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A Kinematic Model of the Nonholonomic n-bar System: Geometry and Flatness
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Abstract: We propose a kinematic model of a system moving in $\mathbb{R}^{m+1}$ and consisting of $n$ rigid bars attached successively to each other and subject to the nonholonomic constraints that the velocity of the source point of each bar is parallel to that bar. We prove that the associated control system is controllable and feedback equivalent to the $m$-chained form around any regular configuration. Hence we deduce that the $n$-bar system is flat and the cartesian position of the source point of the last bar is a flat output. The $n$-bar system is a natural generalization of the $n$-trailer system and we provide a comparison of flatness properties of both systems.

Keywords: $n$-bar system, $m$-chained form, Cartan distribution, feedback equivalence, flatness

1. INTRODUCTION

The well known $n$-trailer system was proposed by Laumond (1991) to model a unicycle-like robot towing trailers. This nonholonomic model has attracted a lot of attention and has been the source of inspiration to study its various properties: controllability (Laumond (1991)), structure (Jean (1998), Pasillas-Lépine and Respondek (2001c), Mormul (2000)), flatness (Fließ et al. (1995), Jakubczyk (1993)), motion planning and tracking (Laumond (1998), Murray and Sastry (1993), Pasillas-Lépine and Respondek (2001d)), optimal control (Laumond (1998)), etc. In this paper we propose its generalization, which we call the $n$-bar system, consisting of a "train" of $n$ rigid bars subject to nonholonomic constraints (see a detailed description in Section 2 below). We study the geometry of the model of the $n$-bar system and prove that around any regular configuration (that is, none of the angles between two consecutive bars is $\pm 2\pi$), the associated control system is feedback equivalent to the $m$-chained form. This implies that the $n$-bar system is flat around any regular configuration and we show that the cartesian position of the source point of the last (from the top) bar is a flat output. We show also that all other minimal flat outputs are equivalent to that one. This is in contrast with the $n$-trailer system for which the position of the last trailer is also a flat output but there is a whole family of non equivalent flat outputs (parameterized by one function of three variables, see Li and Respondek (2010b)). As a by-product of our considerations we deduce the global controllability of the $n$-bar system since it is accessible at any (regular or not) configuration. We send the reader to Li (2010) and Li and Respondek (2010c) for proofs and a geometric analysis of the $n$-bar system and to Slayaman (2008) and Slayaman and Pelletier (2009) for another, although similar, model for the $n$-bar system (called there an articulated arm) and for a detailed analysis of singular configurations.

This paper is organized as follows. We define our model of the $n$-bar system in Section 2. We provide geometric notions and recall a characterization of Cartan distributions $\mathcal{C}^m(\mathbb{R},\mathbb{R}^m)$ in Section 3. Then we give our main results: equivalence of the $n$-bar system in $\mathbb{R}^{m+1}$ to the $m$-chained form and global controllability in Section 4 and its flatness in Section 5.

2. N-BAR SYSTEM IN $\mathbb{R}^{M+1}$

In this section we will consider the $n$-bar system moving in $\mathbb{R}^{m+1}$, as shown on Figure 1, and derive a kinematic model for it. It is assumed that all $n$ components of the $n$-bar system are attached in such a way that $P_i$ is the source point of the $(i+1)$th bar and simultaneously the endpoint of the $i$th bar and that the instantaneous velocity of the point $P_i$ is parallel to the vector $\dot{P}_i\dot{P}_{i+1}$. For $0 \leq i \leq n-1$, furthermore, each rigid bar is assumed to have length one. The coordinates of $P_i$ in $\mathbb{R}^{m+1}$ are given by $P_i = (x^{1}_i, x^{2}_i, \ldots, x^{m+1}_i)$, $0 \leq i \leq n$. Clearly, the configuration of the $n$-bar system can be described completely by the $(n+1)(m+1)$ coordinates

$$x^{1}_0, \ldots, x^{m+1}_0, x^{1}_1, \ldots, x^{m+1}_1, \ldots, x^{1}_n, \ldots, x^{m+1}_n$$

