Sums of values represented by a quadratic form

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Abstract

Len q be a quadratic form over a field K of characteristic different from 2. The properties of the smallest positive integer n such that -1 is a sums of n values represented by q are investigated. The relations of this invariant which is called the q-level of K, with other invariants of K such as the level, the u-invariant and the Pythagoras number of K are studied. The problem of determining the numbers which can be realized as a q-level for particular q or K is also studied. A special emphasis is given to the case where q is a Pfister form. We also observe that the q-level is naturally emerges when one tries to obtain a lower bound for the index of the subgroup of non-zero values represented by a Pfister form q.

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1 Introduction

A celebrated theorem of E. Artin and O. Schreier states that a field K has an ordering if and only if -1 cannot be written as a sum of squares in K. In this case, the field K is called *formally real.* In the situation where K is not formally real, one may wonder how many squares are actually needed to write -1 as a sum of squares in K. This leads to the following definition:

Definition 1.1. The *level* $\mathrm{s}(K)$ of a field K is defined by

$$\mathrm{s}(K) = \min\{n \mid \exists (x_1, \cdots, x_n) \in K^n,\ -1 = x_1^2 + \cdots + x_n^2\}$$

if -1 is a sum of squares in K, otherwise one defines $\mathrm{s}(K) = +\infty$.

A question raised by B. L. van der Waerden in the 1930s asks for the integers that can occur as the level of a field K (cf. [14]). H. Kneser obtained a partial answer to this question in 1934 by showing that the possible values for the level are 1, 2, 4, 8 or the multiples of 16 (cf. [7]). In 1965, A. Pfister developed the theory of multiplicative forms which furnished a complete answer to this question : if finite, the level of a field is always a power of 2 and every prescribed 2-power can be realized as the level of a field, see [16]. We intend to study the following natural generalization of the level of a field.

Definition 1.2. Let (V, q) be a quadratic form over a field K. We define the *level of K with respect to q* (or the *q-level* for short) by

$$\mathrm{s}_q(K) = \min\{n \mid \exists(v_1, \cdots, v_n) \in V^n \text{ such that } -1 = q(v_1) + \cdots + q(v_n)\},$$

if such an n exists and by $\mathrm{s}_q(K) = +\infty$ otherwise.

Under this setting, the (usual) level of K corresponds to the level of K with respect to the quadratic form X^2 over K. More generally, the length (see §2) of a non-zero element $a \in K$ coincides with the q-level of K where q is the quadratic form $-aX^2$. Thus for the case where $\dim q = 1$, the study of q-level is nothing but the investigation of the lengths of elements of K.
To our best knowledge, in the general case of a quadratic form, this notion has not been explicitly defined before, but it appears implicitly in many places and it is closely related to some invariants studied in the literature.

It is already relevant to point out that q-levels are related to some hermitian levels studied by D. W. Lewis. For a ring R with an identity and a non trivial involution σ, recall that the *hermitian level* of R is defined as the least integer n such that -1 is a sum of n hermitian squares in R, i.e., elements of the form $\sigma(x)x$ where $x \in R$. The hermitian level of (R, σ) is denoted by $\mathrm{s}(R, \sigma)$. See [12] and [14].

If L/K (resp. Q) is a quadratic extension with $L = K(\sqrt{a})$ (resp. a K-quaternion algebra $(a, b)_K$) and if $-$ is the canonical involution of L (resp. of Q), it is easy to see that $\mathrm{s}(L, -) = \mathrm{s}_q(K)$ (resp. $\mathrm{s}(Q, -) = \mathrm{s}_q(K)$) where $q = \langle 1, -a\rangle$ (resp. $q = \langle 1, -a, -b, ab\rangle$) is the norm form of L/K (resp. of Q). Note that in both cases the hermitian level is a power of two (see [12, Prop. 1.5]) and the quadratic form q is a Pfister form over K. More generally, we observe that $\mathrm{s}_q(K)$ is always a 2-power or infinite whenever q is a Pfister form (see Proposition 4.1).

Level of a noncommutative ring D, which is denoted by $s(D)$, was introduced in [11] and [13] as the least integer n such that -1 is a sum of n squares in D. If D is the Clifford algebra of a nondegenerate quadratic form q over a field K then we obviously have $\mathrm{s}(D) \leqslant \mathrm{s}_q(K)$; in particular if D is the quaternion algebra $(a, b)_K$, generated by the elements i and j subject to the relations $i^2 = a \in K^\times$, $j^2 = b \in K^\times$ and $ij = -ji$ then we have $s(D) \leqslant \mathrm{s}_q(K)$ where $q = \langle a, b\rangle$.

A substantial part of this paper is devoted to investigating

– the properties of $\mathrm{s}_q(K)$, i.e., the q-level of a field K,
– the relations of $\mathrm{s}_q(K)$ with other invariants of K such as the (usual) level, the u-invariant, the Pythagoras number,
– the calculation of $\mathrm{s}_q(K)$ for particular q or K,
– the behavior of $\mathrm{s}_q(\cdot)$ under field extensions,
– for a given n, the possible values of $\mathrm{s}_q(K)$ that are attained when q runs over all quadratic forms of dimension n over K,
– for a given q, the possible values of $\mathrm{s}_{q_{K'}}(K')$ when K'/K runs over all field extensions of K.

When q is a Pfister form there are several strong analogies between the properties of $\mathrm{s}_q(K)$ and that of $\mathrm{s}(K)$. In particular a 'Pythagoras q-number' which is related to sums of values represented by q is defined and its properties are studied. However it does not mean that when q is a Pfister form, every result on $\mathrm{s}(K)$ can be generalized to $\mathrm{s}_q(K)$, see 4.8.

The paper is structured as follows. In the next section, we collect some definitions and preliminary observations about the q-level of a field and define the q-length of $a \in K$ and the Pythagoras q-number of K which are respective generalizations of the length of a and of the Pythagoras number of K.

Section 3 is devoted to the study of the q-level for an arbitrary quadratic form q. We first give upper bounds for the q-level in terms of some familiar invariants (e.g., the usual level, the Pythagoras number and the u-invariant) before studying the behavior of the q-level and the q-length of an element with respect to purely transcendental field extensions; this leads to a generalization of a theorem due to Cassels, see Proposition 3.7. Next we make the following simple observations (see Proposition 3.9):

– $\{\mathrm{s}_q(K) | \dim q = 1\} = \{1, \cdots, p(K)\}$ if K is non formally real, and
– $\{\mathrm{s}_q(K) | \dim q = 1\} = \{1, \cdots, p(K)\} \cup \{+\infty\}$ if K is formally real.

We then obtain some conclusions about the integers which belong to the set

$$\{\mathrm{s}_q(K) | \dim q = n\},$$

see Corollary 3.10 (2).

We then prove that

– If q is a quadratic form with $\dim q \leqslant 3$ such that $\mathrm{s}_q(K) = +\infty$, then
(a) If $\dim q = 1$ or 2 then for any $k \in \mathbb{N}$ there is a field extension K'/K such that $\mathrm{s}_q(K') = 2^k$.
(b) If $\dim q = 3$ then for any $k \in \mathbb{N}$ there exist field extensions K'/K and K''/K such that $\mathrm{s}_q(K') = \frac{2^{2k}+2}{3}$ and $\mathrm{s}_q(K'') = \frac{2^{2k+1}+1}{3}$.

This result is proved in Corollary 3.12 and Corollary 3.13: for this, the key ingredient is Hoffmann's separation Theorem (see 3.11). Whereas the knowledge of the possible q-levels over a field K is equivalent to the knowledge of the Pythagoras number of K, our result does not give an explicit way to find a quadratic form q with prescribed q-level in general. We make explicit calculations of q-levels for familiar fields, whenever possible.

Section 4 concentrates on the particular case of Pfister forms. We first prove that (see Proposition 4.1 and Theorem 4.2):

– If φ is a Pfister form over K, then $\mathrm{s}_\varphi(K)$ is a 2-power or infinite. Moreover $\{\mathrm{s}_\varphi(K') : K'/K \text{ field extension}\} = \{1, 2, \cdots, 2^i, \cdots, \mathrm{s}_\varphi(K)\}$.

Then, we investigate the behavior of the q-level with respect to quadratic field extensions: it is described in Proposition 4.7. We next show that the q-level, q-length and Pythagoras q-number share similar properties with their respective generalizations (see 4.9 and 4.16).

In 1966, A. Pfister obtained a sharp lower bound for the cardinality of the group $K^\times/K^{\times^2}$ in terms of the level of K. He proved that if K is a field whose level is 2^n then $|\frac{K^\times}{K^{\times^2}}| \geq 2^{n(n+1)/2}$ (see [17, Satz 18 (d)]). The example $K = \mathbb{F}_q$, $q \equiv 3 \mod 4$, shows that this lower bound is attained.
Recall that an element a of $K^\times$ is *represented by* q if there exists $v \in V$ such that $q(v) = a$. Denote by $D_K(q)$ the set of values represented by q. In the case where q is a Pfister form, $D_K(q)$ is a subgroup of $K^\times$ ([8, Proposition X.2.5]) which contains $K^{\times^2}$ hence it is a natural thing to wonder if one may obtain a lower bound for the cardinality of the group $K^\times/D_K(q)$ in terms of the q-level of K. In fact we obtain the following result (see Theorem 4.11):

– Let q be a Pfister form over a field K whose q-level is 2^n. Then $|K^\times/D_K(q)| \geqslant 2^{n(n+1)/2}$ and this lower bound is sharp.

We also draw some direct consequences of this lower bound. We finally study the behavior of the Pythagoras q-number with respect to field extensions L/K of finite degree in Proposition 4.20: we show that $p_q(L)$ is smaller than $[L : K]p_q(K)$ thus generalizing a classical result due to Pfister.

