2), \tag{23}$$

where

$$\mathcal{E}^{-1}(\bar{u}^0) := \int_{\overline{B}} \frac{1}{2} (K^{33}(\bar{u}^0_{,3}) \cdot \bar{u}^0_{,3}) \, dz, \tag{24}$$

$$\mathcal{E}^0(\tilde{u}^0_{\pm}, \bar{u}^0, \bar{u}^1) := \int_{\tilde{\Omega}_{\pm}} \left(\frac{1}{2} a_{\pm}(e(\tilde{u}^0_{\pm})) \cdot e(\tilde{u}^0_{\pm}) - \tilde{f} \cdot \tilde{u}^0_{\pm}\right) dz - \int_{\tilde{\Gamma}_g} \tilde{g} \cdot \tilde{u}^0_{\pm} \, ds_z + \int_{\overline{B}} \left(\frac{1}{2} K^{33}(\bar{u}^0_{,3}) \cdot \bar{u}^1_3 + K^{\alpha 3}(\bar{u}^0_{,\alpha}) \cdot \bar{u}^0_3\right) dz, \tag{25}$$

$$\mathcal{E}^1(\tilde{u}^0_{\pm}, \tilde{u}^1_{\pm}, \bar{u}^0, \bar{u}^1, \bar{u}^2) := \int_{\tilde{\Omega}_{\pm}} (a_{\pm}(e(\tilde{u}^0_{\pm})) \cdot e(\tilde{u}^1_{\pm}) - \tilde{f} \cdot \tilde{u}^1_{\pm}) \, dz - \int_{\tilde{\Gamma}_g} \tilde{g} \cdot \tilde{u}^1_{\pm} \, ds_z + \int_{\overline{B}} \left(K^{33}(\bar{u}^0_{,3}) \cdot \bar{u}^2_{,3} + \frac{1}{2} K^{33}(\bar{u}^1_{,3}) \cdot \bar{u}^1_3\right) dz + \int_{\overline{B}} \left(K^{\alpha 3}(\bar{u}^0_{\alpha}) \cdot \bar{u}^1_{,3} + K^{\alpha 3}(\bar{u}^1_{,\alpha}) \cdot \bar{u}^0_{,3} + \frac{1}{2} K^{\alpha\beta}(\bar{u}^0_{,\alpha}) \cdot \bar{u}^0_{\beta}\right) dz, \tag{26}$$

$$\mathcal{E}^2(\tilde{u}^0_\pm,\tilde{u}^1_\pm,\tilde{u}^2_\pm,\bar{u}^0,\bar{u}^1,\bar{u}^2,\bar{u}^3) := \int_{\tilde{\Omega}_\pm}\left(\frac{1}{2}a_\pm(e(\tilde{u}^1_\pm))\cdot e(\tilde{u}^1_\pm) - \tilde{f}\cdot\tilde{u}^2_\pm\right)dz - \int_{\tilde{\Gamma}_g}\tilde{g}\cdot\tilde{u}^2_\pm\,ds_z + \int_{\tilde{\Omega}_\pm}a_\pm(e(\tilde{u}^0_\pm))\cdot e(\tilde{u}^2_\pm)\,dz + \int_{\bar{B}}K^{33}(\bar{u}^0_{,3})\cdot\bar{u}^3_{,3}\,dz + \int_{\bar{B}}(K^{33}(\bar{u}^1_{,3})\cdot\bar{u}^2_{,3} + K^{\alpha 3}(\bar{u}^0_{,\alpha})\cdot\bar{u}^2_{,3} + K^{\alpha 3}(\bar{u}^1_{,\alpha})\cdot\bar{u}^1_{,3})\,dz + \int_{\bar{B}}(K^{\alpha 3}(\bar{u}^2_{,\alpha})\cdot\bar{u}^0_{,3}\,dz + K^{\alpha\beta}(\bar{u}^0_{,\alpha})\cdot\bar{u}^1_{,\beta})\,dz. \quad (27)$$

We now minimize each of these energies separately. The function class in which we seek the solution of each energy minimization is assumed to be a class of displacements which have finite energy.

Remark. Some considerations on minimization, stationarity and decoupling between orders. We consider a functional $f^\varepsilon(u^\varepsilon)$. We suppose that the following expansions exist:

$$f^\varepsilon(v^\varepsilon) = f^0(v^0) + \varepsilon f^1(v^1) + \cdots \quad (28)$$

In this case the minimization problem $f^\varepsilon(u^\varepsilon) \leqslant f^\varepsilon(v^\varepsilon)$, $\forall v^\varepsilon$ becomes formally

$$f^0(u^0) + \varepsilon f^1(u^1) + \cdots \leqslant f^0(v^0) + \varepsilon f^1(v^1) + \cdots \quad \forall v^0, v^1, \ldots \quad (29)$$

and thus $f^0(u^0) \leqslant f^0(v^0)$, $f^1(u^1) \leqslant f^1(v^1)\ldots$. If we consider the problem (which is not usually equivalent to the minimization problem):

$$\nabla f^\varepsilon(u^\varepsilon) = 0, \quad (30)$$

it becomes formally

$$\nabla f^0(u^0) = 0, \nabla f^1(u^1) = 0, \ldots \quad (31)$$

3.1. Minimization of $\mathcal{E}^{-1}$

The energy is minimized in the class of displacements $\bar{u}^0 \in H(\bar{B}; R^3)$. Since b is a positive definite tensor, the second order tensor K^{33} is also positive definite. Therefore, the energy $\mathcal{E}^{-1}$ is non-negative and the minimizers have the property

$$\bar{u}^0_{,3} = 0, \quad a.e. \text{ in } \bar{B}, \quad (32)$$

i.e., the minimizing displacements are independent of z_3 in the interphase. Based on this result and the continuity conditions (14), we obtain the following condition on $\tilde{u}^0$ evaluated at $S^\pm$

$$\tilde{u}^0\left(z_1, z_2, +\frac{1}{2}\right) = \tilde{u}^0\left(z_1, z_2, -\frac{1}{2}\right), \quad (z_1, z_2) \in \bar{B}, \quad (33)$$

In view of (15) and (16), condition (33) implies that

$$u^0(x_1, x_2, 0^+) = u^0(x_1, x_2, 0^-), \quad (x_1, x_2) \in S. \quad (34)$$

From the mechanical viewpoint, condition (34) gives a *perfect interface condition* for the interphase modeling.