in $X = \mathbb{R}^{(n+1)(m+1)}$. Due to the assumption $|P_i\dot{P}_{i+1}| = 1$, for $0 \leq i \leq n-1$, we have the holonomic constraints $\Psi(x) = 0$, where $\Psi = (\Psi_1, \ldots, \Psi_n)^T : X = \mathbb{R}^{(n+1)(m+1)} \rightarrow \mathbb{R}^n$ is given by

$$\begin{align*}
\Psi_1(x) &= (x^{1}_1 - x^{1}_0)^2 + \cdots + (x^{m+1}_0 - x^{m+1}_n)^2 - 1 \\
\Psi_2(x) &= (x^{2}_2 - x^{2}_1)^2 + \cdots + (x^{m+1}_0 - x^{m+1}_n)^2 - 1 \\
&\vdots \\
\Psi_n(x) &= (x^{n}_n - x^{n}_{n-1})^2 + \cdots + (x^{n+1}_0 - x^{m+1}_{n-1})^2 - 1.
\end{align*}$$

(1)

Under these $n$ holonomic constraints, the true configuration space of the $n$-bar system becomes the regular embedded submanifold $Q = \mathbb{R}^{m+1} \times (S^m)^n \subset X$ defined...
by $Q = \{ x \in X : \Psi(x) = 0 \}$. Moreover, the constraint $\Psi(x) = 0$ implies that for any $1 \leq i \leq n$, there always exists $1 \leq \sigma(i) \leq m + 1$, such that $x_i^\sigma(i) - x_{i-1}^\sigma(i) \neq 0$. Now the assumption that the instantaneous velocity of the point $P_i$ is parallel to the vector $P_iP_{i+1}$, for $0 \leq i \leq n - 1$, imposes the following nonholonomic constraints on the $n$-bar system: the velocity of the system along any trajectory is annihilated by the following differential 1-forms
\[ \Omega_i^j = (x_i^j - x_{i-1}^j)dx_{i-1}^\sigma(i) - (x_i^\sigma(i) - x_{i-1}^\sigma(i))dx_i^j, \]
for $1 \leq i \leq n, 1 \leq j \leq m + 1$ and $j \neq \sigma(i)$. The distribution $E$ annihilated by all forms $\Omega_i^j$ is given by
\[ E = \bigcap_{i,j} \ker \Omega_i^j = \text{span} \{ g_1, \ldots, g_{n+m+1} \}, \]
where
\[ g_1 = (x_1^1 - x_0^1)\frac{\partial}{\partial x_0^1} + \cdots + (x_{m+1}^1 - x_0^1)\frac{\partial}{\partial x_{m+1}^1}, \]
\[ g_2 = (x_2^1 - x_1^1)\frac{\partial}{\partial x_1^1} + \cdots + (x_{m+1}^1 - x_1^1)\frac{\partial}{\partial x_{m+1}^1}, \]
\[ \vdots \]
\[ g_n = (x_n^1 - x_{n-1}^1)\frac{\partial}{\partial x_{n-1}^1} + \cdots + (x_{n+1}^1 - x_{n-1}^1)\frac{\partial}{\partial x_{n+1}^1}, \]
\[ g_{n+i} = \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq m + 1, \]
which defines the control-linear system on $X = \mathbb{R}^{(n+1)(m+1)}$.

3. CHARACTERIZATION OF CARTAN DISTRIBUTION $\mathcal{C}^N(\mathbb{R}, [\mathcal{D}])$

Consider an arbitrary distribution $\mathcal{D}$. The derived flag of $\mathcal{D}$ is the sequence of modules of vector fields $\mathcal{D}^{(0)} \subset \mathcal{D}^{(1)} \subset \cdots$ defined inductively by
\[ \mathcal{D}^{(0)} = \mathcal{D} \quad \text{and} \quad \mathcal{D}^{(i+1)} = \mathcal{D}^{(i)} + [\mathcal{D}^{(i)}, \mathcal{D}^{(i)}], \quad \text{for} \ i \geq 0. \]

The Lie flag of $\mathcal{D}$ is the sequence of modules of vector fields $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots$ defined inductively by
\[ \mathcal{D}_0 = \mathcal{D} \quad \text{and} \quad \mathcal{D}_{i+1} = \mathcal{D}_i + [\mathcal{D}_0, \mathcal{D}_i], \quad \text{for} \ i \geq 0. \]