2 Preliminaries

In this paper, the characteristic of the base field K will always supposed to be different from 2 and all quadratic forms are implicitly supposed to be nondegenerate. The notation $\langle a_1, \cdots, a_n\rangle$ will refer to the diagonal quadratic form $a_1{X_1}^2 + \cdots + a_n{X_n}^2$. Every quadratic form q over K can be diagonalized, that is q is isometric to a diagonal quadratic form $\langle a_1, \cdots, a_n\rangle$ which we denote by $q \simeq \langle a_1, \cdots, a_n\rangle$. We will denote by $W(K)$ the Witt ring of K and by $I(K)$ its fundamental ideal. Its nth power is denoted by $I^n(K)$ and is additively generated by the *n-fold Pfister forms* $\langle\langle a_1, \cdots, a_n\rangle\rangle := \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$. A quadratic form π over K is a *Pfister neighbor* if there exists an n-fold Pfister form φ and $a \in K^\times$ such that $2\dim(\pi) > \dim(\varphi)$ and $a \cdot \pi$ is a subform of φ. In this case, it is known that π is isotropic if and only if φ is isotropic if and only if φ is hyperbolic. For a positive integer n and a quadratic form q, we often use the notations $\sigma_n = n \times \langle 1\rangle$ and $\sigma_{n,q} = n \times q$. We also use $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$.

The *length* of an element $a \in K^\times$ denoted by $\ell(a)$ is the smallest integer n such that a is a sum of n squares; if such an n does not exist, we put $\ell(a) = +\infty$. Note that $\mathrm{s}(K) = \ell(-1)$. Following [19, Ch. 6, p.75], we denote by $\Sigma K^\bullet$ the set of all elements in $K^\times$ which can be written as a sum of squares in K. In [5], [9] the notations $S(K)$ and ΣK^2 are used instead of $\Sigma K^\bullet$.

The *Pythagoras number* of K is defined to be

$$p(K) = \sup\{\ell(a) \mid a \in \Sigma K^\bullet\} \in \mathbb{N}_0 \cup \{+\infty\}.$$

Recall that K is non formally real if and only if $K^\times = \Sigma K^\bullet$; in that case, $p(K)$ is always finite. To see this, if we put $s = \mathrm{s}(K)$ then the form σ_{s+1} is isotropic hence universal and $p(K) \leqslant s+1$. As -1 is not a sum of $s-1$ squares, we obtain that $p(K) \in \{\mathrm{s}(K), \mathrm{s}(K)+1\}$. If K is formally real, $p(K)$ can either be finite or infinite. D. Hoffmann has shown that each integer can in fact be realized as the Pythagoras number of a certain (formally real) field: see [5] or [6, Theorem 5.5]. For further details about the level and the Pythagoras number, the reader may also consult [19, Ch. 3, Ch. 7].

The *u-invariant of* K, which is denoted by $u(K)$, is defined to be $\max(\dim q)$ where q ranges over all anisotropic quadratic forms over K if such a maximum exists, and we define $u(K) = +\infty$ otherwise. Note that $u(K)$ is also the minimal integer n for which all quadratic forms of dimension strictly greater than n (resp. greater or equal than n) are isotropic (resp. universal) over K.

Let (V, q) be a quadratic form over K. We define the q-length of an element of K and the Pythagoras q-number of K as follows.

Definition 2.1. (1) The *q-length* of an element $a \in K^\times$ denoted by $\ell_q(a)$ is defined by

$$\ell_q(a) = \min\{n \mid \exists(v_1, \cdots, v_n) \in V^n,\ a = q(v_1) + \cdots + q(v_n)\}$$

if such an n exists and by $\ell_q(a) = +\infty$ otherwise.
(2) Let $\Sigma_q K^\bullet$ be the set of all elements x in $K^\times$ for which there exists an integer n such that the form $\sigma_{n,q}$ represents x. The Pythagoras q-number is defined by

$$p_q(K) = \sup\{\ell_q(a) \mid a \in \Sigma_q K^\bullet\},$$

if such an n exists and by $p_q(K) = +\infty$ otherwise.

As $\mathrm{s}_{\langle 1\rangle}(K) = \mathrm{s}(K)$, $\ell_{\langle 1\rangle}(a) = \ell(a)$ and $p_{\langle 1\rangle}(K) = p(K)$, one sees that the q-level, the q-length of a and the Pythagoras q-number are respective generalizations of the level, the length of a and the Pythagoras number. Note also that $\mathrm{s}_q(K) = 1$ if and only if q represents -1, that the number $\mathrm{s}_q(K)$ only depends on the isometry class of q and that $\mathrm{s}_{\langle a\rangle}(K) = \ell(-a)$.

Remarks 2.2. (1) For all $a \in K^\times$, we have $p_{a\cdot q}(K) = p_q(K)$, since we obviously have $\ell_{a\cdot q}(b) = \ell_q(a^{-1}b)$ for all $b \in K$.
(2) For every positive integer n, we can easily check that $\ell_{n\times q}(a) = \lceil \frac{1}{n}\ell_q(a)\rceil$ and $p_{n\times q}(a) = \lceil \frac{1}{n}p_q(a)\rceil$.
(3) If $a \in K^\times$ and L/K is a field extension of odd degree then $\ell_q(a) = \ell_{q_L}(a)$.

In the sequel it is convenient to introduce the following notations. If K is a field and n is a positive integer greater or equal than 1, we put

$$L(n, K) = \{\mathrm{s}_q(K) \mid q \text{ is a quadratic form of dimension } n \text{ over } K\}$$

and we set $L(K) = \bigcup_{n\in\mathbb{N}\setminus\{0\}} L(n, K)$. If q is a quadratic form over K, let

$$L_q(K) = \{\mathrm{s}_{q_{K'}}(K') \mid K'/K \text{ field extension}\}.$$

3 General results

3.1 Upper bounds

In the following lemma, we list some properties concerning the q-level of a field.

Lemma 3.1. *Let K be a field and q be a quadratic form over K.*
(1) *We have* $1 \leqslant \mathrm{s}_q(K) \leqslant \mathrm{s}(K) + 1$.
(2) *We have* $1 \leqslant \mathrm{s}_q(K) \leqslant \inf\{\mathrm{s}_{\langle a\rangle}(K) :\ a \in D_K(q)\} = \inf\{\ell(-a) :\ a \in D_K(q)\}$. *More generally, if r is a subform of q, then* $\mathrm{s}_q(K) \leqslant \mathrm{s}_r(K)$.
(3) *If L/K is a field extension then* $\mathrm{s}_q(K) \geqslant \mathrm{s}_{q_L}(L)$.
(4) *If L/K is a field extension whose degree is odd then* $\mathrm{s}_q(K) = \mathrm{s}_{q_L}(L)$.
(5) *For every positive integer n we have* $\mathrm{s}_{n\times q}(K) = \left\lceil \frac{\mathrm{s}_q(K)}{n} \right\rceil$

Proof. To prove (1), we may assume that $\mathrm{s}(K) = n < +\infty$. In this case, the quadratic form σ_{n+1} is isotropic so the quadratic form $\sigma_{n+1,q}$ is isotropic hence universal. In particular, $\sigma_{n+1,q}$ represents -1. This proves (1).

For statement (2), it suffices to prove the second property. If $\sigma_{n,r}$ represents -1 for a certain n, then it is also the case for $\sigma_{n,q}$ hence (2).

Finally, (3) is trivial, (4) follows from the strong version of a theorem of Springer (see [19, Chapter 6, 1.12]) and (5) follows from 2.2 (2). □

Examples 3.2. (1) The upper bound $\mathrm{s}_q(K) \leqslant \mathrm{s}(K)+1$ given in 3.1 (1) is sharp. If $K = \mathbb{Q}(i)$ and $q = \langle 2 \rangle$, we easily see that $\mathrm{s}(K) = 1$ and $\mathrm{s}_q(K) = \ell(-2) = 2$.
(2) In the second statement of 3.1 (2), it is possible to have $\mathrm{s}_r(K) = \mathrm{s}_q(K)$ for a proper subform r of q. For instance consider the forms $r = \langle -3 \rangle$ and $q = \langle -3, X \rangle$ over $\mathbb{Q}(X)$, we have $\mathrm{s}_q(\mathbb{Q}(X)) = \mathrm{s}_r(\mathbb{Q}(X)) = 3$.

The purpose of the following proposition is to give upper bounds for the q-level of a field K in terms of some classical invariants K.

Proposition 3.3. *Let K be a field and let q be a quadratic form over K.*
(1) *If $\mathrm{s}_q(K) < +\infty$ then $\mathrm{s}_q(K) \leqslant p(K)$ (see also Proposition 4.9 (2)).*
(2) *If K is not formally real, we have $\mathrm{s}_q(K) \leqslant \left\lceil \dfrac{u(K)}{\dim(q)} \right\rceil \leqslant u(K)$.*

Proof. To prove (1), one may assume that $p = p(K) < +\infty$. Let $q = \langle a_1, \cdots, a_n \rangle$ be a diagonalization of q. By assumption, there exists an integer m and vectors $v_1, \cdots, v_m$ such that $-1 = \sum_{i=1}^{m} q(v_i)$. It follows that $-1 = a_1\Sigma_1 + \cdots + a_n\Sigma_n$ where $\Sigma_1, \cdots, \Sigma_n \in \Sigma K^{\bullet}$ are sums of at most m squares. By the definition of the Pythagoras number, each Σ_i can be written as a sum of at most p squares. This fact readily implies that $\mathrm{s}_q(K) \leqslant p(K)$.

