



Eigenvalue asymptotics for magnetic fields and degenerate potentials

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Eigenvalue asymptotics for magnetic fields and degenerate potentials

March 23, 2011

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1 Introduction

In this review are presented several results of spectral analysis, based for most of them on the min-max variational principle . These are mainly Weyl-type asymptotics for some "non generic" Schrödinger operators. In the appropriate setup, the Weyl formula describes the asymptotic relationship between the number of eigenvalues less than some fixed value λ and the volume, in phase space, of trajectories with energy less than λ for the corresponding classical problem. To be more precise, let us consider a continuous positive-valued potential V on $\mathbb{R}^m$, and let us make the following assumption for $V(x)$:

$$V(x) \rightarrow +\infty \text{ when } |x| \rightarrow +\infty \quad (1.1)$$

(we call such a $V(x)$ a non degenerate potential). Then for any value of the parameter h in $]0, 1]$, the operator $H_h = -h^2\Delta + V$ defined on $L^2(\mathbb{R}^m)$ is essentially self-adjoint and has a compact resolvent [50]. Moreover, denoting by $N(\lambda, H_h)$ the number of eigenvalues less than some fixed value λ , we get the following semi-classical asymptotic behaviour, when $h \rightarrow 0$:

$$N(\lambda, H_h) \sim h^{-m} (2\pi)^{-m} v_m \int_{\mathbb{R}^m} (\lambda - V(x))_+^{m/2} dx . \quad (1.2)$$

In this so-called semi-classical Weyl asymptotic formula, v_m denotes the volume of the unit ball in $\mathbb{R}^m$, and by W_+ we mean that we take the positive part of W .

If we take $h = 1$ in the previous formula we get the asymptotics for large energies of the operator $H_1 = -\Delta + V$:

$$N(\lambda, H_1) \sim_{\lambda \rightarrow +\infty} (2\pi)^{-m} v_m \int_{\mathbb{R}^m} (\lambda - V(x))_+^{m/2} dx . \quad (1.3)$$

The right-hand side of the formula (1.2) can be seen more generically as the volume, in phase space, of the set $\{(x, \xi), \mathcal{H}(x, \xi) \leq \lambda\}$, where $\mathcal{H}(x, \xi) = \xi^2 + V(x)$ is the principal symbol of H_h and the Hamiltonian of the associated dynamics .

A naturel question is then the following : what can be said of a Schrödinger operator which has a discrete spectrum but does not verify the non-degeneracy condition (1.1) ? In that case the volume of $\{(x, \xi), \xi^2 + V(x) \leq \lambda\}$ may happen to be infinite, so that the formula (1.2) becomes irrelevant. This is the case for instance for the following potential (in $\mathbb{R}^2$)

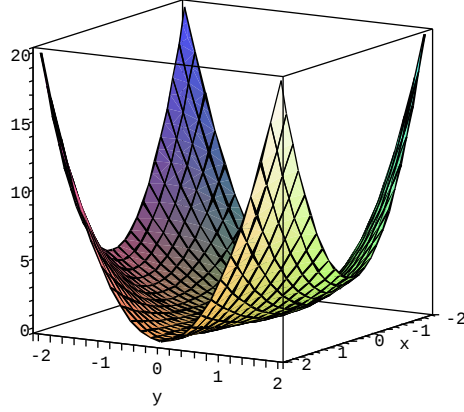


Figure 1: The potential $V(x, y) = (1 + x^2) y^2$.

$V(x, y) = (1 + x^2) y^2$ (Figure 1) .

The problems presented below discuss precisely this question for various situations, and the estimates obtained will be called Weyl-type asymptotics.

First we recall the results obtained in ([39], [41]), for a class of degenerate potentials in the sense we previously defined. These potentials can be seen as a generalization of the preceding example ; they are of the form

$$\begin{aligned} V(x) &= f(y)g(z), \quad x = (y, z) \in \mathbb{R}^n \times \mathbb{R}^p, \\ f &\in C(\mathbb{R}^n; \mathbb{R}_+^*), \quad g \in C(\mathbb{R}^p; \mathbb{R}_+), \quad g \text{ homogeneous of degree } a. \end{aligned}$$

Then we consider Schrödinger operators with magnetic field $H_h(A) = ((h\nabla - iA))^2$. One can call them degenerate in the sense that the principal symbol of $H_h(A)$, which is $\mathcal{H}(x, \xi) = (\xi - A(x))^2$, annihilates on a non compact manifold of $T^*(\mathbb{R}^m)$. If the magnetic field $B = dA$ is such that the counting function $N(\lambda, H_h(A))$ can be defined, then we can look for some alternative to Weyl formula. In particular, when the magnetic field $B = dA$ satisfies some so-called magnetic bottles conditions :

$$\|B(x)\| \rightarrow +\infty \quad \text{quand} \quad |x| \rightarrow +\infty, \quad (1.4)$$

$H_h(A)$ is essentially self-adjoint and has a compact resolvent on $L^2(\mathbb{R}^m)$ [3]. The spectral asymptotics for large energies were computed by Y. Colin de Verdière [6]. Here are discussed the semi-classical version of this result

[60], and the case of magnetic bottles in the hyperbolic context ([42] for the Poincaré half-plane, [43] for geometrically finite hyperbolic surfaces). These Weyl-type asymptotics can be seen as the expression of an integrated density of states on the whole space. For a constant magnetic field $B = \sum_{j=1}^r b_j dx_j \wedge dy_j$, $b_1 \geq b_2 \geq \dots \geq b_r > 0$, the density of states is given by, (for some universal constant C_r) :

$$\nu_B(\lambda) = C_r b_1 b_2 \dots b_r \sum_{n_j \geq 0} \left(\lambda - \sum_{j=1}^r (2n_j + 1) b_j \right)_+^{d/2-r}.$$

In the hyperbolic context $r(x) = 1$ for any x and the intensity $b(x)$ is defined in a slightly different way according to the hyperbolic geometry.

We discuss also another "degenerate" problem in the framework of the superconductivity theory. In order to minimize the associated Ginzburg-Landau functional associated to a given open set Ω in $\mathbb{R}^3$, we study of the spectral properties of the magnetic Laplacian $H = ((\nabla - iA))^2$, with the so-called magnetic Neumann condition at the boundary :

$$\nu(x) \cdot (\nabla - iA)u(x) = 0 \quad \forall x \in \delta\Omega.$$

This comes from the fact that $(0, \sigma A)$ is a trivial critical point for the functional (σ is a parameter related to the magnetic intensity); the magnetic Laplacian previously defined is precisely the Hessian computed at this point.

The magnetic field $B = dA$ is assumed to be constant, and the spectrum contains an absolutely continuous part, which is the whole interval $[b, +\infty[$. ($b = \|B\|$ is the magnetic intensity). However, if we consider the case of the half-space, e.g. for (t, x, y) in $\Omega = \mathbb{R}_+ \times \mathbb{R}^2$, the Neumann realization of the magnetic Laplacian

$$H = (D_t - A_1)^2 + (D_x - A_2)^2 + (D_y - A_3)^2,$$

with $D_s = -i(\frac{\partial}{\partial s})$, and if we assume that the magnetic field is not orthogonal to the boundary $\delta\Omega = \{(0, x, y), (x, y) \in \mathbb{R}^2\}$, we get that the lower bound of the spectrum is strictly less than b , [36], [24]. It can be proved that the part of the spectrum less than b consists of a finite number of eigenvalues, each one having an infinite multiplicity [40].

Furthermore, this number tends to infinity as the angle between the magnetic field and the boundary $\delta\Omega$ tends to zero, and this leads to a Weyl-type

description of the number of eigenvalues less than a fixed real number less than b [40].

The last section is devoted to a problem of magnetic bottle, but in the classical context [59]. This problem is indeed at the origin of the results proved in the semi-classical and quantum mechanics context. The magnetic field considered here is axially symmetric in $\mathbb{R}^3$ and verifies the condition (1.4). Are there bounded trajectories, as suggested by numerical simulations? The operator associated to the Hamiltonian by the Weyl quantification has a discrete spectrum : we are in the case of the magnetic bottles defined previously. However in the classical setup we have to use the results of the KAM theory. The conditions needed to apply Moser's twist theorem have to be checked and then we can conclude that there exists an open set of initial conditions such that the trajectory is bounded. The interesting fact is that the Hamiltonian can be described as a perturbation of an effective Hamiltonian, which is precisely the principal symbol of the Schrödinger operator with the degenerate potential

$$V(x, y) = B^2(x, 0) y^2 \quad .$$

2 Degenerate potentials

2.1 The Tauberian approach

There are a lot of works on the subject, and we refer to [61] for a review. However for the reader's convenience we recall briefly the main results in this approach. Roughly speaking, the Tauberian technique consists on studying the asymptotic behaviour of the Green's function of the operator H_1 and applying a Tauberian theorem. [12] is the first result where (1.3) is proved for a class of non degenerate potentials, then refinements can be found in [57], [33], [31] and [52], where the formula (1.3) is proved under minimal conditions on V .

In [54] Solomyak makes the following remark :

Lemma 2.1 *Let V be a positive a -homogeneous potential :*

$V(x) \geq 0$; $V(tx) = t^a V(x)$ for any $t \geq 0$ ($a > 0$).

If moreover $V(x)$ is strictly positive ($V(x) \neq 0$ if $x \neq 0$) the spectrum of H_1 is discrete and the formula (1.3) takes the form :

$$N(\lambda, H_1) \sim \gamma_{m,a} \lambda^{\frac{2m+am}{2a}} \int_{S^{m-1}} (V(x))^{-m/a} dx \quad (2.1)$$

($\gamma_{m,a}$ is a constant depending only on the parameters m and a .)

From that lemma comes out naturally the idea of investigating the spectrum without the condition of strict positivity (and thus in a case of degeneracy of the potential) ; the two main results are [54] :

Theorem 2.2 *The formula (2.1) still holds for a positive a -homogeneous potential such that $J(V) = \int_{S^{m-1}} (V(x))^{-m/a} dx$ is finite.*

The second result deals with a case where $J(V)$ is infinite :

Theorem 2.3 *Let $V(x) = F(y, z)$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^p$, $n + p = m$, $m \geq 2$, such that $F(sy, tz) = s^b t^{a-b} F(y, z)$ (with $0 < a < b$) and $F(y, z) > 0$ for $|z||y| \neq 0$. Denote by $\lambda_j(y)$ the eigenvalues of the operator $-\Delta_z + F(y, z)$ in $L^2(\mathbb{R}^p)$ and let $s = \frac{2b}{2+a-b}$, then :*

$$\begin{aligned} \text{If } \frac{n}{b} > \frac{m}{a} \quad N(\lambda, H_1) &\sim \gamma_{n,s} \lambda^{\frac{2m+am}{2b}} \int_{S^{m-1}} \Sigma(\lambda_j(y))^{-n/s} dx \\ \text{if } \frac{n}{b} = \frac{m}{a} \quad N(\lambda, H_1) &\sim \frac{a(a+2)}{2b(a-b)} \gamma_{m,a} \lambda^{\frac{2m+am}{2b}} \ln \lambda \int_{S^{n-1} S^{p-1}} F(y, z)^{-m/a} dx. \end{aligned}$$

The proof is based on variational techniques and spectral estimates of [52].