In general, the derived and Lie flags are different; though for any point $p$ in the underlying manifold the inclusion $\mathcal{D}_i(p) \subset \mathcal{D}^{(i)}(p)$ holds, for $i \geq 0$. Two distributions $\mathcal{D}$ and $\mathcal{D}'$ defined on two manifolds $M$ and $\tilde{M}$, respectively, are equivalent if there exists a smooth diffeomorphism $\varphi$ between $M$ and $\tilde{M}$ such that $(\varphi_\ast \mathcal{D})(\tilde{p}) = \mathcal{D}(\tilde{p})$, for each point $\tilde{p}$ in $\tilde{M}$.

An alternative description of the above notions can also be given using the dual language of differential forms. A codistribution $\mathcal{I}$ of rank $s$ on a smooth manifold $M$ (or a Pfaffian system) is a map that assigns smoothly to each
point $p$ in $M$ a linear subspace $T(p) \subset T_p^* M$ of dimension $s$. Such a field of cotangent $s$-planes is spanned locally by $s$ pointwise linearly independent smooth differential 1-forms $\omega_1, \ldots, \omega_s$ on $M$, which will be denoted by $\mathcal{I} = \text{span} \{\omega_1, \ldots, \omega_s\}$. Two codistributions (Pfaffian systems) $\mathcal{I}$ and $\tilde{\mathcal{I}}$ defined on two manifolds $M$ and $\tilde{M}$, respectively, are equivalent if there exists a smooth diffeomorphism $\varphi$ between $M$ and $\tilde{M}$ such that $\mathcal{I}(p) = (\varphi^* \tilde{\mathcal{I}})(p)$ for each point $p$ in $M$. For a codistribution $\mathcal{I}$, its derived flag $\mathcal{I}(0) \supset \mathcal{I}(1) \supset \cdots$ can be defined by $\mathcal{I}(0) = \mathcal{I}$, $\mathcal{I}(i+1) = \{\omega \in \mathcal{I}(i) : d\omega \equiv 0 \mod \mathcal{I}(i)\}$, for $i \geq 0$, provided that each element $\mathcal{I}(i)$ of this sequence has constant rank. In this case, it is immediate to see that the derived flag of the distribution $\mathcal{D} = \mathcal{I}^{\perp}$ coincides with the sequence of distributions that annihilate the elements of the derived flag of $\mathcal{I}$, that is $\mathcal{D}(i) = (\mathcal{I}(0) \cap \mathcal{D})$. Consider $\mathcal{J}^d(\mathbb{R}^n, \mathbb{R}^m)$, the space of $d$-jets of smooth maps from $\mathbb{R}^n$ into $\mathbb{R}^m$ and denote its canonical coordinates by $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_m$, where $\mathcal{X}_0$ represents the independent variable and $\mathcal{X}_i$ for $1 \leq i \leq m$, represent the dependent variables, and $\mathcal{X}_{i+j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, correspond to the ordinary derivatives $\frac{\partial \mathcal{X}_{i+j}}{\partial \mathcal{X}_i}$. The Cartan distribution on $\mathcal{J}^d(\mathbb{R}^n, \mathbb{R}^m)$, which we will denote by $\mathcal{CC}^d(\mathbb{R}^n, \mathbb{R}^m)$, is the completely nonholonomic distribution spanned by the following family of vector fields

$$\frac{\partial}{\partial \mathcal{X}_0} + \sum_{j=1}^{n-1} \sum_{i=1}^{m} \frac{\partial}{\partial \mathcal{X}_i} \mathcal{X}_{i+j+1} \mathcal{X}_{i+j} \mathcal{X}_{i}, \ldots, \frac{\partial}{\partial \mathcal{X}_m} \mathcal{X}_{i}$$

or, equivalently, annihilated by the following family of differential 1-forms $\mathcal{X}_i^1 - \mathcal{X}_{i+1}^1 + \mathcal{X}_i^0$, $0 \leq j \leq n-1$, $1 \leq i \leq m$.

