The resolution of Diophantine equations according to Bhāskara and a justification of the cakravāla by Kṛṣṇadaivajña

François Patte

The resolution of Diophantine equations according to Bhāskara and a justification of the *cakravāla* by Kṛṣṇadaivajña

François Patte

Paris Descartes University
April 9, 2011

Abstract

The main goal of this paper is to present an Indian 'demonstration' of the *cakravāla* rules provided by Bhāskara II. This demonstration was given by Kṛṣṇadaivajña, a mathematician who lived at the end of the sixteenth century, in his commentary on Bhāskara's *Bījagaṇita*: the *Bījapallava*.

The *cakravāla*, or cyclic method, is a procedure to calculate the solutions of a Diophantine quadratic equation of the form: $p\,x^2 + k = y^2$. This cyclic method involves solving linear Diophantine equations, so we will make a presentation, as short as possible, of the rules given by Bhāskara in order to handle these linear and quadratic equations: the *kuṭṭaka* and the *vargaprakṛti*. When necessary, we will present commentators' explanations.

1 The *kuṭṭaka* or linear Diophantine equations

The *kuṭṭaka*, or 'pulveriser', is an algorithm to solve indeterminate equations of the form:

$$a\,u + c = b\,v$$

where all numbers are integers; a, b and c are the coefficients, u and v the unknowns. Bhāskara gives five rules to describe the full procedure. The first rule stipulates that if a common divisor to a and b does not divide c, the equation has no solution.

bhājyo hāraḥ kṣepakaś cāpavartyaḥ
kena apy ādau saṃbhave kuṭṭakārthaṃ |
yena cchinnau bhājyahārau na tena
kṣepaś cet tad duṣṭam uddiṣṭam eva ||

> Firstly, the *dividend*, the *divisor* and the *additive* must be simplified, when possible, by some [number] for the *kuṭṭaka*. If the number by which the *dividend* and the *divisor* are divided does not [divide] the *additive*, the [problem] is impossible.

This stanza tells more than under which circumstances the problem can be solved: It gives the technical vocabulary which will be used to designate the coefficients of the equation.

- a will be the *dividend*
- b the *divisor*
- c the *additive*[1]

[1]We will italicise these terms whenever they are used with their technical meaning.

The word *kuṭṭaka* (pulveriser) which gives its name to the procedure is, in fact, the name of what we are looking for when we solve this equation: a *multiplier*, u in the equation. Problems given as examples in this *kuṭṭaka* chapter emphasise the role of this solution in the complete equation. For instance a problem will be formulated as follows: "O arithmetician! Say quickly the *multiplier* by which two hundred and twenty-one is multiplied then added to sixty-five and divided by one hundred and ninety-five, will leave no remainder."

As for the meaning of the word *kuṭṭaka* —namely pulveriser— it is a common usage for Indian mathematicians to name the multiplication, and the multiplier, by terms that mean 'to hit'.

The name of one quantity is not quoted in this stanza: v, which is called the *quotient*.

We now come to the description of the algorithm which is to be followed to find out the solutions of these equations. We will proceed step by step and explain the meaning at each step, with our modern mathematical vocabulary.

> *mitho bhajet tau dṛḍhabhājyahārau*
> *yāvad vibhājye bhavatīha rūpam* |

> One will *divide mutually* these reduced *dividend* and *divisor*, until the unity will be in the place of the *dividend*.

A new technical expression is used here: 'divide mutually'; it means the Euclidean algorithm. In the *Kriyākramakarī*, a Keralese commentary to Bhāskara's Līlāvatī, from the sixteenth century, this is explained in this way: "This is said: Having divided the two numbers one by the other (the *dividend* by the *divisor*), one will divide the other (the *divisor*) by what remains; having brought [this operation] about again in the same manner, until only one [number] is left ..."

According to this rule, we have to divide a by b, then b by the remainder of the division, the first remainder by the second remainder, "until the number one is in the place of the *dividend*." To understand this last expression, we have to figure out how the Ancients were conducting their calculations on sand: They replaced the numbers by the result of the operation; in a division, they put the quotient aside, then wiping out the dividend, they wrote the remainder in its place.

We can find a mimic of this in the manuscripts. Suppose that one wants to divide 17 by 15, one writes:

17	then:	1	2
15			15

So, the remainder 2 is put "in the place of the dividend" 17.

The fact that, at the end of the procedure, the number one is in the place of the dividend —thus being the last remainder— is normal because, after applying the first stanza, the two numbers, a and b, are relatively prime.

The next step concerns the arrangement of the terms in order to calculate the solutions.

> *phalāny adho 'dhas tad adho niveśyaḥ*
> *kṣepas tathānte kham...* ||

> The quotients [will have to be placed] one under the other; the *additive* must be placed below them, then zero at the last place.

Let us examine these first two steps on the first example given by Bhāskara, who asks to solve the equation: $221\,u + 65 = 195\,v$.

After a reduction of the three numbers by 13, we can begin the process with this equation: $17\,u + 5 = 15\,v$.

Following the first step, we divide 17 by 15: $17 = 1 \times 15 + 2$, then 15 by the remainder 2: $15 = 7 \times 2 + 1$ and we are done because we obtain 1 as the last remainder.

We now put these quotients "one under the other", below them the *additive* and in the last place zero:

quotient:	1
quotient:	7
additive:	5
zero:	0

Indian mathematicians call this setting *'phalavallī'*, the creeper of the results.

Now, we can go through the next step, the very algorithm that calculates the solutions.

... upāntimena ||
svordhve hate 'ntyena yute tad antyaṃ
tyajen muhuḥ syād iti rāśiyugmam |

The last but one having multiplied the [number] which is above and the last [number] having been added, one will remove this last; repeat the operation until there is a couple of numbers.

Before explaining this stanza, we should point out to how it is written. Things are stated with great conciseness and, in the case of an algorithm description, as in this stanza, with expressions that are as general as possible so that they can be used in an iterative process. Here we have: "The last but one having multiplied the one which is above and the last one having been added, one will remove this last", that can be used whatever the length of the creeper, "until there is a couple of numbers", which marks the end of the operations, because three numbers are needed in this process.

We apply this algorithm to the previous example; this is done in two steps.

In the first step, the 'last but one' is 5 and the number above is 7, so we have to multiply 7 by 5 and add the 'last', 0: $7 \times 5 + 0 = 35$ and we remove 0 from the creeper.

In the second step, we have to multiply 1 by the 'last but one' of the new creeper: 35, then add the 'new' last, 5: $1 \times 35 + 5 = 40$. And the process ends because, after removing the last, there remains only two numbers.

1st step				2nd step	
1		1	$1 \times 35 + 5 =$	40
7	$7 \times 5 + 0 =$	35		35
5		5		
0				

At this stage, we have the solution of the equation: 40 is the *quotient* and 35 the *multiplier*: $17 \times 35 + 5 = 600 = 15 \times 40$. But while explaining which of the two numbers found is the *multiplier* and which is the *quotient*, Bhāskara gives the way to find the minimal positive solutions.

ūrdhvo vibhājyena dṛḍhena taṣṭaḥ
phalaṃ guṇaḥ syād aparo hareṇa ||

The uppermost, being reduced to the remainder by the *dividend*, will be the *quotient*; the other, [reduced to the remainder] by the *divisor*, will be the *multiplier*.

There is a new technical expression to explain: 'reduced to the remainder' that we used to translate the Sanskrit word *'taṣṭa'*. The commentator Sūryadāsa says in his *Sūryaprakāśa* —a commentary on Bhāskara's *Bījagaṇita*—: "When the remainder only is needed in a division, the quotient being useless, the conventionally agreed word *'taṣṭa'* is used."

Applying this rule to the example, we have to divide 40 by 17 and 35 by 15 and keep the remainders respectively as the *quotient* and the *multiplier* of the equation:

$$\begin{aligned} 40 &= 2 \times 17 + 6 \\ 35 &= 2 \times 15 + 5 \end{aligned}$$

So, 5 and 6 are the minimal positive solutions of this equation:

$$17 \times 5 + 5 = 15 \times 6$$

A variant of this last rule is given later on, in order to calculate all the solutions for this form of equation.

iṣṭāhatasvasvahareṇa yukte
te vā bhavetāṃ bahudhā guṇāptī ||

> There will be many *multipliers* and *quotients* if they are added to their respective simplifier multiplied by an arbitrary number.

