Describing and Calculating Flat Outputs of Two-input Driftless Control Systems

Shunjie Li, Witold Respondek

Describing and Calculating Flat Outputs of Two-input Driftless Control Systems

Shun-Jie LI * **Witold RESPONDEK** **

* *INSA-Rouen, LMI,* 76801 *Saint-Etienne-du-Rouvray, France (e-mail: shunjie.li@insa-rouen.fr)*
** *INSA-Rouen, LMI,* 76801 *Saint-Etienne-du-Rouvray, France (e-mail: witold.respondek@insa-rouen.fr)*

Abstract: We study the problem of flatness of two-input driftless control systems. Although a characterization of flat systems of that class is known, the problems of describing all flat outputs and of calculating them is open. We show that all x-flat outputs are parameterized by an arbitrary function of three canonically defined variables. We also construct a system of 1^{st} order PDE's whose solutions give all x-flat outputs of 2-input driftless systems. We illustrate our results by describing all flat outputs of models of a rolling disk and a nonholonomic car.

Keywords: Control system, flatness, x-flat, feedback equivalence, characteristic distribution.

1. INTRODUCTION

The notion of flatness has been introduced by Fliess, Lévine, Martin and Rouchon (see Fliess et al. (1992), Fliess et al. (1995), Fliess et al. (1999)) in order to describe the class of control systems, whose set of trajectories can be parameterized by a finite number of functions and their time-derivatives. More formally, a system with m controls is flat if we can find m functions (of the state and control variables and their time-derivatives), called *flat outputs*, such that the evolution in time of the state and control can be expressed in terms of flat outputs and their time derivatives (see Section 2 for a precise definition and references).

This paper is organized as follows. In Section 2, we define the crucial notion of flatness and recall a description of flat driftless 2-input systems. In section 3, we give our main results. We characterize all flat outputs of driftless 2-input systems and give a way of parameterizing them: it turns out that all flat outputs can be parameterized by an arbitrary function of intrinsically defined three variables. We also construct a system of 1^{st} order PDE's whose solutions are flat outputs of a given system. We illustrate our results by describing, in Section 4, all flat outputs of a nonholonomic model of a disk rolling on a plane and of a nonholonomic car (1-trailer system). The latter is well known to be flat, with the position (x, y) of the trailer being a flat output. Based on our results we will find less intuitive choices of flat outputs.

2. FLATNESS OF DRIFTLESS TWO-INPUT CONTROL SYSTEMS

Throughout this paper, the word smooth will always mean C^∞-smooth. Consider a smooth nonlinear control system $\Xi: \ \dot{x} = f(x, u)$, where $x \in X$, an n-dimensional manifold and $u \in U$, an m-dimensional manifold. Given any integer l, we associate to Ξ its l-prolongation Ξ^l given by

$$\Xi^l: \quad \begin{aligned} \dot{x} &= f(x, u^0) \\ \dot{u}^0 &= u^1 \\ &\vdots \\ \dot{u}^l &= u^{l+1}, \end{aligned}$$

which can be considered as a control system on $X^l = X \times U \times \mathbb{R}^{ml}$, whose state variables are $(x, u^0, u^1, \ldots, u^l)$ and whose m controls are the m components of u^{l+1}. Denote $\bar{u}^l = (u^0, u^1, \ldots, u^l)$.

Definition 1. The system Ξ is called flat at a point $(x_0, \bar{u}_0^l) \in X^l = X \times U \times \mathbb{R}^{ml}$, for some $l \geq 0$, if there exist a neighborhood $\mathcal{O}^l$ of $(x_0, \bar{u}_0^l)$ and m smooth functions

$$h_i = h_i(x, u^0, u^1, \ldots, u^l), \quad 1 \leq i \leq m,$$

called *flat outputs*, defined in $\mathcal{O}^l$, having the following property: there exist an integer s and smooth functions γ_i, $1 \leq i \leq n$, and δ_i, $1 \leq i \leq m$, such that we have

$$x_i = \gamma_i(h, \dot{h}, \ldots, h^{(s)})$$
$$u_i = \delta_i(h, \dot{h}, \ldots, h^{(s)}),$$

where $h = (h_1, \ldots, h_m)^\top$, along any trajectory $x(t)$ given by a control $u(t)$ that satisfy $(x(t), u(t), \dot{u}(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l$.

The compositions $\gamma_i(h, \dot{h}, \ldots, h^{(s)})$ and $\delta_i(h, \dot{h}, \ldots, h^{(s)})$ are, a priori, defined in an open set $\mathcal{O}^{s+l} \subset X^{s+l} = X \times U \times \mathbb{R}^{m(s+l)}$. The above definition requires that $\pi(\mathcal{O}^{s+l}) \supset \mathcal{O}^l$, where $\pi(x, \bar{u}^{s+l}) = (x, \bar{u}^l)$, and that for all such $(x, \bar{u}^{s+l})$, the compositions yield, respectively, x_i and u_i. If $h_i = h_i(x, u^0, u^1, \ldots, u^r)$, $r \leq l$, we will say that the system is $(x, u, \ldots, u^r)$-flat and, in particular, x-flat if $h_i = h_i(x)$. In the case $h_i = h_i(x, u^0, u^1, \ldots, u^r)$, we will assume that they are defined on $\mathcal{O}^r \subset X^r = X \times U \times \mathbb{R}^{mr}$, where $\pi^{-1}(\mathcal{O}^r) \supset \mathcal{O}^l$ and π stands for the projection $\pi(x, u^0, \ldots, u^r, \ldots, u^l) = (x, u^0, \ldots, u^r)$.

The notion of flatness has been introduced in control theory by Fliess, Lévine, Martin and Rouchon (Fliess et al. (1992), Fliess et al. (1995), Fliess et al. (1999), see also

Isidori et al. (1995), Jakubczyk (1993), Pomet (1995)) and has attracted a lot of attention because of its extensive applications in constructive controllability and trajectory tracking, compare Martin et al. (2002) and references therein. A similar notion (of underdetermined systems of differential equations that are integrable without integration) has already been studied by Hilbert (1912) and Cartan (1914).