The problem of characterizing distributions that are locally equivalent to the Cartan distribution $\mathcal{CC}^d(\mathbb{R}^n, \mathbb{R}^m)$ has been studied and solved in the following way by Pasillas-Lépine and Respondek (2001b) (see also Yamaguchi (1982), Pasillas-Lépine and Respondek (2001a), Moomr (2004)):

**Theorem 1.** A distribution $\mathcal{D}$ of rank $m+1$, with $m \geq 2$, on a manifold $M$ of dimension $(n+1)m+1$ is equivalent, in a small enough neighborhood of a point $p$ in $M$, to the Cartan distribution $\mathcal{CC}^d(\mathbb{R}^n, \mathbb{R}^m)$ if and only if the following conditions hold:

(i) $\mathcal{D}^{(n)}(p) = T_p M$;

(ii) $\mathcal{D}^{(n-1)}$ is of constant rank $nm + 1$ and contains an involutive subdistribution $\mathcal{L}_{n-1}$ that has constant corank one in $\mathcal{D}^{(n-1)}$;

(iii) $\mathcal{D}(p)$ is not contained in $\mathcal{L}_{n-1}(p)$.

Moreover, if $m \geq 3$, $\mathcal{L}_{n-1}$ exists if and only if the Engel rank of $(\mathcal{D}^{(n-1)})^{\perp}$ equals 1 and rank$\mathcal{L}(\mathcal{D}^{(n-1)}) = (n-1)m$ and is given as $\mathcal{L}_{n-1} = \mathcal{F}_1 + \cdots + \mathcal{F}_m$, where $\mathcal{F}_i = \{f \in \mathcal{D}^{(n-1)} : f \wedge d\omega_i \in (\mathcal{D}^{(n-1)})^{\perp}\}$ and $\omega_i$'s are any differential 1-forms such that $\mathcal{I}^{(n-1)} = (\mathcal{D}^{(n-1)})^{\perp} = \text{span}\{\omega_1, \ldots, \omega_m\}$.

**Remarks 1.** The involutive subdistribution $\mathcal{L}_{n-1}$, whose existence is claimed by (ii), is unique (if it exists) and will be called the canonical involutive subdistribution in $\mathcal{D}^{(n-1)}$. The uniqueness, involutivity, and the explicit form of $\mathcal{L}_{n-1} = \mathcal{F}_1 + \cdots + \mathcal{F}_m$ follow from a result of Bryant (1979) and have been shown in Pasillas-Lépine and Respondek (2001b).

**Remark 2.** Item (i) and (ii) describe the essential geometric property of distributions equivalent to the Cartan distribution $\mathcal{CC}^d(\mathbb{R}^n, \mathbb{R}^m)$ while the condition (iii) distinguishes regular points $p$ at which $\mathcal{D}(p) \not\subset \mathcal{L}_{n-1}(p)$ from singular points, where this last condition is violated.

The case $m = 1$ is excluded from Theorem 1 because if an involutive subdistribution of corank one $\mathcal{L}_{n-1} \subset \mathcal{D}^{(n-1)}$ exists it cannot be unique and therefore there is not a canonical one. However, a “non-canonical” version of Theorem 1 holds for $m = 1$ as well, as proved in Pasillas-Lépine and Respondek (2001b): a rank-two distribution is equivalent to $\mathcal{CC}^d(\mathbb{R}^1, \mathbb{R}^m)$, called also the Goursat normal form or chain form, if and only if there exists a distribution $\mathcal{L}_{n-1}$ satisfying the conditions (i), (ii), and (iii) of Theorem 1.

Let a distribution $\mathcal{D}$ of rank $m+1$, with $m \geq 1$, satisfy the items (i) and (ii) of Theorem 1. The regular locus of $\mathcal{D}$, denoted by $\text{Reg}(\mathcal{D})$, is the subset of $M$ consisting of points at which $\mathcal{D}$ is equivalent to the Cartan distribution $\mathcal{CC}^d(\mathbb{R}^n, \mathbb{R}^m)$ at $0 \in \mathbb{R}^{(n+1)(m+1)}$. It can be proved that $\text{Reg}(\mathcal{D})$ is an open and dense subset of $M$. In the case $m \geq 2$, since $\mathcal{L}_{n-1}$ is unique we clearly have $\text{Reg}(\mathcal{D}) = \{p \in M : \mathcal{D}(p) \not\subset \mathcal{L}_{n-1}(p)\}$.

