Highway traffic model-based density estimation

Irinel-Constantin Morarescu, Carlos Canudas de Wit

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Abstract— **The travel time spent in traffic networks is one of the main concerns of the societies in developed countries. A major requirement for providing traffic control and services is the continuous prediction, for several minutes into the future. This paper focuses on an important ingredient necessary for the traffic forecasting which is the real-time traffic state estimation using only a limited amount of data. Simulation results illustrate the performances of the proposed state-estimation technique.**

Index Terms— **Highway traffic analysis, observer, switching mode model**

I. Introduction

The increased travel time in congested sections has a dramatic social and economic impact. This led to an increasing research on freeway traffic control and development of intelligent transportation systems which are able to provide continuous forecasting of the traffic. A nice survey on the existing techniques for short-term traffic flow prediction can be found in [3]. One of the prerequisites for continuous prediction is an efficient real-time traffic conditions estimation using only a limited amount of data [9], [10]. The real-time density estimation is a challenging problem since the traffic is described by a system which is observable only when a segment situated between two vehicle detector stations (sensors) is entirely congested or entirely free.

Simulation modeling is a popular tool for analyzing transportation problems. Several studies focus on the validation of different microscopic or macroscopic models. Once the models validated they are used for open-loop state-estimation, forecasting and control ([5], [8]). Other papers present imputation techniques to determine the missing on-ramp and off-ramp flows [10].

In [4] it was shown that the traffic dynamics has two globally asymptotically stable equilibrium points which correspond to strictly feasible and infeasible demand (the link is either entirely free or entirely congested). This is the reason why approximating the dynamics using random switches between the mentioned two modes, the density is fairly well estimated. The drawback is, as observed in [6], the system is no longer a conservation law (vehicles may appear/disappear in the network). In order to overcome this inconvenient we use in this paper a deterministic *constrained* model that reduce the number of possible affine dynamics of the system and preserve the number of vehicles in the network. Moreover, we use the vehicles conservation law to guaranty that the estimation error does not increase during the unobservable modes. This model is used to recover the state of the traffic network and precisely localize the eventual congestion front. The state of the network is recovered using what we call forward/backward observers.

The highway network is designed as a sequence of nodes relied by links. Since the sensors are located to the node level, one can easily determine if the node is over saturated or under saturated. Thus, the main concern for the traffic conditions is the estimation of the density inside the links. In order to better locate the congestions appearing into the network, each link is partitioned in several cells. It is worth noting here that the estimation problem is decentralized to the link level. In other words the density of the cells belonging to a link is estimated using only the data provided by the sensors located on the link boundary.

The structure of the paper is as follows: in Section II we introduce the deterministic constrained model that describe the density dynamics. Section III is devoted to observability of the system under consideration and in Section IV we propose an observer design. Section V focuses on the density estimation for each cell of a highway segment. In this section we also study the global observability of the traffic state. Simulation results are provided in Section VI before some concluding remarks.

II. Constrained switching model for traffic estimation

The traffic dynamics models are based on the car conservation principle. The simplest continuous macroscopic traffic model, involving only the density ρ, is the LWR model introduced in [7], [11]. The constitutive assumption of this model, motivated by experimental data, is that the vehicles tend to travel at an equilibrium speed $v = v(\rho)$ where ρ represents the density of a specific section at a specific time. Thus the equilibrium speed depends implicitly on the location and on the time. Since the flow is defined as $\varphi(\rho) = \rho v(\rho)$, one can depict an equilibrium flow function $\varphi = \varphi(\rho)$ called the fundamental diagram in traffic engineering.

In the sequel, we use the macroscopic traffic flow model called the switching mode model (SMM) derived from the cell transmission model (CTM) proposed by Daganzo [1]. The SMM is a piecewise linear state-dependent model in which the flow on each interface is a trade-off between the supply and the demand

$$\varphi_i = \min\{D_{i-1}, S_i\} \qquad (1)$$

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with

$$D_{i-1} = \min\{v_{i-1}\rho_{i-1}, \varphi_{m,i-1}\},$$
$$S_i = \min\{\varphi_{m,i}, w_i(\rho_{m,i} - \rho_i)\}$$

where $\varphi_{m,i}$ is the maximum flow allowed by the capacity of cell i, $\rho_{m,i}$ is the jam density (i.e. the maximum density that can be reached), v_i corresponds to the free flow speed and w_i is the congestion wave speed in cell i. All these parameters can be the same for all cells or allowed to vary for each cell. It is noteworthy that D_{i-1} is the flow that can be delivered by the cell $i-1$ while S_i is the flow that can be received by the cell i.