3.2. Minimization of $\mathcal{E}^0$

Based on (32), the energy $\mathcal{E}^0$ turns out to become independent of $\bar{u}^0, \bar{u}^1$. With a little abuse of notation, we drop the dependence of the these vector fields from the argument of $\mathcal{E}^0$, which becomes

$$\mathcal{E}^0(\tilde{u}^0_\pm) = \int_{\tilde{\Omega}_\pm}\left(\frac{1}{2}a_{\pm ijkl}e_{ij}(\tilde{u}^0_\pm)e_{kl}(\tilde{u}^0_\pm) - \tilde{f}\cdot\tilde{u}^0_\pm\right)dz - \int_{\tilde{\Gamma}_g}\tilde{g}\cdot\tilde{u}^0_\pm\,ds_z. \quad (35)$$

In view of (33), we seek the energy minimizer in the class of displacements

$$V = \left\{(\tilde{u}_\pm) \in H(\Omega_\pm; R^3) : \tilde{u}_+\left(z_1, z_2, +\frac{1}{2}\right) = \tilde{u}_-\left(z_1, z_2, -\frac{1}{2}\right), \times\left(z_1, z_2\right) \in \bar{B},\ \tilde{u}_\pm = 0 \quad \text{on } \tilde{\Gamma}_u\right\}. \quad (36)$$

Using standard arguments, we obtain the equilibrium equations

$$\operatorname{div}(a_\pm(e(\tilde{u}^0_\pm)) + f) = 0 \quad \text{in } \Omega_\pm, \quad (37)$$

$$a_\pm(e(\tilde{u}^0_\pm))n = g \quad \text{on } \tilde{\Gamma}_g, \quad (38)$$

$$a_\pm(e(\tilde{u}^0_\pm))n = 0 \quad \text{on } \partial\Omega_\pm \setminus \tilde{\Gamma}_g, \quad (39)$$

$$a_+(e(\tilde{u}^0_+))i_3 = a_-(e(\tilde{u}^0_-))i_3 \quad \text{on } \bar{S}. \quad (40)$$

The last condition states that as expected, the jump in the traction vector across the rescaled interphase $\bar{B}$ vanishes, and we take $\tilde{\sigma}^0 i_3$ to denote its constant value.

3.3. Minimization of $\mathcal{E}^1$

Condition (32) makes $\mathcal{E}^1$ independent of $\bar{u}^2$. Again with a little abuse of notation, we drop the dependence of this vector field in the argument of $\mathcal{E}^1$, which simplifies as

$$\mathcal{E}^1(\tilde{u}^0_\pm,\tilde{u}^1_\pm,\bar{u}^0,\bar{u}^1) := \int_{\tilde{\Omega}_\pm}(a_\pm(e(\tilde{u}^0_\pm))\cdot e(\tilde{u}^1_\pm) - \tilde{f}\cdot\tilde{u}^1_\pm)\,dz - \int_{\tilde{\Gamma}_g}\tilde{g}\cdot\tilde{u}^1_\pm\,ds_z + \int_{\bar{B}}\left(\frac{1}{2}K^{33}(\bar{u}^1_{,3})\cdot\bar{u}^1_{,3} + K^{\alpha 3}(\bar{u}^0_{,\alpha})\cdot\bar{u}^1_{,3} + \frac{1}{2}K^{\alpha\beta}(\bar{u}^0_{,\alpha})\cdot\bar{u}^0_{,\beta}\right)dz. \quad (41)$$

Applying the divergence theorem and using the equilibrium equations (37)–(39), it turns out that minimizers of $\mathcal{E}^1$ also minimize the functional

$$\int_{\bar{B}}\left(\frac{1}{2}K^{33}(\bar{u}^1_{,3}) + K^{\alpha 3}(\bar{u}^0_{,\alpha}) - \tilde{\sigma}^0 i_3\right)\cdot\bar{u}^1_{,3}\,dz. \quad (42)$$

The corresponding Euler–Lagrange equation takes the form

$$\tilde{\sigma}^0 i_3 = K^{33}(\bar{u}^1_{,3}) + K^{\alpha 3}(\bar{u}^0_{,\alpha}). \quad (43)$$

This relation together with the continuity condition (33) gives the following condition on the jump in the displacement vector field $\bar{u}^1$ across the interphase

$$[\bar{u}^1] = (K^{33})^{-1}(\tilde{\sigma}^0 i_3 - K^{\alpha 3}\bar{u}^0_{,\alpha}). \quad (44)$$

Note that in view of the conditions (14) of continuity of the displacement fields at the interfaces $S^\pm$, the latter condition can be rewritten in the equivalent form

$$[\tilde{u}^1] = (K^{33})^{-1}(\tilde{\sigma}^0 i_3 - K^{\alpha 3}\bar{u}^0_{,\alpha}). \quad (45)$$

3.4. Minimization of $\mathcal{E}^2$

Using the divergence theorem, Eq. (32), the equilibrium equations (37)–(39), and the jump conditions (45), we eliminate $\tilde{u}^0_\pm, \tilde{u}^2_\pm$ and $\bar{u}^3$ from the expression for the energy $\mathcal{E}^2$ and we simplify this expression:

$$\mathcal{E}^2(\tilde{u}^1_\pm,\bar{u}^0,\bar{u}^1) := \int_{\tilde{\Omega}_\pm}\frac{1}{2}a_\pm(e(\tilde{u}^1_\pm))\cdot e(\tilde{u}^1_\pm)\,dz + \int_{\bar{B}}(K^{\alpha 3}(\bar{u}^1_\alpha)\cdot\bar{u}^1_{,3} + K^{\alpha\beta}(\bar{u}^0_{,\alpha})\cdot\bar{u}^1_\beta)\,dz. \quad (46)$$