In this paper, we deal only with two-input driftless (equivalently, control-linear) systems of the form

$$\Sigma : \quad \dot{x} = f_1(x)u_1 + f_2(x)u_2,$$

on an $(n+2)$-dimensional manifold M, where f_1 and f_2 are C^∞-smooth vector fields independent everywhere on M and $u = (u_1, u_2)^\top \in \mathbb{R}^2$. To this system, we associate the distribution $\mathcal{D}$ spanned by the vector fields f_1, f_2, which will be denoted by $\mathcal{D} = \operatorname{span}\{f_1, f_2\}$. Consider another 2-input driftless system

$$\tilde{\Sigma} : \quad \dot{\tilde{x}} = \tilde{f}_1(\tilde{x})\tilde{u}_1 + \tilde{f}_2(\tilde{x})\tilde{u}_2,$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are C^∞-smooth vector fields on $\tilde{M}$. Form the matrices $f(x) = (f_1(x), f_2(x))$ and $\tilde{f}(\tilde{x}) = (\tilde{f}_1(\tilde{x}), \tilde{f}_2(\tilde{x}))$. The systems Σ and $\tilde{\Sigma}$ are feedback equivalent if there exist an invertible 2×2-matrix β, whose entries β_{ij}, $1 \le i, j \le 2$, are C^∞-smooth functions on M, and a diffeomorphism $\varphi : M \to \tilde{M}$ such that $\mathrm{D}\varphi(x) \cdot f(x) \cdot \beta(x) = \tilde{f}(\varphi(x))$. It is easily seen that Σ and $\tilde{\Sigma}$ are locally feedback equivalent if and only if the associated distributions $\mathcal{D} = \operatorname{span}\{f_1, f_2\}$ and $\tilde{\mathcal{D}} = \operatorname{span}\{\tilde{f}_1, \tilde{f}_2\}$ are locally equivalent via φ, i.e., $\mathrm{D}\varphi(x)(\mathcal{D}(x)) = \tilde{\mathcal{D}}(\varphi(x))$.

The *derived flag* of a distribution $\mathcal{D}$ is the sequence of modules of vector fields $\mathcal{D}^{(0)} \subset \mathcal{D}^{(1)} \subset \cdots$ defined inductively by

$$\mathcal{D}^{(0)} = \mathcal{D} \quad \text{and} \quad \mathcal{D}^{(i+1)} = \mathcal{D}^{(i)} + [\mathcal{D}^{(i)}, \mathcal{D}^{(i)}], \quad \text{for} \quad i \ge 0.$$

The *Lie flag* of $\mathcal{D}$ is the sequence of modules of vector fields $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots$ defined inductively by

$$\mathcal{D}_0 = \mathcal{D} \quad \text{and} \quad \mathcal{D}_{i+1} = \mathcal{D}_i + [\mathcal{D}_0, \mathcal{D}_i], \quad \text{for} \quad i \ge 0.$$

In general, the derived and Lie flags are different though for any point x, the inclusion $\mathcal{D}_i(x) \subset \mathcal{D}^{(i)}(x)$ holds, for $i \ge 0$.

A characteristic vector field of a distribution $\mathcal{D}$ is a vector field f that belongs to $\mathcal{D}$ and satisfies $[f, \mathcal{D}] \subset \mathcal{D}$. The characteristic distribution of $\mathcal{D}$, which will be denoted by $\mathcal{C}$, is the subdistribution spanned by all its characteristic vector fields. It follows directly from the Jacobi identity that the characteristic distribution is always involutive but, in general, it need not be of constant rank.

The problem of flatness of driftless 2-input systems has been studied and solved by Martin and Rouchon (1993) (see also Martin and Rouchon (1994) and a related work of Cartan Cartan (1914)). Their important result proves that a system is flat if and only if its associated distribution $\mathcal{D}$ satisfies, on an open and dense subset M' of M, the conditions

$$\operatorname{rank} \mathcal{D}^{(i)} = i + 2, \quad 0 \le i \le n. \tag{1}$$

A distribution $\mathcal{D}$ is called a Goursat structure (also a "système en drapeau" in Kumpera and Ruiz (1982) and a Goursat flag in Mormul (2000)) if it satisfies the conditions (1) at any point $x \in M$. It is known since the work of von Weber (1898), Cartan (1914) and Goursat (1923) that the conditions (1) imply that on an open and dense subset M'' of M, the distribution $\mathcal{D}$ can be brought into the *Goursat normal form*, or equivalently, the corresponding control system is feedback equivalent to the *chained form*:

$$\Sigma_{\text{chain}} : \begin{cases} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= z_3 \cdot v_1 \\ \dot{z}_3 &= z_4 \cdot v_1 \\ &\vdots \\ \dot{z}_{n+1} &= z_{n+2} \cdot v_1 \\ \dot{z}_{n+2} &= v_2. \end{cases}$$

It is easy to see that Σ_{chain} is x-flat with x-flat outputs chosen as $h = (h_1, h_2) = (z_1, z_2)$ and provided that the control $v_1 \ne 0$. Giaro, Kumpera and Ruiz (Giaro et al. (1978)) were the first to observe the existence of singular points in the problem of transforming a distribution of rank two into the Goursat normal form. Murry (1994) proved that the feedback equivalence of Σ to the chained form Σ_{chain} (or, in other words, equivalence of the associated distribution to the Goursat normal form), around an arbitrary point x_0 requires, in addition to (1), the regularity condition (see Theorem 2 below)

$$\dim \mathcal{D}^{(i)}(x_0) = \dim \mathcal{D}_i(x_0), \quad 0 \le i \le n. \tag{2}$$

A natural question arises: can Σ be locally flat at a singular point of $\mathcal{D}$, i.e., at a point not satisfying the regularity condition (2)? In other words, can a driftless 2-input system be flat without being locally equivalent to the chained form? Theorem 2 answers this question (in what concerns x-flatness).