'Simplifiers' refer to the *divisor* and *dividend* used in the previous rule to find out the minimal solutions. By this rule we can calculate all the solutions of the equation $a\,u + c = b\,v$. Once a couple of solutions, $(u_0,\,v_0)$, has been found, the couple:

$$(u_t,\,v_t) = (u_0 + t\,b,\,v_0 + t\,a)$$

will be another couple of solutions for any arbitrary integer t. Indeed:

$$\begin{aligned} a\,u_t + c &= a(u_0 + t\,b) + c \\ &= a\,u_0 + c + t\,ab \\ &= b\,v_0 + t\,ab \\ &= b(v_0 + t\,a) = b\,v_t \end{aligned}$$

For the sake of completeness we should mention a last rule, even if we will not use it explicitly in this paper. One may have recognised in this method for solving this form of equation, an algorithm very close to the Euclid-Bézout algorithm and just as in the latter, there is a sign alternation which concerns the *additive* at each step of computation.

Let us see how this happens, investigating the first step of Bhāskara's process.

We divide a by b: $a = b\,q_1 + r_1, \quad 0 \le r_1 < b$ and replace a in the initial equation:

$$(b\,q_1 + r_1)\,u + c = b\,v \quad \text{so we get the equation:} \quad r_1\,u + c = b\,(v - q_1\,u)$$

Putting $w = v - q_1\,u$ as a new indeterminate, we have a new equation of the same form with $-c$ as the *additive*: $b\,w - c = r_1\,u$.

At the end of the procedure, as the last remainder is $r_n = 1$, the last equation is:

$$r_{n-1}\,y + (-1)^n c = r_n\,x = x$$

So, we alternately have an equation with c or $-c$ for additive (c if the number of quotients is even, $-c$ if it is odd) and Bhāskara's 'creeper algorithm' is a means to compute the two indeterminates u and v going backward from x.

As the coefficient of x is 1, we can choose any integral value for y to have an integral value for x and hence for u and v. Bhāskara has put 0 as the chosen value for y —which makes the calculation of u and v simpler— and, in his description of the 'creeper algorithm', he does not take into consideration that the *additive* could be c or $-c$ according to the parity of the number of operations, but he gives a last rule to modify the found result if the number of operations is odd. This is classic in Sanskrit texts: First a general rule *(utsarga)* is given, then this general rule is corrected by mentioning exceptions *(apavāda)*.

evaṃ tadaivātra yadā samās tāḥ
syur labdhayaś ced viṣamās tadānīm |
yathāgatau labdhiguṇau viśodhyau
svatakṣaṇāc cheṣamitau tu tau staḥ ||

> Thus are exactly [the operations] when the number of quotients is even; if this number is odd the *quotient* and the *multiplier*, as obtained, must be subtracted from their respective Simplifiers and the [correct] *quotient* and *multiplier* are equal to the remainders.

Hence, if the number of quotients is odd, that is to say the number of divisions is odd, we apply the prescribed algorithm to calculate two numbers: u_0 and v_0 and the solutions of the equation are given by $u_1 = b - u_0$ and $v_1 = a - v_0$ because, if u_0 and v_0 are the solutions of $a\,u + c = b\,v$, u_1 and v_1 are the solutions of the equation $a\,u - c = b\,v$.

$$\begin{aligned} a\,u_1 - c &= a\,(b - u_0) - c \\ &= a\,b - (a\,u_0 + c) \\ &= a\,b - b\,v_0 \\ &= b\,(a - v_0) \\ &= b\,v_1 \end{aligned}$$

2 The *vargaprakṛti*
or a study of the properties of the equation: $p\,x^2 \pm k = y^2$

To begin with, let us explain the meaning of the name of this section: *vargaprakṛti*. It is a compound noun *varga-prakṛti*, *varga* means 'square' and *prakṛti* 'origin'. Kṛṣṇadaivajña gives two explanations in his commentary:

> "*Vargaprakṛti*, that is to say: When the original cause *(prakṛti)* is a square *(varga)*, because the original cause of this calculation is the square of unknown quantities.[2]"
>
> "Or *vargaprakṛti*, that is to say: Squares of unknowns are calculated from a number which is at their origin; in this case the number which is the origin for the squares of unknowns is named by the word *'prakṛti'* and this [number] is the very multiplier of the squares of unknowns. Therefore, in this computation of roots, the multiplier of the squares is designated by the word *'prakṛti'*.[3]"

We have two interpretations for the name of this section: Either the origin of the calculation is to find a square quantity and we can think that this refers to a following chapter *(madhyamāharaṇabhedāḥ)* where the construction of a square is needed to solve an equation.

---

[2]Kṛṣṇa's text reads: *yāvadādi*. In Sanskrit mathematical texts, the unknowns are denominated by colour names *kāla*, black, *nīla*, blue … and the first one, our x, is called *yāvattāvat*: so much as; so, *yāvadādi* means: *yāvat[tāvat], kāla, nīla* etc.

[3]See text 1, page 19.

For instance: "What is the number which multiplied by two and added to six times its square gives a square root?"

We have to solve: $6\,x^2 + 2\,x = y^2$. Bhāskara's method is to multiply both sides by 6 and add 1, which yields the following equation:

$$36\,x^2 + 12\,x + 1 = 6\,y^2 + 1 \quad \text{or} \quad (6\,x + 1)^2 = 6\,y^2 + 1$$

In this case the reason for the calculation is to find a square equal to $6\,y^2 + 1$, and he applies the methods of this chapter, then finds the values of x ($x = 3/2, 8, \cdots$).

Or the origin of squares is the coefficient p of the square x^2 in the identity $p\,x^2 \pm k = y^2$. This coefficient plays a central role in the study of this identity because it is the only value which remains fixed throughout the analysis of the properties of three numbers verifying the previous relationship.

iṣṭaṃ hrasvaṃ tasya vargaḥ prakṛtyā
kṣuṇṇo yukto varjito vā sa yena |
mūlaṃ dadyāt kṣepakaṃ taṃ dhanarṇaṃ
mūlaṃ tac ca jyeṣṭhamūlaṃ vadanti ||

> Let an assumed [number] be the *least root*; its square is multiplied by the *prakṛti*; the *additive* is this [number], positive or negative, by which this [square multiplied by the *prakṛti*] is increased or decreased to produce a root and [mathematicians] call this root, the *greatest root*.

As for the *kuṭṭaka*, the first rule gives the definition of the technical words used in this chapter and what the relationships between the different elements involved in this rule are: A number, the square of which is multiplied by a given number, the *prakṛti*, and added or subtracted to another number, 'produces a root', that is to say: is a square.

According to this stanza, in the identity $p\,x^2 \pm k = y^2$,

- p is the *prakṛti*
- x is called the *least root*
- y is the *greatest root*
- k is the *additive* which can be positive or negative. The Sanskrit words used are *dhana* (wealth) and *ṛṇa* (debt).

Regarding the prescription for an operation given in this rule, it is very simple: Choose a number as the *least root*, x, then complete to the nearest square the value of its square multiplied by the *prakṛti*, adding or subtracting the right number, k, to obtain a square, the root of which is the *greatest root*: y. Mostly, the chosen *least root* will be 1, making the *additive* the complement of the *prakṛti* to the nearest square.

The *bhāvanā*

or how to calculate many *least* and *greatest roots*

Once we have found three numbers, x, y and k, using the preceding rule, the *bhāvanā*, which we can translate by 'composition', is a procedure to calculate several triples of numbers which verify the relation $p\,x^2 \pm k = y^2$ with a fixed *prakṛti*. As a convention, we will note such a triple: $[x_n,\, y_n;\, k_n]$.