In [51] D.Robert extends the theory of pseudodifferential operators to pseudodifferential operators with operator symbols. It is thus possible to study cases where the operator has a compact resolvent but the condition $\lim_{\infty} V(x) = +\infty$ is not fulfilled. As an example it gives the asymptotics of $N(\lambda, H_1)$ for the 2-dimensional potential $V(y, z) = y^{2k}(1 + z^2)^l$, where k et l are strictly positive. The asymptotics are the following :

Theorem 2.4

$$\begin{aligned} \text{if } k > l \quad N(\lambda, H_1) &\sim \gamma_1 \lambda^{\frac{l+k+1}{2l}} \\ \text{if } k = l \quad N(\lambda, H_1) &\sim \gamma_2 \lambda^{\frac{2k+1}{2k}} \ln \lambda \\ \text{if } k < l \quad N(\lambda, H_1) &\sim \gamma_3 \lambda^{\frac{2k+1}{2k}}. \end{aligned}$$

The constants γ_i depend only on k and l , but the first one γ_1 takes into account the trace of the operator $(-\Delta_z + z^{2k})^{-(k+1)/2l}$ in $L^2(\mathbb{R})$.

In the 2-dimensional case let us mention the results of B.Simon [53]. He first recalls Weyl's famous result : let H be the Dirichlet Laplacian in a bounded domain Ω in $\mathbb{R}^2$, then the following asymptotics hold :

$$N(\lambda, H) \sim \frac{1}{2}\lambda|\Omega|$$

and then he considers domains Ω for which the volume (denoted by $|\Omega|$) is infinite but the spectrum of the Laplacian is still discrete. These domains are of the type : $\Omega_\mu = \{(y, z); |y||z|^\mu \leq 1\}$.

Actually the problem can be derived from the study of the asymptotics of Schrödinger operators with the homogeneous potential : $V(y, z) = |y|^\alpha |z|^\beta$.

In order to compute Weyl-type asymptotics, he uses the Feynman-Kac formula and the Karamata-Tauberian theorem, but the main tool is what he calls "sliced bread inequalities", which can be seen as a kind of Born-Oppenheimer approximation. More precisely let $H = -\Delta + V(y, z)$ be defined on $\mathbb{R}^{n+p}$, and denote by $\lambda_j(y)$ the eigenvalues of the operator $-\Delta_z + V(y, z)$ in $L^2(\mathbb{R}^p)$. (If the z 's are electron coordinates and the y 's are nuclear coordinates, the $\lambda_j(y)$ are the Born- Oppenheimer curves). He proves the following lemma :

$$\text{Tre}^{-tH} \leq \sum_j e^{-t(-\Delta_y + \lambda_j(y))}$$

(when the second term exists).

Thus he gets the two following coupled results :

Theorem 2.5 *If $H = -\Delta + |y|^\alpha |z|^\beta$ and $\alpha < \beta$, then*

$$N(\lambda, H) \sim c_\nu \lambda^{\frac{2\nu+1}{2}} \quad \left(\nu = \frac{\beta+2}{2\alpha}\right)$$

Corollary 2.6 *if $H = -\Delta_{\Omega_\mu}$ ($\mu > 1$), then $N(\lambda, H) \sim c_\mu \lambda^{\frac{1}{2\mu+1}}$.*

Theorem 2.7 *If $H = -\Delta + |y|^\alpha |z|^\alpha$, then $N(\lambda, H) \sim \frac{1}{\pi} \lambda^{1+\frac{1}{\alpha}} \ln \lambda$*

Corollary 2.8 *if $H = -\Delta_{\Omega_\mu}$ ($\mu = 1$), then $N(\lambda, H) \sim \frac{1}{\pi} \lambda \ln \lambda$.*

The constant c_μ depends only on μ , and the constant c_μ takes in account the trace of the operator $(-\Delta_z + |z|^\beta)^{-\nu}$ in $L^2(\mathbb{R})$.

2.2 The min-max approach

The result presented in this section [39] is based on the method of Courant and Hilbert, the min-max variational principle. .

Thanks to this method, which requires only to study the associated quadratic form, (using appropriate partitions and simplified models) no assumptions on the evolution semi-group are needed, and we get Weyl-type asymptotics for a large class of degenerate potentials, namely potentials of the following form :

$$\begin{aligned} V(x) &= f(y)g(z), \quad x = (y, z) \in \mathbb{R}^n \times \mathbb{R}^p, \quad n + p = m, \quad m \geq 2 \\ f &\in C(\mathbb{R}^n; \mathbb{R}_+^*), \quad g \in C(\mathbb{R}^p; \mathbb{R}_+), \\ \exists a > 0 \quad t.q. \quad g(tz) &= t^a g(z) \quad \forall t > 0, \quad g(z) > 0 \quad \forall z \neq 0. \end{aligned} \quad (2.2)$$

This class contains the potentials studied in [51], [53] and [54].

According to the assumption (2.2) the spectrum of the operator $-\Delta_z + g(z)$ sur $L^2(\mathbb{R}^p)$ is discrete and positive. Let us denote by μ_j its eigenvalues. We have moreover :

Remark 2.9 *If $f(y) \rightarrow +\infty$ when $|y| \rightarrow +\infty$, then $H_h = -h^2\Delta + V$ has a compact resolvent.*

Of course if f was assumed to be homogeneous, the asymptotics would be given by Theorem 2.3. But here the only additional assumption made on f is a locally uniform regularity :

$$\begin{aligned} \exists b, c > 0 \quad t.q. \quad c^{-1} &\leq f(y) \quad \text{and} \quad (2.3) \\ |f(y) - f(y')| &\leq cf(y)|y - y'|^b, \quad \forall (y, y') \quad t.q. \quad |y - y'| \leq 1 \end{aligned}$$

Theorem 2.10 *Let us assume the previous conditions on f and g . Then there exists $\sigma, \tau \in]0, 1[$ such that, for any $\lambda > 0$, one can find $h_0 \in]0, 1[$, $C_1, C_2 > 0$ in order to have*

$$(1 - h^\sigma C_1)n_{h,f}(\lambda - h^\tau C_2) \leq N(\lambda; H_h) \leq (1 + h^\sigma C_1)n_{h,f}(\lambda + h^\tau C_2) \quad \forall h \in]0, h_0[$$

$$\text{with} \quad n_{h,f}(\lambda) = h^{-n}(2\pi)^{-n}v_n \int_{\mathbb{R}^n} \Sigma_{j \in \mathbf{N}} [\lambda - h^{2a/(2+a)} f^{2/(2+a)}(y) \mu_j]_+^{n/2} dy .$$

Provided some additional conditions on f , the previous result can be refined as follows :

Theorem 2.11 *If moreover one can find a constant C_3 such that, for any $\mu > 1$:*

$$\int_{\{y, f(y) < 2\mu\}} f^{-p/a}(y) dy \leq C_3 \int_{\{y, f(y) < \mu\}} f^{-p/a}(y) dy ,$$

then one can take $C_2 = 0$ in Theorem 2.10 :

$$(1 - h^\sigma C_1) n_{h,f}(\lambda) \leq N(\lambda; H_h) \leq (1 + h^\sigma C_1) n_{h,f}(\lambda) \quad \forall h \in]0, h_0[.$$

Remark 2.12 *If $f^{-p/a} \in L^1(\mathbb{R}^n)$ and $g \in C^1(\mathbb{R}^p \setminus \{0\})$, then the formula (1.3) holds.*

The proof of Theorem 2.10 is based on a suitable subdivision of $\mathbb{R}^n$ into cubes $\{Q_r(r\gamma), \gamma \in \mathbb{Z}^n\}$. According to the min-max variational principle we are then reduced to study Dirichlet and Neumann problems in cylinders of $\mathbb{R}^m$:

$$N(\lambda, H_{h,\gamma}^D) \leq N(\lambda, H_h) \leq N(\lambda, H_{h,\gamma}^N)$$

$$H_{h,\gamma}^{D,N} = -h^2 \Delta_y - h^2 \Delta_z + f(y)g(z) \quad \text{on } Q_r(r\gamma) \times \mathbb{R}^p ,$$

with Dirichlet (or Neumann) condition at the boundary.

In each cube $Q_r(r\gamma)$ $f(y)$ is bounded from above by $f(y_\gamma^*)$ where y_γ^* is a minimum for f .

Using the homogeneity of g one gets that the eigenvalues of the operator $-h^2 \Delta_z + f(y_\gamma^*) g(z)$ are of the form $\{(h^a f(y_\gamma^*))^\alpha \mu_j\}$ ($\alpha = 2/(2+a)$).

One gets then a lower bound for $N(\lambda, H_h)$ by taking the sum, for all cubes, of the sum for all j 's of $N(\lambda - (h^a f(y_\gamma^*))^\alpha \mu_j, -h^2 \Delta_{Q_r(r\gamma)}^D)$.

One gets the upper bound following the same procedure.

Concerning the proof of Theorem 2.11, the main tool is an asymptotic formula of the moment of eigenvalues of $-h^2 \Delta_z + g(z)$, which is again obtained using the min-max principle.

As a conclusion, let us notice that if there is some information on the growth of f at infinity, then the asymptotics can be computed in terms of power of h :

Remark 2.13 *If there exist $k > 0$ and $C > 0$ such that $\frac{1}{C}|y|^k \leq f(y) \leq C|y|^k$ for $|y| > 1$, then*

$$\begin{aligned} \text{if } k > a \quad N(\lambda, H_h) &\approx h^{-m} \\ \text{if } k = a \quad N(\lambda, H_h) &\approx h^{-m} \ln \frac{1}{h} \\ \text{if } k < a \quad N(\lambda, H_h) &\approx h^{-n - \frac{pa}{k}} \end{aligned}$$

Remark 2.14 *The Weyl-type formula of the theorem 2.10 gives us a hint of what can be said about the behaviour of the eigenvalues themselves. Actually this question is answered precisely by using Born-Oppenheimer-type methods in [41], where we compute first order approximations for low energies and middle energies, and apply the results to a potential vanishing on a hypersurface.*

3 Magnetic bottles

3.1 General setting

We are now interested in magnetic Laplacians, in various situations when it is possible to look for Weyl-type estimates. This leads us to give a definition of magnetic bottles in a general Riemannian context.

Let us denote by (M, g) a connected Riemannian manifold of dimension d and by $A = \sum_{j=1}^d a_j dx_j$ a real one-form on M . For any $h \in]0, 1[$ we can define the semi-classical magnetic Laplacian

$$\begin{aligned} H_h(A) &= (ih \, d + A)^*(ih \, d + A) , \\ (ih \, d + A)u &= ih \, du + Au , \quad \forall u \in C_0^\infty(M) . \end{aligned} \tag{3.1}$$

The magnetic field is the exact two-form $B = dA$.

The two-form B is associated to a linear operator L_B on the tangent space defined by

$$B(X, Y) = g(L_B X, Y) ; \quad \forall X, Y \in TM \times TM . \tag{3.2}$$

The magnetic intensity $\mathbf{b}$ is given by

$$\mathbf{b} = \frac{1}{2} \text{tr} \left((B^* B)^{1/2} \right) . \tag{3.3}$$

It is possible to define $H_h(A)$ more geometrically, using the Hermitian connection ∇ on a complex-line bundle L over M with curvature equal to iB . This connection exists provided that the cohomology class of $B/2\pi$ is an integer .

It is defined by $\nabla_X f = df(X) - iA(X)f$, where A is a real one-form verifying $B = dA$. One introduces on $C_0^\infty(M; L)$ the quadratic form $q(f) = \int_M \|\nabla f\|^2 dx$, and by Friedrich's process one gets an operator, which is $H_h(A)$.