Let $\mathcal{D}$ be any distribution of rank two such that $\operatorname{rank} \mathcal{D}^{(1)} = 3$ and $\operatorname{rank} \mathcal{D}^{(2)} = 4$. Then there exists a distribution $\mathcal{C}_1 \subset \mathcal{D}$ of corank one which is characteristic for $\mathcal{D}^{(1)}$, i.e., $[\mathcal{C}_1, \mathcal{D}^{(1)}] \subset \mathcal{D}^{(1)}$. Indeed, the above rank assumptions imply that (after permuting f_1 and f_2, if necessary) there exists a smooth function α such that

$$[f_2, [f_1, f_2]] = \alpha [f_1, [f_1, f_2]] \mod \mathcal{D}^{(1)}.$$

It follows that $[f_2 - \alpha f_1, [f_1, f_2]] = 0 \mod \mathcal{D}^{(1)}$ and hence $\mathcal{C}_1 = \operatorname{span}\{f_2 - \alpha f_1\}$. Let $U_{\text{sing}}(x)$ be the 1-dimensional subspace of $\mathbb{R}^2$ such that for any feedback control $(u_1(x), u_2(x))^\top = u(x) \in U_{\text{sing}}(x)$, we have $f_1(x)u_1(x) + f_2(x)u_2(x) \in \mathcal{C}_1(x)$ (clearly, $U_{\text{sing}}(x)$ is spanned by $(\alpha(x), -1)^\top$). Any control $u(t) \in U_{\text{sing}}(x(t))$ will be called *singular* and the trajectories of the system governed by a singular control remain *tangent* to the characteristic subdistribution $\mathcal{C}_1$. We have just given the definition of $U_{\text{sing}}(x)$ for $\dim M \ge 4$ (since we have used $\operatorname{rank} \mathcal{D}^{(2)} = 4$). If $\dim M = 3$, we define $U_{\text{sing}}(x) = 0 \in \mathbb{R}^2$. Note that if $l = 0$, we will denote a fixed control value by u_0 (instead of more complicated u_0^0).

Theorem 2. Consider a 2-input driftless control system $\Sigma : \dot{x} = f_1(x)u_1 + f_2(x)u_2$, where $x \in M$, an $(n+2)$-dimensional manifold, $n \ge 1$. Assume that the distribution $\mathcal{D} = \operatorname{span}\{f_1, f_2\}$ associated to Σ is a Goursat structure, that is, satisfies $\operatorname{rank} \mathcal{D}^{(i)} = i+2$, for $0 \le i \le n$, everywhere on M. Then the following conditions are equivalent:

- (i) Σ is x-flat at $(x_0, \bar{u}_0^l) \in M \times \mathbb{R}^{2(l+1)}$, for a certain $l \ge 0$;
- (ii) Σ is x-flat at $(x_0, u_0) \in M \times \mathbb{R}^2$;

(iii) $\dim \mathcal{D}^{(i)}(x_0) = \dim \mathcal{D}_i(x_0)$, for $0 \le i \le n$, and $u_0 \notin U_{\mathrm{sing}}(x_0)$;
(iv) Σ is locally, around x_0, feedback equivalent to the chained form Σ_{chain} and $u_0 \notin U_{\mathrm{sing}}(x_0)$.

We assume that $\mathcal{D}$ satisfies $\operatorname{rank} \mathcal{D}^{(i)} = i + 2$, for $0 \le i \le n$, so the characteristic distribution $\mathcal{C}_1$ and the set of singular controls U_{sing} are well defined. The above theorem implies that a driftless 2-input system is never flat at (x_0, u_0) such that $u_0 \in U_{\mathrm{sing}}(x_0)$. Therefore any x-flat outputs (φ_1, φ_2) become singular in the control space (at $u_0 \in U_{\mathrm{sing}}$) but they may also exhibit singularities in the state space M. To formalize this, assume that a pair of functions (φ_1, φ_2) defined in an open set $\mathcal{M} \subset M$ is an x-flat output at $(x_0, u_0) \in M \times \mathbb{R}^2$, that is, there exists a neighborhood $\mathcal{O}^0 \subset M \times \mathbb{R}^2$, satisfying $\mathcal{O}^0 \subset \pi^{-1}(\mathcal{M})$, where $\pi(x, u) = x$, in which the conditions of Definition 1 hold. By $Sing(\varphi_1, \varphi_2)$, called the singular locus of (φ_1, φ_2), we will mean the set of points $x \in \mathcal{M}$ such that (φ_1, φ_2) is not x-flat output at (x, u) for any $u \in \mathbb{R}^2$.

The interest of the above theorem is two-fold. First, together with its proof, it will allow us to characterize all x-flat outputs of driftless 2-input systems (see Section 3). Secondly, it shows that a Goursat structure is x-flat at points x_0 satisfying $\dim \mathcal{D}^{(i)}(x_0) = \dim \mathcal{D}_i(x_0)$, for $0 \le i \le n$, only, that is, at regular points of $\mathcal{D}$. Martin and Rouchon (1993) asked (see also Martin and Rouchon (1994)) whether a Goursat structure $\mathcal{D}$ is flat (dynamically linearizable) at points that do not satisfy $\dim \mathcal{D}^{(i)}(x_0) = \dim \mathcal{D}_i(x_0)$. So our result gives a negative answer to their question (for x-flatness). Any Goursat structure can be brought to a generalization of the Goursat normal form, called Kumpera-Ruiz normal form (see Kumpera and Ruiz (1982), Mormul (2000), Pasillas and Respondek (2001)). It follows that none of Kumpera-Ruiz normal forms is x-flat (except for the regular Kumpera-Ruiz normal form, that is, Goursat normal form). In particular, the system

$$\begin{cases} \dot{x}_1 = x_5 u_1 \\ \dot{x}_2 = x_3 x_5 u_1 \\ \dot{x}_3 = x_4 x_5 u_1 \\ \dot{x}_4 = u_1 \\ \dot{x}_5 = u_2 \end{cases}$$

which is the first historically discovered Kumpera-Ruiz normal form (Giaro et al. (1978)), is not x-flat at any point of its singular locus $\{x \in \mathbb{R}^5 : x = 0\}$. This answers negatively another question of Martin and Rouchon (1993).

3. CHARACTERIZATION OF FLAT OUTPUTS

3.1 Main Theorems

Recall a useful result due to Cartan (1914) whose proof can be found in Kumpera and Ruiz (1982) and Martin and Rouchon (1994).