In view of Eq. (43) and of the continuity conditions (14) written for $\tilde{u}^1_\pm$ and $\bar{u}^1$, the vector field $\bar{u}^1$ can be written in the form

$$\bar{u}^1(z_\alpha, z_3) = [\bar{u}^1](z_\alpha)z_3 + \frac{1}{2}S(\tilde{u}^1)(z_\alpha), \quad (47)$$

where $S(\tilde{u}_1)(z_\alpha) := \tilde{u}_1(z_\alpha, 1/2_+) + \tilde{u}^1(z_\alpha, -1/2^-)$. Substituting (47) and (45) into (46), and integrating with respect to z_3 give

$$\mathcal{E}^2(\tilde{u}^1_\pm,\bar{u}^0,\bar{u}^1) := \int_{\tilde{\Omega}_\pm}\frac{1}{2}a_\pm(e(\tilde{u}^1_\pm))\cdot e(\tilde{u}^1_\pm)\,dz + \int_{\bar{S}}\left(\frac{1}{2}K^{\alpha 3}(S(\tilde{u}^1)_{,\alpha})\cdot(K^{33})^{-1}(\tilde{\sigma}^0 i_3 - K^{\beta 3}\bar{u}^0_\beta) + \frac{1}{2}K^{\alpha\beta}(\bar{u}^0_{,\alpha})\cdot S(\tilde{u}^1)_{,\beta}\right)ds_z. \quad (48)$$

The Euler–Lagrange equations for the minimization problem of the latter functional are

$$\operatorname{div}(a_\pm(e(\tilde{u}^1_\pm))) = 0 \quad \text{in } \tilde{\Omega}_\pm, \tag{49}$$

$$a_\pm(e(\tilde{u}^1_\pm))n = 0 \quad \text{on } \tilde{\Gamma}_g, \tag{50}$$

$$a_\pm(e(\tilde{u}^1_\pm))n = 0 \quad \text{on } \partial\tilde{\Omega}_\pm \setminus \tilde{\Gamma}_g, \tag{51}$$

$$\begin{aligned} &- a_+(e(\tilde{u}^1_+))i_3 - \frac{1}{2}(K^{\alpha 3})^T(K^{33})^{-1}(\tilde{\sigma}^0 i_3 - K^{\beta 3}\bar{u}^0_{,\beta})_{,\alpha} \\ &\quad + \frac{1}{2}K^{\alpha\beta}(\bar{u}^0_{,\alpha\beta}) = 0 \quad \text{on } \bar{S}^+, \end{aligned} \tag{52}$$

$$\begin{aligned} &a_-(e(\tilde{u}^1_-))i_3 - \frac{1}{2}(K^{\alpha 3})^T(K^{33})^{-1}(\tilde{\sigma}^0 i_3 - K^{\beta 3}\bar{u}^0_{,\beta})_{,\alpha} \\ &\quad + \frac{1}{2}K^{\alpha\beta}(\bar{u}^0_{,\alpha\beta}) = 0 \quad \text{on } \bar{S}^-, \end{aligned} \tag{53}$$

$$((K^{\alpha 3})^T(K^{33})^{-1}(\tilde{\sigma}^0 i_3 - K^{\beta 3}\bar{u}^0_{,\beta}) + K^{\alpha\beta}(\bar{u}^0_{,\beta}))n_\alpha = 0 \quad \text{on } \partial\bar{S}. \tag{54}$$

We now add Eqs. (52) and (53) together to obtain the following relation for the jump in the traction at order one, defined as $[\tilde{\sigma}^1] := a_+(e(\tilde{u}^1_+))(z_\alpha, 1/2^+)i_3 - a_-(e(\tilde{u}^1_-))(z_\alpha, -1/2^-)i_3$:

$$[\tilde{\sigma}^1] = -(K^{\alpha 3})^T(K^{33})^{-1}(\tilde{\sigma}^0 i_3 - K^{\beta 3}\bar{u}^0_{,\beta})_{,\alpha} - K^{\alpha\beta}(\bar{u}^0_{,\alpha\beta}). \tag{55}$$

Again using (14), we rewrite the latter condition as follows:

$$[\tilde{\sigma}^1] = -(K^{\alpha 3})^T(K^{33})^{-1}(\tilde{\sigma}^0 i_3 - K^{\beta 3}\tilde{u}^0_{,\beta})_{,\alpha} - K^{\alpha\beta}(\tilde{u}^0_{,\alpha\beta}). \tag{56}$$

Relations (45) and (56) are non-local laws for imperfect contact in the minimization problem associated with the rescaled energy (18).

Remark. Condition (54) shows that the asymptotic expansions (21) and (22) do not hold in the neighborhood of $\partial\bar{S}$. More correctly, the energy (48) has to be defined not on the total domain but on a truncated domain defined as $(\tilde{\Omega}_\pm \cup \bar{B}) \setminus T_r$, where T_r is a torus of small radius $r > 0$ enclosing $\bar{S}$. In this case, (54) is replaced by the new condition

$$\begin{aligned} &\int_{\partial T_r \cup (\tilde{\Omega}_\pm)} a_\pm(e(\tilde{u}^1_\pm))n\,ds_z + (((K^{\alpha 3})^T(K^{33})^{-1}(\tilde{\sigma}^0 i_3 - K^{\beta 3}\bar{u}^0_{,\beta}) \\ &\quad + K^{\alpha\beta}(\bar{u}^0_{,\beta})))_{z\in\partial T_r \cap \bar{S}} n_\alpha = 0. \end{aligned} \tag{57}$$

As r tends to zero, there appear concentrated forces on the boundary of $\bar{S}$ (see Abdelmoula et al., 1998, Eq. (10)).

4. Form of the imperfect contact laws with various material symmetries

In this section, the forms of interface laws (45) and (56) for the following classes of material symmetry are deduced: isotropic, orthotropic, transversally isotropic, monoclinic and triclinic.

4.1. Isotropy

The thin layer is assumed to be isotropic and E, v and G are taken to denote the Young's modulus, the Poisson's ratio and the shear modulus, respectively.

Using the following expressions:

$$b_{1111} = b_{2222} = b_{3333} = \frac{E(-1+v)}{-1+v+2v^2}, \tag{58}$$

$$b_{1122} = b_{1133} = b_{2233} = -\frac{Ev}{-1+v+2v^2}, \tag{59}$$

$$b_{1212} = b_{1313} = b_{2323} = G, \tag{60}$$

we obtain the following expressions for the jumps in the displacement components at order one.

$$[\tilde{u}^1_1] = \frac{\tilde{\sigma}^0_{13}}{G} - \tilde{u}^0_{3,1}, \tag{61}$$

$$[\tilde{u}^1_2] = \frac{\tilde{\sigma}^0_{23}}{G} - \tilde{u}^0_{3,2}, \tag{62}$$

$$[\tilde{u}^1_3] = \frac{(-1+v+2v^2)\tilde{\sigma}^0_{33} + Ev(\tilde{u}^0_{1,1} + \tilde{u}^0_{2,2})}{E(-1+v)}. \tag{63}$$