Definition 1: A cell $i \in \{2, \ldots, N\}$ is considered **free** if it is able to accept the flow delivered by its upstream neighboring cell (i.e. $\varphi_i = D_{i-1}$) and is considered **congested** if it is not free (i.e. $\varphi_i = S_i$). The first cell is free if $\varphi_1 \le S_1$ and is congested otherwise.

The state of the system is given by the vector $\rho = (\rho_1, \ldots, \rho_N)$, the measured data used by the system are the upstream and downstream flows (φ_u, φ_d). In order to simplify the analysis we consider that only one congestion wave may exist in the highway segment. Thus one can have only $N+1$ modes since the congestions always appear at cell N and propagate upstream. Furthermore, the front wave moves downstream when the congestion disappears. Denoting by F the free state of a cell and by C the congested one, two adjacent cells can be in one of the following situations: FF, FC or CC.

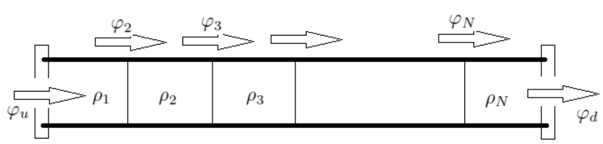

Fig. 1. The switching mode model based on cell transition model.

Let us introduce the index $s(k) \in \{0, 1, \ldots, N\}$ in order to precise the mode of the entire highway segment. This index roughly locates the congestion front. Precisely, $s(k) = i \in \{0, 1, \ldots, N\}$ if and only if the first i cells are free while the last $N-i$ are congested (see Table I for illustration).

$s(k)$	Cell 1	Cell 2	$\ldots$	Cell N
0	C	C	$\ldots$	C
1	F	C	$\ldots$	C
2	F	F	$\ldots$	C
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	F	F	$\ldots$	F

TABLE I
OPERATING MODE TABLE

With this notation the system dynamics writes as:

$$\begin{cases} \rho(k+1) = A_{s(k)}\rho(k) + B\varphi(k) + B_{m,s(k)}\rho_m \\ s(k+1) = s(k) + f(\rho(k), \varphi(k)) \\ y(k) = h(\rho(k)) \end{cases} \tag{2}$$

where $\varphi = (\varphi_u, \varphi_d)$ is the input, $\rho_m = (\rho_{m,1}, \ldots, \rho_{m,N})$,

$$h(\rho(k)) = \begin{cases} w_1(\rho_{m,1} - \rho_1(k)), \text{ if } s(k) = 0 \\ v_N\rho_N(k), \text{ if } s(k) = N \\ 0, \text{ otherwise} \end{cases} \tag{3}$$

and

$$f(\rho(k), u(k)) = \begin{cases} -1 \text{ if } \mathcal{C}^-(\rho(k), s(k)) \\ 0 \text{ if } \mathcal{C}^0(\rho(k), s(k), \varphi(k)) \\ 1 \text{ if } \mathcal{C}^+(\rho(k), s(k)) \end{cases} \tag{4}$$

with

$$\mathcal{C}^-(\rho(k), s(k)) = (s(k) > 0) \wedge$$
$$\left(v_{s(k)-1}\rho_{s(k)-1}(k) > w_{s(k)}(\rho_{m,s(k)} - \rho_{s(k)}(k))\right)$$

$$\mathcal{C}^0(\rho(k), s(k), \varphi(k)) =$$
$$\Big[(s(k) = 0) \wedge \big(\varphi_u(k) = w_1(\rho_{m,1} - \rho_1(k))\big)\Big] \vee$$
$$\Big[(s(k) = N) \wedge \big(v_{N-1}\rho_{N-1}(k) \le w_N(\rho_{m,N} - \rho_N(k))\big)\Big] \vee$$
$$\Big[(0 < s(k) < N) \wedge$$
$$\big(v_{s(k)-1}\rho_{s(k)-1}(k) \le w_{s(k)}(\rho_{m,s(k)} - \rho_{s(k)}(k))\big) \wedge$$
$$\big(v_{s(k)}\rho_{s(k)}(k) \ge w_{s(k)+1}(\rho_{m,s(k)+1} - \rho_{s(k)+1}(k))\big)\Big]$$

$$\mathcal{C}^+(\rho(k), s(k)) = (s(k) < N) \wedge$$
$$\left(v_{s(k)}\rho_{s(k)}(k) < w_{s(k)+1}(\rho_{m,s(k)+1} - \rho_{s(k)+1}(k))\right)$$