(2) One may assume that $u(K) < +\infty$. Then every quadratic form q of dimension greater or equal than $u(K)$ is universal. It follows that if $n \times \dim(q) \geqslant u(K)$ then $\mathrm{s}_q(K) \leqslant n$, hence the result. □

Remark 3.4. In the previous proposition, the bound given in (1) is sharp for any non formally real field K as Proposition 3.9 shows. We now show that the inequality $\mathrm{s}_q(K) \leqslant \left\lceil \frac{u(K)}{\dim(q)} \right\rceil$ is sharp for any prescribed dimension. Let n be a positive integer and choose m such that $n < 2^m$. Let F be a field such that $\mathrm{s}(F) = u(F) = 2^m$ (it is even possible to construct a field F such that $\mathrm{s}(F) = u(F) = p(F) = 2^m$, see [6, §5.2]). If we put $r = \left\lceil \frac{u(F)}{n} \right\rceil$ then

$$\frac{u(F)}{n} \leq r < \frac{u(F)}{n} + 1$$

so $rn \geq u(F)$ which implies that $\mathrm{s}_{\sigma_n}(F) \leq r$. Moreover $(r-1)n < u(F) = s(F)$ so $\sigma_{(r-1)n+1}$ is anisotropic which shows that $\mathrm{s}_{\sigma_n}(F) = r$ as claimed.

Corollary 3.5. *Let K be a field.*
(1) *If q is a quadratic form over K, we have $L_q(K) \subseteq \{1, \cdots, \mathrm{s}_q(K)\}$.*
(2) *If $n \geq 1$ is an integer then $L(n, K) \subseteq \{1, \cdots, h_n\} \cup \{+\infty\}$ where $h_n = \min(p(K), \left\lceil \frac{u(K)}{n} \right\rceil)$. In particular, if K is non formally real and $n \geq u(K)$ then $L(n, K) = \{1\}$.*

Proof. This follows from Lemma 3.1 (3) and Proposition 3.3. □

3.2 Behavior under purely transcendental extensions

The following two results generalize two classical theorems concerning quadratic forms under transcendental field extensions in the framework of q-levels.

We begin by generalizing the classical fact that the level of a field is invariant under a purely transcendental extension.

Proposition 3.6. *Let q be a quadratic form over K and let K'/K be a purely transcendental field extension of K. Then $\mathrm{s}_q(K) = \mathrm{s}_q(K') = \mathrm{s}_q(K((t)))$.*

Proof. By Lemma 3.1 (3), we have $\mathrm{s}_q(K) \geqslant \mathrm{s}_q(K')$ hence we may assume that $\mathrm{s}_q(K') = n < +\infty$. The quadratic form $\sigma_{n,q}$ represents -1 over K' and by Cassels-Pfister's Theorem (see [8, Theorem IX.1.3]), $\sigma_{n,q}$ already represents -1 over K hence the first equality. The second equality is proved similarly. □

We now come to the generalization of a Theorem due to J. W. S. Cassels which asserts that the polynomial $P = 1 + X_1^2 + \cdots + X_n^2$ is not a sum of n squares over $L = K(X_1, \cdots, X_n)$ if K is formally real (see [8, Corollary IX.2.4]). Note that this is equivalent to $\ell(P) = n + 1$ over L.

Proposition 3.7. *For $m \geq 1$ and $n \geq 0$, let $q = \langle a_1, \cdots, a_m \rangle$ be a quadratic form over a field K and set $L = K(X_j^{(i)}, i = 1, \cdots, n, j = 1, \cdots, m)$. If $\sigma_{n,q}$ is anisotropic, then*

$$\ell_{q_L}(a_1 + \sum_{i,j} a_j (X_j^{(i)})^2) = n + 1.$$

In particular, this is the case if $\mathrm{s}_q(K) = +\infty$.

Proof. If $n = 0$, we have to prove that $\ell_q(a_1) = 1$, which is obvious since a_1 is represented by q. Assume now that $n \geq 1$, and set $a = a_1 + \sum_{i,j} a_j (X_j^{(i)})^2$.

Since a_1 is represented by q, and thus by q_L, we have $\ell_{q_L}(a) \leq n+1$. Assume that $\ell_{q_L}(a) < n + 1$, so that a is represented by σ_{n,q_L}. Since $\sigma_{n,q}$ is anisotropic over K by assumption, any subform q' of $\sigma_{n,q}$ is also anisotropic over K. It implies that q'_L is anisotropic over L, since L/K is purely transcendental. Consequently, q'_M is anisotropic over M for any subfield M of L, and any subform q' of $\sigma_{n,q}$. A repeated application of a theorem due to Cassels (cf. [21, Theorem 3.4, p.150]) then shows that $a_1 + a_1(X_1^{(1)})^2$ is represented by $\langle a_1 \rangle$ over $K(X_1^{(1)})$. This implies that $1 + (X_1^{(1)})^2$ is a square in $K(X_1^{(1)})$, hence a contradiction.

Let us prove the last part of the proposition. If $\mathrm{s}_q(K) = +\infty$ but $\sigma_{n,q}$ is isotropic for $n \geq 1$ then $\sigma_{n,q}$ is universal, hence represents -1, so $\mathrm{s}_q(K) \leq n$, and we have a contradiction. Now apply the first part to conclude. □

Corollary 3.8. *With the same hypotheses as in 3.7, $\ell_{q_L}(\sum_{i,j} a_j (X_j^{(i)})^2) = n$.*

3.3 Values of the q-level

3.3.1 About $L(n, K)$

Proposition 3.9. *If K is non formally real then $L(1, K) = L(K) = \{1, \cdots, p(K)\}$. If K is formally real then $L(1, K) = L(K) = \{1, \cdots, p(K)\} \cup \{+\infty\}$*

Proof. As $L(1, K) \subseteq L(K)$, the direct inclusions come from Proposition 3.3 (1) in both cases. It remains to show that the sets on the right-hand sides are included in $L(1, K)$. This is a conclusion of the fact $\mathrm{s}_q(\langle\rangle)$

For this, we distinguish between the cases $p(K) < +\infty$ and $p(K) = +\infty$. Note that $+\infty = \mathrm{s}_{\langle 1 \rangle}(K) \in L(1, K)$ in the formally real case.

If $p = p(K) < +\infty$, there exists $a \in \Sigma K^\bullet$ such that $\ell(a) = p$. Write $a = x_1^2 + \cdots + x_p^2$ where $x_i \in K$. For $1 \leqslant i \leqslant \ell(a)$ put $\beta_i = x_1^2 + \cdots + x_i^2$. Then $\ell(\beta_i) = i = \mathrm{s}_{\langle -\beta_i \rangle}(K)$ hence the result.

Suppose now that $p(K) = +\infty$ and let n be a fixed integer. By definition of the Pythagoras number, there exists $a \in \Sigma K^\bullet$ such that $q = \ell(a) > n$. Put $a = x_1^2 + \cdots x_q^2$ where $x_i \in K$. If we choose $b = x_1^2 + \cdots x_n^2$ then $\ell(b) = n = \mathrm{s}_{\langle -b \rangle}(K)$ and this concludes the proof. $\square$

This result has several immediate consequences.

Corollary 3.10. (1) *There exists a field K such that for every integer n, there exists a quadratic form q over K with $\mathrm{s}_q(K) = n$.*
(2) *If $n \geq 1$ is an integer then $\{1, \cdots, l_n\} \subseteq L(n, K)$ where $l_n = \lfloor p(K)/n \rfloor$. In particular, if $p(K) = +\infty$ then $L(n, K) = \mathbb{N}_0 \cup \{+\infty\}$ for any $n \geq 1$.*

Proof. (1) By Proposition 3.9, it suffices to find K with $p(K) = +\infty$. Following the proof of [6, Theorem 5.5], put $K = F(X_1, X_2, \cdots)$ with an infinite number of variables X_i over a formally real field F. If $n \in \mathbb{N}_0$, put $P_n = 1 + X_1^2 + \cdots + X_n^2$. Then $\ell(P_n) = n+1$ over $F(X_1, \cdots, X_n)$ (by Cassels' theorem mentionned above) and K (as K/F is purely transcendental). Thus $p(K) = +\infty$.

(2) It suffices to prove the first assertion. If we fix $1 \leq i \leq l_n$ then $n \times i \leq p(K)$. By Proposition 3.9, there exists a form $q_i = \langle a_i \rangle$ with $\mathrm{s}_{q_i}(K) = n \times i$. By Lemma 3.1 (5), the form $\sigma_{n,q_i} \in L(n, K)$ then has level i hence the result. $\square$

3.3.2 About $L_q(K)$

We now focus on the proof of Corollary 3.13, mentioned in the Introduction, for which the crucial ingredient is the following result due to D. W. Hoffmann in [4, Theorem 1].

Theorem 3.11 (Hoffmann's Separation Theorem)**.** *Let q and q' be anisotropic quadratic forms over K such that $dim(q) \leqslant 2^n < \dim(q')$. Then q is anisotropic over $K(q')$, the function field of the projective quadric defined by $q' = 0$.*

The key fact for us is the second assertion of the following proposition. The first assertion is a direct consequence of Corollary 3.10 (1) but can also be proved independently using Hoffmann's result.

Proposition 3.12. *Let n be a positive integer and K be a formally real field.*
(1) *There exist a field extension K'/K and a quadratic form q over K' such that $\mathrm{s}_q(K') = n$.*
(2) *Let q be a quadratic form over K with $\mathrm{s}_q(K) = +\infty$. If there exists a positive integer k such that $1 + (n-1)\dim q \leqslant 2^k < 1 + n \dim q$ then there exists a field extension K'/K such that $\mathrm{s}_q(K') = n$. In fact one can choose $K' = K(\langle 1 \rangle \perp \sigma_{n,q})$.*

Proof. For (1), we can suppose that $n > 1$. Let m and r be two integers such that
$$\frac{2^r - 1}{n} < m \leqslant \frac{2^r - 1}{n-1}.$$
These two integers do exist: in fact, it suffices to choose m and r satisfying $r > \dfrac{\ln(n^2 - n + 1)}{\ln(2)}$ which implies that $\dfrac{2^r - 1}{n-1} - \dfrac{2^r - 1}{n} = \dfrac{2^r - 1}{n(n-1)} > 1$.