To express the jumps of the stress components at order one, we have the relations

$$[\tilde{\sigma}^1_{13}] = \frac{-2E\tilde{u}^0_{1,11} + E(-1+v)\tilde{u}^0_{1,22} - (1+v)(E\tilde{u}^0_{2,12} + 2v\tilde{\sigma}^0_{33,1})}{2(-1+v^2)}, \tag{64}$$

$$[\tilde{\sigma}^1_{23}] = \frac{-E(1+v)\tilde{u}^0_{1,12} + E(-1+v)\tilde{u}^0_{2,11} - 2(E\tilde{u}^0_{2,22} + v(1+v)\tilde{\sigma}^0_{33,2})}{2(-1+v^2)}, \tag{65}$$

$$[\tilde{\sigma}^1_{33}] = -\tilde{\sigma}^0_{13,1} - \tilde{\sigma}^0_{23,2}. \tag{66}$$

4.2. Orthotropic symmetry

It is now assumed that the thin layer is orthotropic and we take $E_i (i = 1,2,3)$, $v_{ij}(i, j = 1,2,3)$ and $G_{ij}((i,j) = (1,2), (1,3), (2,3))$ to denote the Young's moduli, the Poisson's ratios and the shear moduli, respectively. We also recall that $\frac{v_{12}}{E_1} = \frac{v_{21}}{E_2}$, $\frac{v_{13}}{E_1} = \frac{v_{31}}{E_3}$ and $\frac{v_{23}}{E_2} = \frac{v_{32}}{E_3}$. By taking the expressions

$$b_{1111} = \frac{E_1^2(E_2 - E_3 v_{23}^2)}{E_1(E_2 - E_3 v_{23}^2) - E_2(E_2 v_{12}^2 + E_3 v_{13}(v_{13} + 2v_{12}v_{23}))}, \tag{67}$$

$$b_{2222} = \frac{E_2^2(E_1 - E_3 v_{13}^2)}{E_1(E_2 - E_3 v_{23}^2) - E_2(E_2 v_{12}^2 + E_3 v_{13}(v_{13} + 2v_{12}v_{23}))}, \tag{68}$$

$$b_{3333} = \frac{E_2 E_3(E_1 - E_2 v_{12}^2)}{E_1(E_2 - E_3 v_{23}^2) - E_2(E_2 v_{12}^2 + E_3 v_{13}(v_{13} + 2v_{12}v_{23}))}, \tag{69}$$

$$b_{1122} = b_{2211} = \frac{E_1 E_2(E_2 v_{12} + E_3 v_{13}v_{23})}{E_1(E_2 - E_3 v_{23}^2) - E_2(E_2 v_{12}^2 + E_3 v_{13}(v_{13} + 2v_{12}v_{23}))}, \tag{70}$$

$$b_{1133} = b_{3311} = \frac{E_1 E_2 E_3(v_{13} + v_{12}v_{23})}{E_1(E_2 - E_3 v_{23}^2) - E_2(E_2 v_{12}^2 + E_3 v_{13}(v_{13} + 2v_{12}v_{23}))}, \tag{71}$$

$$b_{2233} = b_{3322} = \frac{E_2 E_3(E_1 v_{23} + E_2 v_{12}v_{13})}{E_1(E_2 - E_3 v_{23}^2) - E_2(E_2 v_{12}^2 + E_3 v_{13}(v_{13} + 2v_{12}v_{23}))}, \tag{72}$$

$$b_{1212} = G_{12}; b_{1313} = G_{13}; b_{2323} = G_{23}, \tag{73}$$

the elastic constants b_{ijkl} are replaced by constants E_i, v_{ij} and G_{ij} and the following jumps in the displacements at order one are obtained:

$$[\tilde{u}^1_1] = \frac{\tilde{\sigma}^0_{13}}{G_{13}} - \tilde{u}^0_{3,1}, \tag{74}$$

$$[\tilde{u}^1_2] = \frac{\tilde{\sigma}^0_{23}}{G_{23}} - \tilde{u}^0_{3,2}, \tag{75}$$

$$[\tilde{u}^1_3] = \alpha_o\tilde{\sigma}^0_{33} - \beta_o\tilde{u}^0_{1,1} - \gamma_o\tilde{u}^0_{2,2}, \tag{76}$$

where

$$\alpha_o = \frac{E_1(E_2 - E_3 v_{23}^2) - E_2(E_2 v_{12}^2 + E_3 v_{13}(v_{13} + 2v_{12}v_{23}))}{E_2 E_3(E_1 - E_2 v_{12}^2)}, \tag{77}$$

$$\beta_o = \frac{E_1(\nu_{13}+\nu_{12}\nu_{23})}{E_1 - E_2\nu_{12}^2}, \tag{78}$$

$$\gamma_o = \frac{E_2\nu_{12}\nu_{13}+E_1\nu_{23}}{E_1 - E_2\nu_{12}^2}. \tag{79}$$

The jumps in the stress components at order one are

$$[\tilde{\sigma}_{13}^1] = \bar{\alpha}_o\tilde{u}_{1,11}^0 + G_{12}\bar{u}_{1,22}^0 + \tilde{\beta}_o\tilde{u}_{2,12}^0 - \frac{\nu_{13}+\nu_{12}\nu_{23}}{-1+\nu_{12}\nu_{21}}\sigma_{33,1}^0 \tag{80}$$

$$[\tilde{\sigma}_{23}^1] = \bar{\beta}_o\tilde{u}_{1,12}^0 + G_{12}\bar{u}_{2,11}^0 + \tilde{\gamma}_o\tilde{u}_{2,22}^0 - \frac{\nu_{23}+\nu_{13}\nu_{21}}{-1+\nu_{12}\nu_{21}}\sigma_{33,2}^0 \tag{81}$$

$$[\tilde{\sigma}_{33}^1] = -\tilde{\sigma}_{13,1}^0 - \tilde{\sigma}_{23,2}^0, \tag{82}$$

where

$$\bar{\alpha}_o = \frac{E_3(\nu_{13}+\nu_{12}\nu_{23})^2 - E_1(-1+\nu_{12}\nu_{21})(-1+\nu_{23}\nu_{32})}{(-1+\nu_{12}\nu_{21})(-1+\nu_{12}(\nu_{21}+\nu_{23}\nu_{31})+\nu_{23}\nu_{32}+\nu_{13}(\nu_{31}+\nu_{21}\nu_{32}))}, \tag{83}$$