Bhāskara gives the *bhāvanā* rule in two parts: The first part gives the setting of the numbers in order to make the calculation, the second is the description of how to proceed.

hrasvajyeṣṭhakṣepakān nyasya teṣāṃ
tān anyān vādho niveśya krameṇa |
sādhyāny ebhyo bhāvanābhir bahūni
mūlāny eṣāṃ bhāvanā procyate 'taḥ ||

vajrābhyāso jyeṣṭhalaghvos tadaikyaṃ
hrasvaṃ laghvor āhatiś ca prakṛtyā |
kṣuṇṇā jyeṣṭhābhyāsayug jyeṣṭhamūlaṃ
tatrābhyāsaḥ kṣepayoḥ kṣepakaḥ syāt ||

> Having set down a *least root* and a *greatest root* and an *additive*, then successively placed under them, the same ones or others, many roots can be calculated by compositions *(bhāvanā)*; that is why the composition is taught.
>
> Given the cross products of the *greatest* and *least roots*, their sum is a *least root.* And the product of the *least roots*, multiplied by the *prakṛti* and added to the product of the *greatest root*, is a *greatest root.* The product of the *additives* will be an *additive.*

The arrangement is very simple: We put two triples on two lines, one under the other; if we have only one triple we can put the same triple on the second line.

To calculate a new *least root*, we have to make a 'cross product' of the *least* and *greatest roots*: We multiply the *least root* on the first line by the *greatest root* on the second line and add the product of the *greatest root* on the first line by the *least root* on the second line. The expression 'cross product' is the translation of the Sanskrit technical term: *vajrābhyāsa*, meaning 'a multiplication like a thunderbolt'.

For a new *greatest root*, we multiply the product of the two *least roots* on the two lines by the *prakṛti* and add the product of the two *greatest roots.*

And for a new *additive*, we multiply the two *additives* on the two lines.

We can summarise these calculations in the following way: Arrows connect numbers to be multiplied, then results are added:

$$\begin{array}{ccc} x_1 & y_1 & k_1 \\ & \times & \\ x_2 & y_2 & k_2 \end{array} \qquad \begin{array}{cccc} p \longrightarrow & x_1 & y_1 & k_1 \\ & \downarrow & \downarrow & \\ & x_2 & y_2 & k_2 \end{array} \qquad \begin{array}{ccc} x_1 & y_1 & k_1 \\ & & \downarrow \\ x_2 & y_2 & k_2 \end{array}$$

$$x_3 = x_1 y_2 + x_2 y_1 \qquad y_3 = p x_1 x_2 + y_1 y_2 \qquad k_3 = k_1 k_2$$

It is easy to demonstrate that the new triple $[x_3,\, y_3\,; k_3]$ verifies the same relation as the two triples $[x_1,\, y_1\,; k_1]$ and $[x_2,\, y_2\,; k_2]$.

Bhāskara also gives the same rule of composition by replacing the addition by the difference of the products. Commentators explain that the use of this latter rule is useful if we need smaller numbers as roots.

Solving 'simple' Pell's equations

Before coming to an example, we need a last rule given by Bhāskara in this *vargaprakṛti* chapter.

iṣṭavargahṛtaḥ kṣepaḥ
kṣepaḥ syād iṣṭabhājite |
mūle te sto 'thavā kṣepaḥ
kṣuṇṇaḥ kṣuṇṇe tadā pade ||

> The *additive* divided by the square of an assumed number will an *additive*; the two *roots* divided by this assumed number, are the [*roots*]; or the *additive* multiplied is an *additive*; in this case the *roots* are multiplied.

This rule is to be used in order to reduce triples obtained by the *bhāvanā*.

It is quite obvious: If we have a triple $[x_1,\, y_1\,; k_1]$, suppose that the square of a number d divides the *additive* k_1, then we can write: $k_1 = d^2 k_1'$ and if we write the relationship between the three numbers of the triple as: $k_1 = y_1 - p\, x_1^2$ we have:

$$\begin{aligned} d^2 k_1' &= y_1^2 - p\, x_1^2 \\ k_1' &= \left(\frac{y_1}{d}\right)^2 - p\left(\frac{x_1}{d}\right)^2 \end{aligned}$$

So, if we can simplify the *additive* by the square of the number, then we have to simplify the *least* and *greatest roots* by the number itself. Similarly, if we multiply the *additive* by a square, the two *roots* are multiplied by the number:

$$d^2 k_1 = (d\, y_1)^2 - p\,(d\, x_1)^2$$

Now, let us see how to use this material to solve a simple Pell's equation. Bhāskara gives this example:

> Which square multiplied by eleven and increased by one is a square? O my friend!

And the solution runs as follows:

1. According to the first rule, we choose 1 as *least root* and complete the *prakṛti*, 11, to the nearest square with the *additive* -2:

   $$[1,\ 3;\ -2] \qquad 11 \times 1^2 - 2 = 3^2$$

2. We use the *bhāvanā* to find another triple; as we have only one triple, we put it on the two lines:

   $$\begin{matrix} 1 & 3 & -2 \\ 1 & 3 & -2 \end{matrix} \qquad 1 \times 3 + 3 \times 1 = 6\ ; \quad 11 \times 1 \times 1 + 3 \times 3 = 20\ ; \quad (-2) \times (-2) = 4$$

   And we obtain a new triple for the same *prakṛti*:

   $$[6,\ 20;\ 4] \qquad 11 \times 6^2 + 4 = 20^2$$

3. As the *additive* of this last triple is a square we can use the simplification rule and divide it by 4. So we have to divide the *least* and *greatest roots* by 2. Fortunately these two are even! Thus we have the solution of the question asked by Bhāskara:

   $$[3,\ 10;\ 1] \qquad 11 \times 3^2 + 1 = 10^2$$

4. And now, because the *additive* is 1, we can use repetitively the *bhāvanā* to find all the triples which are solutions to the equation:

   $$\begin{matrix} 3 & 10 & 1 \\ 3 & 10 & 1 \end{matrix} \qquad [60,\ 199;\ 1] : \quad 11 \times 60^2 + 1 = 199^2$$

   $$\begin{matrix} 3 & 10 & 1 \\ 60 & 199 & 1 \end{matrix} \qquad [1197,\ 3970;\ 1] : \quad 11 \times 1197^2 + 1 = 3970^2$$

Evam ānantyam, 'thus is infinity' is generally the conclusion of the commentators.

3 The *cakravāla*

or the cyclic method to solve Pell's equations

From the example in the previous section, we can have an idea of what the method to solve a Pell's equation will be. Unfortunately when reading the third step things do not go as smoothly because there is no knowing that the calculated *roots* will be divisible by the number whose square divides the *additive* found by way of the *bhāvanā*.

The *cakravāla* is a method to solve this problem.

One rule is given in two parts; the first part expounds the way to proceed.

hrasvajyeṣṭhapadakṣepān
bhājyaprakṣepabhājakān |
kṛtvā kalpyo guṇas tatra
tathā prakṛtitaś cyute ||
guṇavarge prakṛtyone
'thavālpaṃ śeṣakaṃ yathā |

> Having made the *least* and *greatest roots* and the *additive*, a *dividend*, a *k-additive*[4] and a *divisor*, a *multiplier* must be produced so that, in this procedure, the remainder will be small when the square of the *multiplier* is removed from the *prakṛti*, or when it is decreased by the *prakṛti*.

Given a triple $[x_1,\, y_1;\, k_1]$ of integers found using the rules from the *vargaprakṛti* chapter, we have to solve a *kuṭṭaka* defining the *least root*, x_1, as the *dividend*, the *greatest root*, y_1, as the *k-additive* and the *additive* as the divisor:

$$x_1 u + y_1 = k_1 v$$

The rule stipulates to choose a solution $(u,\, v) = (\alpha,\, \beta)$ such that the *multiplier*, i.e. α, minimises the difference between its square and the *prakṛti*.

The second part of the rule explains how to build a new triple $[x_2,\, y_2;\, k_2]$ from the solutions of the *kuṭṭaka*.

tat tu kṣepahṛtaṃ kṣepo
vyastaḥ prakṛtitaś cyute |
guṇalabdhiḥ padaṃ hrasvaṃ
tato jyeṣṭham ato 'sakṛt |
tyaktvā pūrvapadakṣepāc
cakravālam idaṃ jaguḥ ||

> This remainder, divided by the *additive* is an *additive*, reversed if there was subtraction from the *prakṛti*. The *quotient* associated to the *multiplier* is a *least root*, whence a *greatest root*. Putting them aside again and again from the previous *roots* and *additives*, [mathematicians] call this [procedure] the *cakravāla*.