**4. FIRST MAIN RESULT: EQUIVALENCE OF THE N-BAR SYSTEM TO THE M-CHAINED FORM**

Consider two driftless control systems

$$\Sigma : \dot{x} = \sum_{i=0}^{m} f_i(x)u_i = f(x)u, \quad x \in M,$$

and $\tilde{\Sigma} : \dot{x} = \sum_{i=0}^{m} \tilde{f}_i(x)\tilde{u}_i = \tilde{f}(x)\tilde{u}, \quad x \in \tilde{M},$

where $u = (u_0, \ldots, u_m)^T \in \mathbb{R}^{m+1}$, $\tilde{u} = (\tilde{u}_0, \ldots, \tilde{u}_m)^T \in \mathbb{R}^{m+1}$ and the rows $f = (f_0, \ldots, f_m)$ and $\tilde{f} = (\tilde{f}_0, \ldots, \tilde{f}_m)$ are formed by $C^\infty$-smooth vector fields $f_i$ and $\tilde{f}_i$, $0 \leq i \leq m$, on $M$ and $\tilde{M}$, respectively. We say that $\Sigma$ and $\tilde{\Sigma}$ are feedback equivalent if there exists a diffeomorphism $\varphi : M \to \tilde{M}$, $\varphi = \varphi(x)$ and a feedback transformation $u = \beta(x)\tilde{u}$, where $\beta(x)$ is an invertible $C^\infty$-smooth $(m+1) \times (m+1)$-matrix such that $D\varphi(x) \cdot f(x)\beta(x) = \tilde{f}(\varphi(x))$. $\varphi$ is called $\varphi(x)$.

**Definition 2.** An $(m+1)$-input driftless control system $\Sigma : \dot{x} = \sum_{i=0}^{m} u_i f_i(x)$, defined on $\mathbb{R}^{(n+1)(m+1)}$, is said to be in the $m$-chained form if it is represented by

$$\dot{x}_0 = u_0 \quad \dot{x}_1 = x_1^1 u_0 \quad \cdots \quad \dot{x}_m^1 = x_1^m u_0$$

$$\cdots \quad \dot{x}_m^{m-1} = x_1^m u_0 \quad \cdots \quad \dot{x}_m = u_m.$$

A system in the $m$-chained form is also called the canonical contact system on $\mathcal{J}^d(\mathbb{R}^n, \mathbb{R}^m)$. In fact, the vector fields

...
Remark 1. The system satisfies the condition (i) and (ii) of Theorem 1.

Theorem 3. The n-bar system Π moving in $\mathbb{R}^{n+1}$, for $m \geq 1$, defined by (6), is locally feedback equivalent to the $m$-chained form at any point $x \in X = \mathbb{R}^{(n+1)(m+1)}$ satisfying $\Psi(x) = 0$ (that is, at $x$ corresponding to a point $q \in Q$) such that

(R1) $\sum_{j=1}^{m+1} (x_i^j - x_{i-1}^j)(x_{i+1}^j - x_i^j) \neq 0$, for $1 \leq i \leq n-1$, if $m \geq 2$;

(R2) $\sum_{j=1}^{m+1} (x_i^j - x_{i-1}^j)(x_{i+1}^j - x_i^j) \neq 0$, for $2 \leq i \leq n-1$, if $m = 1$.

Moreover, at any point $q \in Q$ (equivalently, at any point $x \in X = \mathbb{R}^{(n+1)(m+1)}$ satisfying $\Psi(x) = 0$), the $n$-bar system satisfies the condition (i) and (ii) of Theorem 1.

Remark 2. The regularity condition $\sum_{j=1}^{m+1} (x_i^j - x_{i-1}^j)(x_{i+1}^j - x_i^j) \neq 0$ has a clear interpretation for the $n$-bar system. Let $\theta_i$, for $1 \leq i \leq n-1$, denote the angles of the $(i+1)$-th bar with respect to the $i$-th bar, i.e., the angle between the vectors $P_{i-1}P_i$ and $P_iP_{i+1}$. Then clearly the regularity conditions mean that $\theta_i$ are different from $\pm \pi/2$, in other words, the $i$-th bar is not perpendicular to the $(i+1)$-th one. Using the angles $\theta_i$, the regular locus can be rewritten as

$$\text{Reg}(\Gamma)_{m \geq 2} = \{ q \in Q : \theta_i \neq \pm \pi/2 \, \, \, 1 \leq i \leq n-1 \}$$

$$\text{Reg}(\Gamma)_{m = 1} = \{ q \in Q : \theta_i \neq \pm \pi/2 \, \, \, 2 \leq i \leq n-1 \}$$