Lemma 3. (E. Cartan) Consider a rank two distribution $\mathcal{D}$ defined on a manifold M of dimension $n+2$, for $n \ge 2$. If $\mathcal{D}$ satisfies $\operatorname{rank} \mathcal{D}^{(i)} = i + 2$, for $0 \le i \le n$, everywhere on M, then each distribution $\mathcal{D}^{(i)}$, for $0 \le i \le n-2$, contains a unique involutive subdistribution $\mathcal{C}_{i+1}$ that is characteristic for $\mathcal{D}^{(i+1)}$ and of corank one in $\mathcal{D}^{(i)}$.

Theorem 2 implies that the only Goursat structures that are x-flat are those equivalent to the chained form (equivalently, whose associated distribution $\mathcal{D}$ is equivalent to the Goursat normal form). For this reason, we will consider in two theorems below such distributions only. Moreover, any distribution equivalent to the Goursat normal form obviously satisfies the assumptions of Lemma 3 and defines the involutive distribution $\mathcal{C}_{n-1}$ that is characteristic distribution for $\mathcal{D}^{(n-1)}$ and of corank one in $\mathcal{D}^{(n-2)}$.

Theorem 4. (Characterization of flat outputs, first version) Consider a driftless 2-input smooth control system Σ defined on a manifold M of dimension $n+2$ whose associated distribution $\mathcal{D}$ satisfies $\operatorname{rank} \mathcal{D}^{(i)} = \operatorname{rank} \mathcal{D}_i = i+2$, for $0 \le i \le n$. Fix $x_0 \in M$ and let g be an arbitrary vector field in $\mathcal{D}$ such that $g(x_0) \notin \mathcal{C}_{n-1}(x_0)$ and φ_1, φ_2 be two smooth functions defined in a neighborhood $\mathcal{M}$ of x_0. Then (φ_1, φ_2) is an x-flat output of Σ at (x_0, u_0), $u_0 \notin U_{\mathrm{sing}}(x_0)$, if and only if the following conditions hold:

(i) $\mathrm{d}\varphi_1(x_0) \wedge \mathrm{d}\varphi_2(x_0) \neq 0$, i.e., $\mathrm{d}\varphi_1$ and $\mathrm{d}\varphi_2$ are independent at x_0;
(ii) $L_c\varphi_1 \equiv L_c\varphi_2 \equiv L_c(\frac{L_g\varphi_2}{L_g\varphi_1}) \equiv 0$, for any $c \in \mathcal{C}_{n-1}$, where φ_1 and φ_2 are ordered such that $L_g\varphi_1(x_0) \neq 0$ which is always possible due to item (iii);
(iii) $(L_g\varphi_1(x_0), L_g\varphi_2(x_0)) \neq (0, 0)$.

Moreover, if a pair of functions (φ_1, φ_2) satisfies (i) everywhere in $\mathcal{M}$ and forms an x-flat output at (x, u) for any $x \in \tilde{\mathcal{M}}$ and certain $u = u(x)$, where $\tilde{\mathcal{M}}$ is open and dense in $\mathcal{M}$, then

$$Sing(\varphi_1, \varphi_2) = \{x \in \mathcal{M} : (L_g\varphi_1(x), L_g\varphi_2(x)) = (0, 0)\}.$$

Theorem 5. (Characterization of flat outputs, second version) Consider a driftless 2-input smooth control system Σ defined on a manifold M of dimension $n+2$ whose associated distribution $\mathcal{D}$ satisfies $\operatorname{rank} \mathcal{D}^{(i)} = \operatorname{rank} \mathcal{D}_i = i+2$, for $0 \le i \le n$. Fix $x_0 \in M$ and let φ_1, φ_2 be two smooth functions defined in a neighborhood $\mathcal{M}$ of x_0. Then (φ_1, φ_2) is an x-flat output of Σ at (x_0, u_0), $u_0 \notin U_{\mathrm{sing}}(x_0)$, if and only if the following conditions hold:

(i)′ $\mathrm{d}\varphi_1(x_0) \wedge \mathrm{d}\varphi_2(x_0) \neq 0$;
(ii)′ $\mathcal{L} = (\mathrm{span}\,\{\mathrm{d}\varphi_1, \mathrm{d}\varphi_2\})^{\perp} \subset \mathcal{D}^{n-1}$ in $\mathcal{M}$;
(iii)′ $\mathcal{D}(x_0)$ is not contained in $\mathcal{L}(x_0)$.

Moreover, if a pair of functions (φ_1, φ_2) satisfies (i)′ everywhere in $\mathcal{M}$ and forms an x-flat output at (x, u) for any $x \in \tilde{\mathcal{M}}$ and certain $u = u(x)$, where $\tilde{\mathcal{M}}$ is open and dense in $\mathcal{M}$, then

$$Sing(\varphi_1, \varphi_2) = \{x \in \mathcal{M} : \mathcal{D}(x) \subset \mathcal{L}(x)\}.$$

Remark 1. Notice that Theorem 5 is valid for any $n \ge 1$ (i.e., $\dim M \ge 3$) while Theorem 4 is true for $n \ge 2$ only (i.e., $\dim M \ge 4$). In fact, in Theorem 4 we use the characteristic distribution $\mathcal{C}_{n-1}$ of $\mathcal{D}^{(n-1)}$ but if $\dim M = 3$, such a distribution does not exist and therefore Theorem 4 can not be applied in that case.

Remark 2. The two items (iii) and (iii)′ describing the singular locus of an x-flat output (φ_1, φ_2) are equivalent under the condition $\operatorname{rank} \mathcal{D}^{(i)} = \operatorname{rank} \mathcal{D}_i = i + 2$, for $0 \le i \le n$.

Remark 3. The conditions of both theorems are verifiable, i.e., given a pair of functions (φ_1, φ_2) in a neighborhood of a point x_0, we can easily verify whether (φ_1, φ_2) forms an x-flat output of a control system under considerations and verification involves derivations and algebraic operations only (without solving PDE's or bringing the system to a normal form). Moreover, the theorems allow us to find the singular locus of a given flat output (φ_1, φ_2).

A natural question to ask is if there is a lot of pairs (φ_1, φ_2) which satisfy the conditions of Theorem 4 or 5? In other words, is there a lot of pairs (φ_1, φ_2) which are x-flat outputs for a 2-input driftless control system? This question has an elegant answer given by the following theorem.