'This remainder' is the remainder of the subtraction between the square of the *multiplier*, which is a solution of the *kuṭṭaka*, and the *prakṛti*. So we have: $k_2 = \dfrac{\alpha^2 - p}{k_1}$, or $k_2 = -\dfrac{p - \alpha^2}{k_1}$ because it is mentioned: "Reversed if there was subtraction from the *prakṛti*."

The *quotient* obtained as a solution of the *kuṭṭaka* is a new *least root*: $x_2 = \beta = \dfrac{x_1\alpha + y_1}{k_1}$. And we calculate from these two, k_2 and x_2, the *greatest root*, using the relation: $y_2^2 = p\,x_2^2 + k_2$.

[4]We introduce here a new technical notation because in both, the *vargaprakṛti* and *kuṭṭaka* chapters, the term *'additive'* is used. We will note the *additive* related to the *kuṭṭaka* as: *'k-additive'* to avoid confusion.

Cakravāla means 'a circle'; commentators say that this method is thus called because from *roots* arises a *kuṭṭaka*, the solutions of which give new *roots* from which we solve a new *kuṭṭaka* and so on until we find the solution of a Pell's equation.

We now give a last rule by Bhāskara before going into the explanations given by Kṛṣṇadaivajña to justify the *cakravāla* as a method leading to integral solutions of a Pell's equation.

> *caturdvyekayutāv evam abhinne bhavataḥ pade* |
> *caturdvikṣepamūlābhyāṃ rūpakṣepārthabhāvanā* ||

> Thus, there are two non-fractional *roots* when the *additive* is four, two or one. From two *roots* associated with the *additives* four and two, a composition whose goal is *additive* one [must to be carried out].

Kṛṣṇadaivajña provides this commentary to the first verse:

> "'Thus', that is to say: with the *cakravāla*. If the *additive* is four and if the *additive* is two and if the *additive* is one, there are two non-fractional *roots*; this is a way to imply that there are two non-fractional *roots* whatever the *additive*. *'Additive'* also is a synecdoche, with this word, *subtractive* is also to be understood.[5]"

Kṛṣṇa's interpretation of this verse is a clear cut affirmation that the *cakravāla* will yield integral solutions to a Pell's equation whatever the *additive*.

He also explains the second verse:

> "Now, in order to calculate roots associated to an *additive* one, [the author] says that there is also another method: A composition must be performed and there are two *roots* from *additive* one, if the *additive* is four, [directly] using [the rule]: "The additive divided by the square of an assumed number, etc.", if the *additive* is two, after calculating two roots associated to *additive* four by an equal composition, and applying the same rule afterwards.[6]"

This rule claims that by an iterative use of the *bhāvanā*, solutions of the equation with *additive* 1, can be obtained from equations with *additives* ± 2 or ± 4. This result has been known since Brahmagupta's times.

If the additive is ± 2 and x_1 and y_1 are the *least* and *greatest roots* of the equation, then the *bhāvanā* gives $x_2 = 2x_1y_1$, which is even, as the *least root* and $y_2 = p\,x_1^2 + y_1^2 = 2\,y_1^2 \pm 2$ (for $p\,x_1^2 = y_1^2 \pm 2$), which is also even, as the *greatest root* of the equation $p\,x^2 + 4 = y^2$. Then the simplification rule applies and $x_2/2$ and $y_2/2$ are integral solutions of $p\,x^2 + 1 = y^2$.

In his *Brahmasphuṭasiddhānta*, Brahmagupta adds these rules if the additive is ± 4:

> "If the *additive* is four, the square of the *last root*[7] less three, halved and multiplied by the *last root* is a *last root*. The square of the *last root* less one, divided by two and multiplied by the *first root* is a *first root*.
>
> If the additive is minus four, let two squares of the *last root* be added to three and one; let half of their product separately put: Then, minus one and multiplied by the first [term of the product] decreased by one, it is a *last root*, and multiplied by the product of the *roots* it is a *first [root]* associated to this last *root*.[8]"

[5]See text 2 page 19.

[6]See text 3 page 19.

[7]Brahmagupta calls *last root (antyapada)* what Bhāskara calls *greatest root (jyeṣṭhapada)* and he calls *first root (ādyapada)* what Bhāskara calls *least root (kaniṣṭhapada)*.

[8]See text 5 page 20.

According to the first rule, if x_1 and y_1 are solutions of equation $p\,x^2 + 4 = y^2$, then:

$$v_1 = \frac{y_1^2 - 3}{2} y_1 \quad \text{and} \quad u_1 = \frac{y_1^2 - 1}{2} x_1$$

are solutions of equation $p\,u^2 + 1 = v^2$.

This is true but one can remark that if x_1 is odd and y_1 is even, then u_1 is not an integer, as shown by example $60\,x^2 + 4 = y^2$. Taking $x_1 = 1$ and $y_1 = 8$, $u_1 = \frac{63}{2}$ and one may wonder whether the goal of solving this form of equation has always been to find integral solutions. Bhāskara also gives examples with rational solutions and, while solving the famous equation $67\,x^2 + 1 = y^2$, which is given as an example in the *cakravāla* chapter, at one step, rational solutions are found and composed using the *bhāvanā* to find integral solutions at the end of the procedure.

The second rule yields integral solutions, whatever the parity of x_1 and y_1 is. If x_1 and y_1 are solutions of equation $p\,x^2 - 4 = y^2$, then $v_1 = \big(\frac{(y_1^2+3)(y_1^2+1)}{2} - 1\big)(y_1^2 + 2)$ and $u_1 = \big(\frac{(y_1^2+3)(y_1^2+1)}{2}\big)x_1 y_1$ are solutions of equation $p\,u^2 + 1 = v^2$.

These formulæ can be found using the *bhāvanā* recursively, starting from:

$$p\Big(\frac{x_1}{2}\Big)^2 \pm 1 = \Big(\frac{y_1}{2}\Big)^2$$

For the first formula we should apply the *bhāvanā* twice, first composing $y_1/2$ and $x_1/2$ with themselves, then with the result of this composition.

The second formula is more difficult to establish because the *additive* is alternately -1 and 1 and five successive compositions are needed.

From a modern point of view, the *bhāvanā* expresses that the norm in the quadratic field $\mathbb{Q}[\sqrt{p}]$ is multiplicative and Brahmagupta's formulæ are obtained expanding $\big(\frac{y_1}{2} + \frac{x_1}{2}\sqrt{p}\big)^3$, in the first case and $\big(\frac{y_1}{2} + \frac{x_1}{2}\sqrt{p}\big)^6$, in the second case, the norm of $\frac{y_1}{2} + \frac{x_1}{2}\sqrt{p}$ being respectively 1 and -1.

4 Kṛṣṇa's *upapatti*

Kṛṣṇadaivajña

He is from an important family of astronomers who emigrated from Vidharba, in the eastern part of Maharashtra, to Varanasi during the sixteenth century. He was a protégé of the Mughal emperor Jahāngir and was an astrologer at the Mughal court. His commentary on Bhāskara's *Bījagaṇita*: the *Bījapallava* is dated "Saturday, the fourth *tithi* of the dark fortnight of the *Caitra* month, 1523 *Śaka* year": Saturday April 21, 1601.[9]

According to Professor Sreeramula Rajeswara Sarma,[10] in the following illustration, Kṛṣṇa might be the astrologer in the centre of this miniature, seated between two Muslim astrologers, and casting a horoscope for the birth of Salim, the future emperor Jahāngir. The painting is in the Museum of Fine Arts, Boston (courtesy of Pr. Sreeramula Rajeswara Sarma).

[9]Date conversion was done using Michio Yano's pancanga program.