For $j > 1$, let $\varphi_j = \langle 1 \rangle \perp \sigma_{j,q}$ where q is the anisotropic quadratic form $q = \sigma_m$. The quadratic forms φ_n and φ_{n-1} are anisotropic over K. If $K' = K(\varphi_n)$ then, by Theorem 3.11 and by the choice of m and r, $(\varphi_{n-1})_{K'}$ is anisotropic hence $s_q(K') \geqslant n$. As $(\varphi_n)_{K'}$ is isotropic, we obtain $s_q(K') = n$ hence (1). For (2), the fact that $s_q(K) = +\infty$ implies similarly that $s_q(K') = n$ for $K' = K(\psi_n)$ where $\psi_n = \langle 1\rangle \perp \sigma_{n,q}$. □

Corollary 3.13. *Let q be a quadratic form of dimension at most 3 such that $s_q(K) = +\infty$.*
(a) If q has dimension 1 or 2 then $2^k \in L_q(K)$ for any $k \in \mathbb{N}$.
(b) If q has dimension 3 then $\frac{2^{2k}+2}{3}$, $\frac{2^{2k+1}+1}{3} \in L_q(K)$ for any $k \in \mathbb{N}$.

Proof. (1) Consider the form $\varphi_{2^k} = \langle 1\rangle \perp \sigma_{2^k,q}$. As $\dim q = 1$ or 2, we have $1 + (2^k - 1)\dim q \leqslant 2^{k+\dim q-1} < 1 + 2^k \dim q$. By Proposition 3.12 (2), we obtain $s_q(K') = 2^k$ where $K' = K(\varphi_{2^k})$.

(2) If $n = \frac{2^{2k}+2}{3}$, we have $1+(n-1)\dim q \leqslant 2^{2k} < 1+(n-1)\dim q$, therefore the existence of K' follows from Proposition 3.12 (2). The proof of the second assertion is similar and is left to the reader. □

Corollary 3.14. *Let K be a field and let n be a nonnegative integer.*
(1) If $s(K) = +\infty$ then there exists a field extension K'/K with $s(K') = 2^n$.
(2) If $a \in K$ and $\ell(a) = +\infty$ over K then there exists a field extension K'/K such that $\ell(a) = 2^n$ over K'.

3.4 Some examples and calculations

We now make explicit calculations of q-levels in many familiar fields. Note that $s_{\langle -1\rangle}(K) = 1$ for any field K and that $s_{\langle 1\rangle}(K) = +\infty$ when K is formally real. The results 3.9, 3.5 (2) and 3.10 (2) are used without further mention.

3.4.1 Non formally real fields

Algebraically closed fields : in such a field K, $L(n, K) = L_q(K) = \{1\}$ for any n and q.

Finite fields : if K is a finite field, $L(K) = \{1, 2\}$. As $u(K) = 2$, we have $L(n, K) = L_q(K) = \{1\}$ for $n \geq 2$, $\dim(q) \geq 2$. Moreover, the quadratic form $q = \langle a\rangle$ has q-level 1 if and only if $-a$ is a square, otherwise it has q-level 2.

Non dyadic local fields : in such a field K with residue field $\overline{K}$, denote by U the group of units and choose a uniformizer π. Any quadratic form q can be written $\partial_1(q) \perp \pi\partial_2(q)$ where $\partial_1(q) = \langle u_1, \cdots, u_r\rangle$, $\partial_2(q) = \langle u_{r+1}, \cdots u_n\rangle$ for $u_i \in U$, $i = 1, \cdots, n$. The forms $\partial_1(q)$ and $\partial_2(q)$ are respectively called the first and second residue forms of q. By a Theorem of T. A. Springer, q is anisotropic over K if and only if $\overline{\partial_1(q)}$ and $\overline{\partial_2(q)}$ are anisotropic over $\overline{K}$ (see [8, Proposition VI.1.9]). This reduces the calculation of the q-levels over K to calculations of some levels over $\overline{K}$.

Now, $p(K) = \min(s(K) + 1, 4)$ (see [19, Chapter 7, Examples 1.4 (1)]) hence $L(K) = \{1, 2\}$ (resp. $\{1, 2, 3\}$) if $|\overline{K}| \equiv 1 \mod 4$ (resp. $|\overline{K}| \equiv 3 \mod 4$).

Take $u \in U$ with $\overline{u} \notin \overline{K}^2$ and put $q = \langle -u\rangle$. Then $\langle 1, -u\rangle$ is anisotropic but $\langle 1, -u, -u\rangle$ is isotropic by Springer's Theorem hence $s_q(K) = 2$. If $|\overline{K}| \equiv 3 \mod 4$, take $q' = \langle \pi\rangle$. As -1 is not a square in $\overline{K}$, Springer's Theorem shows that $s_{q'}(K) > 2$ hence $s_{q'}(K) = 3$.

Dyadic local fields : we have $p(K) = \min(\mathrm{s}(K)+1, 4)$ and $\mathrm{s}(K) = 1, 2$ or 4. If $K = \mathbb{Q}_2$, we have $\mathrm{s}(K) = 4$ (see [8, Examples XI.2.4]), hence $L(\mathbb{Q}_2) = \{1, 2, 3, 4\}$. We have $\mathrm{s}_{\langle -2\rangle}(\mathbb{Q}_2) = 2$, $\mathrm{s}_{\langle 2\rangle}(\mathbb{Q}_2) = 3$ and $\mathrm{s}_{\langle 1\rangle}(K) = 4$.

The fields $L_n = K(X_1, \cdots, X_n)$ **and** $M_n = K((X_1)) \cdots ((X_n))$: if $\mathrm{s}(K) = 2^m$ then $p(L_n) = p(M_n) = 2^m + 1$ ([19, Chapter 7, Proposition 1.5]) hence $L(L_n) = L(M_n) = \{1, \cdots, 2^m + 1\}$. Recall that $p(K) \in \{2^m, 2^m + 1\}$.

Each value in $\{1, \cdots, p(K)\}$ is attained as a q-level over K and $\mathrm{s}_q(K) = \mathrm{s}_q(L_n) = \mathrm{s}_q(M_n)$ by Proposition 3.6. If $p(K) = \mathrm{s}(K) = 2^m$, let $q = \langle X_n\rangle$ over L_n. As M_n is a local field with residue field $K((X_1)) \cdots ((X_{n-1}))$ of level 2^m and uniformizer X_n, we have $\mathrm{s}_q(L_n) = \mathrm{s}_q(M_n) = 2^m + 1$ by Springer's theorem.

Non formally real global fields : if K is a number field, we have $p(K) = \min(\mathrm{s}(K) + 1, 4)$ by Hasse-Minkowski principle (see [19, Examples 1.4 (2)]) and $\mathrm{s}(K) = 1, 2$ or 4 ([8, Theorem XI.1.4]). If K is a function field, the previous examples show that $L(K) \subseteq \{1, 2, 3\}$.

3.4.2 Formally real fields

Real closed fields : over such a field, a quadratic form q has q-level $+\infty$ if and only if q is positive definite, otherwise it has q-level one.

Formally real global fields : we have $p(K) = 4$ (resp. $p(K) = 3$) if K has a dyadic place P such that $[K_P : \mathbb{Q}]$ is odd (resp. otherwise) by [19, Chapter 7, Examples 1.4 (3)].

In particular, $L(\mathbb{Q}) = \{1, 2, 3, 4\}$. By Hasse-Minkowski principle, any indefinite quadratic form q of dimension ≥ 5 is isotropic hence has q-level one, see [8, Chapter VI.3]. Any positive definite quadratic form q has an infinite q-level. A quadratic form q that is not positive definite has a finite q-level. For example $\mathrm{s}_{\langle -5\rangle}(\mathbb{Q}) = 2$, $\mathrm{s}_{\langle -6\rangle}(\mathbb{Q}) = 3$ and $\mathrm{s}_{\langle -7\rangle}(\mathbb{Q}) = 4$ as 6 (resp. 7) is a sum of three squares (resp. four squares) but not a sum of two squares (resp. three squares) in $\mathbb{Q}$.

The field $\mathbb{R}(X_1, \cdots, X_n)$: its Pythagoras number is 2 if $n = 1$, is 4 if $n = 2$ and is in the interval $[n + 2; 2^n]$ for $n \geq 3$ (the two last results are due to J. W. S. Cassels, W. J Ellison and A. Pfister, see [2] or [8, Examples XI.5.9 (4)]).

If $n = 1$, we thus have $L(\mathbb{R}(X)) = \{1, 2\}$. By a Theorem due to E. Witt, we know that any totally indefinite quadratic form over $\mathbb{R}(X)$ of dimension ≥ 3 is isotropic hence has q level one. Consider $q = \langle -(1 + X^2)\rangle$. As $1 + X^2$ is not a square in $\mathbb{R}(X)$, q does not represent -1. But $\langle 1\rangle \perp 2 \cdot q$ is totally indefinite as 1 is totally positive whereas $-(1 + X^2)$ is totally negative hence it is isotropic. This proves that $\mathrm{s}_q(\mathbb{R}(X)) = 2$.