Remark 3.1 *Gauge invariance*

If $A' = A + d\phi$ is another magnetic potential associated to B , the operators $H_h(A)$ and $H_h(A')$ are unitarily equivalent.

This property implies that $H_h(A)$ and $H_h(A')$ have the same spectrum. Therefore we give the following definition, which does not depend from the choice of the magnetic potential A :

Definition 3.2 *(M, h, B) is called a magnetic bottle if*

- 1) $H_h(A)$ *is essentially self-adjoint with domain $C_0^\infty(M; L)$*
- 2) $H_h(A)$ *has a compact resolvent*

In [3], which is the first paper on the subject, and also in [14], [30] one can find necessary conditions or sufficient conditions for $H_h(A)$ to have a compact resolvent.

3.2 The Euclidean case

3.2.1 The results

Let us take for (M, g) the Euclidean space $\mathbb{R}^d$. The operator defined in (3.1) is

$$H_h(A) = \sum_{j=1}^d \left(\frac{h}{i} \frac{\partial}{\partial x_j} - a_j \right)^2 \quad .$$

Furthermore there exists, for any $x \in \mathbb{R}^d$, an orthonormal basis $(e_j(x))$ of $\mathbb{R}^d$ such that $B(x)$ has the following expression

$$B(x) = \sum_{j=1}^{r(x)} b_j(x) dx_j \wedge dy_j, \quad b_1(x) \geq b_2(x) \geq \dots \geq b_r(x) > 0 \quad . \quad (3.4)$$

The magnetic intensity is equal to the norm of the vector $B(x) = (b_j(x))_j$. The $b_j(x)$ are the moduli of the non zero eigenvalues of the endomorphism L_B associated to $B(x)$ and $2r(x)$ is the rank of L_B . For odd dimension in particular 0 is always an eigenvalue. We assume moreover the following properties for B :

- (B_1) $\lim_{\|x\| \rightarrow \infty} \|B(x)\| = \infty$
- (B_2) there exists $C > 0$ such that, for every x and x' verifying :
 $\|x - x'\| \leq 1, \|B(x)\| \leq C\|B(x')\|$
- (B_3) $M(x) = o(\|B(x)\|^{\frac{3}{2}})$ when $\|x\| \rightarrow \infty$
 where $M(x) = \max_{|\beta|=2} \left(\sup_{\|x-x'\| \leq 1} \|D^\beta A(x')\| \right)$.

The high energy behaviour of $N(\lambda, H_1(A))$, ($h = 1, \lambda \rightarrow +\infty$), is given by Y. Colin de Verdière in [6] :

Theorem 3.3 *Under the conditions (B_1-B_3) , $(\mathbb{R}^d, 1, B)$ is a magnetic bottle and*

$$N_B^{as}[\lambda(1 - o(1))] \leq N(\lambda, H_1(A)) \leq N_B^{as}[\lambda(1 + o(1))] \quad (\lambda \rightarrow +\infty) .$$

The expression for N_B^{as} is the following :

$$N_B^{as}(\lambda) = \sum_{r=1}^{[d/2]} C_{k,r} \sum_{(n_1, \dots, n_r) \in \mathbb{Z}_r^+} \int_{A_r} \left(\lambda - \sum_{i=1}^r (2n_i + 1)b_i(x) \right)_+^{k/2} \prod_{i=1}^r b_i(x) dx .$$

We used the following notations :

- $A_r = \{x \in \mathbb{R}^d; r(x) = r\}$
- $C_{k,r} = \frac{\gamma_k}{(2\pi)^{k+r}}$, γ_k = volume of the unit ball of $\mathbb{R}^k$.

In [60] we give an equivalent of $N(E, H_h(A))$ for a fixed energy E when h tends to zero. This is the semi-classical version of the previous asymptotics.

We first notice that $H_h(A) = h^2 H_1(A/h)$. $H_1(A/h)$ is the (non semi-classical) Schrödinger operator associated to the magnetic field $\frac{B}{h}$:

$$H_1(A/h) = \sum_{j=1}^d \left(\frac{1}{i} \frac{\partial}{\partial x_j} - \frac{a_j}{h} \right)^2 . \quad (3.5)$$

Consequently, we get $N(E, H_h(A)) = N(\frac{E}{h^2}, H_1(A/h))$ for any fixed energy E .

Using an adaptation of the method explained in [6], we get the following asymptotics [60] :

Theorem 3.4 *Under the conditions $(B_1 - B_3)$, $(\mathbb{R}^d, h, B)$ is a magnetic bottle and we have, for any energy E :*

$$\frac{1}{h^d} N_{hB}^{as}[E(1 - o(1))] \leq N(E, H_h(A)) \leq \frac{1}{h^d} N_{hB}^{as}[E(1 + o(1))] \quad (h \rightarrow 0) .$$

Remark 3.5 *The expression for N_B^{as} becomes more explicit when $d = 2$. We have then*

$b_1(x) = \|B(x)\| = \mathbf{b}(x)$, and :

$$\frac{1}{h^d} N_{hB}^{as}(E) = \frac{1}{2\pi h^2} \int_{\mathbb{R}^2} \mathbf{b}(x) \sum_{n \in \mathbb{N}} [E - (2n + 1)h\mathbf{b}(x)]_0^+ dx . \quad (3.6)$$

$[\rho]_+^0$ is the Heaviside function :

$$[\rho]_+^0 = \begin{cases} 1, & \text{if } \rho > 0 \\ 0, & \text{if } \rho \leq 0 . \end{cases}$$

Remark 3.6 *H. Matsumoto recovers the conclusions of this theorem by studying the semi-group $\exp(-tH_h)$ [38]. The following equivalent is obtained :*

$$\text{Tr}(\exp(-tH_h)) = \frac{1}{h^d} Z_{hB}(t)$$

where $Z_{hB}(t) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} \prod_{i=1}^{r(x)} \frac{htb_i(x)}{\sinh htb_i(x)} dx$ is the Laplace transform of the function $N_{hB}^{as}(\lambda)$ introduced previously. In the 3-dimensional case Tamura [56] obtains a result of the same kind, involving only the norm of the magnetic field. However they both require stronger conditions for B , in order to make sure that $\exp(-tH_h)$ is a trace semi-group. This comes from the philosophy of the min-max method, which does not deal with the evolution semi-group but requires only to study the quadratic form, using partitions and asymptotic formulas for simplified operators (namely here for constant fields in cubes), so that we can get a formula with the minimal assumptions.

Remark 3.7 *The operators verifying the assumptions of the theorem are special examples of the hypoelliptic operators $\sum_{k=1}^d X_k^* X_k$ introduced by L. Hörmander [25], in the case of real vector fields X_k .*

Remark 3.8 *Let us set*

$$\nu_{B(x)}(\lambda) = C_{k,r} \sum_{(n_1, \dots, n_r) \in \mathbb{Z}_r^+} \left(\lambda - \sum_{i=1}^r (2n_i + 1)b_i(x) \right)_+^{k/2} \prod_{i=1}^r b_i(x) .$$

(In this definition, the numbers k et r depend on x .) The function $N_B^{as}(\lambda)$ has then the following expression :

$$N_B^{as}(\lambda) = \int_{\mathbb{R}^d} \nu_{B(x)}(\lambda) dx .$$

In the case of a constant magnetic field, the function $\nu_B(\lambda)$ can be seen as a density of states for the Schrödinger operator in $\mathbb{R}^d$.

The proof consists of two main parts which we develop in next sections : the asymptotic spectral estimate for the Dirichlet problem in the cube $[0, R]^d$ in the case of a constant field , and the appropriate subdivision in cubes which makes possible the reduction to the simplified problem .

3.2.2 The Dirichlet problem in a cube for a constant magnetic field

When the field B is constant, the function $\nu_B(\lambda)$ is used to estimate $N_{B,R}(\lambda)$, the counting function of the spectrum concerning Dirichlet problem for the Schrödinger operator with the magnetic field B in the cube $[0, R]^d$. We recall the precise estimate, given in [6]:

Theorem 3.9 *There exists a constant c depending only on d such that, for any A with $0 < A < R/2$, the following inequalities hold :*

- $N_{B,R}(\lambda) \leq R^d \nu_B(\lambda).$
- $N_{B,R}(\lambda) \geq (R - A)^d \nu_B(\lambda - C/A^2).$

The proof of this result uses the spectrum for constant fields on a torus, and a method due to Polya, which consists in subdividing $\mathbb{R}^d$ into cubes and taking an approximation by a “large” torus ([6], [8] and [9]).

To be more precise we have :

Lemma 3.10 *Constant field on a torus*

Let $B = \sum_{j=1}^r b_j dx_j \wedge dy_j$, $b_1 \geq \dots \geq b_r > 0$ on the torus $M = \mathbb{R}^d / \Gamma$, where $d = r + k$ and Γ_0 is a lattice on $\mathbb{R}^k$.

It is assumed that $\Gamma = \bigoplus_{j=1}^r \rho_j \mathbb{Z}^2 \oplus \Gamma_0$ and that $b_j \rho_j^2 \in 2\pi\mathbb{Z}$. Then

1) the cohomology class of $B/2\pi$ is an integer,

2) the spectrum of $H_h(B)$ is constituted of the eigenvalues

$\lambda = \sum_{i=1}^r (2n_i + 1)b_i + \mu$, where $n_i \in \mathbb{N}^*$, and μ is an eigenvalue of the Laplacian on $\mathbb{R}^k / \Gamma_0$,

3) the multiplicity of λ is equal to the sum of the multiplicity of μ and of $\prod_{i=1}^r \frac{b_i \rho_i^2}{2\pi}$.

3.2.3 A subdivision of R^d into appropriate cubes

Lemma 3.11 *Under the assumptions $(B_1 - B_3)$, and for a fixed $\varepsilon > 0$, there exists for any h a subdivision of R^d in cubes $(\Omega_i)_{i \geq 0}$ of sides r_i , and numbers $(a_i)_{i \geq 1}$ ($0 < a_i \leq r_i/2$) such that, if we set $M_i = \max_{\|\beta\|=2} \sup_{x \in \Omega_i} \|D^\beta a(x)\|$, the following inequalities hold, for any x in Ω_i and for any integer $i \geq 1$:*

$$i) r_i^2 M_i \leq \varepsilon h \|B(x)\|^{1/2}$$

$$ii) M_i \leq \varepsilon^3 \|B(x)\|^{3/2}$$

$$iii) 1/a_i^2 \leq M_{x,\varepsilon} = \max\left(\frac{4\varepsilon \|B(x)\|}{h}, 1/\varepsilon\right).$$

3.3 The hyperbolic half-plane

3.3.1 The setup

Now, we consider the case where $M = \mathbb{H}$ is the hyperbolic plane : $\mathbb{H} = \mathbb{R} \times]0, +\infty[$, endowed with the hyperbolic metric $g = \frac{dx^2 + dy^2}{y^2}$. We have

$\rho_B = \tilde{\mathbf{b}} dv$, where $dv = y^{-2} dx dy$ is the Riemannian measure on M . Thus we have $\mathbf{b} = y^2 (\partial_x A_2 - \partial_y A_1)$ and

$$H_1(A) = y^2 (D_x - A_1)^2 + y^2 (D_y - A_2)^2, \quad (3.7)$$

We define $\mathbf{b} = |\tilde{\mathbf{b}}|$.