[10]"Astronomical Instruments in Mughal Miniatures" in *Studien in Indologie und Iranistik* 16-17 (1992) 235-276. Reprinted in *The Archaic and the Exotic: Studies in the History of Indian Astronomical Instruments*, Manohar, New Delhi 2008.

Figure 1: Kṛṣṇadaivajña[11]

The word *upapatti*

This word is used by commentators when they want to give an explanation or a justification of a rule given in the work they are commenting upon. In mathematical texts, rather than a full demonstration such as we may know nowadays, this word indicates that the operation or the procedure formulated by the author is coherent and achieves the result which it has been created for.

Here Kṛṣṇadaivajña justifies the *cakaravāla* rule by showing why the use of the *kuṭṭaka* is necessary to find integral solutions of a Pell's equation.

Notations for operations

To support his reasoning he makes some calculations and uses formal notations. Here is a page of a manuscript with an example of these notations.

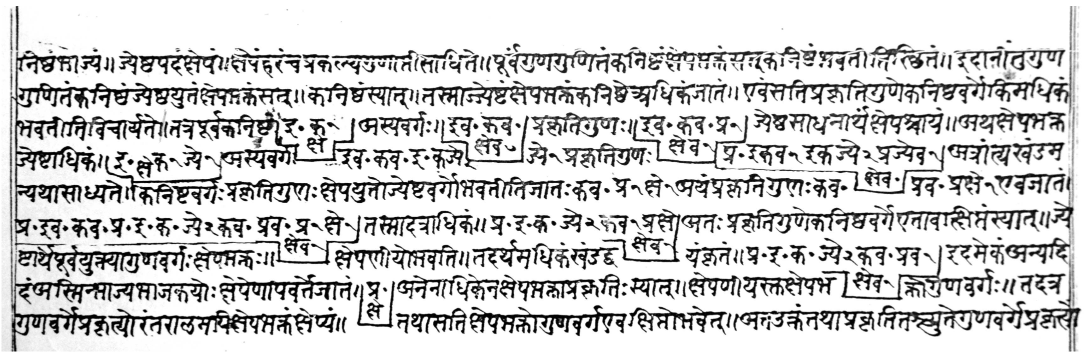

Figure 2: A manuscript page of the *Bījapallava*

We can read on lines 3 to 5 (we have supplied between square brackets some words or signs which are missing but which can be found in other manuscripts):

[11]"Astrologers casting the Horoscope," detail from the "Birth of Salim," Museum of Fine Arts, Boston, 17.3112. Cf. Stuart Cary Welch, Imperial Mughal Painting, London 1978, Pl. 16, pp. 70-71.

tatra pūrvakaniṣṭaṃ | *i • ka 1* / *kṣe* | *asya vargaḥ* | *iva • kava* / *kṣeva 1* | *prakṛtiguṇaḥ* | *iva • kava • pra 1* / *kṣeva 1* | *jyeṣṭhasādhārthaṃ kṣepaś cāyaṃ* [| *iva 1* / *kṣe 1* |]

atha kṣepabhaktajyeṣṭhādhikaṃ [kaniṣṭhaṃ] | *i • ka 1 jye 1* / *kṣe [1]* | *asya vargaḥ* | *iva • kava [1] i • ka [•] jye [2] jye[va] 1* / *[kṣeva 1]* | *prakṛtiguṇaḥ* | *pra • i[va •] kava 1 [pra •] i [•] ka [•] jye 2 pra [•] jyeva 1* / *kṣeva [1]* |

The explanation of this system is as follows: It uses the first syllable of a word as algebraic symbols: *i* stands for *iṣṭa* (assumed number), *ka* for *kaniṣṭha (least root)*, *jye* is *jyeṣṭha (greatest root)*, *pra* for *prakṛti* and *kṣe* for *kṣepa (additive).* The Sanskrit word for square is *varga*, so to denote the square of one quantity, the syllable *va* is postponed to the syllable representing the quantity: *kava* means the square of the *kaniṣṭha*, x^2.

A bullet is a multiplication sign, but not in all manuscripts. Fractions are noted by putting the numerator above the denominator, without any fraction line. There is no sign for addition, only a number is placed after a symbol as a counting indication; for instance if we want to note the square of the sum of the *least (ka)* and *greatest roots (jye)*, it will be written like this: *kava 1 ka•jye 2 jyeva 1* $(x^2 + 2xy + y^2)$.

Finally the calculations are separated from the text by a frame.

Here is the translation of this passage:

> Now, the preceding *least root* is $\dfrac{\alpha x}{k}$; its square $\dfrac{\alpha^2 x^2}{k^2}$ multiplied by the *prakṛti* is: $\dfrac{\alpha^2 x^2 p}{k^2}$ and, in order to calculate the *greatest root*, the *additive* is this one: $\left[\dfrac{\alpha^2}{k}\right]$.
>
> Then, [the *least root*], added to the *greatest* one divided by the *additive* is: $\dfrac{\alpha x + y}{k}$; its square, $\dfrac{\alpha^2 x^2 + 2\alpha x\, y + y^2}{k^2}$, multiplied by the *prakṛti* is: $\dfrac{p\alpha^2 x^2 + 2p\alpha x\, y + py^2}{k^2}$.

A preliminary study

As a starting point to his justification, Kṛṣṇa mentions the rule for the additive simplification by a square: "If the *additive* is divided by the square of an assumed number…" (see page 7) and he uses it twice: First in the 'multiplicative form', multiplying the *least root* by an arbitrary number, he says that the *additive* must be multiplied by the square of that number. If we denote α the arbitrary number and if we have a triple $[x_1,\ y_1;\ k_1]$, we have a new *least root* αx_1 and a new *additive* $\alpha^2 k_1$.

Then he uses the same rule, choosing as assumed number the *additive* of the triple and dividing the just obtained *least root* and *additive.* Doing this, he concludes that we have again a new *least root* and *additive*:

$$x_2 = \frac{\alpha x_1}{k_1} \qquad k_2 = \frac{\alpha^2 k_1}{k_1^2} = \frac{\alpha^2}{k_1}$$

And he remarks:

> "In these conditions, one must imagine a number chosen in this way: Once the *least root* is multiplied by this number, there will be a simplification if it is divided by the *additive*, if not how could the *least root* be non-fractional?
>
> For this purpose —that is to say: What is the number by which the *least root* being multiplied then divided by the *additive* will be without remainder?— a *multiplier*

> and a *quotient* must be calculated after making the *least root* a *dividend*, the *additive* a *divisor* without any *k-additive*.[12]"

Indeed, if we want x_2 to be an integer we must choose α such that k_1 divides αx_1 so, we must solve the following *kuṭṭaka*:

$$x_1\, u = k_1\, v$$

And Kṛṣṇa concludes:

> "In that case, the *quotient* will be the *least root*. The square of the *multiplier* —the very [number] sought, which is the *multiplier* in this *[kuṭṭaka]*— divided by the previous *additive* will be the *additive*. Then, the *greatest root* multiplied by the *multiplier* and divided by the *additive* will be the *greatest root*.[13]"

If the couple $(\alpha,\ \beta)$ is a solution of the *kuṭṭaka* $x_1\, u = k_1\, v$, the *quotient* is $\beta = \dfrac{\alpha x_1}{k_1}$ and we recognise the value calculated above by Kṛṣṇa as the new *least root*, x_2; the associated values for the *additive*, k_2, and the *greatest root*, y_2, follow. From the general solution of this *kuṭṭaka* without *k-additive*, namely: $u = k_1\, t, \quad v = x_1\, t$ with t an arbitrary integer, we can see that the new triple $[x_2,\ y_2;\ k_2]$ is composed of integral values:

$$x_2 = \beta = \frac{\alpha\, x_1}{k_1} = \frac{k_1 t\, x_1}{k_1} = t\, x_1\,; \quad k_2 = \frac{\alpha^2}{k_1} = \frac{k_1^2 t^2}{k_1} = k_1 t^2\,; \quad y_2 = \frac{\alpha\, y_1}{k_1} = \frac{k_1 t\, y_1}{k_1} = t\, y_1$$

Of course we recognise the 'simplification rule' (page 7) under its multiplicative form, but what is interesting in Kṛṣṇa's explanation is the reason for introducing the *kuṭṭaka* and the distribution of the roles: The *least root* is a *dividend* and the *additive* a *divisor*. He will never explain why the *greatest root* is chosen as the *k-additive* but will show by a calculation that this choice allows the new *additive* to be minimised.