It is interesting to observe the difference between the planar ($m = 1$) and all other cases ($m \geq 2$). Namely, the angle $\pm \pi/2$ between the bars $P_{i-1}P_i$ and $P_iP_{i+1}$ (the two most far from the controlled one) is a singularity for $m \geq 2$ but is not for the planar case. The latter implies, in particular, that the 2-bar system in $\mathbb{R}^2$ is transformable into the chained form even if the bars are perpendicular. This is not true any longer if we consider the 2-bar system in the space $\mathbb{R}^{m+1}$, $m \geq 2$ (in $\mathbb{R}^3$, for instance). Of course, the 2-bar system in $\mathbb{R}^2$ is just the 1-trailer system (a unicycle-like mobile robot towing one trailer or, equivalently, a nonholonomic car) and it is well known that the system can be brought into the chained form even if the axles are perpendicular. In other words, the rank 2 distributions on 4-dimensional manifolds with the growth vector $(2,3,4)$ have no singularities, a result that goes back to Engel (1890).

The property of controllability of the $n$-bar system can also be obtained from Theorem 3.

Corollary 4. The $n$-bar system $\Gamma$ is globally controllable on $\mathbb{R}^{m+1} \times (S^m)^n$.

5. SECOND MAIN RESULT: FLATNESS OF THE N-BAR SYSTEM IN $\mathbb{R}^{m+1}$

Consider a smooth nonlinear control system $\Xi : \dot{x} = f(x,u)$, where $x \in X$, an $n$-dimensional manifold and $u \in U$, an $m$-dimensional manifold. Given any integer $l$, we associate to $\Xi$ its $l$-prolongation $\Xi^l$ by

$$\dot{x} = f(x,u^0)$$

$$u^0 = u^1$$

which can be considered as a control system on $X^l = X \times U \times \mathbb{R}^m_l$, whose state variables are $(x,u^0,u^1,\ldots,u^l)$ and whose $m$ controls are the $m$ components of $u^{l+1}$. Denote $u^l = (u^0,u^1,\ldots,u^l)$.

Definition 5. The system $\Xi$ is called flat at a point $(x_0,u_0^0) \in X^l = X \times U \times \mathbb{R}^m_l$, for some $l \geq 0$, if there exist a neighborhood $O(X^l)$ of $(x_0,u_0^0)$ and $m$ smooth functions

$$h_i = h_i(x,u^0,u^1,\ldots,u^l), \, \, \, 1 \leq i \leq m,$$

called flat outputs, defined in $O(X^l)$, having the following property: there exist an integer $s$ and smooth functions $\gamma_i$, $1 \leq i \leq n$, and $\delta_i$, $1 \leq i \leq m$, such that we have

$$x_i = \gamma_i(h,\bar{h},\ldots,h^s), \, \, \, 1 \leq i \leq n$$

$$u_i = \delta_i(h,\bar{h},\ldots,h^s), \, \, \, 1 \leq i \leq m,$$

where $h = (h_1,\ldots,h_m)^T$, along any trajectory $x(t)$ given by a control $u(t)$ that satisfies $(x(t),u(t),\dot{u}(t),u^{l+1}(t)) \in O(X^l)$.

The compositions $\gamma_i(h,\bar{h},\ldots,h^s)$ and $\delta_i(h,\bar{h},\ldots,h^s)$ are, a priori, defined in an open set $O^{l+1} \subset X^{s+l} = X \times U \times \mathbb{R}^{m+l}$. The above definition requires that $\pi(O^{l+1}) \supset O^l$, where $\pi(x,\bar{u}^{l+1}) = (x,\bar{u}^l)$, and that for all such $(x,\bar{u}^{l+1})$, the compositions yield, respectively, $x_1$ and $u_1$. If $h_i = h_i(x,u^0,u^1,\ldots,u^l)$, we will assume that they are defined on $O^l \subset X^l = X \times U \times \mathbb{R}^m_l$, where $\pi^{-1}(O^l) \supset O^l$ and $\pi$ stands for the projection $\pi(x,u^0,u^1,\ldots,u^l) = (x,u^0,u^1,\ldots,u^l)$.