The hyperbolic framework has been used mainly for studying the Maass Laplacian, which corresponds to the constant magnetic field case. This case has been studied by many authors [21], [15], [11] [13]. In [27] Y.Inahama and S.Shirai consider asymptotically constant magnetic fields and in [29] they

deal with Pauli operators. In [26] N.Ikeda studies the relationship between Maass Laplacian and Schrödinger operators with Morse potentials.

From an other point of view , the asymptotic distribution of large eigenvalues in the hyperbolic context has already been studied for Schrödinger operators (without magnetic field) [28] . The method is based on Feynman-Kac representation of the heat kernel and the Tauberian theorem. As already mentioned our own method involves only min-max techniques so it does not require to study properties of the evolution semigroup. We get the asymptotic distribution of large eigenvalues for a certain type of magnetic bottles following the method used in the euclidean case, but replacing cubes by rectangles adapted to the hyperbolic geometry.

The result is very similar to (3.6), according to the hyperbolic definition of the intensity $\tilde{\mathbf{b}}$ of the magnetic field. Moreover the techniques are local, so they have been successfully applied to geometrically finite hyperbolic surfaces of infinite area [43].

We give first the basic results in the case of a constant magnetic field.

3.3.2 The Maass Laplacian

The first paper on Maass Laplacian is due to J. Elstrodt [15]

We consider the case where $\tilde{\mathbf{b}} = y^2(\partial_x A_2(x, y) - \partial_y A_1(x, y))$ is constant. We choose a gauge such that $A_2 = 0$, so $A_1(x, y) = \tilde{\mathbf{b}}y^{-1}$. We can assume that $A_1(x, y) = \mathbf{b}y^{-1}$, by eventually performing the change $x \rightarrow -x$, which is a unitary operator on $L^2(\mathbb{H})$. The operator we are interested in is

$$H_1 A^{\mathbf{b}} = y^2(D_x - \mathbf{b}y^{-1})^2 + y^2 D_y^2, \quad \text{with } \mathbf{b} \geq 0 \text{ constant.} \quad (3.8)$$

Let U be the unitary operator

$$U : L^2(\mathbb{H}) \rightarrow L^2(\mathbb{R} \times \mathbb{R}_+), \quad Uf = y^{-1}f; \quad (3.9)$$

$\mathbb{R} \times \mathbb{R}_+$ is endowed with the standard Lebesgue measure $dx dy$. Then

$$P_{\mathbf{b}} = U(-\Delta_{A^{\mathbf{b}}})U^* = (D_x - \mathbf{b}y^{-1})y^2(D_x - \mathbf{b}y^{-1}) + D_y y^2 D_y. \quad (3.10)$$

Using partial Fourier transform we get that $\text{sp}(P_{\mathbf{b}}) = \bigcup_{\xi \in \mathbb{R}} \text{sp}(P_{\mathbf{b}}(\xi))$, where

$P_{\mathbf{b}}(\xi)$ is the self-adjoint operator on $L^2(\mathbb{R}_+)$ defined by

$$P_{\mathbf{b}}(\xi)f = (y\xi - \mathbf{b})^2 f(y) + D_y(y^2 D_y f)(y); \quad \forall f \in C_0^\infty(\mathbb{R}_+). \quad (3.11)$$

Moreover we have

$$\mathrm{sp}(P_{\mathbf{b}}(\xi)) = \mathrm{sp}(P_{\mathbf{b}}(1)) , \quad \text{if } \xi > 0 .$$

$$\mathrm{sp}(P_{\mathbf{b}}(\xi)) = \mathrm{sp}(P_{\mathbf{b}}(-1)) , \quad \text{if } \xi < 0 .$$

This leads to the well-known following theorem:

Theorem 3.12 *The spectrum of $P_{\mathbf{b}}(\pm 1)$ is formed by its absolutely continuous part and its discret part, and*

$$\begin{aligned} \mathrm{sp}(P_{\mathbf{b}}(-1)) &= \mathrm{sp}_{ac}(P_{\mathbf{b}}(-1)) = \mathrm{sp}_{ac}(P_{\mathbf{b}}(1)) = [\mathbf{b}^2 + \frac{1}{4}, +\infty[\\ \mathrm{sp}(P_{\mathbf{b}}(1)) &= \mathrm{sp}_{ac}(P_{\mathbf{b}}(1)) \quad , \quad \text{if } \mathbf{b} \leq \frac{1}{2} \\ \mathrm{sp}_d(P_{\mathbf{b}}(1)) &= \{(2j+1)\mathbf{b} - j(j+1) ; j \in \mathbb{N}, j < \mathbf{b} - \frac{1}{2}\} \quad \text{if } \mathbf{b} > \frac{1}{2} . \end{aligned}$$

Corollary 3.13 *The spectrum of $-\Delta_{A^{\mathbf{b}}}$ is essential: $\mathrm{sp}(-\Delta_{A^{\mathbf{b}}}) = \mathrm{sp}_{es}(-\Delta_{A^{\mathbf{b}}})$. Its absolutely continuous part is given by $\mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) = [\mathbf{b}^2 + \frac{1}{4}, +\infty[$. The remaining part of its spectrum is empty if $0 \leq \mathbf{b} \leq 1/2$, otherwise it is formed by a finite number of eigenvalues of infinite multiplicity given by*

$$\mathrm{sp}_p(-\Delta_{A^{\mathbf{b}}}) = \{(2j+1)\mathbf{b} - j(j+1) ; j \in \mathbb{N}, j < \mathbf{b} - \frac{1}{2}\} , \quad (\text{if } \frac{1}{2} < \mathbf{b} .)$$

3.3.3 Weyl-type asymptotics (high energy)

Let us assume that

$$A_j(x, y) \in C^2(\mathbb{H}; \mathbb{R}) , \quad \forall j . \tag{3.12}$$

It is well known that $H_1(A)$ defined by (3.7) is then essentially self-adjoint on $L^2(\mathbb{H})$, see for example [55]. We assume moreover the following magnetic bottles-type assumptions.

•

$$\mathbf{b}(x, y) \rightarrow +\infty \quad \text{as} \quad d(x, y) \rightarrow +\infty , \tag{3.13}$$

($d(x, y)$ denotes the hyperbolic distance from (x, y) to the point $(0, 1)$).

- $\exists C_0 > 0$ such that, for any vector field X on $\mathbb{H}$,

$$|X\tilde{\mathbf{b}}| \leq C_0(|\tilde{\mathbf{b}}| + 1)\sqrt{g(X, X)}; \quad (3.14)$$

Theorem 3.14 *Under the assumptions (3.12), (3.13) et (3.14)*

1) *the operator $H_1(A)$ has a compact resolvent.*

2) *for any $\delta \in]\frac{1}{3}, \frac{2}{5}[$, there exists a constant $C > 0$ such that*

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{H}} \left(1 - \frac{C}{(\mathbf{b}(m) + 1)^{(2-5\delta)/2}}\right) \mathbf{b}(m) \sum_{k=0}^{+\infty} [\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0 dv \\ \leq N(\lambda, H_1(A)) \leq \end{aligned} \quad (3.15)$$

$$\frac{1}{2\pi} \int_{\mathbb{H}} \left(1 + \frac{C}{(\mathbf{b}(m) + 1)^{(2-5\delta)/2}}\right) \mathbf{b}(m) \sum_{k=0}^{+\infty} [\lambda(1 + C\lambda^{-3\delta+1}) - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0 dv$$

Remark 3.15 *Comparing this result with the one obtained in [6] and in particular with the formula (3.6), it turns out that they differ only by the additional term $-\frac{1}{4}$, which comes from the geometry of the problem. This term becomes really significant in the following corollary :*

Corollary 3.16 *Under the assumptions of Theorem 3.14 and if the function $\omega(\mu) = \int_{\mathbb{H}} [\mu - \mathbf{b}(m)]_+^0 dv$ verifies*

$$\exists C_1 > 0 \text{ s.t. } \forall \mu > C_1, \forall \tau \in]0, 1[, \quad \omega((1 + \tau)\mu) - \omega(\mu) \leq C_1 \tau \omega(\mu), \quad (3.16)$$

then

$$N(\lambda; H_1(A)) \sim \frac{1}{2\pi} \int_{\mathbb{H}} \mathbf{b}(m) \sum_{k \in \mathbb{N}} [\lambda - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0 dv. \quad (3.17)$$

The assumption (3.29) is satisfied for example when $\omega(\lambda) \sim \alpha \lambda^k \ln^j(\lambda)$ when $\lambda \rightarrow +\infty$, with $k > 0$, or $k = 0$ and $j > 0$.

For example this allows us to consider magnetic fields of the type

$$\mathbf{b}(x, y) = \left(\frac{x}{y}\right)^{2j} + g(y), \text{ with } j \in \mathbb{N}^* \text{ and } g(y) = p_1(y) + p_2(1/y),$$

where $p_1(s)$ and $p_2(s)$ are, for large s , polynomial functions of order ≥ 1 .

The function $\omega(\lambda)$ indeed verifies in that case

$\omega(\lambda) \sim \alpha \lambda^{\frac{1}{2j}} \ln(\lambda)$ when $\lambda \rightarrow +\infty$. and

$$N(\lambda; H_1(A)) \sim \frac{C}{2\pi} \lambda^{1+1/2j} \ln(\lambda).$$

In next section we give an outline of the proof of theorem 3.14, by describing the techniques specific to the hyperbolic context : definition of a diffeomorphism from $\mathbb{R}^2$ to $\mathbb{H}$, control of the magnetic field by a constant one in a suitable rectangle, partition of $\mathbb{R}^2$ into such appropriate rectangles, so that we can apply the min-max variational method.

3.3.4 Outline of the proof

- A diffeomorphism from $\mathbb{R}^2$ to $\mathbb{H}$

Let us consider the diffeomorphism

$$\begin{aligned} \phi : \mathbb{R}^2 &\rightarrow \mathbb{H} \\ (x, y) = \phi(x, t) &:= (x, e^t) \\ \text{which induces a unitary operator} \\ \widehat{U} : L^2(\mathbb{H}; dv) &\rightarrow L^2(\mathbb{R}^2; dxdt) \\ (\widehat{U}f)(x, t) &= e^{-t/2} f(x, e^t) \text{ for any } f \in L^2(\mathbb{H}). \end{aligned}$$

The quadratic form associated to $H_1(A)$ is given by ($\forall u \in L^2(\mathbb{H})$)

$$q(u) = \int_{\mathbb{H}} [|y(D_x - A_1)u|^2 + |y(D_y - A_2)u|^2] \frac{dxdy}{y^2}.$$

Writing $\tilde{A}_i(x, t) = A_i(x, e^t)$, $i = 1, 2$, and $w = \widehat{U}u$, we get after computation

$$q(u) = \widehat{q}^{\tilde{A}}(w) = \int_{\mathbb{R}^2} [|e^t(D_x - \tilde{A}_1)w|^2 + |(e^{-t/2}D_te^{t/2} - e^t\tilde{A}_2)w|^2] dxdt.$$

The operator associated to $\widehat{q}^{\tilde{A}}$ is $\widehat{H}(\tilde{A}) = \widehat{U}H_1(A)\widehat{U}^{-1}$.

This gives $\widehat{H}(\tilde{A}) = e^{2t}(D_x - \tilde{A}_1)^2 + (D_t - e^t\tilde{A}_2)^2 + 1/4$.

The additional term $1/4$ appears here naturally as a by-product of the transformation which allows us to deal with a problem in $\mathbb{R}^2$ instead of the initial problem in $\mathbb{H}$.