Explanation of Bhāskara's rule for the *cakravāla*

> "The master endeavoured to calculate differently because in the [preceding calculation] the *additive* is too large. A *multiplier* and a *quotient* are calculated putting the *least root* (x_1) as the *dividend*, the *greatest root* (y_1) as the *k-additive* and the *additive* (k_1) as the *divisor*.[14]"

The 'preceding calculation' is the one done by Kṛṣṇa in his preliminary study; according to it, the new *additive* obtained after applying the *kuṭṭaka* is: $k_2 = \dfrac{\alpha^2}{k_1}$, α being the *multiplier* of the resolved *kuṭṭaka*.

Of course, even if the preliminary study clearly shows how to produce integral solutions for a Pell's equation, as the goal of the *cakravāla* is to produce an *additive* equal to ± 1 or ± 2 or ± 4, as stated in the Bhāskara's last rule: "If the *additive* is two or four ..." (see page 10), and from there to use the *bhāvanā* as a shortcut to find the integral solutions of $p\, x^2 + 1 = y^2$, in Kṛṣṇa's study the size of the *additive* cannot be mastered.

Kṛṣṇa will now justify that the new *additive* proposed by Bhāskara, $k_2 = \pm \dfrac{\alpha^2 - p}{k_1}$, which can be minimised by the choice of α, is the right *additive*, if the new *least root* is set to be the *quotient* obtained in the *kuṭṭaka* with *k-additive* fixed as the previous *greatest root*, y_1.

[12] See text 6, page 20.

[13] Text 7, page 20.

[14] See the rule page 9.

Comparing the *least root* in Kṛṣṇa's study with Bhāskara's *least root*

Kṛṣṇa introduces his calculations like this:

> "A [new] *least root* has been previously put as the *least root* multiplied by the *multiplier*[15] and divided by the *additive* but now the *least root* multiplied by the *multiplier* and added to the *greatest root* then divided by the *additive* will be a [new] *least root.* Therefore the *greatest root* divided by the *additive* is produced as an additional [number] to the *least root.* Thus, let us see what number is added to the square of the *least root* multiplied by the *prakṛti.*[16]"

Applying Bhāskara's rule, let $(\alpha,\ \beta)$ be a solution of the *kuṭṭaka*: $x_1\,u + y_1 = k_1\,v$, a new *least root* is $\beta = \dfrac{\alpha\, x_1 + y_1}{k_1}$ and Kṛṣṇa makes this calculation:

$$
\begin{aligned}
p\Big(\frac{\alpha\, x_1 + y_1}{k_1}\Big)^2 &= \frac{p\,\alpha^2\, x_1^2 + 2\,p\,\alpha x_1 y_1 + p\, y_1^2}{k_1^2} \\
&= \frac{p\,\alpha^2\, x_1^2 + 2\,p\,\alpha x_1 y_1 + p\,(p\, x_1^2 + k_1)}{k_1^2} \qquad \text{using} \quad y_1^2 = p\, x_1^2 + k_1 \\
&= \frac{p\,\alpha^2\, x_1^2 + 2\,p\,\alpha x_1 y_1 + p^2 x_1^2 + p\, k_1}{k_1^2} \\
&= p\Big(\frac{\alpha\, x_1}{k_1}\Big)^2 + \boxed{\frac{2\,p\,\alpha x_1 y_1 + p^2 x_1^2 + p\, k_1}{k_1^2}}
\end{aligned}
$$

So, the boxed number is the sought one.

Then Kṛṣṇa remarks that in his previous reasoning the square of the *multiplier* (α) divided by the *additive* has to be added in order to find the *greatest root* and, for this purpose, he splits the number he has just calculated in two components:

$$
\frac{2\,p\,\alpha x_1 y_1 + p^2 x_1^2 + p\, k_1}{k_1^2} = \frac{2\,p\,\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{p}{k_1} \qquad (*)
$$

And now the argument is:

> "With this additional number, the *prakṛti* divided by the *additive* is added, but the square of the *multiplier* divided by the *additive* must be added, thus in this [rule] the difference between the square of the *multiplier* and the *prakṛti* divided by the *additive* must be also added, because doing this, only the square of the *multiplier* divided by the *additive* will be added.[17]"

With this argument the new *additive*, $\dfrac{\alpha^2 - p}{k_1}$, given by Bhāskara's rule is now justified:

$$
\frac{p}{k_1} + \frac{\alpha^2 - p}{k_1} = \frac{\alpha^2}{k_1}
$$

since this last result will enable us to find a *greatest root* if we add it to $p\big(\dfrac{\alpha\, x_1}{k_1}\big)^2$ as shown in the preliminary study.

Kṛṣṇa also explains why the new *additive* has to be 'reversed' if the *prakṛti* is greater than the square of the *multiplier*, the result being the same in both cases. Let us summarise these reasons using modern notations:

[15] Calculated by the *kuṭṭaka*.

[16] Text 4, page 20.

[17] Text 8, page 20.

$$\text{if } \alpha^2 \geq p \quad \text{then} \quad \frac{p}{k_1} + \frac{\alpha^2 - p}{k_1} = \frac{\alpha^2}{k_1}$$
$$\text{if } \alpha^2 \leq p \quad \text{then} \quad \frac{p}{k_1} + \Big(-\frac{p - \alpha^2}{k_1}\Big) = \frac{\alpha^2}{k_1}$$

Putting things together

After this separate study about the *additive* according to Bhāskara's rule, Kṛṣṇa takes into account the first member of the number $(*)$ he had split in two parts and says: "No doubt then that this very number, $\dfrac{2\,p\,\alpha x_1 y_1 + p^2 x_1^2}{k_1^2}$, is added to the square of a *greatest root*, namely the square of $\dfrac{\alpha\, y_1}{k_1}$."

To understand what is meant here, let us summarise the full calculation made by Kṛṣṇa

1. Firstly he develops $p\Big(\dfrac{\alpha\, x_1 + y_1}{k_1}\Big)^2$ and, using the identity: $y_1^2 = p\, x_1^2 + k_1$, he obtains this equality:
$$p\Big(\frac{\alpha\, x_1 + y_1}{k_1}\Big)^2 = p\Big(\frac{\alpha\, x_1}{k_1}\Big)^2 + \frac{2\,p\,\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{p}{k_1}$$
2. He then adds, or subtracts, the *additive* given by the rule: $\Big|\dfrac{\alpha^2 - p}{k_1}\Big|$:
$$p\Big(\frac{\alpha\, x_1 + y_1}{k_1}\Big)^2 \pm \Big|\frac{\alpha^2 - p}{k_1}\Big| = p\Big(\frac{\alpha\, x_1}{k_1}\Big)^2 + \frac{2\,p\,\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{\alpha^2}{k_1}$$
3. He adds the first and the last terms of the second member of the equality, after remarking that this sum is a *greatest root*:
$$p\Big(\frac{\alpha\, x_1 + y_1}{k_1}\Big)^2 \pm \Big|\frac{\alpha^2 - p}{k_1}\Big| = \Big(\frac{\alpha\, y_1}{k_1}\Big)^2 + \frac{2\,p\,\alpha x_1 y_1 + p^2 x_1^2}{k_1^2}$$

Now the justification of the *cakravāla* rule is complete because the last result is a square:

$$\Big(\frac{\alpha\, y_1}{k_1}\Big)^2 + \frac{2\,p\,\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} = \Big(\frac{\alpha\, y_1 + p\, x_1}{k_1}\Big)^2$$

And thus we have a new triple that verifies the relation $p\, x_2^2 + k_2 = y_2^2$:

$$x_2 = \frac{\alpha\, x_1 + y_1}{k_1} \qquad k_2 = \Big|\frac{\alpha^2 - p}{k_1}\Big| \qquad y_2 = \frac{\alpha\, y_1 + p\, x_1}{k_1}$$

Kṛṣṇa concludes his *upapatti* like this:

> "When a *least root* multiplied by an assumed number and added to a *greatest root* [the result] being divided by an *additive* is put as a *least root*, then the difference between the square of the assumed number and the *prakṛti* divided by the *additive* is an *additive*. The *greatest root* multiplied by the assumed number and added to the *least root* multiplied by the *prakṛti*, [the result] being divided by the *additive* is then the [new] *greatest root*.
>
> In this procedure, even if there is no requirement of the *kuṭṭaka* —*roots* being obtained only by force of an assumed number— a *kuṭṭaka* is nevertheless performed for a state of non-fractionation [of the *roots*]; hence the statement: "Having made the *least* and *greatest roots* and the *additive*...[18] is justified"

[18]Text 9, page 20.