Let $h_1,\ldots,h_m$ be flat outputs of the system $\Xi$. It has been observed in Respondek (2003) that there exist integers $k_1,\ldots,k_m$ such that span $\{dx_1,\ldots,dx_n,du_1,\ldots,du_m,du_{m+1},\ldots,du_{m+l}\}$ is contained in $\{dh_i^1,1 \leq i \leq m, \, \, \, 0 \leq j \leq k_i\}$, and if at the same time span $\{dx_1,\ldots,dx_n,du_1,\ldots,du_m\}$ is span $\{dh_i^1,1 \leq \}$
Notice that on the configuration space $Q$, we have always that $\Phi^* x^0_0 = x^0_0$, for $1 \leq m + 1$. Thus according to Theorem 7, the coordinates of $P_0 = (x^0_0, x^0_1, \ldots, x^0_{m+1})$ are minimal $x$-flat outputs of $\Gamma$ around $q_0$ which implies immediately that $\Gamma$ is $x$-flat at $(q_0, u^0)$ for some control $u^0$. Before we characterize the control $u^0$, notice that Theorem 7 implies that for control systems that are feedback equivalent to the $m$-chained form, for $n \geq 2, m \geq 2$, the minimal flat outputs are equivalent in the sense that for any two families of minimal flat outputs $(h_0, \ldots, h_m)$ and $(\bar{h}_0, \ldots, \bar{h}_m)$, we have span \{ $dh_0, \ldots, dh_m$ \} = span \{ $d\bar{h}_0, \ldots, d\bar{h}_m$ \}. In view of this and the item (ii) of Theorem 8, any minimal flat outputs $(h_0, \ldots, h_m)$ of the $n$-bar system in $\mathbb{R}^{m+1}$ for $n \geq 2, m \geq 2$ satisfy span \{ $dx^1_0, \ldots, dx^m_0$ \} = span \{ $dx^1_0, \ldots, dx^m_0$ \}. This proves (iii).

Now we are going to characterize the control $u^0$. According to the definition of the flat output, the entire state and all the controls should be functions of the coordinates $x^1_0, \ldots, x^m_0$ and their time-derivatives. Recall the system $\Delta$ given be (2) and (3) and consider the system of equation for the $x^0_0$-variables

\[ \begin{align*}
    \dot{x}^1_0 &= v_1(x^1_0 - x^0_1) \\
    \dot{x}^m_0 &= v_1(x^m_0 - x^0_{m+1}) \\
    \Psi_1(x) &= \sum_{j=1}^{m+1} (x^j_0 - x^j_{m+1})^2 = 1
\end{align*} \]

A direct computation shows that

\[ v_1 = \left( (x^0_1)^2 + \cdots + (x^{m+1}_1)^2 \right)^{1/2} = \eta_1(P_0, \bar{P}_0), \]

for some function $\eta_1$. Substituting (8) into (7), we get

\[ x^1_0 = x^1_0 + \frac{\dot{x}^1_0}{v_1} = \phi^1_1(P_0, \bar{P}_0) \]

\[ x^m_0 = x^m_0 + \frac{\dot{x}^m_0}{v_1} = \phi^{m+1}_1(P_0, \bar{P}_0), \]

for some functions $\phi^i_1$, for $1 \leq i \leq m + 1$. Put $\varphi_1 = (\phi^1_1, \ldots, \phi^{m+1}_1)$, we thus have $P_1 = (x^1_0, \ldots, x^{m+1}_0) = (\varphi^1_1, \ldots, \varphi^{m+1}_1)(P_0, \bar{P}_0) = \varphi_1(P_0, \bar{P}_0).