The field $\mathbb{Q}(X_1, \cdots, X_n)$: its Pythagoras number is 5 if $n = 1$ (by a result due to Y. Pourchet, see [20]). For $n \geqslant 2$, we only know that $p(\mathbb{Q}(X_1, X_2)) \leqslant 8$, $p(\mathbb{Q}(X_1, X_2, X_3)) \leqslant 16$ and $p(\mathbb{Q}(X_1, \cdots, X_n)) \leqslant 2^{n+2}$. The first two results are due to Colliot-Thélène and Jannsen in [3] and the last one is due to Arason.

If $n = 1$, the study done for $\mathbb{Q}$ together with Proposition 3.6 show that $\mathrm{s}_{\langle -5\rangle}(\mathbb{Q}(X)) = 2$, $\mathrm{s}_{\langle -6\rangle}(\mathbb{Q}(X)) = 3$ and $\mathrm{s}_{\langle -7\rangle}(\mathbb{Q}(X)) = 4$. By a result of Y. Pourchet (see [20, Proposition 10]), for $d \in \mathbb{Z}$, the polynomial $X^2 + d$ is a sum of exactly five squares of $\mathbb{Q}[X]$ if and only if $\mathrm{s}(\mathbb{Q}(\sqrt{-d})) = 4$ which in turn is equivalent to $d > 0$ and $d \equiv -1 (\mathrm{mod}\ 8)$ (see [8, Remark XI.2.10]). For $q = \langle -(X^2 + 7)\rangle$, this readily implies that $\mathrm{s}_q(\mathbb{Q}(X)) = 5$.

3.4.3 A remark

Suppose that K is a non formally real field. In all the above examples in which we know $\mathrm{s}(K)$, $u(K)$ and $p(K)$, we have $p(K) = \min(\mathrm{s}(K) + 1, u(K))$. We always have $p(K) \leqslant \min(\mathrm{s}(K) + 1, u(K))$ but there is no equality in general.

To see this, we use the construction of fields with prescribed even u-invariant due to Merkurjev (see [15] or [6, Section 5]).

Theorem 3.15 (Merkurjev)**.** *Let m be an even number and E be a field. There exists a non formally real field F over E such that $u(F) = m$ and $I^3(F) = 0$.*

Put $m = 2n+2$. The proof of Merkurjev's result is based upon a construction of an infinite tower of fields F_i. More precisely $F_0 = E(X_1, Y_1 \cdots, X_n, Y_n)$ and if F_i is constructed then F_{i+1} is the free compositum over F_i of all function fields $F_i(\psi)$ where ψ ranges over (1) all quadratic forms in $I^3(F_i)$ (2) all quadratic forms of dimension $2n + 3$ over F_i. Then $F = \cup_{i=0}^{\infty} F_i$ is the desired field.

Choose a field E with $\mathrm{s}(E) = 4$. Then there exists a field F over E such that $u(F) = 2n + 2$ and $I^3(F) = 0$. If we consider the 2-fold anisotropic Pfister form $\varphi = \langle 1, 1, 1, 1\rangle$ over E, the form φ stays anisotropic over any F_i of the tower as it cannot become hyperbolic over the function field of a quadratic form of dimension strictly greater than 4 by [8, Theorem X.4.5] hence $\mathrm{s}(F) = 4$. Now, choose $n = 2$ so that $\min(\mathrm{s}(F) + 1, u(F)) = 5$. Let $x \in K^\times$ and consider the quadratic form $\varphi \perp \langle -x\rangle$: it is isotropic since it is the Pfister neighbor of $\langle\langle -1, -1, x\rangle\rangle$ which is hyperbolic as $I^3(F) = 0$. As $\mathrm{s}(F) = 4$, this means that φ is anisotropic over F hence every $x \in K^\times$ is a sum of at most 4 squares in K. Now, -1 is a sum of 4 squares and is not a sum of three squares which shows that $p(F) = 4 < \min(\mathrm{s}(F) + 1, u(F)) = 5$.

4 The case of Pfister forms

4.1 Values of the q-level : the case of Pfister forms

Proposition 4.1. *Let φ be a Pfister form over a field K. Then $\mathrm{s}_\varphi(K)$ is either infinite or a power of two.*

Proof. This proof is similar to Pfister's proof of the fact that the level of a field, when finite, is a power of two. As pointed out in the Introduction, when φ is a one or two-fold Pfister form, the result is proved in [12, Prop. 1.5] in the context of hermitian levels of fields and quaternion algebras. As $s_{\langle 1\rangle}(K) = \mathrm{s}(K)$, one may assume that $\dim(\varphi) = 2^n$ with $n \geqslant 1$ and that $\mathrm{s}_\varphi(K) > 1$. Let $\mathrm{s}_\varphi(K) \geqslant m$ and let r be a positive integer such that $2^r < m \leqslant 2^{r+1}$. By hypothesis, the quadratic form $\langle 1\rangle \perp \sigma_{m-1,\varphi}$ is anisotropic (otherwise we would have $\mathrm{s}_\varphi(K) < m$). Moreover,

$$\dim\left(\langle 1\rangle \perp \sigma_{m-1,\varphi}\right) = 1 + 2^n(m-1) \geqslant 1 + 2^{n+r} > 2^{n+r} = \frac{1}{2} \times 2^{n+r+1}.$$

This form is a Pfister neighbor of $\sigma_{2^{r+1},\varphi}$ which is anisotropic. The subform $\langle 1\rangle \perp \sigma_{2^{r+1}-1,\varphi}$ is also anisotropic so $\mathrm{s}_\varphi(K) \geqslant 2^{r+1}$, hence the result. □

Theorem 4.2. *If φ is a Pfister form over K, then $\mathrm{s}_\varphi(K)$ is a 2-power or infinite. Moreover $L_\varphi(K) = \{1, 2, \cdots, 2^i, \cdots, \mathrm{s}_\varphi(K)\}$.*

Proof. The direct inclusion comes from Proposition 4.1 together with Lemma 3.1(3). To prove the converse, first note that the two numbers 1 and $\mathrm{s}_\varphi(K)$ are respectively attained over $K(\varphi)$ and by the φ-level of K. Let $n > 0$ be such that $2^n < \mathrm{s}_\varphi(K)$ and for an integer k, put $\varphi_k = \sigma_{2^k,\varphi}$. Then φ_k is a Pfister form for any k. Put $K' = K(\varphi_{n+1})$. As $2^n < \mathrm{s}_\varphi(K)$, φ_n is not hyperbolic over K, Cassels-Pfister subform Theorem shows that $(\varphi_n)_{K'}$ is anisotropic which implies that $\mathrm{s}_\varphi(K') > 2^{n-1}$ (and is a 2-power by the previous proposition). Moreover, the form $\psi_n = \langle 1 \rangle \perp \varphi_n$ is a Pfister neighbor of φ_{n+1}. As φ_{n+1} is hyperbolic over K', ψ_n is isotropic over K' so $\mathrm{s}_\varphi(K') = 2^n$ hence the result. □

Corollary 4.3. *Let φ be a n-fold Pfister form over K and q be a subform of φ such that $\dim(q) > 2^{n-1}$. Then $\mathrm{s}_\varphi(K) \leqslant \mathrm{s}_q(K) \leqslant 2\,\mathrm{s}_\varphi(K)$.*

Proof. If $\mathrm{s}_\varphi(K) = +\infty$, we have $\mathrm{s}_q(K) = +\infty$ by Lemma 3.1 (2) so suppose that $\mathrm{s}_\varphi(K) < +\infty$. By the previous proposition $\mathrm{s}_\varphi(K) = 2^r$ for an integer r and we have $\mathrm{s}_\varphi(K) = 2^r \leqslant \mathrm{s}_q(K)$ by Lemma 3.1 (2). If $n = 1$, $q = \varphi$ and the result is clear. Suppose $n \geqslant 2$. By the definition of the φ-level the quadratic form $\langle 1 \rangle \perp \sigma_{2^r,\varphi}$ is isotropic. This form is a Pfister neighbor of $\sigma_{2^{r+1},\varphi}$ which is thus hyperbolic. Finally, $\sigma_{2^{r+1},q}$ is isotropic being a Pfister neighbor of $\sigma_{2^{r+1},\varphi}$. Hence -1 is represented by $\sigma_{2^{r+1},q}$ and thus $\mathrm{s}_q(K) \leqslant 2^{r+1} = 2\,\mathrm{s}_\varphi(K)$. □

Example 4.4. The upper bound given in 4.3 is sharp. Let $p \neq 2$ be a prime number and $K = \mathbb{Q}_p$. If φ is the unique 4-dimensional anisotropic form over K and q is the pure subform of φ, we have $\mathrm{s}_q(K) = 2$ and $\mathrm{s}_\varphi(K) = 1$.

4.2 Behavior under quadratic extensions

We now investigate the behavior of the q-level under quadratic extensions in the case of Pfister forms.

Lemma 4.5. *Let K be a field, (V, φ) a Pfister form over K and let $L = K(\sqrt{d})$ be a quadratic field extension of K. Then we have $\ell_\varphi(-d) \leqslant 2s$ where $s = \mathrm{s}_\varphi(L)$.*

Proof. By hypothesis, there exist $2s$ vectors $v_1, w_1, \cdots, v_s, w_s$ in V such that

$$-1 = \varphi(v_1 \otimes 1 + w_1 \otimes \sqrt{d}) + \cdots + \varphi(v_s \otimes 1 + w_s \otimes \sqrt{d}). \tag{1}$$

Denote by b_φ the bilinear form associated to φ. Then for $v,\ w \in V$ we have $\varphi(v \otimes 1 + w \otimes \sqrt{d}) = \varphi(v) + d\varphi(w) + 2b_\varphi(v, w)\sqrt{d}$. >From equation (1) we obtain the following equation

$$-1 = (\varphi(v_1) + \cdots + \varphi(v_s)) + d(\varphi(w_1) + \cdots + d\varphi(w_s)), \tag{2}$$

Thus $-d(\varphi(w_1) + \cdots + \varphi(w_s))^2$ is equal to

$$(\varphi(v_1) + \cdots + \varphi(v_s))(\varphi(w_1) + \cdots + \varphi(w_s)) + (\varphi(w_1) + \cdots + \varphi(w_s)). \tag{3}$$

As $\sigma_{s,\varphi}$ is a Pfister form, it is multiplicative hence the first term of the expression (3) is represented by $\sigma_{s,\varphi}$. The expression (3) can therefore be represented by the form $\sigma_{2s,\varphi}$, hence the result. □

Example 4.6. Note that the bound obtained in the previous lemma is optimal. Take $K = \mathbb{Q}$, $d = -3$ and $L = \mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{-3})$. Let $\varphi = \langle 1, 1\rangle$. As $-1 = (\frac{1+\sqrt{-3}}{2})^2 + (\frac{1-\sqrt{-3}}{2})^2$, the element -1 is represented by φ, hence $\mathrm{s}_\varphi(L) = 1$. As $-d = 3$ is represented by $2\times\varphi$ but it is not represented by φ we have $\ell_\varphi(3) = 2$.