Theorem 6. (Uniqueness of flat outputs) Consider a driftless 2-input smooth control system Σ whose associated distribution $\mathcal{D}$ satisfies $\operatorname{rank} \mathcal{D}^{(i)} = \operatorname{rank} \mathcal{D}_i = i+2$, for $0 \le i \le n$, locally around a point $x_0 \in M$, a manifold of dimension $n+2$. Let g be an arbitrary vector field in $\mathcal{D}$ such that $g(x_0) \notin \mathcal{C}_{n-1}(x_0)$. Then for a given arbitrary smooth function φ_1 such that $L_c \varphi_1 = 0$, for any $c \in \mathcal{C}_{n-1}$, and $L_g \varphi_1(x_0) \neq 0$, there always exists a function φ_2 such that (φ_1, φ_2) is an x-flat output of Σ at (x_0, u_0), $u_0 \notin U_{\text{sing}}(x_0)$. Moreover, if for a given function φ_1 as above, the pairs (φ_1, φ_2) and $(\varphi_1, \tilde{\varphi}_2)$ are both x-flat outputs of Σ at (x_0, u_0), then

$$\operatorname{span}\{\mathrm{d}\varphi_1, \mathrm{d}\varphi_2\}(x) = \operatorname{span}\{\mathrm{d}\varphi_1, \mathrm{d}\tilde{\varphi}_2\}(x),$$

for any x in a neighborhood of x_0.

Remark. Observe that x-flat outputs $(h_1, \ldots, h_m)$ and $(\tilde{h}_1, \ldots, \tilde{h}_m)$ of a system with m controls such that $\operatorname{span}\{\mathrm{d}h_1, \ldots, \mathrm{d}h_m\} = \operatorname{span}\{\mathrm{d}\tilde{h}_1, \ldots, \mathrm{d}\tilde{h}_m\}$ can be considered as statically equivalent. Indeed, in that case there exist smooth functions H_i and $\tilde{H}_i$ of m variables such that $h_i = H_i(\tilde{h}_1, \ldots, \tilde{h}_m)$ and $\tilde{h}_i = \tilde{H}_i(h_1, \ldots, h_m)$. It thus follows from Theorem 6 that for a given arbitrary φ_1 (satisfying the assumptions of the theorem), the choice of φ_2 is unique in the sense that all functions φ_2 giving x-flat outputs (φ_1, φ_2) yield, actually, statically equivalent x-flat outputs.

3.2 Finding x-flat outputs

The importance of Theorem 4 is that it not only allows to check whether a given pair of functions forms an x-flat output but also, together with Theorem 6, to express explicitly a system of 1st order PDE's to be solved in order to calculate all x-flat outputs for a given 2-input driftless system. Recall that the characteristic distribution $\mathcal{C}_{n-1}$ of $\mathcal{D}^{(n-1)}$ can be easily calculated as (see Bryant et al. (1991))

$$\mathcal{C}_{n-1} = \{f \in \mathcal{D}^{(n-1)} : f \lrcorner d\omega \in (\mathcal{D}^{(n-1)})^{\perp}\},$$

where ω is any non-zero differential 1-form annihilating $\mathcal{D}^{(n-1)}$.

Theorem 7. Assume that a control system Σ is x-flat at (x_0, u_0), $u_0 \notin U_{\text{sing}}(x_0)$, that is, the associated distribution $\mathcal{D}$ is, locally at x_0, equivalent to the Goursat normal form on an $(n+2)$-dimensional manifold M. Let $\mathcal{C}_{n-1} = \operatorname{span}\{c_1, \ldots, c_{n-1}\}$ be the characteristic distribution of $\mathcal{D}^{(n-1)}$ such that $c_{n-1}(x_0) \notin \mathcal{C}_{n-2}(x_0)$ and g any vector field in $\mathcal{D}$ such that $g(x_0) \notin \mathcal{C}_{n-1}(x_0)$. Then

(i) For any smooth function φ_1 such that

$$\text{(Flat 1)} \qquad \begin{aligned} L_{c_i}\varphi_1 &= 0, \quad 1 \le i \le n-1, \\ L_g \varphi_1(x_0) &\neq 0, \end{aligned}$$

the distribution $\mathcal{L} = \operatorname{span}\{c_1, \ldots, c_{n-1}, v\}$ is involutive, where $v = (L_g\varphi_1)[c_{n-1}, g] - (L_{[c_{n-1},g]}\varphi_1)g$.

(ii) A pair of functions (φ_1, φ_2) forms an x-flat output of Σ at (x_0, u_0), $u_0 \notin U_{\text{sing}}(x_0)$, if and only if after a permutation (if necessary) φ_1 satisfies (Flat 1), $\mathrm{d}\varphi_1(x_0) \wedge \mathrm{d}\varphi_2(x_0) \neq 0$, and φ_2 satisfies

$$\text{(Flat 2)} \qquad \begin{aligned} L_{c_i}\varphi_2 &= 0, \quad 1 \le i \le n-1 \\ L_v \varphi_2 &= 0. \end{aligned}$$

Remark. In (ii) only one implication may need permuting φ_1 and φ_2. Indeed, if (φ_1, φ_2) satisfies (Flat 1) and (Flat 2), then it is an x-flat output (and no permutation is needed). If (φ_1, φ_2) is an x-flat output, then at least one φ_i, $1 \le i \le 2$, satisfies $L_g \varphi_i(x_0) \neq 0$ and we choose φ_1 such that $L_g \varphi_1(x_0) \neq 0$.