- Gauge

We want to work with a gauge such that $A_2 = 0$. Since

$$\tilde{\mathbf{b}} = y^2 (\partial_x A_2 - \partial_y A_1)$$

one can take

$$A_1(x, y) = - \int_1^y \frac{\tilde{\mathbf{b}}(x, s)}{s^2} ds$$

The associated quadratic form is

$$\hat{q}^{\tilde{A}}(w) = \int_{\mathbb{R}^2} \left[|e^t(D_x - \tilde{A}_1)w|^2 + |D_t w|^2 + 1/4|w|^2 \right] dx dt .$$

- Localization

According to the assumption (3.14) we can control the magnetic field by a constant one on an appropriate rectangle :

We set $\Omega(x_0, y_0, a, \varepsilon_0) := \{(x, y) / |x - x_0| \leq a\varepsilon_0, |y - y_0| \leq \varepsilon_0 y_0\}$.
($a > 0$ and $\varepsilon_0 > 0$ small enough)

Lemma 3.17 *There exists $C_1 > 0$ such that, for any $(x_0, y_0) \in \mathbb{H}$ with $\mathbf{b}(x_0, y_0) > 1$, the following holds*

$$\frac{1}{C_1} \mathbf{b}(x_0, y_0) \leq \mathbf{b}(x, y) \leq C_1 \mathbf{b}(x_0, y_0) \quad \forall (x, y) \in \Omega(x_0, y_0, a, \varepsilon_0).$$

- Partition of $\mathbb{R}^2$

For any $\alpha \in \mathbb{Z}^2$, let us denote by $K(\alpha)$ the rectangle

$$K(\alpha) =] - \frac{e^{\alpha_2}}{2} + e^{\alpha_2} \alpha_1, e^{\alpha_2} \alpha_1 + \frac{e^{\alpha_2}}{2} [\times] - \frac{1}{2} + \alpha_2, \alpha_2 + \frac{1}{2} [. \quad (3.18)$$

Therefore $\mathbb{R}^2 = \cup_{\alpha} \overline{K}(\alpha)$ and $K(\alpha) \cap K(\beta) = \emptyset$ for any $\alpha \neq \beta$. According to lemma 3.17, it is possible to subdivide each $K(\alpha)$, (if necessary), into $M(\alpha)$ rectangles:

$$\overline{K}(\alpha) = \cup_{j=1}^{M(\alpha)} \overline{K}_{\alpha,j} \quad (3.19)$$

$$K_{\alpha,j} =] - \frac{\epsilon_{\alpha,j} e^{t_{\alpha,j}}}{2} + x_{\alpha,j}, x_{\alpha,j} + \frac{\epsilon_{\alpha,j} e^{t_{\alpha,j}}}{2} [\times] - \frac{\epsilon_{\alpha,j}}{2} + t_{\alpha,j}, t_{\alpha,j} + \frac{\epsilon_{\alpha,j}}{2} [,$$

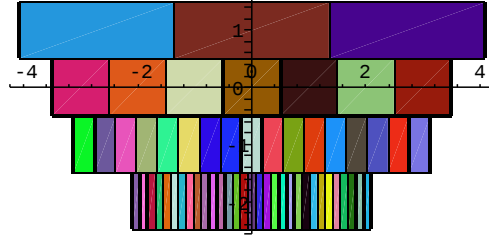


Figure 2: Partition of $\mathbb{R}^2$ by the rectangles $\overline{K}(\alpha)$

with

$$\frac{1}{a_0(1 + \mathbf{b}^{\delta_0}(x_{\alpha,j}, e^{t_{\alpha,j}}))} \leq \epsilon_{\alpha,j} \leq \frac{a_0}{(1 + \mathbf{b}^{\delta_0}(x_{\alpha,j}, e^{t_{\alpha,j}}))}, \quad (3.20)$$

and such that $K_{\alpha,k} \cap K_{\alpha,j} = \emptyset$ if $k \neq j$.

This lemma is the hyperbolic version of the lemma 3.11 for the euclidean case.

The partition $\mathbb{R}^2 = \cup_{\alpha} \overline{K}(\alpha)$ and the partition on $\mathbb{H}$ obtained after applying the diffeomorphism ϕ are represented on figures 2 and 3.

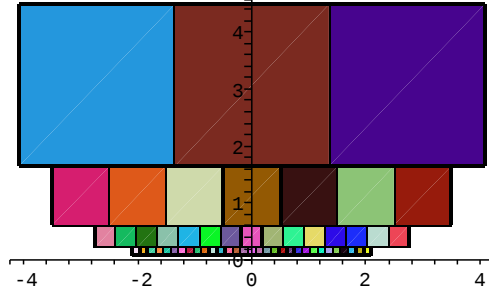


Figure 3: Partition of $\mathbb{H}$ by the cubes $\phi(\overline{K}(\alpha))$

3.4 Geometrically finite hyperbolic surfaces

3.4.1 Introduction

Concerning magnetic bottles in hyperbolic geometry the min-max method can be generalized to the geometrically finite hyperbolic surfaces, in the case when these manifolds are of infinite volume. Such manifolds contain cusps and funnels [43].

Actually, when the hyperbolic manifolds are compact the result is given by [7] in the more general context of compact Riemannian manifolds, where it is shown that the Weyl asymptotics hold. For the case of non compact manifolds of finite volume we refer to [20], where the authors study examples

for which the Weyl formula is still valid : $N(\lambda) \sim_{+\infty} \frac{\lambda}{4\pi} |\mathbf{M}|$.

It seems to be the standard result in this context.

In the case of the Poincaré half-plane, $\mathbf{M} = \mathbb{H}$, we have seen previously that the Weyl formula does not hold : $\lim_{\lambda \rightarrow +\infty} \lambda^{-1} N(\lambda) = +\infty$.

For example when $\mathbf{b}(z) = a_0^2(x/y)^{2m_0} + a_1^2 y^{m_1} + a_2^2/y^{m_2}$, $a_j > 0$ and $m_j \in \mathbb{N}^*$,

then

$$N(\lambda) \sim_{+\infty} \lambda^{1+1/(2m_0)} \ln(\lambda) \alpha(m_0, m_1, m_2) .$$

It turns out [43] that it is still the case when $\mathbf{M}$ has an infinite area and is geometrically finite, and if we adapt the preceding example to this new situation, i.e. m_0 is absent, m_1 appears in the cusps and m_2 in the funnels, we get

$$N(\lambda) \sim_{+\infty} \lambda^{1+1/m_2} \alpha(m_2) :$$

The interesting point is that the cusps do not contribute to the leading part of $N(\lambda)$.

Let us explain the result, and first what is such a surface.

3.4.2 Definition

If $(\mathbf{M}, g)$ is a smooth connected Riemannian manifold of dimension two, it is called a geometrically finite hyperbolic surface of infinite area if it can be decomposed in the following way :

$$\mathbf{M} = \left(\bigcup_{j=0}^{J_1} M_j \right) \cup \left(\bigcup_{k=1}^{J_2} F_k \right) ; \quad (3.21)$$

where the M_j and the F_k are open sets of $\mathbf{M}$, such that the closure of M_0 is compact, and if $J_1 > 0$, the other M_j are cuspidal ends of $\mathbf{M}$, and the F_k are funnel ends of $\mathbf{M}$.

This means that, for any j , $1 \leq j \leq J_1$, there exist strictly positive constants a_j and L_j such that M_j is isometric to $\mathbb{S} \times]a_j^2, +\infty[$, equipped with the metric

$$ds_j^2 = y^{-2} (L_j^2 d\theta^2 + dy^2) ; \quad (3.22)$$

($\mathbb{S} = \mathbb{S}^1$ is the unit circle.)

In the same way, for any k , $1 \leq k \leq J_2$, there exist strictly positive constants α_k and τ_k such that F_k is isometric to $\mathbb{S} \times]\alpha_k^2, +\infty[$, equipped with the metric

$$ds_k^2 = \tau_k^2 \cosh^2(t) d\theta^2 + dt^2 . \quad (3.23)$$

Moreover, for any two integers $j, k > 0$, we have $M_j \cap F_k = \emptyset$ and $M_j \cap M_k = F_j \cap F_k = \emptyset$ if $j \neq k$.

3.4.3 Assumptions on the magnetic field

Let us choose some $z_0 \in M_0$ and let us define

$$d : \mathbf{M} \rightarrow \mathbb{R}_+ ; \quad d(z) = d_g(z, z_0) ; \quad (3.24)$$

$d_g(\cdot, \cdot)$ denotes the distance with respect to the metric g .

We assume the smooth one-form A to be given such that the magnetic field $\tilde{\mathbf{b}}$ satisfies

$$\lim_{d(z) \rightarrow \infty} \mathbf{b}(z) = +\infty . \quad (3.25)$$

If $J_1 > 0$, there exists a constant $C_1 > 0$ such

$$|X\tilde{\mathbf{b}}(z)| \leq C_1(\mathbf{b}(z) + 1)e^{d(z)}|X|_g ; \quad (3.26)$$

$$\forall z \in M_j , \forall X \in T_z\mathbf{M} \text{ and } \forall j = 1, \dots, J_1 .$$

There exists a constant $C_2 > 0$ such

$$|X\tilde{\mathbf{b}}(z)| \leq C_2(\mathbf{b}(z) + 1)|X|_g ; \quad (3.27)$$

$$\forall z \in F_k , \forall X \in T_z\mathbf{M} \text{ and } \forall k = 1, \dots, J_2 .$$

3.4.4 Asymptotics for large energies

Theorem 3.18 *Under the above assumptions, $-\Delta_A$ has a compact resolvent and for any $\delta \in]\frac{1}{3}, \frac{2}{5}[$, there exists a constant $C > 0$ such that*

$$\begin{aligned} & \frac{1}{2\pi} \int_{\mathbf{M}} \left(1 - \frac{C}{(\mathbf{b}(m) + 1)^{(2-5\delta)/2}}\right) \mathcal{N}(\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m)) \, dm \\ & \leq N(\lambda, -\Delta_A) \leq \\ & \frac{1}{2\pi} \int_{\mathbf{M}} \left(1 + \frac{C}{(\mathbf{b}(m) + 1)^{(2-5\delta)/2}}\right) \mathcal{N}(\lambda(1 + C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m)) \, dm \end{aligned} \quad (3.28)$$

where

$$\mathcal{N}(\mu, \mathbf{b}(m)) = \mathbf{b}(m) \sum_{k=0}^{+\infty} [\mu - (2k+1)\mathbf{b}(m)]_+^0 \quad \text{if } \mathbf{b}(m) > 0 ,$$

and

$$\mathcal{N}(\mu, \mathbf{b}(m)) = \mu/2 \quad \text{if } \mathbf{b}(m) = 0 .$$

$[\rho]_+^0$ is the Heaviside function:

$$[\rho]_+^0 = \begin{cases} 1, & \text{if } \rho > 0 \\ 0, & \text{if } \rho \leq 0. \end{cases}$$

The Theorem remains true if we replace $\int_{\mathbf{M}}$ by $\sum_{k=1}^{J_2} \int_{F_k}$, due to the fact that the other parts are bounded by $C\lambda$.