It is worth noting here that the function $f(\rho(k), \varphi(k))$ formalize the conditions characterizing the forward/backward motion of the congestion front. Precisely, the cell $s(k)$ becomes congested in the moment when the conditions $\mathcal{C}^-(\rho(k), s(k))$ hold true. The cell $s(k)+1$ becomes free when the conditions $\mathcal{C}^+(\rho(k), s(k))$ hold true. When the conditions $\mathcal{C}^0(\rho(k), s(k))$ are verified the front of congestion is kept inside the cell $s(k)$ sufficiently far from the interface between cells $s(k)-1$ and $s(k)$. All these conditions are based on the interface flows adjoint to the cell $s(k)$.
In order to define the matrices $A_i \in \mathbb{R}^{N\times N}$, $B_i \in \mathbb{R}^{N\times 2}$, $B_{m,i} \in \mathbb{R}^{N\times N}$, $\forall i \in \{0, 1, \ldots, N\}$ used in (2) we introduce the following notation:

$$\Gamma_i := \begin{pmatrix} 1 - \frac{T}{l_1}v_1 & 0 & \ldots & 0 & 0 \\ \frac{T}{l_2}v_1 & 1 - \frac{T}{l_2}v_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \frac{T}{l_i}v_{i-1} & 1 \end{pmatrix} \in \mathbb{R}^{i\times i},$$
$$\forall i = 1, \ldots, N$$

For all $i = 1, \ldots, N-1$ one defines the matrix $\Delta_i \in \mathbb{R}^{(N-i)\times(N-i)}$ by

$$\begin{pmatrix} 1 - \frac{T}{l_{i+1}}w_{i+1} & \frac{T}{l_{i+1}}w_{i+2} & 0 & \ldots & 0 \\ 0 & 1 - \frac{T}{l_{i+2}}w_{i+2} & \frac{T}{l_{i+2}}w_{i+3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{T}{l_{N-1}}w_N \\ 0 & 0 & 0 & \ldots & 1 - \frac{T}{l_N}w_N \end{pmatrix}$$

$$\begin{cases} A_N = \Gamma_N, \quad A_0 = \begin{pmatrix} 1 & (\frac{T}{l_1}w_2, \mathbf{0}_{N-2}^\top) \\ \mathbf{0}_{N-1} & \Delta_1 \end{pmatrix} \\ A_i = \begin{pmatrix} \Gamma_i & \begin{pmatrix} \mathbf{0}_{i-1,N-i} \\ (\frac{T}{l_i}w_{i+1}, \mathbf{0}_{N-i-1}^\top) \end{pmatrix} \\ \mathbf{0}_{N-i,i} & \Delta_i \end{pmatrix}, \\ \qquad \forall\ 1 \leq i \leq N-1 \end{cases}$$

$$B = \begin{pmatrix} \frac{T}{l_1} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & -\frac{T}{l_N} \end{pmatrix},$$

$$\begin{cases} B_{m,N} = \mathbf{0}_{N,N}, \qquad B_{m,0} = \mathbf{I}_N - A_0 \\ B_{m,i} = \begin{pmatrix} \mathbf{0}_{i-1,i} & \mathbf{0}_{i-1,N-i} \\ \mathbf{0}_i^\top & (-\frac{T}{l_i}w_{i+1}, \mathbf{0}_{N-i-1}^\top) \\ \mathbf{0}_{N-i,i} & \mathbf{I}_{N-i} - \Delta_i \end{pmatrix}, \\ \qquad \forall\ 1 \leq i \leq N-1 \end{cases}$$

where $\mathbf{0}$ represents either a column vector (one index) or a matrix (two indices) having all the components equal zero.