Example 8. Consider a 2-input driftless control system

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$

on a 4-dimensional manifold M. Assume that the system is x-flat, that is, the associated distributions $\mathcal{D} = \operatorname{span}\{f_1, f_2\}$ satisfies the conditions of Theorem 7. Choose a vector field $c \in \mathcal{C}_1$ characteristic for $\mathcal{D}^{(1)}$ and $g \in \mathcal{D}$ such that $g(x_0) \wedge c(x_0) \neq 0$. According to Theorem 7 we take as φ_1 an arbitrary solution of $L_c\varphi_1 = 0$, $L_g\varphi_1(x_0) \neq 0$ and, in order to find φ_2, we have to solve $L_c\varphi_2 = 0$, $L_v\varphi_2 = 0$, where $v = (L_g\varphi_1)[c, g] - (L_{[c,\,g]}\varphi_1)g$. Notice that the above system of three 1st order PDE's contains a fourth one; indeed we have

$$L_v\varphi_1 = (L_g\varphi_1)L_{[c,g]}\varphi_1 - (L_{[c,g]}\varphi_1)L_g\varphi_1 = 0.$$

The system

$$L_c\varphi_i = L_v\varphi_i = 0, \quad 1 \le i \le 2, \tag{3}$$

admits two independent functions φ_1 and φ_2 as solutions if and only if the distribution $\operatorname{span}\{c, v\}$ is integrable. A direct calculation shows that this is the case (see Li (2010), Li and Respondek (2010)). All becomes clear: the involutive distribution $\operatorname{span}\{c, v\}$ is just the distribution $\mathcal{L}$ of Theorem 5 while φ_1 and φ_2 satisfying (3) are x-flat outputs since their differentials span $\mathcal{L}^{\perp}$. We also see that $\mathcal{L}$ is not unique: different choices of φ_1 lead to different vector fields v which, in turn, give different distributions $\mathcal{L} = \operatorname{span}\{c, v\}$, although all of them are involutive and thus define (via $\operatorname{span}\{\mathrm{d}\varphi_1, \mathrm{d}\varphi_2\} = \mathcal{L}^{\perp}$) non equivalent flat outputs. This is in a perfect accordance with Theorem 6.

4. EXAMPLES

Example 9. (Vertical rolling disk) Consider a vertical disk of radius R rolling without slipping on a horizontal plane. Denote by (x, y) the position of the contact point in the xy-plane, and by θ and ϕ, respectively, the rotation angle of the disk and the orientation of the disk. The controls u_1 and u_2 allow the disk to rotate and turn. This leads to the following model given by a driftless system on $Q = \mathbb{R}^2 \times S^1 \times S^1$:

$$\Sigma_{\text{disk}} : \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} R\cos\phi \\ R\sin\phi \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2 = f_1 u_1 + f_2 u_2.$$

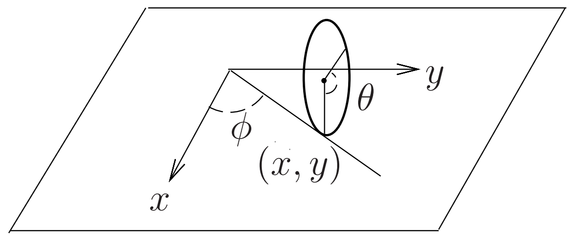

Fig. 1. the rolling disk

A direct computation shows that $\operatorname{rank}\mathcal{D}^{(i)} = \operatorname{rank}\mathcal{D}_i = i+2$, for $0 \le i \le 2$, and $\mathcal{C}_1 = \operatorname{span}\{f_1\}$. Therefore by Theorem 2, the model Σ_{disk} is x-flat at any point of its configuration space Q. Moreover, it satisfies the hypothesis of Theorems 4, 5 and 6 and U_{sing} is given by $U_{\text{sing}} = \{u = (u_1, u_2)^\top : u_2 = 0\}$. Thus the singular control corresponds to rolling the disk along a straight line. Now let us calculate all its x-flat outputs by using the procedure given in Section 3.2. We choose $c = f_1 = R\cos\phi\frac{\partial}{\partial x} + R\sin\phi\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta}$, and take $g = f_2 = \frac{\partial}{\partial\phi}$. Then as a first flat output we can take any function φ_1 satisfying the following system of equations

$$L_c\varphi_1 = R\cos\phi\frac{\partial\varphi_1}{\partial x} + R\sin\phi\frac{\partial\varphi_1}{\partial y} + \frac{\partial\varphi_1}{\partial\theta} \equiv 0$$
$$L_g\varphi_1(q_0) \neq 0.$$

Solving this system of equations, we get that φ_1 is any function of the form

$$\varphi_1 = \varphi_1(\phi, x - R\theta\cos\phi, y - R\theta\sin\phi)$$

satisfying $\frac{\partial\varphi_1}{\partial\phi}(q_0) \neq 0$. Choose one such φ_1 and then φ_2 is any function independent with φ_1 that satisfies $L_c\varphi_2 = L_v\varphi_2 = 0$, where the vector field v is given by

$$v = (L_g\varphi_1)[c, g] - (L_{[c,g]}\varphi_1)g.$$

To illustrate this, choose the function $\varphi_1 = x - R\theta\cos\phi$ around a point q_0 such that $L_g\varphi_1(q_0) = R\theta\sin\phi \neq 0$ and then $v = R^2\theta\sin^2\phi\frac{\partial}{\partial x} - R^2\theta\sin\phi\cos\phi\frac{\partial}{\partial y} - R\sin\phi\frac{\partial}{\partial\phi}$. Solving the system of equations $L_c\varphi_2 = L_v\varphi_2 = 0$, we get $\varphi_2 = \varphi_2(x - R\theta\cos\phi, y - R\theta\sin\phi)$ satisfying $(\mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2)(q_0) \neq 0$. All such functions satisfy $\operatorname{span}\{\mathrm{d}\varphi_1, \mathrm{d}\varphi_2\} = \operatorname{span}\{\mathrm{d}\varphi_1, \mathrm{d}\tilde\varphi_2\}$ and we can take, for instance, $\varphi_2 = y - R\theta\sin\phi$. Moreover, the singular locus of the x-flat output $(x - R\theta\cos\phi,\ y - R\theta\sin\phi)$ is given by

$$\begin{aligned} Sing(\varphi_1,\varphi_2) &= \{q \in Q : (L_g\varphi_1(q), L_g\varphi_2(q)) = (0,0)\} \\ &= \{q \in Q : \theta = 0\}. \end{aligned}$$

To see that $\sin\phi = 0$ is, indeed, not a singularity, we just permute φ_1 and φ_2.