Corollary 3.19 *Under the assumptions of Theorem 3.18 and if the function*

$$\omega(\mu) = \int_{\mathbf{M}} [\mu - \mathbf{b}(m)]_+^0 dm$$

satisfies, $\exists C_1 > 0$ s.t. $\forall \mu > C_1, \forall \tau \in]0, 1[$,

$$\omega((1 + \tau)\mu) - \omega(\mu) \leq C_1 \tau \omega(\mu), \quad (3.29)$$

then

$$N(\lambda; -\Delta_A) \sim \frac{1}{2\pi} \int_{\mathbf{M}} \mathcal{N}(\lambda - \frac{1}{4}, \mathbf{b}(m)) dm. \quad (3.30)$$

For example this allows us to consider magnetic fields of the following type:

$$\begin{aligned} & \text{on } F_k, \quad \mathbf{b}(\theta, t) = p_k(1/\cosh(t)), \\ & \text{and on } M_j, \quad j > 0, \quad \mathbf{b}(\theta, y) = q_j(y), \end{aligned}$$

where the $p_k(s)$ and the $q_j(s)$ are, for large s , polynomial functions of order ≥ 1 . In this case, if d is the largest order of the $p_k(s)$, then

$$N(\lambda; -\Delta_A) \sim \alpha \lambda^{1+1/d},$$

for some constant $\alpha > 0$, depending only on the funnels F_k where the order of $p_k(s)$ is d .

4 A Neumann problem with magnetic field

4.1 A problem arising from super-conductivity

In super-conductivity theory the following question has to be answered : can we minimize, for a given open set Ω of $\mathbb{R}^d$, and a given potential $A \in H^1(\Omega; \mathbb{R}^d)$, the Ginzburg-Landau functional :

$$\begin{aligned} \mathcal{G}(\psi, \tilde{A}) = & \int_{\Omega} |(\nabla - i\kappa\tilde{A})\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2 dx \dots \\ & \dots + \kappa^2 \int_{\Omega} |\overrightarrow{rot}\tilde{A} - \sigma\overrightarrow{rot}A|^2 dx \end{aligned}$$

on the set $(\psi, \tilde{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^d)$. κ and σ denote parameters related to the intensity of the magnetic field, and the dimension d considered is either 2 or 3. $(0, \sigma A)$ turns out to be a trivial critical point of $\mathcal{G}(\psi, \tilde{A})$; in order to study the Hessian matrix of the functional at this point we are then reduced to investigate the spectral properties, (modulo the parameters κ et σ) of the magnetic Laplacian $H_h = ((h\nabla - iA))^2$, with the Neumann-type boundary conditions :

$$\nu(x) \cdot (h\nabla - iA)u(x) = 0 \quad \forall x \in \delta\Omega .$$

A lot has been done to understand the properties of this operator . In particular it is known from the works by K.Lu et X.B.Pan [34]-[36] and by B.Helffer et A.Morame [22]-[24] that unlike the Dirichlet case, the lower bound of its spectrum can be less than $hb = h \inf_{\Omega} \|B\|$ if $B = dA$ denotes the magnetic field. This comes from the following fundamental fact, which makes the basic difference between Dirichlet and Neumann problems :

If we consider the Neumann operator on $L^2(\mathbb{R}_+)$ defined by $Q_x = D_t^2 + (t - x)^2$, and if we denote by $\mu(x)$ its first eigenvalue, then

$$\inf_{x \in \mathbb{R}} \mu(x) = \mu(x_0) = \Theta_0 < 1 .$$

If we consider the Dirichlet operator the corresponding quantity is equal to 1.

As a consequence, in the case where $b \geq \Theta_0 b'$, (with $b' = \inf_{\delta\Omega} \|B\|$), we get that the lower bound of the spectrum for the Neumann problem is $h\Theta_0 b'$.

Furthermore, for a constant non zero B , any normalized fundamental eigenfunction is localized exponentially (for h going to 0) in the neighbourhood of points of the boundary with maximal curvature (see [22] for dimension 2 and [23] for dimension 3). Superconductivity comes precisely from this crucial feature. We should also mention the papers by B.Helffer et S.Fournais, in particular [16] for asymptotic estimates of low eigenvalues in dimension 2, and [17] about the “critical fields”, which are responsible for the transitions from superconducting states to normal states. This list of papers on the subject is far from being exhaustive...

4.2 The spectrum in the case of the half-space, for a constant field and for $h=1$

Let us consider, for (t, x, y) in $\Omega = \mathbb{R}_+ \times \mathbb{R}^2$, the Neumann realization of the magnetic Laplacian

$$H = (D_t - A_1)^2 + (D_x - A_2)^2 + (D_y - A_3)^2$$

where $D_s = -i(\frac{\partial}{\partial s})$.

Let us denote by b the norm of B , and by θ the angle between $B = dA$, seen as a 3-dimensional vector field, and the boundary $\partial\Omega$.

This implies that a suitable choice for the gauge A is the 1-form

$$A = b(x \sin \theta - t \cos \theta)dy$$

so that the operator H can be written as

$$H_\theta^b = D_t^2 + D_x^2 + (D_y - b(x \sin \theta - t \cos \theta))^2 .$$

By homogeneity we get :

$$\sigma(H_\theta^b) = b\sigma(H_\theta)$$

with

$$H_\theta = D_t^2 + D_x^2 + (D_y - (x \sin \theta - t \cos \theta))^2 .$$

- $\theta = 0$

The spectrum of the Neumann operator H_0 is absolutely continuous. More precisely one has :

$$\sigma(H_0) = \sigma_{ac}(H_0) = [b\theta_0, +\infty[. \quad (4.1)$$

- $\theta = \frac{\pi}{2}$

The spectrum of $H_{\frac{\pi}{2}}$ is still absolutely continuous but

$$\sigma(H_{\frac{\pi}{2}}) = \sigma_{ac} = [b, +\infty[. \quad (4.2)$$

- $\theta \in]0, \frac{\pi}{2}[$

The spectrum of H_θ is no longer absolutely continuous as proved by K. Lu and X-B. Pan [35], (see also [23]).

We prove in [41] the following :

Theorem 4.1

$$\text{If } \theta \in]0, \frac{\pi}{2}[, \quad \sigma(H_\theta^b) \cap]-\infty, b[= \{b\nu_1(\theta), b\nu_2(\theta), \dots, b\nu_j(\theta), b\nu_{j+1}(\theta), \dots\}. \quad (4.3)$$

(Each $b\nu_j(\theta)$ is an eigenvalue of infinite multiplicity of H_θ).

To prove this result we first observe that $\sigma(H_\theta) = \bigcup_{\tau \in \mathbb{R}} \sigma(H_{\theta, \tau})$, where $H_{\theta, \tau}$ denotes the Neumann realization in the half plane $F = \mathbb{R}_+ \times \mathbb{R}$ of the operator $H_{\theta, \tau} = D_t^2 + D_x^2 + (\tau - (x \sin \theta - t \cos \theta))^2$.

Furthermore using for any τ the change of coordinates $x \rightarrow x - \frac{\tau}{b \sin \theta}$, we see that $\sigma(H_{\theta, \tau}) = \sigma(P_\theta)$, with

$$P_\theta = D_t^2 + D_x^2 + (t \cos \theta - x \sin \theta)^2 ,$$

and thus the spectrum of H_θ is essential and given by :

$$\sigma(H_\theta) = \sigma_{ess}(H_\theta) = \sigma(P_\theta). \quad (4.4)$$

In [35], (see also [23]), it was proved that

$$\inf \sigma(P_\theta) = \nu(\theta) < 1 = \inf \sigma_{ess}(P_\theta), \quad (4.5)$$

so there exists a countable set of eigenvalues $(\nu_j(\theta))_{j \in I}$, ($I \subset \mathbb{N}$), contained in $[\nu(\theta), 1[$.

4.3 Weyl-type asymptotics when the field is nearly tangent to the boundary

For any $d \leq 1$ let us denote by $N(d, P_\theta)$ the number of eigenvalues of P_θ in $] - \infty, d[$:

$$N(d, P_\theta) = \text{Tr}(E_{]-\infty, d[}(P_\theta)) = \#\{j; \nu_j(\theta) < d\} . \quad (4.6)$$

We prove the following results [41] :

Theorem 4.2 *For any $\theta \in]0, \frac{\pi}{2}[$, P_θ admits a finite number of eigenvalues in $] - \infty, 1[$, and there exists a constant $C \geq 1$ such that*

$$N(1, P_\theta) \leq \frac{C}{\sin \theta} . \quad (4.7)$$

It can be noticed that the upper bound of $N(1, P_\theta)$ goes to infinity when the angle θ between the magnetic field and the boundary tends to zero. Therefore we can consider θ (or more precisely $\frac{\sin \theta}{\sqrt{\cos \theta}}$) as a semi-classical parameter, and using once more min-max techniques we give a Weyl-type asymptotic estimate of $N(d, P_\theta)$ for $d < 1$ [41] :

Theorem 4.3 *If $d \in]\Theta_0, 1[$, there exists a constant $C_d > 0$ such that*

$$\left| N(d, P_\theta) - \frac{1}{2\pi \sin \theta} \int_{\mathbb{R}} [d - \mu(x)]_+^{1/2} dx \right| \leq C_d . \quad (4.8)$$

In this expression clearly appears an "effective" potential, as in the previous case of degenerate potentials. This is not so surprising, since the operator we finally study is

$$P_\theta = D_t^2 + D_x^2 + (t \cos \theta - x \sin \theta)^2 ,$$

which turns out to be a Schrödinger operator with the degenerate potential $V_\theta(x, t) = (t \cos \theta - x \sin \theta)^2$. The "effective" potential here is the function of one variable $\mu(x)$ previously introduced, which is responsible for the superconductivity.

5 A problem of magnetic bottle in classical mechanics

5.1 The Lorentz equation

The motion in $\mathbb{R}^3$ of a particle of mass m and charge e in a magnetic field $\vec{B}$ can be described by Lorentz equation :

$$m\ddot{x} = e\dot{x} \wedge \vec{B} .$$

To this equation corresponds the following Lagrangian (for $m = e = 1$) :

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2} \dot{x}^2 + \dot{x} \cdot A(x) , \quad (5.1)$$

where A denotes a magnetic potential : $\overrightarrow{rot}A = \vec{B}$. To obtain the associated Hamiltonian , we compute the conjugate momenta

$$\xi_j = \frac{\partial \mathcal{L}}{\partial \dot{x}_j} \quad (5.2)$$

which writes $\xi = \dot{x} + A(x)$ so we get :

$$\mathcal{H}(x, \xi) = \xi \dot{x} - \mathcal{L}(x, \dot{x}) = \frac{1}{2}(\xi - A(x))^2 . \quad (5.3)$$

When $\vec{B}$ is a constant field (in time and position) the Hamiltonian is integrable, the trajectories are helicoidal and the axis of the motion is the direction of the field. There are three numbers conserved during the motion (the integrals of motion) which are the energy (e.g. the Hamiltonian itself), the Larmor radius $\rho = \frac{v_{\perp}}{B}$ and the magnetic moment $I = \frac{v_{\perp}^2}{2B}$. We denote respectively by $v_{\perp}$ the orthogonal component (to field lines) of the velocity and by B the norm of $\vec{B}$.

Remark 5.1 *Applying Weyl quantification to $\mathcal{H}(x, \xi)$ we obtain the Schrödinger operator defined on $L^2(\mathbb{R}^3)$ by*

$$H_h(A) = \sum_{j=1}^3 \left(\frac{h}{i} \frac{\partial}{\partial x_j} - A(x_j) \right)^2 .$$

In the case when $\vec{B}$ is a constant vector field the spectrum of the operator $H_h(A)$ is composed of eigenvalues of infinite multiplicity, which are the Landau levels $\lambda_j(h) = h(2j+1)B$.