The first paragraph is a summary of Bhāskara's rule with a slight difference: while the rule says: "The *quotient* associated to the *multiplier* is a *least root*, whence a *greatest root*", that is to say that once we have a *least root* and an *additive*, we can calculate the associated *greatest root* by the general relation: $p\,x^2 + k = y^2$, we can nevertheless calculate the *greatest root* using the result of Kṛṣṇa's calculations: $y_2 = \dfrac{\alpha\, y_1 + p\, x_1}{k_1}$ where α is a solution of the *kuṭṭaka* laid down for the *cakravāla*.

In the second paragraph, we have an interesting observation: Whatever the number d is, if $[x_1, y_1; k_1]$ is a triple, solution of a Pell's equation, $[dx_1, dy_1; d^2k_1]$ is another such triple. Kṛṣṇa uses this to prove that the result in the *cakravāla* rule is a square when he remarks that the *additive* $\dfrac{\alpha^2 - p}{k_1}$ eliminates $\dfrac{p}{k_1}$ and that, in fact, we get a *greatest root* while combining $p\Big(\dfrac{\alpha\, x_1}{k_1}\Big)^2$ and what remains: $\dfrac{\alpha^2}{k_1}$.

Final remarks

What Kṛṣṇa really demonstrates here is that if we follow Bhāskara's rule, putting as a *least root* $x_2 = \dfrac{\alpha\, x_1 + y_1}{k_1}$, the *quotient* of a well-chosen *kuṭṭaka*, and as *additive* $\left|\dfrac{\alpha^2 - p}{k_1}\right|$, the result is a square.

Another very interesting point is his attempt, in the preliminary study, to justify the use of the *kuṭṭaka* if we want integral solutions. The use of the full *kuṭṭaka*, with the *greatest root* as a *k-additive*, is not explained though, but it is certainly not obvious!

The expression found for the *greatest root* allows to make an iterative description of the *cakravāla* process: Let $[x_1,\ y_1;\ k_1]$ be a triple of integers such as $p\,x_1^2 + k_1 = y_1^2$ and let u_1 and v_1 be integral solutions of $x_1\, u + y_1 = k_1\, v$, such that $|u_1^2 - p|$ is minimal, then:

$$x_2 = \frac{x_1\, u_1 + y_1}{k_1} \qquad k_2 = \left|\frac{u_1^2 - p}{k_1}\right| \qquad y_2 = \frac{u_1\, y_1 + p\, x_1}{k_1}$$

is a new triple of integers verifying the same relation.

From there, other triples can be calculated by induction; but, as the difference between the square of the solution of the *kuṭṭaka* and the *prakṛti* is minimised —that is to say: the *additive*— at each step, the process will come to an end with an *additive* equal to 1. This also is not obvious.

A question which is not approached by Kṛṣṇa —nor by Bhāskara— is: Why is the *additive*, $\dfrac{\alpha^2 - p}{k_1}$ an integer? We can answer this question, supposing that the *least root*, x_1 and the *additive*, k_1 , are relatively prime —if they are not, the equation could be transformed into an equation with *additive* equal to 1, and the *cakravāla* is useless in that case.

We multiply $\alpha^2 - p$ by x_1^2 and obtain the following identities:

$$\begin{aligned}(\alpha^2 - p)\, x_1^2 &= \alpha^2\, x_1^2 - p\, x_1^2\\ &= \alpha^2\, x_1^2 - y_1^2 + k_1 \quad (\text{because } p\, x_1^2 = y_1^2 - k_1)\\ &= (\alpha\, x_1 + y_1)(\alpha\, x_1 - y_1) + k_1\end{aligned}$$

Then k_1 divides the right member of the last identity, because α had been chosen for this purpose, so k_1 must divide the left member and as it does not divide x_1, it must divide $\alpha^2 - p$.

The last remark we can make is: How has such a sophisticated method been developed? The answer might be found in some unknown works or commentaries in the multitude of manuscripts stored in libraries in India.

5 Examples

Bhāskara puts forward examples in order to illustrate the theoretical part of the *cakravāla*.

He asks to solve these two equations:

$$67\,x^2 + 1 = y^2 \quad \text{and} \quad 61\,x^2 + 1 = y^2$$

We give briefly the solutions according to Kṛṣṇa's commentary but using our modern notations (starred items indicate the beginning of a cycle).

$67\,x^2 + 1 = y^2$

*1. Choose a suitable triple: $[x_1,\, y_1;\, k_1] = [1,\, 8;\, -3]$ $\quad 67 \times 1^2 - 3 = 8^2$

2. Solve the *kuṭṭaka*: $u + 8 = -3\,v:$ $\quad u_0 = 1 \quad v_0 = -3$

3. Calculate $p - u_0^2 = 67 - 1 = 66$

4. Calculate $p - u_0^2 = 67 - 1 = 66$ The result is not small.

5. Other solutions of the *kuṭṭaka* are: $u = 1 - 3t \quad v = -3 + t$. Choose $t = -2:$

$$u_1 = u_0 - 2 \times -3 = 7 \qquad v_1 = -3 - 2 = -5$$

6. Calculate $p - u_1^2 = 67 - 49 = 18$ which minimises the difference.

   The additive is: $k_2 = -\dfrac{p - u_1^2}{k_1} = -\dfrac{18}{-3} = 6$

   The least root is: $x_2 = v_1 = -5$

   The greatest root is: $y_2 = 41$

*7. A new triple is: $[x_2,\, y_2;\, k_2] = [5,\, 41;\, 6]$ The new *vargaprakṛti* is: $67\,x^2 + 6 = y^2$

8. Solve the *kuṭṭaka*: $5\,u + 41 = 6\,v:$ $\quad u_0 = 5 \quad v_0 = 11$

9. Calculate $p - u_0^2 = 67 - 25 = 42$

   The additive is: $k_3 = -\dfrac{p - u_0^2}{k_2} = -\dfrac{42}{6} = -7$

   The least root is: $x_3 = v_0 = 11$

   The greatest root is: $y_3 = 90$

*10. A new triple is: $[x_3,\, y_3;\, k_3] = [11,\, 90;\, -7]$ The new *vargaprakṛti* is: $67\,x^2 - 7 = y^2$

11. Solve the *kuṭṭaka*: $11\,u + 90 = -7\,v:$ $\quad u_0 = 9 \quad v_0 = -27$

12. Calculate $u_0^2 - p = 81 - 67 = 14$

    The additive is: $k_4 = \dfrac{u_0^2 - p}{k_3} = \dfrac{14}{-7} = -2$

    The least root is: $x_4 = -27$

    The greatest root is: $y_4 = 221 \qquad 67 \times 27^2 - 2 = 221^2$

*13. A new triple is: $[x_4,\, y_4;\, k_4] = [27,\, 221;\, -2]$ The new *vargaprakṛti* is: $67\,x^2 - 2 = y^2$

14. The *additive* is now -2 and we can use the *bhāvanā* as a shortcut:

    27 221 −2
    27 221 −2

    The least root is: $x_5 = 2 \times 27 \times 221 = 11934$

    The greatest root is: $y_5 = 67 \times 27^2 + 221^2 = 97684$

    The additive is: $k_5 = (-2)^2 = 4$

15. The *additive* being a square, we can now use the simplification rule, dividing k_5 by 4 we have to divide the *roots* by 2 and find the solution:

$$67 \times 5967^2 + 1 = 48842^2$$

The second example is more impressive and uses fractional intermediary *roots*.
$61\,x^2 + 1 = y^2$.