$$\bar{\beta}_o = G_{12} - \frac{E_3(\nu_{13}\nu_{21}+\nu_{23})(\nu_{13}+\nu_{12}\nu_{23})}{(-1+\nu_{12}\nu_{21})(-1+\nu_{12}(\nu_{21}+\nu_{23}\nu_{31})+\nu_{23}\nu_{32}+\nu_{13}(\nu_{31}+\nu_{21}\nu_{32}))} \tag{84}$$

$$-\frac{E_2(\nu_{12}+\nu_{13}\nu_{32})}{-1+\nu_{12}(\nu_{21}+\nu_{23}\nu_{31})+\nu_{23}\nu_{32}+\nu_{13}(\nu_{31}+\nu_{21}\nu_{32})}, \tag{85}$$

$$\bar{\gamma}_o = \frac{-E_3(\nu_{13}\nu_{21}+\nu_{23})^2 + E_2(-1+\nu_{12}\nu_{21})(-1+\nu_{13}\nu_{31})}{(-1+\nu_{12}\nu_{21})(-1+\nu_{12}(\nu_{21}+\nu_{23}\nu_{31})+\nu_{23}\nu_{32}+\nu_{13}(\nu_{31}+\nu_{21}\nu_{32}))}. \tag{86}$$

4.3. Transverse isotropy (axis 1)

The thin layer is assumed to be transversally isotropic in one of the directions in the plane of the glue (for example along the 1-axis), and we take E_1, $E_2 = E_3$, ν_{12}, ν_{13} and G_{13} to denote the Young's modulus, the Poisson's ratio and the shear modulus, respectively. Using the following expressions:

$$b_{1111} = \frac{E_1^2(-1+\nu_{23})}{2E_3\nu_{12}^2 + E_1(-1+\nu_{23})}, \tag{87}$$

$$b_{2222} = \frac{E_3(-E_1+E_3\nu_{12}^2)}{(2E_3\nu_{12}^2+E_1(-1+\nu_{23}))(1+\nu_{23})}, \tag{88}$$

$$b_{3333} = \frac{E_3(-E_1+E_3\nu_{12}^2)}{(2E_3\nu_{12}^2+E_1(-1+\nu_{23}))(1+\nu_{23})}, \tag{89}$$

$$b_{1122} = b_{2211} = \frac{E_1E_3\nu_{12}}{E_1 - 2E_3\nu_{12}^2 - E_1\nu_{23}}, \tag{90}$$

$$b_{1133} = b_{3311} = \frac{E_1E_3\nu_{12}}{E_1 - 2E_3\nu_{12}^2 - E_1\nu_{23}}, \tag{91}$$

$$b_{2233} = b_{3322} = -\frac{E_3(E_3\nu_{12}^2+E_1\nu_{23})}{(2E_3n12^2+E_1(-1+\nu_{23}))(1+\nu_{23})}, \tag{92}$$

$$b_{1212} = G_{12}, \tag{93}$$

$$b_{1313} = G_{12}, \tag{94}$$

$$b_{2323} = \frac{2E_3}{1+\nu_{23}}, \tag{95}$$

we obtain the jump in the displacement at order one

$$[\tilde{u}_1^1] = \frac{\tilde{\sigma}_{13}^0}{G_{12}} - \tilde{u}_{3,1}^0, \tag{96}$$

$$[\tilde{u}_2^1] = \frac{\tilde{\sigma}_{23}^0(1+\nu_{23})}{2E_3} - \tilde{u}_{3,2}^0, \tag{97}$$

$$[\tilde{u}_3^1] = -\frac{(E_1-E_3\nu_{12}^2)(2E_3\nu_{12}^2+E_1(-1+\nu_{23}))(1+\nu_{23})}{E_3}\tilde{\sigma}_{33}^0 \tag{98}$$

$$-\frac{E_1\nu_{12}(1+\nu_{23})}{E_1-E_3\nu_{12}^2}\tilde{u}_{1,1}^0 - \frac{E_3\nu_{12}^2+E_1\nu_{23}}{E_1-E_3\nu_{12}^2}\tilde{u}_{2,2}^0, \tag{99}$$

and the jump in the stress vector at order one

$$[\sigma_{13}^1] = \frac{E_1^2}{E_1-E_3\nu_{12}^2}\tilde{u}_{1,11}^0 + G_{12}\tilde{u}_{1,22}^0 \tag{100}$$

$$+\frac{E_1G_{12}+E_1E_3\nu_{12}-E_3G_{12}\nu_{12}^2}{E_1-E_3\nu_{12}^2}\tilde{u}_{2,12}^0 + \frac{E_1\nu_{12}(1+\nu_{23})}{E_1-E_3\nu_{12}^2}\tilde{\sigma}_{33,1}^0, \tag{101}$$

$$[\tilde{\sigma}_{23}^1] = \frac{-E_3G_{12}\nu_{12}^2+E_1(G_{12}+E_3\nu_{12})}{E_1-E_3\nu_{12}^2}\tilde{u}_{1,12}^0 + G_{12}\tilde{u}_{2,11}^0 \tag{102}$$

$$+\frac{E_1E_3}{E_1-E_3\nu_{12}^2}\tilde{u}_{2,22}^0 + \frac{E_3\nu_{12}^2+E_1\nu_{23}}{E_1-E_3\nu_{12}^2}\tilde{\sigma}_{33,2}^0, \tag{103}$$

$$[\tilde{\sigma}_{33}^1] = -\tilde{\sigma}_{13,1}^0 - \tilde{\sigma}_{23,2}^0 \tag{104}$$