Proposition 4.7. *Let φ be a Pfister form over a field K. Let $d \in K$ be an element such that $\ell_\varphi(-d) = n$. If $L = K(\sqrt{d})$, we have $\mathrm{s}_\varphi(L) = 2^k$ or 2^{k-1} where k is determined by $2^k \leqslant n < 2^{k+1}$.*

Proof. As $\ell_\varphi(-d) = n$, there exist vectors $v_1, \cdots, v_n$ such that $-d = \varphi(v_1) + \cdots + \varphi(v_n)$. We thus have

$$-1 = \varphi(v_1 \otimes \frac{1}{\sqrt{d}}) + \cdots + \varphi(v_n \otimes \frac{1}{\sqrt{d}}),$$

so $\mathrm{s}_\varphi(L) \leqslant n$ and $\mathrm{s}_\varphi(L) \leqslant 2^k$ by Proposition 4.1. It suffices to prove that the case $\mathrm{s}_\varphi(L) \leqslant 2^{k-2}$ cannot occur. If $\mathrm{s}_\varphi(L) \leqslant 2^{k-2}$ the previous lemma implies that $\ell_\varphi(-d) \leqslant 2^{k-1} < n$, which is a contradiction. □

Remark 4.8. In Proposition 4.7, if $\varphi = \langle 1\rangle$ the possibility $\mathrm{s}_\varphi(K) = 2^{k-1}$ is ruled out, see [21, Ch. 4, Thm. 4.3] or [9, Prop. 3.3]. In general, both values 2^k and 2^{k-1} can happen as we now show. For instance, by taking $K = \mathbb{Q}$, $d = -3$ and $\varphi = \langle 1, 1\rangle$ we obtain $n = \ell_\varphi(3) = 2$ and so $k = 1$. In this case $\mathrm{s}_\varphi(L) = 1 = 2^{k-1}$. Now take $d = -1$ and $\varphi = \langle 1\rangle$. We obtain $n = \ell_\varphi(1) = \ell(1) = 1$ and so $k = 0$. In this case we have $\mathrm{s}_\varphi(L) = \mathrm{s}(\mathbb{Q}(i)) = 1 = 2^k$.

4.3 Values represented by a Pfister form

Recall that $D_K(\varphi)$ is a subgroup of $K^\times$ if φ is a Pfister form over K. In this case, we first generalize some facts concerning $\mathrm{s}(K)$ or $p(K)$ to $\mathrm{s}_\varphi(K)$ or $p_\varphi(K)$.

Proposition 4.9. *Let (V, φ) be a Pfister form over K with $\mathrm{s}_\varphi(K) < +\infty$. Then:*
(1) For every $a \in K$, there exists an integer n such that $a \in D_K(\sigma_{n,\varphi})$.
(2) We have $p_\varphi(K) \in \{\mathrm{s}_\varphi(K), \mathrm{s}_\varphi(K) + 1\}$.
(3) If $t = p_\varphi(K)$ then $D_K(\sigma_{t,\varphi}) = K^\times$.
(4) If K is non formally real and $\mathrm{s}_\varphi(K) < p_\varphi(K)$ then $2\,\mathrm{s}_\varphi(K)\dim(\varphi) \leqslant u(K)$.

Proof. In the proof, we will use the notations $s = \mathrm{s}_\varphi(K)$ and $p = p_\varphi(K)$.

(1) Note that $a = (\frac{a+1}{2})^2 + (-1)(\frac{a-1}{2})^2$ and that the Pfister form $\sigma_{s,\varphi}$ represents -1. As Pfister forms are multiplicative (see [8, Theorem X.2.8]), $\sigma_{s,\varphi}$ represents $(-1)(\frac{a-1}{2})^2$ hence $\sigma_{s+1,\varphi}$ represents $(\frac{a+1}{2})^2 + (-1)(\frac{a-1}{2})^2 = a$.

(2) comes from the fact that $s = \ell_\varphi(-1) \leqslant p$ and from (1) as $\sigma_{s+1,\varphi}$ is universal and (3) is a consequence of (1).

(4) One may assume that $u(K) < +\infty$. As $s < p$, the quadratic form $\sigma_{s,\varphi}$ is not universal (otherwise, we would have $s = p$). So there exists an element $-a \in K^\times$ which is not represented by this form. Define $\psi = \langle 1, a\rangle \otimes \sigma_{s,\varphi}$. We claim that ψ is anisotropic. If it is isotropic, as $\sigma_{s,\varphi}$ is anisotropic, there exist elements $b,\ c \in K^\times$ both represented by $\sigma_{s,\varphi}$ such that $b = -ac$. As $\sigma_{s,\varphi}$ is multiplicative, it represents bc hence $-a$ which is a contradiction. The form ψ is thus anisotropic with dimension $2s \times \dim(\varphi)$, hence the result. □

Remark 4.10. If $K = \mathbb{Q}$, $q = \langle -5 \rangle$ then by subsection 3.4, $\mathrm{s}_q(K) = 2$ and $p_q(K) = 4$ (as $\ell_q(-35) = 4$, $p(\mathbb{Q}) = 4$ and $D_{\mathbb{Q}}(\sigma_{n,q}) = -5\,\mathbb{Q}^+$ if $n \geq 4$). Hence statements (1), (2), (3) in Proposition 4.9 are not valid if φ is not a Pfister form.

Consider a field F with $u(F) = s(F) = 4$ (given by the construction in 3.4.3 for example) and take $q = \langle -1, -1, -1 \rangle$. Then $\mathrm{s}_q(F) = 1$, $p_q(F) = 2$ (as q does not represent 1 but $2 \times q$ is universal) and $2\,\mathrm{s}_q(F)\dim(q) > u(F) = 4$, hence statement (4) is also false in general.

Theorem 4.11. *Let φ be a Pfister form over a field K whose φ-level is 2^n. Then $|K^\times / D_K(\varphi)| \geqslant 2^{n(n+1)/2}$.*

Proof. To prove this result, we adapt Pfister's proof of the fact that $|K^\times / {K^\times}^2| \geqslant 2^{n(n+1)/2}$ whenever $\mathrm{s}(K) = 2^n$. In the sequel, we set $G_j := D_K(\sigma_{j,\varphi}) \subset K^\times$ for any integer j and write s for $\mathrm{s}_\varphi(K)$. We have :

$$-1 = \varphi(e_1) + \cdots + \varphi(e_s) \tag{4}$$

Note that the conclusion is clear if $n = 0$ and is true if $n = 1$ (in this case, $-1 \notin D_K(\varphi)$ hence $|K^\times / D_K(\varphi)| \geqslant 2$). One may assume that $n \geqslant 2$. Let $j = 2^i$ where $0 \leqslant i < n$. We claim that the elements $a_1 = \varphi(e_1) + \cdots + \varphi(e_{2j})$, $a_2 = \varphi(e_{2j+1}) + \cdots + \varphi(e_{4j})$, $\cdots$ are pairwise non congruent modulo G_j. Indeed, if we would have $\varphi(e_{2j+1}) + \cdots + \varphi(e_{4j}) = c(\varphi(e_1) + \cdots + \varphi(e_{2j}))$ for some $c \in G_j$ then we would obtain $\varphi(e_1) + \cdots + \varphi(e_{4j}) = (1+c)(\varphi(e_1) + \cdots + \varphi(e_{2j})) \in G_{2j}$ which would contradict the minimality of s in (4), hence the claim.

The elements $a_1, a_2, \cdots$ are not in G_j (otherwise, this would also contradict the minimality in (4)), hence there are at least $1 + \frac{s}{2j} = 1 + 2^{n-i-1}$ elements in G_{2j}/G_j (the $\frac{n}{2j}$ elements a_i and the element 1). Now, G_{2j}/G_j injects in $K^\times / D_K(\varphi)$. If G_{2j}/G_j has infinite order, there is nothing to prove. Otherwise, G_{2j}/G_j is a 2-group and what we have done above shows that $[G_{2j} : G_j] \geqslant 2^{n-i}$. We have the following sequence of inclusions

$$D_K(\varphi) = G_1 \subset G_2 \subset \cdots \subset G_{2^{n-1}} \subset G_{2^n} \subset K^\times.$$

We thus obtain $|K^\times / D_K(\varphi)| \geqslant |G_{2^n}/G_1|$ and

$$|G_{2^n}/G_1| = |G_{2^n}/G_{2^{n-1}}| \times \cdots \times |G_2/G_1| \geqslant \prod_{i=0}^{n-1} 2^{n-i} = 2^{\frac{n(n+1)}{2}}.$$

□

Example 4.12. The lower bound indicated in 4.11 is attained as we see by taking $K = \mathbb{Q}_p$, $p \neq 2$ and φ the unique 4-dimensional anisotropic form over K since $|K^\times / D_K(\varphi)| = 1$ and $\mathrm{s}_\varphi(K) = 1$.