*1. Choose a suitable triple: $[x_1,\ y_1;\ k_1] = [1,\ 8;\ 3]$ $\qquad 61 \times 1^2 + 3 = 8^2$

2. Solve the *kuṭṭaka*: $u + 8 = 3\,v$: $u_0 = 1 \quad v_0 = 3$ and, as in the previous example, choose other solutions $u_1 = 1 + 2 \times 3 = 7 \qquad v_1 = 3 + 2 = 5$, which minimises $p - u_1^2$

3. Calculate $p - u_1^2 = 61 - 49 = 12$

   The additive is: $k_2 = -\dfrac{p - u_1^2}{k_1} = -\dfrac{12}{3} = 4$

   The least root is: $x_2 = v_1 = 5$

   The greatest root is: $y_2 = 39$

*4. A new triple is: $[x_2,\ y_2;\ k_2] = [5,\ 39;\ -4]$ The new *vargaprakṛti* is: $61\,x^2 - 4 = y^2$

5. The *additive* being a square, we can now use the simplification rule, dividing k_2 by 4 we have to divide the *roots* by 2 and find fractional solutions with triple: $[x_3,\ y_3;\ k_3] = [\frac{5}{2},\ \frac{39}{2};\ -1]$

6. Using an equal *bhāvanā*, we get a new triple: $[x_3,\ y_3;\ k_3] = [\frac{195}{2},\ \frac{1523}{2};\ 1]$

7. Compose this triple with the preceding one and obtain the triple:

   $[x_4,\ y_4;\ k_4] = [3805,\ 29718\,;-1]$

8. An equal *bhāvanā* yields the solution:

$$61 \times 226153980^2 + 1 = 1766319049^2$$

Sanskrit texts

Text 1.
vargaḥ prakṛtir yatreti vargaprakṛtiḥ | yato 'sya gaṇitasya yāvadādivargaḥ prakṛtiḥ | yadvā yāvadādivargeṣu prakṛtibhūtād aṅkād idaṃ gaṇitaṃ pravartata iti vargaprakṛtiḥ | atra yāvadvargādiṣu prakṛtibhūto yo 'ṅkaḥ sa prakṛtiśabdenocyate | sa cāvyaktavargaguṇaka eva | ato 'tra padasādhane vargasya yo guṇaḥ sa prakṛtiśabdena vyavahṛyate |

Text 2.
evaṃ cakravālena caturdvyekayutau catuḥkṣepe dvikṣepa ekakṣepe cābhinne pade bhavataḥ | idam upalakṣaṇam | yatra kutrāpi kṣepe 'bhinne pade bhavataḥ | yutāv ity apy upalakṣanam tena śuddhāv apīti jñeyam |

Text 3.
atha rūpakṣepapadānayane prakārāntaram apy astīty āha ``caturdvikṣepamūlābhyām'' iti | catuḥkṣepamūlābhyāṃ dvikṣepamūlābhyāṃ ca rūpakṣepārthaṃ bhāvanā rūpakṣepārthabhāvanā kāryeti śeṣaḥ | catuḥkṣepe ``iṣṭavargahṛtaḥ kṣepa'' ityādinā dvikṣepe tu tulyabhāvanayā

catuḥkṣepapade prasādhya paścād ``iṣṭavargahṛtaḥ kṣepa'' ityādinā rūpakṣepaje pade vā bhavata itiyarthaḥ ।

Text 4.
pūrvaṃ tu guṇaguṇitaṃ kaniṣṭhaṃ kṣepabhaktaṃ sat kaniṣṭhaṃ bhavatīti sthitam । idānīṃ tu guṇaguṇitaṃ kaniṣṭhaṃ jyeṣṭhayutaṃ kṣepabhaktaṃ sat kaniṣṭhaṃ syāt । tasmād jyeṣṭhaṃ kṣepabhaktaṃ kaniṣṭhe 'dhikaṃ jātam । evaṃ sati prakṛtiguṇe kaniṣṭhavarge kim adhikaṃ bhavatīti vicāryate ।

Text 5.
caturadhike antyapadakṛtis tryūnā dalitā antyapadaguṇā antyapadam ।
antyapadakṛtis vyekā dvihṛtā ādyapadāhatā ādyapadam ॥
caturūne antyapadakṛtī tryekayute vadhadalam pṛthak vyekam ।
vyekāḍyāhatam antyam padavadhaguṇam ādyam āntyapadam ॥

Text 6.
atreṣṭaṃ tādṛśaṃ kalpanīyaṃ yena guṇitaṃ kaniṣṭhaṃ kṣepabhaktaṃ śuddhyet । anyathā kaniṣṭham abhinnaṃ kathaṃ syāt ।
tadarthaṃ kaniṣṭhaṃ kena guṇitaṃ kṣepabhaktaṃ niḥśeṣaṃ syād iti kaniṣṭhaṃ bhājyaṃ prakalpya kṣepaṃ haraṃ ca prakalpya kṣepābhāve guṇāptī sādhye ।

Text 7.
atra yā labdhis tat kaniṣṭhaṃ padam । yo 'tra guṇas tad eveṣṭam iti guṇakavargaḥ pūrvakṣepabhaktaḥ kṣepaḥ syāt । jyeṣṭham api guṇaguṇitaṃ kṣepabhaktaṃ jyeṣṭhaṃ syāt ।

Text 8.
anenādhikena kṣepabhaktā prakṛtiḥ kṣiptā syāt kṣepaṇīyas tu kṣepabhakto guṇavargaḥ । tad atra guṇavargaprakṛtyor antarālam api kṣepabhaktaṃ kṣepyam । tathā sati kṣepabhakto guṇavarga eva kṣipto bhavet ।

Text 9.
yadā tv iṣṭaguṇaṃ kaniṣṭaṃ jyeṣṭhayutaṃ kṣepabhaktaṃ sat kaniṣṭhaṃ kalpyate tadā guṇavargaprakṛtyor antaraṃ kṣepabhaktaṃ sat kṣepo bhavatīṣṭaguṇaṃ jyeṣṭhaṃ prakṛtiguṇakaniṣṭena yutaṃ kṣepabhaktaṃ sat tatra jyeṣṭhaṃ bhavatīti ।
atra yady apy iṣṭavaśād eva padasiddhir astīti kuṭṭakasya nāpekṣā tathāpy abhinnatvārthaṃ kuṭṭakaḥ kṛtaḥ । ata upapannaṃ hrasvajyeṣṭhapadakṣepān ityādi ।

References

[1] André Weil, *Number Theory, An approach through history from Hammurapi to Legendre*, Reprint of the 1984 Edition, Birkhäuser, Boston, 2007

[2] Clas-Olof Selenius, *Rationale of the chakravāla process of Jayadeva and Bhāskara II*, in Historia Mathematica 2, 1975 pp 167-184

[3] Kṛṣṇadaivajña, *Bīja Pallavam a commentary on Bīja Ganita, the Algebra in Sanskrit*, Edited with Introduction by T.V. Radhakrishna Sastri, Tanjore Saraswati Mahal Series N° 78, Tanjore, 1958

[4] A. A. Krishnaswami Ayyangar, *New Light on Bhaskara's Chakravala or Cyclic Method of solving Indeterminate Equation of the Second Degree in Two Variables*, Journal of the Indian Mathematical Society, 1929-30, Vol.18

[5] Edward Everett Whitford, *The Pell Equation*, College of the city of New York, 1912

[6] Edward Strachey, *Bija Ganita or the Algebra of the Hindus*, London, 1813 *(partly translated from a Persian translation by Ata Alla Rasheedee in 1634)*

Manuscripts

[1] *Bījapallava*, Bhandarkar Oriental Institute, Puṇe, No 287 of Viśrāma (1), foll. 133

[2] *Bījapallava*, Bhandarkar Oriental Institute, Puṇe, No 538 of 1895-1902, foll. 116

[3] *Bījapallava*, Maharaja Maan Singh Pustak Prakash Research Center, Jodhpur, foll. 243