4.4. Transverse isotropy (axis 3)

The thin layer is assumed to be transversally isotropic in the orthogonal direction with respect to the plane of the glue and we take $E_1 = E_2$, E_3, ν_{12}, ν_{13} and G_{13} to denote the Young's modulus, the Poisson's ratio and the shear modulus, respectively. Using the following expressions:

$$b_{1111} = b_{2222} = -\frac{E_1(E_1-E_3\nu_{13}^2)}{(1+\nu_{12})(E_1(-1+\nu_{12})+2E_3\nu_{13}^2)}, \tag{105}$$

$$b_{3333} = \frac{E_1E_3(-1+\nu_{12})}{E_1(-1+\nu_{12})+2E_3\nu_{13}^2}, \tag{106}$$

$$b_{1122} = b_{2211} = -\frac{E_1(E_1\nu_{12}+E_3\nu_{13}^2)}{E_1(-1+\nu_{12})+2E_3\nu_{13}^2}, \tag{107}$$

$$b_{1133} = b_{3311} = \frac{E_1E_3\nu_{13}}{E_1-2E_3\nu_{13}^2-E_1\nu_{12}}, \tag{108}$$

$$b_{2233} = b_{3322} = \frac{E_1E_3\nu_{13}}{E_1-2E_3\nu_{13}^2-E_1\nu_{12}}, \tag{109}$$

$$b_{1212} = \frac{2E_1}{1+\nu_{12}}, \tag{110}$$

$$b_{1313} = G_{13}, \tag{111}$$

$$b_{2323} = G_{13}, \tag{112}$$

we obtain the jump in the displacement at order one

$$[\tilde{u}_1^1] = \frac{\tilde{\sigma}_{13}^0}{G_{13}} - \tilde{u}_{3,1}^0, \tag{113}$$

$$[\tilde{u}_2^1] = \frac{\tilde{\sigma}_{23}^0}{G_{13}} - \tilde{u}_{3,2}^0, \tag{114}$$

$$[\tilde{u}_3^1] = \frac{E_1(-1+\nu_{12})+2E_3\nu_{13}^2}{E_1E_3(-1+\nu_{12})}\tilde{\sigma}_{33}^0 + \frac{\nu_{13}}{-1+\nu_{12}}(\tilde{u}_{1,1}^0+\tilde{u}_{2,2}^0), \tag{115}$$

and the jump in the stress vector at order one

$$[\tilde{\sigma}^1_{13}] = -\frac{E_1}{-1+\nu_{12}^2}\tilde{u}^0_{1,11} - \frac{2E_1(-1+\nu_{12})}{-1+\nu_{12}^2}\tilde{u}^0_{1,22} \tag{116}$$

$$+\frac{2E_1 - E_1\nu_{12}}{-1+\nu_{12}^2}\tilde{u}^0_{2,12} + \frac{\nu_{13}(1+\nu_{12})}{-1+\nu_{12}^2}\tilde{\sigma}^0_{33,1}, \tag{117}$$

$$[\tilde{\sigma}^1_{23}] = -\frac{2E_1 - E_1\nu_{12}}{-1+\nu_{12}^2}\tilde{u}^0_{1,12} + \frac{2E_1(-1+\nu_{12})}{-1+\nu_{12}^2}\tilde{u}^0_{2,11} \tag{118}$$

$$+\frac{E_1}{-1+\nu_{12}^2}\tilde{u}^0_{2,22} + \frac{\nu_{13}(1+\nu_{12})}{-1+\nu_{12}^2}\tilde{\sigma}^0_{33,2}, \tag{119}$$

$$[\tilde{\sigma}^1_{33}] = -\tilde{\sigma}^0_{13,1} - \tilde{\sigma}^0_{23,2}. \tag{120}$$

4.5. Monoclinic symmetry

The thin layer is now assumed to be monoclinic. The jump in the displacement at order one is given by

$$[\tilde{u}^1] = A^m\tilde{\sigma}^0_3 + B^m D\tilde{u}^0, \tag{121}$$

where A^m is a 3×3-matrix, B^m is a 3×6-matrix,

$$\tilde{\sigma}^0_3 = (\tilde{\sigma}^0_{13}, \tilde{\sigma}^0_{23}, \tilde{\sigma}^0_{33})^T \quad \text{and} \quad D\tilde{u}^0 = (\tilde{u}^0_{3,1}, \tilde{u}^0_{3,2}, \tilde{u}^0_{1,1}, \tilde{u}^0_{1,2}, \tilde{u}^0_{2,1}, \tilde{u}^0_{2,2})^T.$$

The non-equal to zero coefficients of A^m (5 coefficients) and B^m (6 coefficients) are given in Appendix. The jump in the stress vector at order one is given by the relation

$$[\tilde{\sigma}^1]e_3 = C^m D^2\tilde{u}^0 + D^m D\tilde{\sigma}^0_3, \tag{122}$$

where C^m is a 3×6-matrix, D^m is a 3×4-matrix,

$$D\tilde{\sigma}^0_3 = (\tilde{\sigma}^0_{33,1}, \tilde{\sigma}^0_{33,2}, \tilde{\sigma}^0_{13,1}, \tilde{\sigma}^0_{23,1})^T \quad \text{and}$$
$$D^2\tilde{u}^0 = (\tilde{u}^0_{1,11}, \tilde{u}^0_{1,12}, \tilde{u}^0_{1,11}, \tilde{u}^0_{1,12}, \tilde{u}^0_{2,1}, \tilde{u}^0_{2,2})^T.$$

The non-equal to zero coefficients of C^m (8 coefficients) and D^m (6 coefficients) are given in Appendix.

4.6. Triclinic symmetry

The thin layer is assumed to be fully anisotropic. The jump in the displacement at order one is given by the relation

$$[\tilde{u}^1] = A^{an}\tilde{\sigma}^0_3 + B^{an} D\tilde{u}^0, \tag{123}$$

where A^{an} is a 3×3-matrix, B^{an} is a 3×6-matrix,

$$\tilde{\sigma}^0_3 = (\tilde{\sigma}^0_{13}, \tilde{\sigma}^0_{23}, \tilde{\sigma}^0_{33})^T \quad \text{and} \quad D\tilde{u}^0 = (\tilde{u}^0_{1,1}, \tilde{u}^0_{1,2}, \tilde{u}^0_{2,1}, \tilde{u}^0_{2,2}, \tilde{u}^0_{3,1}, \tilde{u}^0_{3,2})^T.$$

The non-equal to zero coefficients of A^{an} (9 coefficients) and B^{an} (14 coefficients) are given in Appendix. The jump in the stress vector at order one is given by the relation

$$[\tilde{\sigma}^1]e_3 = C^{an} D^2\tilde{u}^0 + D^{an} D\tilde{\sigma}^0_3, \tag{124}$$

where C^{an} is a 3×7-matrix, D^{an} is a 3×6-matrix,

$$D\tilde{\sigma}^0_3 = (\tilde{\sigma}^0_{33,1}, \tilde{\sigma}^0_{33,2}, \tilde{\sigma}^0_{13,1}, \tilde{\sigma}^0_{13,2}, \tilde{\sigma}^0_{23,1}, \tilde{\sigma}^0_{23,2})^T \quad \text{and}$$
$$D^2\tilde{u}^0 = (\tilde{u}^0_{1,11}, \tilde{u}^0_{1,12}, \tilde{u}^0_{1,11}, \tilde{u}^0_{1,12}, \tilde{u}^0_{2,11}, \tilde{u}^0_{2,12}, \tilde{u}^0_{2,22}, \tilde{u}^0_{3,12})^T.$$

The non-equal to zero coefficients of C^{an} (13 coefficients) and D^{an} (6 coefficients) are given in Appendix.