To consider another possibility, we choose $\varphi_1 = \phi$ and then we have $v = R\sin\phi\frac{\partial}{\partial x} - R\cos\phi\frac{\partial}{\partial y}$. Solving the system of equations $L_c\varphi_2 = L_v\varphi_2 = 0$, we get $\varphi_2 = \varphi_2(\phi, R\theta - x\cos\phi - y\sin\phi)$ satisfying $(\mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2)(q_0) \neq 0$. We can take, for instance, $\varphi_2 = R\theta - x\cos\phi - y\sin\phi$ and a simple calculation shows that there does not exist singular point of the x-flat output $(\phi,\ R\theta - x\cos\phi - y\sin\phi)$ in the state space Q. In other words, $(\phi,\ R\theta - x\cos\phi - y\sin\phi)$ is an x-flat output at any point $(q, u) \in Q \times \mathbb{R}^2$ provided that $u \notin U_{\text{sing}}(q)$.

For various choices of functions, our result allows to eliminate them as candidates for x-flat outputs. For example, we can conclude that if (φ_1, φ_2) is an x-flat output, then $L_c\varphi_i \equiv 0$, for $i = 1, 2$, where $c = R\cos\phi\frac{\partial}{\partial x} + R\sin\phi\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta}$ is a characteristic vector field of $\mathcal{D}^{(1)}$. It follows that independently of the choice of φ_2, neither (x, φ_2), nor (y, φ_2), nor (θ, φ_2) can serve as an x-flat output.

Example 10. (Nonholonomic car)

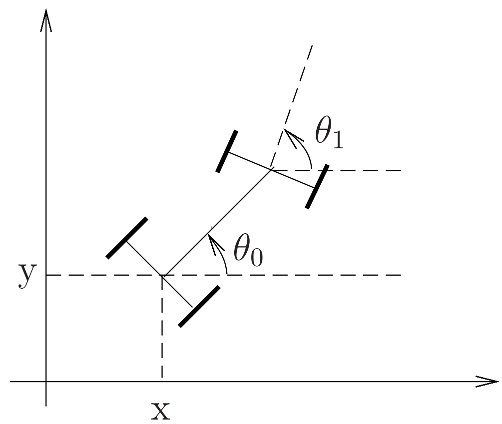

Fig. 2. nonholonomic car

Consider a model of a nonholonomic car Σ_{car}, equivalently of a unicycle-like robot towing a trailer (see Jean (1998), Laumond (1997)),

$$\Sigma_{\text{car}} : \begin{pmatrix} \dot x \\ \dot y \\ \dot\theta_0 \\ \dot\theta_1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1 - \theta_0)\cos\theta_0 \\ \cos(\theta_1 - \theta_0)\sin\theta_0 \\ \sin(\theta_1 - \theta_0) \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2,$$

where $q = (x, y, \theta_0, \theta_1) \in \mathbb{R}^2 \times S^1 \times S^1$. It is well known that the system is x-flat and that the position (x, y) of the mid-point of the rear wheels is an x-flat output (see Fliess et al. (1995) and Jakubczyk (1993)). We will illustrate our results by providing other (statically non-equivalent) x-flat outputs that are less intuitive. We choose as a characteristic vector field $c = \frac{\partial}{\partial\theta_1}$ and take $g = \cos(\theta_1 - \theta_0)\cos\theta_0\frac{\partial}{\partial x} + \cos(\theta_1 - \theta_0)\sin\theta_0\frac{\partial}{\partial y} + \sin(\theta_1 - \theta_0)\frac{\partial}{\partial\theta_0}$.

As a first x-flat output we can take any function φ_1 satisfying $L_c\varphi_1 = \frac{\partial\varphi_1}{\partial\theta_1} \equiv 0$ and $L_g\varphi_1(q) \neq 0$, that is any function $\varphi_1 = \varphi_1(x, y, \theta_0)$ such that $L_g\varphi_1(q) \neq 0$. Let us choose one such φ_1 then φ_2 satisfies $L_c\varphi_2 = L_v\varphi_2 = 0$, where the vector field v is given by $v = (L_g\varphi_1)[c, g] - (L_{[c,g]}\varphi_1)g = -\frac{\partial\varphi_1}{\partial\theta_0}\cos\theta_0\frac{\partial}{\partial x} - \frac{\partial\varphi_1}{\partial\theta_0}\sin\theta_0\frac{\partial}{\partial y} + (\frac{\partial\varphi_1}{\partial x}\cos\theta_0 + \frac{\partial\varphi_1}{\partial y}\sin\theta_0)\frac{\partial}{\partial\theta_0}$. Therefore φ_2 can be taken as any functions $\varphi_2(x, y, \theta_0)$ satisfying $L_v\varphi_2 = 0$ and $(\mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2)(q) \neq 0$. Given φ_1 as above, the space of solutions for φ_2 is thus parameterized by one function of two variables but any two solutions φ_2 and $\tilde{\varphi}_2$ give statically equivalent flat outputs, that is $\text{span}\{\mathrm{d}\varphi_1, \mathrm{d}\varphi_2\} = \text{span}\{\mathrm{d}\varphi_1, \mathrm{d}\tilde{\varphi}_2\}$. On the other hand, different choices of φ_1 will lead to nonequivalent pairs (φ_1, φ_2) of x-flat outputs.

To illustrate this, take $\varphi_1 = x$, then $v = \cos\theta_0 \frac{\partial}{\partial\theta_0}$ and $L_c\varphi_2 = L_v\varphi_2 = 0$ imply that φ_2 is any function of the form $\varphi_2(x, y)$ satisfying $\frac{\partial\varphi_2}{\partial y}(q) \neq 0$ (because of $(\mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2)(q) \neq 0$). All such functions satisfy $\text{span}\{\mathrm{d}x, \mathrm{d}\varphi_2\} = \text{span}\{\mathrm{d}x, \mathrm{d}\tilde{\varphi}_2\}$ and we can take, for instance, $\varphi_2 = y$. This gives the well-known x-flat output (x, y).