III. Observability

The observability for different SMM modes (see [9] and the references therein) is summarized in Table II. Based

Upstream Cells	Downstream Cells	Observable with
Free-flow	Free-flow	Downstream measurement
Congested	Congested	Upstream measurement
Congested	Free-flow	Up. and Down. measurement
Free-flow	Congested	Unobservable

TABLE II

Observability for different SMM modes

on this table we define the notions of forward/backward observability. In order to better explain these concepts let us consider a simple case study where the highway segment has only one cell.

$$\begin{aligned} \rho(k+1) &= \begin{cases} \rho(k) + \dfrac{T}{l}\Big(\varphi_u(k) - v\rho(k)\Big), \text{ if FF} \\ \rho(k) + \dfrac{T}{l}\Big(w(\rho_m - \rho(k)) - \varphi_d(k)\Big), \text{ if CC} \\ \rho(k) + \dfrac{T}{l}\Big(\varphi_u(k) - \varphi_d(k)\Big), \text{ if FC} \end{cases} \\ y(k) &= h(\rho(k)) \end{aligned}$$

where FF means the upstream and the downstream flows are free, CC means the upstream and the downstream flows are congested while FC means the upstream flow is free and the downstream flow is congested.

In FF case φ_u is not restricted by the density of the cell while $\varphi_d = v\rho$ which is state dependent. In other words φ_u is the input and φ_d the output of the system. The observability matrix is equal with v. Since $v \neq 0$, the observability condition is satisfied and we say the system is *backward observable* (i.e. using downstream measurements).

In CC case φ_d is measured and $\varphi_u = w(\rho_m - \rho)$. Thus we have a reversed situation in which φ_d is the input and φ_u is the output of the system. In this case the observability matrix is w. Since $w \neq 0$, the observability condition is satisfied and we say the system is *forward observable* (i.e. using upstream measurements).

Finally in FC case neither φ_u, nor φ_d depend on the density of the cell. The observability matrix is 0 and the density is unobservable but as we shall see the system is open-loop stable.

IV. Observer design

An open-loop estimator for system (2) should be described by

$$\begin{cases} \hat{\rho}(k+1) = A_{\hat{s}(k)}\hat{\rho}(k) + B\varphi(k) + B_{m,\hat{s}(k)}\rho_m \\ \hat{s}(k+1) = \hat{s}(k) + f(\hat{\rho}(k), \varphi(k)) \end{cases} \tag{5}$$

But, computations similar with those provided in the proof of Proposition 3 below, show that using this scheme the estimation error will remain always constant. Therefore, if the initial estimation is bad the algorithm is worthless.

In the next section we propose an observer that use the measures (v_u, v_d) to detect the moment when the congestion front pases over the sensor. Doing so, we design the input and the output of the system according to the operating mode in order to assure the estimation error decreasing. The observer design is illustrated in figure 2 (see also (6) below).

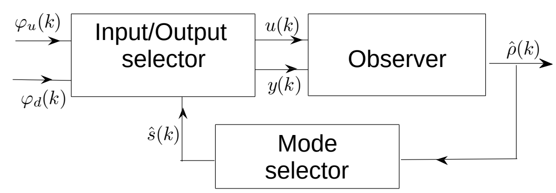

Fig. 2. The proposed observer architecture.

V. Traffic state estimation

A. Density estimation for a highway segment

In the sequel we consider a highway segment partitioned in N cells. We suppose that all the cells have the same length (i.e. $l_i = l$). We also consider the estimation dynamics

$$\begin{cases} \hat{\rho}(k+1) = A_{\hat{s}(k)}\hat{\rho}(k) + B\hat{\varphi}(k) + B_{m,\hat{s}(k)}\rho_m \\ \qquad\qquad + L_{\hat{s}(k)}(y(k) - \hat{y}(k)) \\ \hat{s}(k+1) = \hat{s}(k) + f(\hat{\rho}(k), \hat{\varphi}(k)) \\ \hat{y}(k) = h(\hat{\rho}(k)) \end{cases} \tag{6}$$

where $\hat{\varphi}$ will be defined according to different possible situations. As in the single cell case we study three situations.

Situation 1: backward observer for the FF mode. The upstream and the downstream flows are free. In this case φ_u is measured and not restricted by the density of the first cell while $\varphi_d = v_N\rho_N$. Thus, the density dynamics is given by:

$$\rho(k+1) = A_N\rho(k) + B\varphi(k) + B_{m,N}\rho_m \tag{7}$$

Let us consider $\hat{\varphi}(k) = (\varphi_u, v_N\hat{\rho}_N)$ and the initial estimation $\hat{\rho}(0) = (0, 0, \ldots, 0)^\top$.

Proposition 1: If the upstream and the downstream flows are free the open-loop estimation error is exponentially decreasing and the decreasing rate is given by $\max_{i=\overline{1,n}}\left(1-\frac{T}{l_i}v_i\right)$.