5. Conclusion

In this paper, a method is presented for obtaining interface law based on a model for a composite consisting of adherents separated by a thin interphase with a similar stiffness to that of the two adherents. This method is based on two main assumptions: the possible existence of expansions in series in terms of the interphase thickness of the displacement vector fields and stress tensor fields, and the assumption that the minimizations of the energies at each order is equivalent to the minimization of the energy of the initial three-dimensional problem. This yields a family of non-local imperfect interface laws, which define a jump in the displacements and in the traction vector fields. Several cases of interphase material symmetry are studied here, resulting in various types of interface laws. In future studies, it is proposed to test the validity of these laws by comparing the results obtained with experimental data, and to implement them in a computational software program. Shear tests and the possible use of digital image correlation method for full-field displacement measurements would allow a validation of the interface laws (Cognard et al., 2008; Nunes, 2010). A first comparison with results obtained by using the finite element method is performed in Lebon and Ronel (2007).

Appendix A. Coefficients for monoclinic materials

$$A^m_{11} = \frac{b_{2323}}{-b_{1323}b_{2313} + b_{1313}b_{2323}}; \quad A^m_{12} = -\frac{b_{2323}}{-b_{1323}b_{2313} + b_{1313}b_{2313}},$$
$$A^m_{21} = -\frac{b_{1323}}{-b_{1323}b_{2313} + b_{1313}b_{2323}}; \quad A^m_{22} = \frac{b_{1313}}{-b_{1323}b_{2313} + b_{1313}b_{2313}},$$
$$A^m_{33} = \frac{1}{b_{3333}},$$
$$B^m_{11} = \frac{b_{1323}b_{2313} - b_{1313}b_{2323}}{-b_{1323}b_{2313} + b_{1313}b_{2323}},$$
$$B^m_{22} = \frac{b_{1323}b_{2313} - b_{1313}b_{2323}}{-b_{1323}b_{2313} + b_{1313}b_{2323}},$$
$$B^m_{33} = -\frac{b_{1133}}{b_{3333}}; \quad B^m_{34} = -\frac{b_{1233}}{b_{3333}},$$
$$B^m_{35} = -\frac{b_{1233}}{b_{3333}}; \quad B^m_{36} = -\frac{b_{2233}}{b_{3333}},$$

$$C^m_{11} = \frac{-b_{1133}^2 + b_{1111}b_{3333}}{b_{3333}}; \quad C^m_{12} = \frac{-2b_{1133}b_{1233} + 2b_{1112}b_{3333}}{b_{3333}},$$
$$C^m_{13} = \frac{-b_{1233}^2 + b_{1212}b_{3333}}{b_{3333}}; \quad C^m_{14} = \frac{-b_{1133}b_{1233} + b_{1211}b_{3333}}{b_{3333}},$$
$$C^m_{15} = -\frac{b_{1233}^2 + b_{1133}b_{2233} - b_{1122}b_{3333} - b_{1212}b_{3333}}{b_{3333}},$$
$$C^m_{16} = \frac{-b_{1233}b_{2233} + b_{2212}b_{3333}}{b_{3333}}; \quad C^m_{21} = \frac{-b_{1133}b_{1233} + b_{1112}b_{3333}}{b_{3333}},$$
$$C^m_{22} = \frac{-b_{1233}^2 - b_{1133}b_{2233} + (b_{1122} + b_{1212})b_{3333}}{b_{3333}},$$
$$C^m_{23} = \frac{-b_{1233}b_{2233} + b_{1222}b_{3333}}{b_{3333}}; \quad C^m_{24} = \frac{-b_{1233}^2 + b_{1212}b_{3333}}{b_{3333}},$$
$$C^m_{25} = \frac{-2b_{1233}b_{2233} + 2b_{1222}b_{3333}}{b_{3333}}; \quad C^m_{26} = \frac{-b_{2233}^2 + b_{2222}b_{3333}}{b_{3333}},$$
$$D^m_{11} = \frac{b_{1133}}{b_{3333}}; \quad D^m_{12} = \frac{b_{1233}}{b_{3333}},$$
$$D^m_{21} = \frac{b_{1233}}{b_{3333}}; \quad D^m_{22} = \frac{b_{2233}}{b_{3333}},$$
$$D^m_{33} = -1; \quad D^m_{34} = -1$$

Appendix B. Coefficients for anisotropic materials

$$
\begin{aligned}
A_{11}^{an} &= \frac{b_{2333}^2 - b_{2323}b_{3333}}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
A_{12}^{an} &= -\frac{-b_{1333}b_{2333} + b_{1323}b_{3333}}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
A_{13}^{an} &= -\frac{b_{1333}b_{2323} - b_{1323}b_{2333}}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
A_{21}^{an} &= A_{12}^{an},\\
A_{22}^{an} &= \frac{b_{1333}^2 - b_{1313}b_{3333}}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
A_{23}^{an} &= \frac{b_{1313}b_{2333} - b_{1323}b_{1333}}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
A_{31}^{an} &= A_{13}^{an},\\
A_{32}^{an} &= \frac{-b_{1323}b_{1333} + b_{1313}b_{2333}}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
A_{33}^{an} &= \frac{b_{1323}^2 - b_{1313}b_{2323}}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})}.
\end{aligned}
$$