To see another choice, take $\varphi_1 = \theta_0$, then $v = -\cos\theta_0 \frac{\partial}{\partial x} - \sin\theta_0 \frac{\partial}{\partial y}$ and the general solution of $L_c\varphi_2 = L_v\varphi_2 = 0$ is $\varphi_2 = \varphi_2(\theta_0, x\sin\theta_0 - y\cos\theta_0)$, which gives as an x-flat output $(\theta_0, x\sin\theta_0 - y\cos\theta_0)$. Notice that the singular loci of the two choices of x-flat outputs are different. In fact, $Sing(x, y) = \{\theta_1 - \theta_0 = \pm\frac{\pi}{2}\}$ and $Sing(\theta_0, x\sin\theta_0 - y\cos\theta_0) = \{\theta_1 - \theta_0 = 0, \pm\pi\}$.

Now take $\varphi_1 = x + \theta_0$ around $\cos\theta_0 \neq 0$, then $v = -\cos\theta_0 \frac{\partial}{\partial x} - \sin\theta_0 \frac{\partial}{\partial y} + \cos\theta_0 \frac{\partial}{\partial\theta_0}$. Thus the general solution of $L_c\varphi_2 = L_v\varphi_2 = 0$ is $\varphi_2 = \varphi_2(x+\theta_0, y - \ln|\cos\theta_0|)$. We can take, for instance, $\varphi_2 = y - \ln|\cos\theta_0|$ which gives a third x-flat output $(x+\theta_0, y - \ln|\cos\theta_0|)$ of Σ_{car} and its singular locus is defined by $Sing(x + \theta_0, y - \ln|\cos\theta_0|) = \{\cos\theta_0 = 0\} \cup \{\cos(\theta_1 - \theta_0)\cos\theta_0 + \sin(\theta_1 - \theta_0) = 0\}$.

REFERENCES

R. Bryant, S-S. Chern, R. Gardner, H. Goldschmidt, and P. Griffiths. *Exterior Differential Systems.* Mathematical Sciences Research Institute Publications. Springer-Verlag, New York, 1991.

E. Cartan. Sur l'équivalence absolue de certains systèmes d'équations différentielles et sur certaines familles de courbes. *Bulletin de la Société Mathématique de France*, 42:12–48, *Œuvres complètes*, Part. II, Vol. 2, Gauthiers-Villars, Paris,1914.

M. Fliess, J. Lévine, P. Martin and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. *Comptes Rendus de l'Académie des Sciences*, 315:619–624, 1992.

M. Fliess, J. Lévine, P. Martin and P. Rouchon. Flatness and defect of nonlinear systems: Introductory theory and examples. *International Journal of Control*, 61(6):1327–1361, 1995.

M. Fliess, J. Lévine, P. Martin and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems, *IEEE Trans. Automat. Control*, 61:1327–1361, 1999.

F. Jean. The car with n trailers: Characterisation of the singular configurations, *Control, Optimisaton and Calculus of Variations*, Vol. 1, pp. 241–266, 1996.

A. Giaro, A. Kumpera and C. Ruiz. Sur la lecture correcte d'un resultat d'Élie Cartan. *Comptes Rendus de l'Académie des Sciences de Paris*, 287:241–244, 1978.

E. Goursat. *Leçons sur le problème de Pfaff.* Hermann, Paris, 1923.

D. Hilbert. Über den begriff der klasse von differentialgleichungen. *Math. Ann.*, 73(1):95–108, 1912.

A. Isidori, C. H. Moog, and A. de Luca. A sufficient condition for full linearization via dynamic state feedback. *In Proc. 25th IEEE Conf. on Decision & Control*, Athens, pp. 203–207, 1986.

B. Jakubczyk. Invariants of dynamic feedback and free systems. In *Proceedings of the European Control Conference*, Groningen, 1993.

A. Kumpera and C. Ruiz. Sur l'équivalence locale des systèmes de Pfaff en drapeau. In F. Gherardelli, editor, *Monge-Ampère equations and related topics*, pp. 201–247. Instituto Nazionale di Alta Matematica Francesco Severi, Rome, 1982.

J. P. Laumond, *Robot Motion Planning and Control.* Lecture Notes on Control and Information Sciences. Springer-Verkag, Berlin, 1997.

S-J. Li. *Géométrie et classification des systèmes de contact: Applications au contrôle des systèmes mécaniques non holonomes*, INSA-Rouen, Laboratoire de Mathématiques (LMI), Ph.D. thesis, 2010.

S-J. Li and W. Respondek. Flat outputs of two-input driftless control system, submitted, 2010.

P. Martin, R. M. Murray, and P. Rouchon. Flat systems. *In Mathematical control theory,* Part 1, 2 (Trieste, 2001), ICTP Lect. Notes, VIII, pp. 705–768 (electronic). Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.

P. Martin and P. Rouchon. Feedback linearization and driftless systems. *CAS internal report* No. 446, 1993.

P. Martin and P. Rouchon. Feedback linearization and driftless systems. *Mathematics of Control, Signals, and Systems*, 7:235–254, 1994.

P. Mormul. Goursat flags: classification of codimension-one singularities. *Journal of dynamical and control systems* vol.6, pp. 311–330, 2000.

R. Murray. Nilpotent bases for a class of nonintegrable distributions with applications to trajectory generation for nonholonomic systems. *Mathematics of Control, Signals, and Systems*, 7:58–75, 1994.

W. Pasillas-Lépine and W. Respondek. On the geometry of control systems equivalent to canonical contact systems: regular points, singular points and flatness, *Proceedings of the 39th IEEE Conference of Decision and Control,* Sydney, Australia, 5151–5156, 2000.

W. Pasillas-Lépine and W. Respondek. On the geometry of Goursat structures. *ESAIM Control, Optimisation, and Calculus of Variations*, 6:119–181, 2001.

W. Respondek. Symmetries and minimal flat outputs of nonlinear control systems. In *New Trends in Nonlinear Dynamics and Control, and their Applications*, W. Kang, M. Xiao, and C. Borges (eds.), LNCIS 295, pp. 65-86Springer, 2003.

J.-B. Pomet. A differential geometric setting for dynamic equivalence and dynamic linearization. In *Geometry in Nonlinear Control and Differential Inclusions*, volume 32, pp. 319–339. Banach Center Publications, Warszawa, 1995.

E. von Weber. Zur Invariantentheorie der Systeme Pfaff'scher Gleichungen. *Berichte Verhandlungen der Koniglich Sachsischen Gesellshaft der Wissenshaften Mathematisch-Physikalische Klasse, Leipzig*, 50:207–229, 1898.