Proof: As before the estimation error at step k will be denoted by $\tilde{\rho}(k) := \rho(k) - \hat{\rho}(k)$. Straightforward computation shows that

$$\tilde{\rho}(k+1) = E_1\tilde{\rho}(k),$$

$$E_1 \triangleq \begin{pmatrix} 1-\frac{T}{l_1}v_1 & 0 & \dots & 0 & 0 \\ \frac{T}{l_2}v_1 & 1-\frac{T}{l_2}v_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{T}{l_N}v_{N-1} & 1-\frac{T}{l_N}v_N \end{pmatrix}$$

The eigenvalues of E_1, and of $E_1^\top$ as well, are $1-\frac{T}{l_i}v_i, i \in \{1,\dots,N\}$. Therefore, denoting by $||E_1||$ the spectral norm of E_1 and by $|x|$ the Euclidean norm of the vector x, one obtains:

$$\begin{aligned} |\tilde{\rho}(k+1)| \le ||E_1|||\tilde{\rho}(k)| &= \sqrt{\lambda_{\max}(E_1^\top E_1)}|\tilde{\rho}(k)| \\ &\le \max_{i=\overline{1,n}}\left(1-\frac{T}{l_i}v_i\right)|\tilde{\rho}(k)| \end{aligned}$$

where $\lambda_{\max}(E_1^\top E_1)$ stands for the largest eigenvalue of the symmetric matrix $E_1^\top E_1$. ■

Remark 1: Using the definition of $h(\cdot)$ (see (3)), when $s(k) = N$ one gets $y(k) - \hat{y}(k) = v_N\tilde{\rho}(k)$. Thus, in the FF case the closed-loop error dynamics is given by

$$\tilde{\rho}(k+1) = (E_1 - L_F \cdot C_F)\tilde{\rho}(k)$$

where $L_F = L_N v_N := (\ell_{F,1}, \ell_{F,2}, \dots, \ell_{F,N})^\top$ and $C_F = (0,\dots,0,1)$. The eigenvalues and the spectral norm of $E_1 - L_F \cdot C_F =$

$$\begin{pmatrix} 1-\frac{T}{l_1}v_1 & 0 & \dots & 0 & -\ell_{F,1} \\ \frac{T}{l_2}v_1 & 1-\frac{T}{l_2}v_2 & \dots & 0 & -\ell_{F,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{T}{l_N}v_{N-1} & 1-\frac{T}{l_N}v_N-\ell_{F,N} \end{pmatrix}$$

can be arbitrarily decreased by choosing an appropriate observer gain L_F.

It is worth noting that E_1 is a non-negative matrix and the choice of $\hat{\rho}(0) = (0,0,\dots,0)^\top$ will assure the non-negativity of the vector $\tilde{\rho}$. Therefore, we always underestimate the traffic state in the FF mode.

Situation 2: forward observer for the CC mode. The upstream and the downstream flows are congested. In this case φ_d is measured and does not depend on ρ_N. On the other hand $\varphi_u = w_1(\rho_{m,1} - \rho_1)$. The density dynamics is given by:

$$\rho(k+1) = A_0\rho(k) + B\varphi(k) + B_{m,0}\rho_m \tag{8}$$

In this situation we consider $\hat{\varphi}(k) = (w_1(\rho_{m,1} - \hat{\rho}_1), \varphi_d)$ and the initial estimation $\hat{\rho}(0) = \rho_m$.

Proposition 2: If the upstream and the downstream flows are congested the open-loop estimation error is exponentially decreasing and the decreasing rate is given by $\max_{i=\overline{1,n}}\left(1-\frac{T}{l_i}w_i\right)$.

Proof: The dynamics of the estimation error is given in this case by

$$\tilde{\rho}(k+1) = E_2\tilde{\rho}(k),$$

$$E_2 \triangleq \begin{pmatrix} 1-\frac{T}{l_1}w_1 & \frac{T}{l_1}w_2 & 0 & \dots & 0 \\ 0 & 1-\frac{T}{l_2}w_2 & \frac{T}{l_2}w_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{T}{l_{N-1}}w_N \\ 0 & 0 & 0 & \dots & 1-\frac{T}{l_N}w_N \end{pmatrix}$$