$$
\begin{aligned}
B_{11}^{an} &= \frac{b_{1133}(b_{1333}b_{2323} + b_{1323}b_{2333}) + b_{1123}(b_{1333}b_{2333} - b_{1323}b_{3333}) - b_{1113}(b_{2333}^2 - b_{2323}b_{3333})}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{12}^{an} &= \frac{-b_{1233}(b_{1333}b_{2323} - b_{1323}b_{2333}) + b_{1223}(b_{1333}b_{2333} - b_{1323}b_{3333}) - b_{1213}(b_{2333}^2 - b_{2323}b_{3333})}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{13}^{an} &= \frac{-b_{1233}(b_{1333}b_{2323} - b_{1323}b_{2333}) + b_{1223}(b_{1333}b_{2333} - b_{1323}b_{3333}) - b_{1213}(b_{2333}^2 - b_{2323}b_{3333})}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{14}^{an} &= \frac{-b_{1333}(b_{2233}b_{2323} + b_{2223}b_{2333}) + b_{1323}(b_{2233}b_{2333} - b_{2223}b_{3333}) - b_{1322}(b_{2333}^2 - b_{2323}b_{3333})}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{15}^{an} &= \frac{-b_{1333}^2 b_{2323} + 2b_{1323}b_{1333}b_{2333} - b_{1313}b_{2333}^2 - b_{1323}^2 b_{3333} + b_{1313}b_{2323}b_{3333}}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{16}^{an} &= 0,\\
B_{21}^{an} &= \frac{b_{1133}(b_{1323}b_{1333} - b_{1313}b_{2333}) + b_{1113}(b_{1333}b_{2333} + b_{1313}b_{3333}) - b_{1113}(b_{1323}b_{3333} - b_{1333}^2)}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{22}^{an} &= \frac{b_{1233}(b_{1323}b_{1333} - b_{1313}b_{2333}) + b_{1213}(b_{1333}b_{2333} - b_{1323}b_{3333}) + b_{1223}(b_{1313}b_{3333} - b_{1333}^2)}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{23}^{an} &= \frac{b_{1233}(b_{1323}b_{1333} - b_{1313}b_{2333}) - b_{1223}(b_{1333}^2 + b_{1313}b_{3333}) + b_{1213}(b_{1333}b_{2333} - b_{1323}b_{3333})}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{24}^{an} &= \frac{b_{2223}(b_{1323}b_{1333} - b_{1333}^2) + b_{1322}(b_{1333}b_{2333} - b_{1323}b_{3333}) - b_{1313}(b_{2233}b_{2333} - b_{2223}b_{3333})}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{25}^{an} &= 0,\\
B_{26}^{an} &= \frac{-b_{1333}^2 b_{2323} + 2b_{1323}b_{1333}b_{2333} - b_{1313}b_{2333}^2 - b_{1323}^2 b_{3333} + b_{1313}b_{2323}b_{3333}}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{31}^{an} &= \frac{b_{1133}(b_{1313}b_{2323} - b_{1323}^2) + b_{1123}(b_{1323}b_{1333} - b_{1313}b_{2333}) - b_{1113}(b_{1333}b_{2323} - b_{1323}b_{2333})}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{32}^{an} &= \frac{b_{1233}(b_{1313}b_{2323} - b_{1323}^2) + b_{1223}(b_{1323}b_{1333} - b_{1313}b_{2333}) - b_{1213}(b_{1333}b_{2323} - b_{1323}b_{2333})}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{33}^{an} &= \frac{b_{1233}(b_{1313}b_{2323} - b_{1323}^2) + b_{1223}(b_{1323}b_{1333} - b_{1313}b_{2333}) - b_{1213}(b_{1333}b_{2323} - b_{1323}b_{2333})}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{34}^{an} &= \frac{b_{1323}(b_{1333}b_{2223} - b_{1323}b_{2233}) - b_{1322}(b_{1333}b_{2323} - b_{1323}b_{2333}) + b_{1313}(b_{2233}b_{2323} - b_{2223}b_{2333})}{b_{1333}^2 b_{2323} - 2b_{1323}b_{1333}b_{2333} + b_{1323}^2 b_{3333} + b_{1313}(b_{2333}^2 - b_{2323}b_{3333})},\\
B_{35}^{an} &= 0,\\
B_{36}^{an} &= 0.
\end{aligned}
$$

$$C_{11}^{an} = \frac{b_{1133}^2 + b_{1111}b_{3333}}{b_{3333}},$$
$$C_{12}^{an} = \frac{-2b_{1133}b_{1233} + 2b_{1112}b_{3333}}{b_{3333}},$$
$$C_{13}^{an} = \frac{-b_{1233}^2 + b_{1212}b_{3333}}{b_{3333}},$$
$$C_{14}^{an} = \frac{-b_{1133}b_{1233} + b_{1211}b_{3333}}{b_{3333}},$$
$$C_{15}^{an} = \frac{-b_{1233}^2 - b_{1133}b_{2233} + b_{1122}b_{3333} + b_{1212}b_{3333}}{b_{3333}},$$
$$C_{16}^{an} = \frac{-b_{1233}b_{2233} + b_{2212}b_{3333}}{b_{3333}},$$
$$C_{21}^{an} = \frac{-b_{1133}b_{1233} + b_{1112}b_{3333}}{b_{3333}},$$
$$C_{22}^{an} = \frac{-b_{1233}^2 - b_{1133}b_{2233} + (b_{1122} + b_{1212})b_{3333}}{b_{3333}},$$
$$C_{23}^{an} = \frac{-b_{1233}b_{2233} + b_{1222}b_{3333}}{b_{3333}},$$
$$C_{24}^{an} = \frac{-b_{1233}^2 + b_{1212}b_{3333}}{b_{3333}},$$
$$C_{25}^{an} = \frac{-2b_{1233}b_{2233} + 2b_{1222}b_{3333}}{b_{3333}},$$
$$C_{26}^{an} = \frac{-b_{2233}^2 + b_{2222}b_{3333}}{b_{3333}},$$
$$C_{37}^{an} = b_{1323} - b_{2313}.$$

$$D_{11}^{an} = \frac{b_{1133}}{b_{3333}},$$
$$D_{12}^{an} = \frac{b_{1233}}{b_{3333}},$$
$$D_{21}^{an} = \frac{b_{1233}}{b_{3333}},$$
$$D_{22}^{an} = \frac{b_{2233}}{b_{3333}},$$
$$D_{33}^{an} = \frac{b_{1323}^2 - b_{1313}b_{2323}}{b_{1323}b_{2313} - b_{1313}b_{2323}},$$
$$D_{34}^{an} = \frac{b_{1323}b_{2323} - b_{2313}b_{2323}}{b_{1323}b_{2313} - b_{1313}b_{2323}},$$
$$D_{35}^{an} = \frac{-b_{1313}b_{1323} + b_{1313}b_{2313}}{b_{1323}b_{2313} - b_{1313}b_{2323}},$$
$$D_{36}^{an} = \frac{b_{2313}^2 - b_{1313}b_{2323}}{b_{1323}b_{2313} - b_{1313}b_{2323}}.$$

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