Corollary 4.13. *Let K be a field with level 2^n and $t = 2^m \geq 0$. Consider the subgroup $G = \{a_1^2 + \cdots + a_t^2 \neq 0 \mid a_i \in K\} \subset K^\times$. If $t > \mathrm{s}(K)$ then we have $G = K^\times$. If $t \leqslant \mathrm{s}(K)$ then we have $|K^\times / G| \geqslant 2^{k(k+1)/2}$ where $k = n - m$.*

Proof. Consider the Pfister form $\varphi = \sigma_t$. We have $G = D_K(\varphi)$. If $t > \mathrm{s}(K)$, φ is isotropic, hence $G = K^\times$. If $t \leqslant \mathrm{s}(K)$ then by Lemma 3.1 (5), we have $\mathrm{s}_\varphi(K) = \left\lceil \frac{\mathrm{s}(K)}{t} \right\rceil = 2^{n-m} = 2^k$ and the result follows from Theorem 4.11 (2). □

Corollary 4.14. *Let $L = K(\sqrt{d})$ be a quadratic extension of K where $d \in K^\times \setminus K^{\times^2}$ and $\varphi = \langle 1, -d\rangle$. If $\mathrm{s}_\varphi(K) = 2^n$ then $|K^\times/N_{L/K}(L^\times)| \geqslant 2^{n(n+1)/2}$.*

Proof. As $N_{L/K}(L^\times) = D_K(\varphi)$, the result follows from Theorem 4.11 (2). □

Remark 4.15. It is relevant to mention the following formula obtained by D. Lewis in [10, Cor. after Prop. 3]: if L/K is a quadratic field extension,

$$|L^\times/L^{\times 2}||K^\times/N_{L/K}(L^\times)| = \frac{1}{2}|K^\times/K^{\times 2}|^2.$$

The below result has first been proved by A. Pfister for the form $\varphi = \langle 1\rangle$ in [16]. Our reformulation is taken from [1].

Proposition 4.16. *Let φ be a Pfister form over a field K.*
(1) *For every $x,\ y \in K^\times$ we have $\ell_\varphi(xy) \leqslant \ell_\varphi(x) + \ell_\varphi(y) - 1$.*
(2) *For every $x \in K^\times$ we have $\mathrm{s}_\varphi(K) \leqslant \ell_\varphi(x) + \ell_\varphi(-x) - 1$.*

Proof. First note that (2) is a consequence of (1) as $\mathrm{s}_\varphi(K) = \ell_\varphi(-x^2)$. If $\ell_\varphi(x) = +\infty$ or $\ell_\varphi(y) = +\infty$, the result is trivial. One may assume that $\ell_\varphi(x)$ and $\ell_\varphi(y)$ are both finite and that $\ell_\varphi(x) \leqslant \ell_\varphi(y)$ without loss of generality. We prove the result by induction over $\ell_\varphi(y)$. If $\ell_\varphi(y) = 1$ then we obtain $\ell_\varphi(x) = 1$ and the result holds, because φ is multiplicative. Define r to be the positive integer such that $2^{r-1} < \ell_\varphi(x) \leqslant 2^r$. One may write $\ell_\varphi(y) = 2^r k + l$ where $k \geqslant 0$ and $1 \leqslant l \leqslant 2^r$. Two cases are to be considered: $k = 0$ or $k \geqslant 1$.

If $k = 0$ then $\ell_\varphi(y) = l \leqslant 2^r$ and y is represented by $\sigma_{2^r,\varphi}$. As x is also represented by $\sigma_{2^r,\varphi}$, we deduce that xy is represented by $\sigma_{2^r,\varphi}$ (because φ is multiplicative). Consequently, $\ell_\varphi(xy) \leqslant 2^r$. But $2^{r-1} < \ell_\varphi(x) \leqslant \ell_\varphi(y)$ so $2^r + 2 \leqslant \ell_\varphi(x) + \ell_\varphi(y)$ which shows that $\ell_\varphi(xy) \leqslant \ell_\varphi(x) + \ell_\varphi(y) - 1$.

If $k \geqslant 1$, y is represented by the form $\sigma_{2^r k,\varphi} \perp \sigma_{l,\varphi}$ so we can write $y = y' + y''$ where $\ell_\varphi(y') = 2^r k$ and $\ell_\varphi(y'') = l$. As x and y' are respectively represented by $\sigma_{2^r,\varphi}$ and $\sigma_{2^r k,\varphi}$, xy' is represented by $\sigma_{2^r k,\varphi}$, that is $\ell_\varphi(xy') \leqslant 2^r k$. As $\ell_\varphi(y'') < \ell_\varphi(y)$, the induction hypothesis implies that $\ell_\varphi(xy'') \leqslant \ell_\varphi(x) + \ell_\varphi(y'') - 1 = \ell_\varphi(x) + l - 1$. Hence $\ell_\varphi(xy) = \ell_\varphi(xy' + xy'') \leqslant 2^r k + (\ell_\varphi(x) + l - 1) = \ell_\varphi(x) + \ell_\varphi(y) - 1$. □

Remark 4.17. The two bounds given in the previous proposition are sharp for any Pfister form φ: choose $x = 1$, $y = -1$ so that $\ell_\varphi(1) = 1$ and $\ell_\varphi(-1) = \mathrm{s}_\varphi(K)$.

The first inequality does not hold in general. In the formally real case, choose $\varphi = \langle -1\rangle$ and $x = y = -1$: then $\ell_\varphi(-1) = 1$ but $\ell_\varphi((-1) \times (-1)) = +\infty$. In the non formally real case, take $K = \mathbb{Q}_p$ (p odd) and choose $x = u$ where u is a unit such that $\overline{u} \notin \overline{K}^2$ and $y = \pi$ where π is a uniformizer. Choose $\varphi = \langle u, \pi\rangle$ over K. Then $\ell_\varphi(u) = \ell_\varphi(\pi) = 1$. Now $\ell_\varphi(u\pi) = 1$ if and only if the quadratic form $\langle -u, -\pi, u\pi\rangle$ is isotropic. As this latter form is a Pfister neighbor of $\langle\langle u, \pi\rangle\rangle$ which is anisotropic, it follows that $\ell_\varphi(u\pi) = 2 > \ell_\varphi(u) + \ell_\varphi(\pi) - 1 = 1$.

4.4 Pythagoras q-number and field extensions

If (V, φ) is a Pfister form over a field K then for every $x,\ y \in V$ there exists $z \in V$ such that $\varphi(x) \cdot \varphi(y) = \varphi(z)$. As A. Pfister observed, it is also known that z can be chosen in such a way that its first component is of the form $b_\varphi(x, y)$ where b_φ is the bilinear form associated to φ, see [19, Chapter 2, Cor. 2.3] or

[18, Satz 1]. For the case where $\varphi = \sigma_{2^r}$ $(r \geqslant 0)$, this implies in particular that for every $x_1, \cdots, x_{2^r}, y_1, \cdots, y_{2^r} \in K$ there exists an identity like

$$(x_1^2 + \cdots + x_{2^r}^2) \cdot (y_1^2 + \cdots + y_{2^r}^2) = (x_1 y_1 + \cdot + x_{2^r} y_{2^r})^2 + z_2^2 + \cdots + z_{2^r}^2$$

where $z_2, \cdots, z_{2^r}$ are suitable elements of K. We obtain the following lemma as an immediate consequence of Pfister's observation.

Lemma 4.18. *Let (V, ψ) be a Pfister form over a field K. Let $\psi = \langle 1 \rangle \perp \psi_0$ where ψ_0 is a subform of ψ of codimension 1. Let $r \in \mathbb{N}$, $m = 2^r$ and $x_1, \cdots, x_m, y_1, \cdots, y_m \in V$. Then there exist $z_1, \cdots, z_m \in V$ such that*

$$(\psi(x_1) + \cdots + \psi(x_m)) \cdot (\psi(y_1) + \cdots + \psi(y_m)) = \psi(z_1) + \cdots + \psi(z_m)$$

and there exists z_1' in the underlying vector space of ψ_0 such that

$$\psi(z_1) = (b_\psi(x_1, y_1) + \cdots + b_\psi(x_m, y_m))^2 + \psi_0(z_1')$$

where b_ψ is the bilinear form associated to ψ.

Proof. Apply [19, Chapter 2, Cor. 2.3] to the Pfister form $\varphi = \sigma_{2^r, \psi}$. □

The following result was proved for the form $\varphi = \langle 1 \rangle$ in [19, Chapter 7, 1.12].