The eigenvalues of E_2, and of $E_2^\top$ as well, are $1-\frac{T}{l_i}w_i, i \in \{1,\dots,N\}$. Denoting by $||E_2||$ the spectral norm of E_2 one obtains:

$$\begin{aligned} |\tilde{\rho}(k+1)| \le ||E_2|||\tilde{\rho}(k)| &= \sqrt{\lambda_{\max}(E_2^\top E_2)}|\tilde{\rho}(k)| \\ &\le \max_{i=\overline{1,n}}\left(1-\frac{T}{l_i}w_i\right)|\tilde{\rho}(k)| \end{aligned}$$

where as before $\lambda_{\max}(E_2^\top E_2)$ stands for the largest eigenvalue of the symmetric matrix $E_2^\top E_2$. ■

Remark 2: Using the definition of $h(\cdot)$ (see (3)), when $s(k) = 0$ one obtains $y(k) - \hat{y}(k) = -w_1\tilde{\rho}(k)$. Thus, in the CC case the closed-loop error dynamics is given by

$$\tilde{\rho}(k+1) = (E_2 - L_C \cdot C_C)\,\tilde{\rho}(k)$$

where $L_C = -L_0 w_1 := (\ell_{C,1}, \ell_{C,2}, \dots, \ell_{C,N})^\top$ and $C_C = (1,0,\dots,0)$. The eigenvalues and the spectral norm of $E_2 - L_C \cdot C_C =$

$$\begin{pmatrix} 1-\frac{T}{l_1}w_1-\ell_{C,1} & \frac{T}{l_1}w_2 & 0 & \dots & 0 \\ -\ell_{C,2} & 1-\frac{T}{l_2}w_2 & \frac{T}{l_2}w_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\ell_{C,N-1} & 0 & 0 & \dots & \frac{T}{l_{N-1}}w_N \\ -\ell_{C,N} & 0 & 0 & \dots & 1-\frac{T}{l_N}w_N \end{pmatrix}$$

can be arbitrarily decreased by choosing an appropriate observer gain L_C.

It is worth noting that E_2 is a non-negative matrix and the choice of $\hat{\rho}(0) = \rho_m$ will assure the non-positivity of the vector $\tilde{\rho}$. Therefore, we always overestimate the traffic state in congested flow mode.

Situation 3: Open-loop estimator for the coupled FC mode. The upstream flow is free and the downstream flow is congested. In this case neither φ_u, nor φ_d depend on the density of the first cell or last cell. Furthermore, in this situation we cannot detect how many cells are congested. As even in a single cell case we are not able to provide an algorithm for the density estimation, we show that (6) keeps the estimation error constant during this transition phase between Situation 1 and Situation 2.

Remark 3: If we initialize the system when both the upstream and downstream flows are either free or congested,

the density of each cell can be very well approximated before encountering the situation 3. On the other hand, if the length of the highway segment is not very large we can propagate the estimation error for a short period before switching to Situation 1 or Situation 2 and starting to decrease it.
In this situation we consider $\hat{\varphi}(k) = \varphi(k)$ and we initialize with 0 the densities of the cells assumed free and with the corresponding component of the vector ρ_m the densities of the cells assumed congested. As we shall see our assumption on the number of free/congested has no impact on the state estimation algorithm.

Proposition 3: If the upstream flow is free and the downstream flow is congested the estimation error provided by the equations (2) and (6) does not increase.

Proof: For all i belonging to $\{1, \dots, N\}$ let us introduce the following quantity:

$$K_i(k) = \frac{T}{l}\Big(v_i\rho_i(k) - w_{i+1}(\rho_{m,i+1} - \rho_{i+1}(k))\Big) \qquad (9)$$

Without any loss of generality let us suppose that $s(k) = i \in \{1, \dots, N\}$ and $\hat{s}(k) = i + j \in \{1, \dots, N\}, j \geq 0$. Therefore, (2) and (6) lead to

$$\begin{cases} \rho(k+1) = A_i\rho(k) + B\varphi(k) + B_{m,i}\rho_m \\ \hat{\rho}(k+1) = A_{i+j}\hat{\rho}(k) + B\hat{\varphi}(k) + B_{m,i+j}\rho_m \end{cases}$$