In the same way, we obtain that, for $2 \leq i \leq n,$

\[ v_i = \eta_i(P_{i-1}, \bar{P}_{i-1}) = \tilde{\eta}_i(P_0, \tilde{P}_0, P^{(i)}_0, \ldots, P^{(i)}_0) \]

and $P_i = \varphi_i(P_{i-1}, \bar{P}_{i-1}) = \tilde{\varphi}_i(P_0, \tilde{P}_0, P^{(i)}_0, \ldots, P^{(i)}_0)$, for some functions $\tilde{\eta}_i$ and $\tilde{\varphi}_i$. Finally, the controls $v_{n+j}$, for $1 \leq j \leq m + 1$, can be expressed by

\[ v_{n+j} = \frac{d}{dt} \left( \varphi_i^j(P_0, \tilde{P}_0, P^{(i)}_0, \ldots, P^{(i)}_0) \right) = \tau_j(P_0, \tilde{P}_0, P^{(i)}_0, \ldots, P^{(i)}_0), \]

for some functions $\tau_j$. In this way, the entire state and all controls $v_i$, for $1 \leq i \leq n + m + 1$, are expressed as functions of the coordinates of $P_0 = (x^1_0, \ldots, x^{m+1}_0)$ and their derivatives up to order $n + 1$. The $n$-bar system $\Gamma$ has $m + 1$ controls while the system $\Delta$ has $n + m + 1$ controls.
So there must be \( n \) relations between controls of \( \Delta \) when restricted to \( Q = \{ x \in X : \Psi(x) = 0 \} \). We will see this below and at the same time we will clarify the problem of singularities. Clearly, in order that the above computations hold, all the controls \( v_i \), for \( 1 \leq i \leq n \), cannot vanish. It is sufficient, however, to assume that the control \( v_1 \) is nonzero since around any point \( x \) satisfying \( \Psi(x) = 0 \) and the regularity condition \( \sum_{j=1}^{m+1} (x_i^j - x_{i-1}^j)(x_{i+1}^j - x_i^j) \neq 0 \), the condition \( v_1 \neq 0 \) implies that \( v_i \neq 0 \), for \( 2 \leq i \leq n \). In fact, differentiating the constraint 

\[
\Psi_1(x) = (x_1^1 - x_0^1)^2 + (x_1^2 - x_0^2)^2 + \cdots + (x_1^{m+1} - x_0^{m+1})^2 - 1 = 0,
\]

we get \( \sum_{j=1}^{m+1} 2((x_1^j - x_0^j)x_1^j - (x_1^j - x_0^j)x_0^j) = 0 \).

Substituting \( x_0^j = v_1(x_1^j - x_0^j) \) and \( x_1^j = v_2(x_1^j - x_1^j) \), for \( 1 \leq j \leq m + 1 \), into the above equation, by a simple calculation we get \( v_i = v_2 \sum_{j=1}^{m+1} (x_1^j - x_0^j)(x_2^j - x_1^j) \). Recall that around any regular point \( x_0 \), we have always that \( \sum_{j=1}^{m+1} (x_1^j - x_0^j)(x_2^j - x_1^j) \neq 0 \). Therefore, the condition \( v_1 \neq 0 \) implies that \( v_2 \neq 0 \) and similarly, it can be shown that 

\[
v_i = v_{i+1} \sum_{j=1}^{m+1} (x_{i+1}^j - x_{i-1}^j)(x_{i+1}^1 - x_{i}^1),
\]

for \( 1 \leq i \leq n-1 \) and \( v_n = \sum_{j=1}^{m+1} (x_n^j - x_{n-1}^j)v_{n+j} \).

The above equations show, first, that \( v_1 \neq 0 \) is equivalent to \( v_2 \neq 0 \), \( 1 \leq i \leq n \). Secondly, they imply that there exist \( n \) relations between the controls \( v_i \), \( 1 \leq i \leq n + m + 1 \), of \( \Delta \) implying that the \( n \)-bar system possesses, indeed, \( m + 1 \) controls. Moreover, the condition \( v_1 \neq 0 \) (which yields \( v_i \neq 0 \), \( 1 \leq i \leq n \)) implies that the instantaneous velocity \( \dot{P}_0 \) of the point \( P_0 \) is nonzero (and, consequently, the instantaneous velocities \( \dot{P}_i \) of the points \( P_i \), \( 0 \leq i \leq n-1 \), are nonzero). Therefore the condition (b) holds.

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