Proposition 4.19. *Let (V, φ) be a Pfister form over a field K. Suppose that $\mathrm{s}_\varphi(K) = +\infty$. Let $f(x) \in K[x]$ be such that $\ell_{\varphi|_{K(x)}}(f(x)) < +\infty$. Then $\deg(f(x)) = 2n$ is even and $\ell_{\varphi|_{K(x)}}(f(x)) \leqslant p_\varphi(K)(n+1)$.*

Proof. If $p_\varphi(K) = +\infty$ the result is trivial. Assume now that $p_\varphi(K) < +\infty$ and consider the smallest positive integer m such that $f(x)$ is represented by $\sigma_{m,\varphi}$ over $K(x)$. Using the first representation Theorem of Cassels-Pfister (see [8, Theorem IX.1.3]) we obtain that $f(x)$ is represented by $\sigma_{m,\varphi}$ over $K[x]$. This implies that the degree of $f(x)$ can not be odd, otherwise $\sigma_{m,\varphi}$ would be isotropic over K which contradicts the hypothesis $\mathrm{s}_\varphi(K) = +\infty$. So the degree of $f(x)$ is even and we may suppose that $\deg(f(x)) = 2n$. We proceed by induction on n. If $n = 0$, then $f(x) = c \in K$ is a constant polynomial. As $f(x) = c$ is a represented by $\sigma_{m,\varphi}$ over $K(x)$, it is represented by $\sigma_{m,\varphi}$ over K by Substitution Principle (see [19, Chapter 1, 3.1]). It follows that $\ell_{K(x)}(f(x)) \leqslant p_\varphi(K)$, hence the result. Suppose now that $n \geqslant 1$. Take $f(x) = a_{2n} x^{2n} + \cdots + a_0$ where $a_{2n}, \cdots, a_0 \in K$. As $f(x)$ is represented by $\sigma_{m,\varphi}$ over $K(x)$ it follows that a_{2n} is represented by $\sigma_{m,\varphi}$ over K. Thus the polynomial $\frac{f(x)}{a_{2n}}$ is also represented by $\sigma_{m,\varphi}$ over $K[x]$. Let k be a positive integer such that $m \leqslant 2^k$. The polynomial $\frac{f(x)}{a_{2n}}$ is also represented by $\sigma_{2^k,\varphi}$. It follows that

$$\frac{f(x)}{a_{2n}} = g_1(x) + \cdots + g_{2^k}(x) \tag{5}$$

where the polynomials $g_1(x), \cdots, g_{2^k}(x)$ are represented by the form φ over $K[x]$. Let $v_1(x), \cdots, v_{2^k}(x) \in V[x]$ be the elements such that $\varphi(v_i(x)) = g_i(x)$ for every $i = 1, \cdots, 2^k$. By comparing the leading coefficients of the equation (5) we obtain a relation

$$1 = b_1 + \cdots + b_{2^k} \tag{6}$$

where $b_1, \cdots, b_{2^k} \in K$ are represented by φ. Let $w_1, \cdots, w_{2^k} \in V$ be the elements such that $\varphi(w_i) = b_i$ for every $i = 1, \cdots, 2^k$. By multiplying the relations of the equations (5) and (6) and applying Lemma 4.18 we obtain:

$$\frac{f(x)}{a_{2n}} = \mathrm{s}(x) + r(x) \tag{7}$$

where $\mathrm{s}(x) = (b_\varphi(v_1(x), w_1) + \cdots + b_\varphi(v_{2^k}(x), w_{2^k}))^2$ and $r(x)$ is represented by $\sigma_{m,\varphi}$ over $K[x]$. Note that $\mathrm{s}(x)$ is a monic polynomial with the same degree as $f(x)$. It follows that $r(x)$ is a polynomial whose degree satisfies $\deg(r(x)) < 2n$. As $r(x)$ is represented by $\sigma_{m,\varphi}$, the degree of $r(x)$ should be even, otherwise $\sigma_{m,\varphi}$ would be isotropic over K which is a contradiction. We then have $\deg(r(x)) \leqslant 2n-2$. The relation (7) implies that $f(x) = a_{2n}\,\mathrm{s}(x) + a_{2n}r(x)$. As a_{2n} is represented by $\sigma_{m,\varphi}$ over K, $a_{2n}r(x)$ is also represented by $\sigma_{m,\varphi}$ over $K[x]$. By the induction hypothesis we obtain $\ell_{\varphi|K[x]}(a_{2n}r(x)) \leqslant p_\varphi(K)(n)$. We obviously have $\ell_{\varphi|K[x]}(a_{2n}\,\mathrm{s}(x)) = \ell_\varphi(a_{2n}) \leqslant p_\varphi(K)$. We so obtain $\ell_{\varphi|_{K[x]}}(f(x)) \leqslant p_\varphi(K)(n) + p_\varphi(K) = p_\varphi(K)(n+1)$. □

Proposition 4.20. (1) *(Pfister) Let K be a real field and let L/K be a field extension of finite degree. Then $p(L) \leqslant p(K)[L:K]$.*
(2) *Let L/K be a field extension of finite degree. Let φ be a Pfister form over K. Suppose that $\mathrm{s}_\varphi(K) < +\infty$. Then $p_\varphi(L) \leqslant p_\varphi(K)[L:K]$.*

Proof. The statement (1) is proved in [19, Chapter 7, 1.13]. It is clear that (1) is a particular case of (2) by taking $\varphi = \langle 1 \rangle$. To prove (2), it suffices to prove the result for the case where $L = K(\alpha)$ is a simple extension. Let V be the underlying vector space of φ. Suppose that $[L:K] = n$. If $p_\varphi(K) = +\infty$ the conclusion is trivial. Assume that $p_\varphi(K) < +\infty$. Let $\beta \in L$ be an element such that $r := \ell_{\varphi|_L}(\beta) < +\infty$. In order to prove the result we have to show that $r \leqslant p_\varphi(K)[L:K]$. Every element of the vector space $V \otimes_K L$ can be written as $v_0 \otimes 1 + v_1 \otimes \alpha + \cdots + v_{n-1} \otimes \alpha^{n-1}$ where $v_i \in V$ for every $i = 1, \cdots, n$. There exist $w_1, \cdots, w_r \in V \otimes L$ such that $\beta = \varphi(w_1) + \cdots + \varphi(w_r)$. Let $w_j = \sum_{i=0}^{n-1} v_{ij} \otimes \alpha^i$ where $v_{ij} \in V$ and $j = 1, \cdots, r$. Put $w_j(x) = \sum_{i=0}^{n-1} v_{ij} \otimes x^i \in V[x] = V \otimes_K K[x]$. We so have $w_j = w_j(\alpha)$ for every $j = 1, \cdots, r$. Consider the polynomial $f(x) = \varphi(w_1(x)) + \cdots + \varphi(w_r(x))$. The degree of $f(x)$ is even and satisfies $\deg(f(x)) \leqslant 2(n-1)$. According to Proposition 4.19, we have $\ell_{\varphi|_{K[x]}} \leqslant p_\varphi(K)n$. By substituting $x := \alpha$, we obtain $\ell_\varphi(\beta) \leqslant p_\varphi(K)n$. □

References

[1] Becher K.; On the number of square classes of a field of finite level. Proceedings of the Conference on Quadratic Forms and Related Topics (Baton Rouge, LA, 2001). Doc. Math. 2001, Extra Vol., 65–84

[2] Cassels J. W. S., Ellison W. J., Pfister A.; On sums of squares and on elliptic curves over function fields. J. Number Theory **3** (1971), 125–149.

[3] Colliot-Thélène J.-L., Jannsen U.; Sommes des carrés dans les corps de fonctions, C.R. Acad. Sci. Paris, Sér. I **312** (1991), 759–762.

[4] Hoffmann D. W.; Isotropy of quadratic forms over the function field of a quadric. Math. Z. **220** (1995), 461–476.

[5] Hoffmann D. W.; Pythagoras numbers of fields. J. Amer. Math. Soc. **12** (1999), 839–848.

[6] Hoffmann D. W.; Isotropy of quadratic forms and field invariants. Quadratic forms and their applications (Dublin 1999), 73–102, Contemp. Math. **272**, Amer. Math. Soc., Providence, RI, (2000).

[7] Kneser H.; Verschwindende Quadratsummen in Körpern. Jahresber. Dtsch. Math.-Ver. **44** (1934), 143–146.

[8] Lam T.-Y.; *Introduction to quadratic forms over fields.* Graduate Studies in Mathematics, 67. Providence, RI: American Mathematical Society (AMS), (2005).

[9] Leep D. B.; A historical view of the Pythagoras numbers of fields. Quadratic forms—algebra, arithmetic, and geometry, 271–288, Contemp. Math., 493, Amer. Math. Soc., Providence, RI, 2009.

[10] Lewis D. W.; New improved exact sequences of Witt groups. J. Algebra **74** no. 1 (1982), 206–210.

[11] Lewis D. W.; Levels and sublevels of division algebras. Proc. Roy. Irish Acad. Sect. A **87** no. 1 (1987), 103–106.

[12] Lewis D. W.; Sums of hermitian squares. J. Algebra **115** (1988), 466–480.

[13] Lewis D. W.; Levels of quaternion algebras. Quadratic forms and real algebraic geometry (Corvallis, OR, 1986). Rocky Mountain J. Math. **19** no. 3 (1989), 787–792.

[14] Lewis D. W.; Levels of fields and quaternion algebras – a survey. Théorie des nombres, Années 1996/97–1997/98, 9 pp., Publ. Math. UFR Sci. Tech. Besançon, Univ. Franche-Comté, Besançon, 1999.

[15] Merkurjev A. S.; On the norm residue symbol of degree 2. Dokladi Akad. Nauk. SSSR 261 (1981) 542–547. (English translation: Soviet Math. Doklady 24 (1981) 546–551.)

[16] Pfister A.; Zur Darstellung von -1 als Summe von Quadraten in einem Körper. J. London Math. Soc. **40** (1965), 159–165.

[17] Pfister A.; Quadratische Formen in beliebigen Körpern. Invent. Math., **1** (1966), 116–132.

[18] Pfister A.; Zur Darstellung definiter Funktionen als Summe von Quadraten. Invent. Math., **4** (1967), 229–237.

[19] Pfister A.; *Quadratic forms with applications to algebraic geometry and topology.* London Mathematical Society Lecture Note Series, 217. Cambridge: Cambridge Univ. Press, (1995).

[20] Pourchet Y.; Sur la représentation en somme de carrés des polynômes à une indéterminée sur un corps de nombres algébriques. Acta Arith. **19** (1971), 89–104.

[21] Scharlau W.; *Quadratic and Hermitian forms.* Grundlehren der Mathematischen Wissenschaften, 270. Springer-Verlag, Berlin, (1985).

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