Therefore, since $B_i = B_{i+j}$ and $\varphi(k) = \hat{\varphi}(k)$ one obtains

$$\begin{aligned} \tilde{\rho}(k+1) &= A_i\rho(k) - A_{i+j}\hat{\rho}(k) + B_{m,i}\rho_m - B_{m,i+j}\rho_m \\ &= A_{i+j}\tilde{\rho}(k) + \sum_{\ell=1}^{j}(A_{i+\ell-1} - A_{i+\ell})\rho(k) + \\ &+ \sum_{\ell=1}^{j}(B_{m,i+\ell-1} - B_{m,i+\ell})\rho_m \\ &= A_{i+j}\tilde{\rho}(k) + \begin{pmatrix} \mathbf{0}_{i-2} \\ K_{i-1}(k) \\ K_i(k) - K_{i-1}(k) \\ K_{i+1}(k) - K_i(k) \\ \vdots \\ K_{i+j}(k) - K_{i+j-1}(k) \\ -K_{i+j}(k) \\ \mathbf{0}_{N-i-j} \end{pmatrix} \end{aligned}$$

Thus, $Sum(\tilde{\rho}(k+1)) = Sum(A_{i+j}\tilde{\rho}(k))$. On the other hand

$$\begin{aligned} Sum(A_i x) &= Sum(\Gamma_i(x_1, \dots, x_i)^\top) + \frac{T}{l}w_{i+1}x_{i+1} \\ &+ Sum(\Delta_i(x_{i+1}, \dots, x_N)^\top) \\ &= Sum((x_1, \dots, x_i)^\top) + Sum((x_{i+1}, \dots, x_N)^\top) \\ &= Sum(x), \quad \forall x \in \mathbb{R}^n, \forall i \in \{1, \dots, N\} \end{aligned}$$

We conclude that $Sum(\tilde{\rho}(k+1)) = Sum(\tilde{\rho}(k))$ which means that the estimation error is constant in average inside the highway segment but it may be distributed in different way at each time-step. Precisely, the estimation error will decrease inside the cells where both the inflow and outflow are either free or congested for the real and the estimation model in the same time and it will accumulate in the other cells. ■

B. Global observability for traffic state

From the previous sections it is clear that the hybrid system (2) is not observable since some of its mode are unobservable. Nevertheless, during the unobservable mode the state estimation error remains constant. This means that using only partial data we are able to asymptotically reconstruct the state of the system if the following Assumption is satisfied.

Assumption 1: The coupled FC mode periods are shorter than δ.
It is noteworthy that Assumption 1 is satisfied in practice and δ can be fixed by studying the behavior of the network during several weeks.

Definition 2: We say that a system is globally observable if there exists a non-increasing function $V : \mathbb{R} \mapsto \mathbb{R}_+$ characterizing the error estimation and some fixed strictly positive constants $\delta \in \mathbb{Z}$ and $\alpha < 1$ such that $V(k + p\delta) < \alpha^p V(k), \ \forall p \in \mathbb{Z}$.

Proposition 4: If Assumption 1 holds, the system (2) is globally observable using the observer (6).

Proof: Let us consider $V(k) = |\tilde{\rho}(k)|$ and $\alpha = \max\{||E_1 - LC_N||, \ ||E_2 - LC_0||\} < 1$. Since we are not able to assure the decreasing of V during the coupled FC mode periods we shall consider the observer gain $L = 0$ during these periods. Doing so we get $V(k+1) \leq V(k)$ for all $k \geq 0$. On the other hand Assumption 1 assures as that during δ steps at least one time the forward or the backward observability situation is encountered. Taking into account the definition of α one obtains $V(k + \delta) < \alpha V(k)$ which leads straightforwardly to the global observability of the system (2). ■

VI. Simulation results

In the sequel, the theoretical results are illustrated by some simulations. The fundamental diagram of the network has been done using the technique described in [2]. Precisely, we consider a highway segment with five identical cells. The jam density is $\rho_{m,i} = 200\ Veh/Km$, the free flow speed is $v_i = 90\ Km/h$ and the front of congestion speed is $w_i = 16\ Km/h$.

The macroscopic simulation has been done during 140 minutes. The upstream flow was sequentially increased while the downstream flow was set constant to a half of the maximal capacity of the road. This has been done in order to create a congestion which expand backwards. When all the cells have been congested we have drastically decreased the upstream flow inducing the congestion vanishing. It is worth noting here that the macroscopic simulation accurately reproduce the measured densities(see Figure 3).

Figure 4 emphasize the behavior of the network and the congestion front motion. Figure 5 shows the estimation results when the estimation starts after 50 minutes when the fifth cell is already congested. Therefore, we have initialized the densities of the first four cells to zero and the density of the last cell to the jam density $\rho_{m,5} = 200\ Veh/Km$. The estimated densities approach the simulated ones during the CC mode.