5.2 Adiabatic invariants

Let us consider a magnetic field slowly varying in position so that it is almost constant throughout a whole rotation of the particle : the motion is approximately a circle and the center of this circle (the guiding center) is slowly moving along the direction of the field, with a very small rotation period.

Under the previous conditions the Hamiltonian slowly depends on the position variables, except for one (denoted by x_0); it can be written as $\mathcal{H} = \mathcal{H}(x, \epsilon x_0, \xi, \xi_0)$, where ϵ is a small parameter.

Let us assume that, for the Hamiltonian $\mathcal{H}_0(x, \xi)$ obtained by fixing the value of ϵx_0 et ξ_0 , there exist closed trajectories in phase space (with a non vanishing frequency). Then one can introduce the action-angle variables (I, ϕ) . The action variable $I((x, \epsilon x_0, \xi, \xi_0))$ corresponds to the magnetic moment ; it is not an integral of motion as it is in the constant case, but it turns out from the method of moyennisation [1] that it is an adiabatic invariant, which means more precisely :

$$\exists c > 0 \text{ t.q. } |I((x(t), \epsilon x_0(t), \xi(t), \xi_0(t))) - I((x(0), \epsilon x_0(0), \xi(0), \xi_0(0)))| < c \epsilon$$

for $0 \leq t \leq 1/\epsilon$.

Performing symplectic transformations one can get the invariance of I to all orders ([32],[46]). Furthermore, if the magnetic field is a convex function along the field lines (seen as a function of the arc length s) there exists another invariant, which is longitudinal and given by $J = \oint \frac{v_{\parallel}^2}{B} ds$. The trajectory is actually reflected at the points s_1, s_2 verifying $IB(s_i) = \mathcal{H}$, and the integral is computed on a whole oscillation [48].

M.Gardner [18] proves the invariance of this quantity to all orders.

V.I.Arnold [2] considers the following question : is it possible to get the particles not only adiabatically but really confined? Considering a magnetic field with symmetry axis he writes the corresponding Hamiltonian, with two degrees of freedom, as a perturbation of an integrable one, for which the motion in phase space is on a torus.

Under a non-degeneracy condition (the ratio of frequencies varies in time) he applies KAM theorem to get that the invariant tori are not all destroyed under the perturbation and he concludes by a dimension argument that the action I is a perpetually adiabatic invariant.

The difficult point here is to check the non-degeneracy condition, according to the fact that the ratio of frequencies is of small order. Arnold

checks that condition only in the special case when the Hamiltonian writes $\mathcal{H} = \frac{1}{2}(\xi_1^2 + \xi_0^2) + U(x_1, x_0)$ with : $U(x_1, x_0) = \frac{1}{2}x_1^2(1 + \epsilon^2 x_0^2)$.

Another method consists in applying a theorem of J.Moser [44] which gives the existence of periodic solutions for systems next to an integrable one. M.Braun [5] proves by this way the existence of a region where the particles submitted to the earth magnetic field are indefinitely retained.

In [59] we apply Moser's theorem in the case of a magnetic field which is linear and symmetric in position

$$\vec{B}(x, y, z) = (x, y, -2z) , \quad (5.4)$$

and we get an open set of initial conditions for which the motion remains bounded.

This field is actually of the "magnetic bottle " type, since its norm tends to infinity as position tends to infinity. As already mentioned, the operator obtained from the Hamiltonian by Weyl's quantification has a compact resolvent . The result of quantum mechanics seems to be stronger compared with the classical one. This is a well-known fact that it is harder to get informations on the classical motion than on the spectrum of the quantum operator...

5.3 Bounded trajectories

5.3.1 The Hamiltonian in cylindrical coordinates

According to the symmetry of the magnetic field (5.4), we introduce cylindrical coordinates. Considering B as a 2-form we have $B = d(-r^2 z) \wedge d\theta$, so we can choose the following gauge $A = (A_r, A_\theta, A_z) = (0, -rz, 0)$. The field lines are characterized by $\theta = \text{constant}$ et $r^2 z = \text{constant}$. The conjugate moments, (defined by (5.2)) have the following expression

$$(\xi_r, \xi_\theta, \xi_z) = (\dot{r}, r^2(\dot{\theta} - z), \dot{z}) ,$$

so we get, using formula (5.3)

$$\mathcal{H}(r, \theta, z, \xi_r, \xi_\theta, \xi_z) = \frac{1}{2}(\xi_r^2 + \xi_z^2) + \frac{1}{2r^2}(\xi_\theta + r^2 z)^2 . \quad (5.5)$$

Writing the second equation of Hamilton

$$\dot{\xi}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} \quad (5.6)$$

one gets the existence of an integral of motion, which is ξ_θ .

Dimension has been reduced according to the symmetry of the problem. We have to study a Hamiltonian with 2 degrees of freedom which is defined as follows :

$$\mathcal{H}_M(r, z, \xi_r, \xi_z) = \frac{1}{2}(\xi_r^2 + \xi_z^2) + \frac{(M + r^2 z)^2}{2r^2} . \quad (5.7)$$

5.3.2 The reduced Hamiltonian and the magnetic field lines

The value of $M = \xi_\theta$ is fixed by the initial conditions, so is the energy E (E is the value taken by the Hamiltonian, and it is a constant of motion too). M may be negative. let us consider the magnetic field line (L_M) defined by $r^2 z = -M$. There exists a unique point Ω_M such that the norm B at that point is minimal on (L_M) . Thus, to any point $P(r, z)$ we associate new coordinates (u, v) as follows :

- v is the distance from $P(r, z)$ to the magnetic field line (L_M)
- u is the arc length between the projection of P on (L_M) to Ω_M .

We denote by $k(u)$ the curvature of (L_M) at the point $(u, 0)$ and we set :

$$\mathcal{H}_0(u, v, \xi_u, \xi_v) = \frac{1}{2} \left(\frac{\xi_u^2}{[1 + vk(u)]^2} + \xi_v^2 + B^2(u, 0)v^2 \right) . \quad (5.8)$$

According to the new coordinates the Hamiltonian $\mathcal{H}_M(u, v, \xi_u, \xi_v)$ obtained from the expression (5.7) can be reexpressed as a perturbation of $\mathcal{H}_0(u, v, \xi_u, \xi_v)$ in the following way :

Proposition 5.2 *For any initial conditions satisfying*

$$E < 2|M|^{4/3} \quad \text{et} \quad E < \frac{\epsilon^2}{16}|M|^{2/3} \quad (\text{CI})$$

- *the distance v from the trajectory to the magnetic field line (L_M) is less than ϵ*
- *there exists a constant C (depending only on E and M) such that*

$$|\mathcal{H}_M(u, v, \xi_u, \xi_v) - \mathcal{H}_0(u, v, \xi_u, \xi_v)| < C\epsilon^3$$

Remarks

1) The first condition is a consequence of the second one for ϵ small enough.

2) The proposition comes from the fact that according to the inequality $\frac{(M+r^2z)^2}{2r^2} < E$ the motion has to remain inside a strip $\mathcal{B}$ around (L_M) (see figure 4).

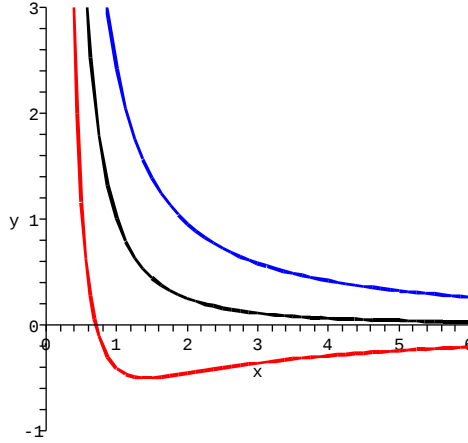


Figure 4: The strip $\mathcal{B}$ and the magnetic field line (L_M)

3) Since v is small the original Hamiltonian can be seen indeed as a perturbation of the Hamiltonian $\frac{1}{2} (\xi_u^2 + \xi_v^2 + B^2(u, 0)v^2)$ which corresponds quantically to an operator with a degenerate potential, precisely of the form described in section 2.2.

4) The Hamiltonian $\mathcal{H}_u = \frac{1}{2} (\xi_v^2 + B^2(u, 0)v^2)$ represents for a fixed value of u the energy of a harmonic oscillator. The point (v, ξ_v) moves along the ellipse $\mathcal{H}_u = \text{constant}$ for some constant depending on u and the corresponding action, which is (up to a factor 2π) the area enclosed by the ellipse, has the following expression

$$I_u = \frac{\xi_v^2 + B^2(u, 0)v^2}{2 B(u, 0)} .$$

5) Conditions (CI) entail the following inequality

$$B(u, 0)I_u < \frac{\epsilon^2}{16}|M|^{2/3} + C\epsilon^3 .$$

As a consequence, if I_u is bounded from below by a constant independent of time, we get an upper bound for $B(u, 0)$ and hence for the trajectory itself since $B(u, 0)$ is an increasing function of u .

5.3.3 Action variables

We set

$$u = \epsilon u_1, \quad v = \epsilon v_1, \quad \xi_u = \epsilon \xi_{u_1}, \quad \xi_v = \epsilon \xi_{v_1}, \quad \mathcal{H}_M = \epsilon^2 \mathcal{K}_M, \quad E = \epsilon^2 E' .$$

We perform some symplectic transformations (in the language of mechanics, we perform some changes of canonical variables) in order to get explicit action-angle coordinates (I, J, ϕ, ψ) such that

Theorem 5.3

$$\mathcal{K}_M(u_1, v_1, \xi_{u_1}, \xi_{v_1}) = \mathcal{K}_M(I, J, \phi, \psi) = E' + B(\epsilon u_1, 0) [I + c(\epsilon J) + \mathcal{O}(\epsilon)] , \quad (5.9)$$

where the second derivative of the function $c(\epsilon J)$ does not vanish on an interval of the type $]A, +\infty[$.

The action variable I is, up to a multiplicative factor ϵ^{-2} , the variable I_u we defined previously. For the points of the motion situated on the magnetic field line, we recognize the magnetic moment $I = \frac{v_\perp^2}{2B}$ mentioned in the introduction. The second action variable is a function $J(c)$, which represents the area enclosed by the curve $\mathcal{C}_c$ defined as follows : $\frac{1}{2} \xi_{u_1}^2 - c B(\epsilon u_1, 0) = E'$. If we denote by B_M the minimal value of B on (L_M) , $\mathcal{C}_c$ is a closed curve for $c \in] - \frac{E'}{B_M}, 0[$ (see Figures 5 , 6).

We have

$$J(c) = \oint_{\mathcal{C}_c} \xi_{u_1} du_1 = \epsilon^{-1} \int_{u_{\min}}^{u_{\max}} \sqrt{2[E' + cB(u, 0)]} du$$

. To obtain (5.9) it remains to check that $(\epsilon J)(c)$ is an increasing function on $] - \frac{E'}{B_M}, 0[$, and that its derivative is also increasing on an interval of the type $]a, 0[$. This is due to the asymptotic behaviour of $B(u)$ at infinity. On figure 6 we set $E' = B_M = 1$; it can be seen that the area enclosed by the curve $\mathcal{C}_c$ is increasing as c grows from $-0,9$ to $-0,1$.