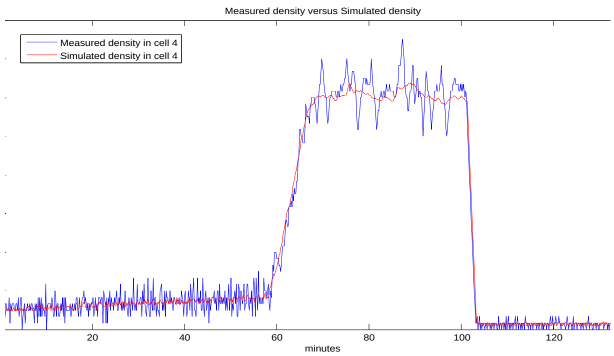

Fig. 3. The density evolution given by the macroscopic simulation is smoother but it accurately reproduces the behavior of the measured density.

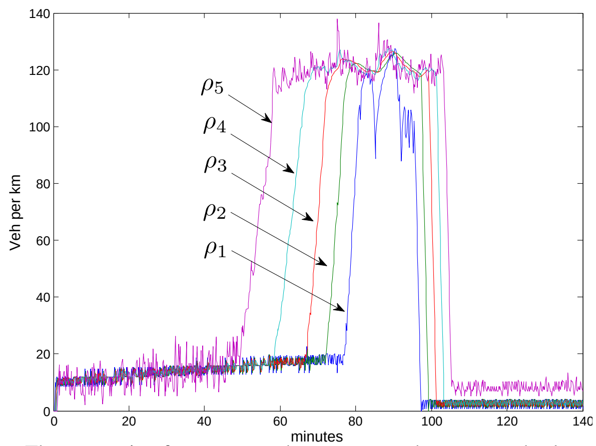

Fig. 4. The congestion front appears downstream and propagates backward from the fifth cell to the first one.

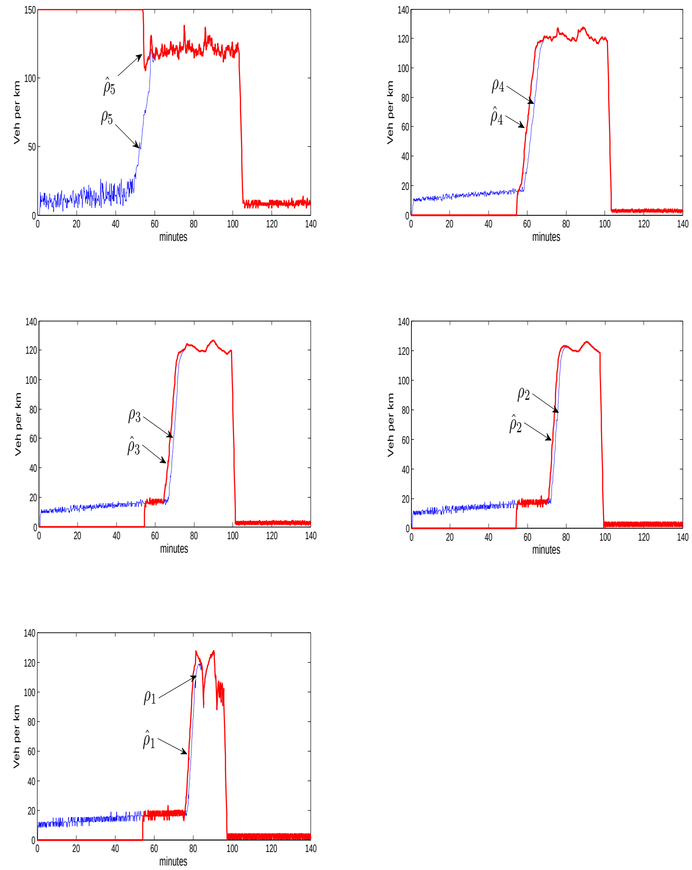

Fig. 5. During the transition period when some cells are congested and some other are free the total estimation error propagates but the distribution changes. When all the cells are congested the estimation error disappears.

VII. Conclusions

In this paper we proposed a strategy for real-time density estimation for traffic networks. To this aim, we introduced a deterministic constrained macroscopic model which reduce the number of possible affine dynamics of the system and preserve the number of vehicles in the network. This model is used to recover the state of the traffic network and precisely localize the eventual congestion front. The state of the network is recovered using what we call forward/backward observers. We pointed out that during unobservable modes the estimation error is preserved due to vehicle conservation law. Numerical simulations show the efficiency of the proposed strategy.

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