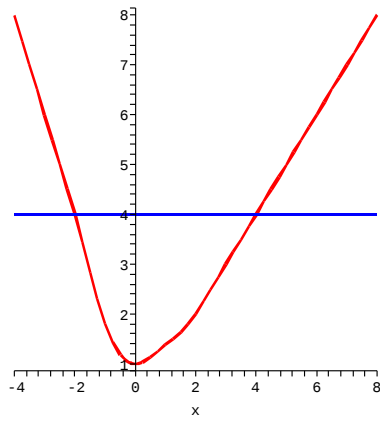


Figure 5: The function $B(u)$

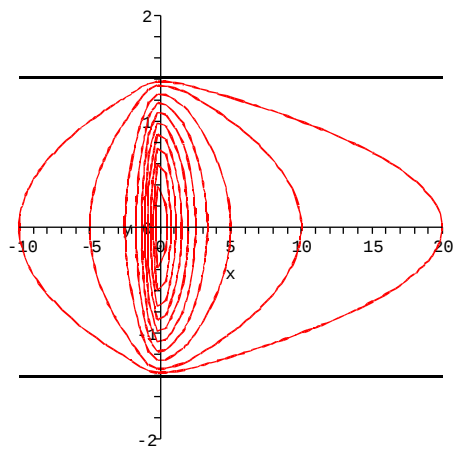


Figure 6: Curves $\mathcal{C}_c$

The theorem (5.3) entails that $c''(\epsilon J)$ vanishes only on a finite number of values $J_1, \dots, J_p$. It is possible then to apply Moser's theorem on each annulus of the type $\mathcal{A}(\epsilon J_k, \epsilon J_{k+1})$ and on each annulus exterior to the circles $J = J_1$ et $J = J_p$.

On such an annulus the action J can be expressed as a function of the new time ϕ . One shows that the diffeomorphism of the annulus $(J(0), \psi(0)) \rightarrow (J(2\pi), \psi(2\pi))$ verifies Moser's condition (see [44], [45]). In fact this condition expresses the fact that $c''(\epsilon J)$ does not vanish. Let us notice that Moser's condition can be shown to be equivalent to the weak non-degeneracy condition introduced by Arnold in [2]. The interesting fact is that it is possible to check it explicitly in this setup.

Therefore we obtain the existence of an infinite number of curves which are invariant by this diffeomorphism. The curves generate invariant tori ; they foliate the surface of energy $\mathcal{H} = E$ so that any trajectory starting between two tori remains between those tori. Consequently the quantity I_u is a perpetual adiabatic invariant.

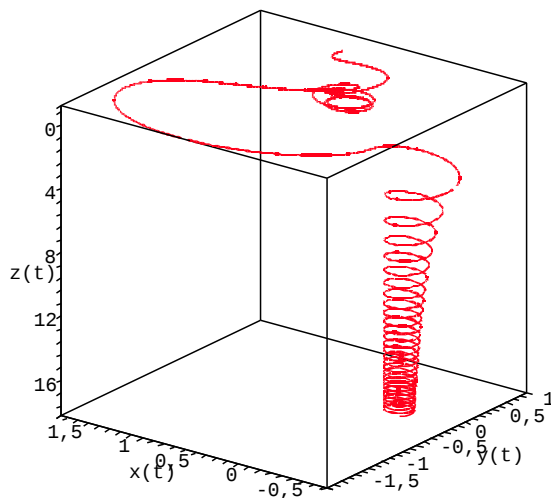


Figure 7: A trajectory

According to Remark 5 of the previous section we get then the following result :

Theorem 5.4 *Let M be fixed. There exist $\epsilon_0 > 0$ and $K > 0$ such that the trajectory of the solution is bounded, provided the following conditions are fulfilled*

$$\begin{aligned} (CI) \quad E &< \frac{\epsilon^2}{16} |M|^{2/3} \\ (CI)' \quad I_u(0) &> 2K\epsilon^3 \end{aligned}$$

for an $\epsilon < \epsilon_0$.

The conditions (CI) et $(CI)'$ are compatible. They express the fact that the velocities have to be small compared with positions whereas the component of the velocity normal to the magnetic field line has to be not too small compared with the total energy. In particular this excludes the case $z(0) = \dot{z}(0) = \dot{\theta}(0) = 0$, which corresponds to a particle starting on the magnetic field line (L_0) (defined by $\theta = \theta(0), z = 0$) with a velocity parallel to this line. Obviously in that case the motion is not bounded since the particle rolls up along the line (L_0) , which can be seen as the bottom of the well for the degenerate potential $V(u, v) = B(u, 0) v^2$.

The figures 7 and 8 represent the motion of a particle computed by numerical simulation. The initial conditions are $M_0(1, 1, 0)$ and $V_0(0, 15; -0, 25; 0, 25)$. When the particle goes away from the origin, the strip $\mathcal{B}$ in which it is contained (considering a vertical section) clearly appears. The numerical simulation seems to suggest that the motion downwards is bounded : the radius of the helix decreases but the motion is stabilized and reflected and the particle is returning towards the origin to an other helicoidal-like motion .

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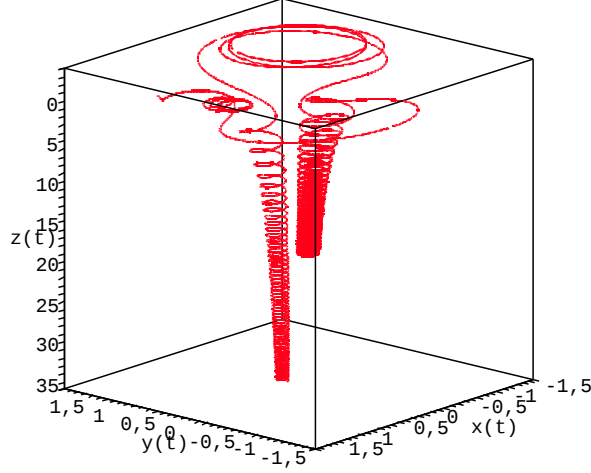


Figure 8: The same trajectory, continued

6 Open problems and conclusion

- The last work presented gives of course only a partial answer. The symmetry argument is crucial, because the invariant tori generated by the unperturbed Hamiltonian prevent the particle to go away. In the non symmetric case the phase space has dimension 6, the energy surface has dimension 5 and the tori do not play a limitative role any longer. It would be of real interest to investigate if there exists a drift exponentially small for that situation, as suggested by the works of Nekhoroshev and Georgilli [47],[19]. There is a paper by G.Benettin et P.Sempio [4] which goes in that direction, requiring three time scales for such a motion.
- Concerning the magnetic Neumann operator there exist asymptotics for the first eigenvalue in the case of a constant magnetic field. If the domain Ω is an open set of $\mathbb{R}^2$ one gets ([16] [22]) the estimate

$$\lambda_1(h) = \Theta_0 b h - k_{max} C_1 h^{3/2} + \mathcal{O}(h^{7/4}) .$$

In this formula Θ_0 denotes the minimum of the function $\mu(x)$ defined

in section 4.1, b is the norm of the magnetic field, C_1 is a universal constant and k_{max} is the maximal value taken by the curvature on the boundary $\partial\Omega$. But what is the situation for a variable magnetic field? Here is the result given in [22] : in the case $b > \Theta_0 b'$ (keeping the definitions of the section 4.1 and assuming that there is a unique non degenerate minimum on the boundary) :

$$\lambda_1(h) = \Theta_0 b' h + \mathcal{O}(h^{3/2}) .$$

More recently N. Raymond in [49] improved that result by computing the term of next order in the expansion. This term contains the curvature, the normal derivative of B and the tangential derivative of order 2. Is it still the case in dimension 3?

- Concerning magnetic bottles in hyperbolic geometry, we gave a generalization to the geometrically finite hyperbolic surfaces, in the case when these manifolds are of infinite volume.

Another generalization of the result would be to investigate the 3-dimensional case. What properties should we require on the magnetic field to get a Weyl-type estimate in the hyperbolic half-space?

Finally, from a more geometrical point of view it would be nice to understand what term is replacing the additional term $1/4$, which is a feature of the hyperbolic geometry, when another metric is considered.

- Another natural problem about magnetic bottles has been considered in [10].

Let us consider a particle in a domain Ω in $\mathbb{R}^d$ ($d \geq 2$) in the presence of a magnetic field B . The topological boundary $\partial\Omega$ of Ω is assumed to be compact. At the classical level, if the strength of the field tends to infinity as x approaches the boundary $\partial\Omega$, the charged particle is expected to be confined and never visit the boundary: the Hamiltonian dynamics is complete. At the quantum level the fact that the particle never feels the boundary amounts to saying that the magnetic field completely determines the motion, so there is no need for boundary conditions. At the mathematical level, the problem is to find conditions on the behavior of $B(x)$ as x tends to $\partial\Omega$ which ensure that the magnetic operator H_A is essentially self-adjoint on $C_0^\infty(\Omega)$. These conditions will not depend on the gauge A , but only on the field B . This

question may be of technological interest in the construction of tokamacs for the nuclear fusion [58]. The ionized plasma which is heated is confined thanks to magnetic fields.

The result is the following: under some continuity assumption on the direction of $B(x)$ at the boundary, for any $\epsilon > 0$ and $R > 0$, there exists a constant $C_{\epsilon,R} \in \mathbb{R}$ such that, $\forall u \in C_o^\infty(\Omega)$, the quadratic form h_A satisfies the quite optimal bound

$$h_A(u) \geq (1 - \epsilon) \int_{\Omega \cap \{x \mid |x| \leq R\}} |B|_{\text{sp}} |u|^2 |dx| - C_{\epsilon,R} \|u\|^2. \quad (6.1)$$

Here $|B(x)|_{\text{sp}}$ is a suitable norm on the space of bi-linear antisymmetric forms on $\mathbb{R}^d$, called the *spectral norm*. This implies that H_A is essentially self-adjoint if $|B(x)|_{\text{sp}} \geq (1 + \eta)D(x)^{-2}$ where $\eta > 0$ and D is the distance to the boundary of Ω .

Examples of such magnetic bottles are given in the following cases:

- The domain Ω is a polytope
- The boundary is smooth and the Euler characteristic vanishes (toroidal domain)
- The boundary is smooth and the Euler characteristic does not vanish (non toroidal domain)
- Monopoles and dipoles in $\Omega = \mathbb{R}^3 \setminus 0$

For any $\epsilon > 0$ and when Ω is the unit disk, an example of a non essentially self-adjoint operator H_A is given with $|B(x)|_{\text{sp}} \sim (\sqrt{3}/2 - \epsilon)D(x)^{-2}$ showing that the previous bound is rather sharp.

The following questions seem to be quite interesting:

- What are the properties of a *classical* charged particle in a confining magnetic box? Are almost all trajectories not hitting the boundary?
- What is the *optimal* constant C in the estimates $|B(x)|_{\text{sp}} \geq CD(x)^{-2}$? (We know that the optimal constant lies in the interval $[\sqrt{3}/2, 1]$).

- In conclusion, we tried in this paper to highlight the relationship between magnetic bottles and degenerate potentials, as well in the classical mechanics context as in the quantum mechanics one. The Weyl asymptotics have to be revisited in both cases, and the classical Hamiltonian induced by a magnetic bottle can be seen as a perturbation of the Hamiltonian derived from an operator with a degenerate potential. It could be nice to go further in that comparison, by trying to express the Weyl-type asymptotics in a unified way.

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