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LIMIT THEOREMS FOR ONE AND TWO-DIMENSIONAL RANDOM
WALKS IN RANDOM SCENERY

FABIENNE CASTELL, NADINE GUILLOTIN-PLANTARD, AND FRANÇOISE PÈNE

Abstract. Random walks in random scenery are processes defined by
\[ Z_n := \sum_{k=1}^{n} \xi_{X_1 + \ldots + X_k}, \]
where \((X_k, k \geq 1)\) and \((\xi_y, y \in \mathbb{Z}^d)\) are two independent sequences of i.i.d. random variables with
values in \(\mathbb{Z}^d\) and \(\mathbb{R}\) respectively. We suppose that the distributions of \(X_1\) and \(\xi_0\) belong to the
normal basin of attraction of stable distribution of index \(\alpha \in (0, 2)\) and \(\beta \in (0, 2)\). When \(d = 1\)
and \(\alpha \neq 1\), a functional limit theorem has been established in [11] and a local limit theorem
in [5]. In this paper, we establish the convergence of the finite-dimensional distributions and a
local limit theorem when \(\alpha = d\) (i.e. \(\alpha = d = 1\) or \(\alpha = d = 2\)) and \(\beta \in (0, 2)\). Let us mention
that functional limit theorems have been established in [2] and recently in [8] in the particular
case where \(\beta = 2\) (respectively for \(\alpha = d = 2\) and \(\alpha = d = 1\)).

1. Introduction

Random walks in random scenery (RWRS) are simple models of processes in disordered media
with long-range correlations. They have been used in a wide variety of models in physics to
study anomalous dispersion in layered random flows [14], diffusion with random sources, or spin
depolarization in random fields (we refer the reader to Le Doussal’s review paper [12] for a
discussion of these models).

On the mathematical side, motivated by the construction of new self-similar processes with
stationary increments, Kesten and Spitzer [11] and Borodin [3, 4] introduced RWRS in dimension
one and proved functional limit theorems. This study has been completed in many works, in
particular in [2] and [8]. These processes are defined as follows. Let \(\xi := (\xi_y, y \in \mathbb{Z}^d)\) and
\(X := (X_k, k \geq 1)\) be two independent sequences of independent identically distributed random
variables taking values in \(\mathbb{R}\) and \(\mathbb{Z}^d\) respectively. The sequence \(\xi\) is called the random scenery.
The sequence \(X\) is the sequence of increments of the random walk \((S_n, n \geq 0)\) defined by \(S_0 := 0\)
and \(S_n := \sum_{i=1}^{n} X_i\), for \(n \geq 1\). The random walk in random scenery \(Z\) is then defined by
\[ Z_0 := 0 \text{ and } \forall n \geq 1, \quad Z_n := \sum_{k=0}^{n-1} \xi_{S_k}. \]

Denoting by \(N_n(y)\) the local time of the random walk \(S\) :
\[ N_n(y) := \#\{k = 0, \ldots, n - 1 : S_k = y\}, \]
it is straightforward to see that \(Z_n\) can be rewritten as \(Z_n = \sum_{y} \xi_y N_n(y)\).

As in [11], the distribution of \(\xi_0\) is assumed to belong to the normal domain of attraction of
a strictly stable distribution \(S_\beta\) of index \(\beta \in (0, 2]\), with characteristic function \(\phi\) given by
\[ \phi(u) = e^{-|u|^\beta (A_1 + i A_2 \text{sgn}(u))} \quad u \in \mathbb{R}, \]

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\[ \text{F unctional limit theorem.} \]

Our first result gives a limit theorem for the distribution of attraction of a stable distribution on \( [0, \infty) \). Then the following weak convergences hold in the space of càdlàg real-valued functions defined

\[ \xi \]

where

\[ \sup_{t > 0} |E [\xi_0 \mathbf{1}_{\{\xi_0 \leq t\}}]| < +\infty. \]  

(1)

Under these conditions (for \( \beta \in (0, 2) \)), there exists \( C_\xi > 0 \) such that we have

\[ \forall t > 0, \; \mathbb{P} (|\xi_0| \geq t) \leq C_\xi t^{-\beta}. \]  

(2)

Concerning the random walk, the distribution of \( X_1 \) is assumed to belong to the normal basin of attraction of a stable distribution \( S_\alpha \) with index \( \alpha \in (0, 2] \).

Then the following weak convergences hold in the space of càdlàg real-valued functions defined on \([0, \infty)\) and on \( \mathbb{R} \) respectively, endowed with the Skorohod \( J_1 \)-topology (see [1, chapter 3]):

\[
\left(n^{-1/\alpha} S_{[nt]} \right)_{t \geq 0} \xrightarrow{\mathcal{L}} (U(t))_{t \geq 0}
\]

and

\[
\left(n^{-\frac{1}{\beta}} \sum_{k=0}^{[nx]} \xi_{ke_1} \right)_{x \in \mathbb{R}} \xrightarrow{\mathcal{L}} (Y(x))_{x \in \mathbb{R}}, \text{ with } e_1 = (1, 0, \cdots, 0) \in \mathbb{Z}^d,
\]

where \( U \) and \( Y \) are two independent Lévy processes such that \( U(0) = 0, Y(0) = 0, U(1) \) has distribution \( S_\alpha' \), \( Y(1) \) and \( Y(-1) \) have distribution \( S_\beta \).

**Functional limit theorem.**

Our first result is concerned with a functional limit theorem for \( (Z_{[nt]})_{t \geq 0} \). Intuitively speaking,

- when \( \alpha < d \), the random walk \( S_n \) is transient, its range is of order \( n \), and \( Z_n \) has the same behaviour as a sum of about \( n \) independent random variables with the same distribution as the variables \( \xi_x \). Therefore, \( n^{-1/\beta}(Z_{[nt]})_{t \geq 0} \) weakly converges in the space \( D([0, \infty)) \) of càdlàg functions endowed with the Skorohod \( J_1 \)-topology, to a multiple of the process \( (Y_t) \), as proved in [4];
- when \( \alpha > d \) (i.e. \( d = 1 \) and \( 1 < \alpha \leq 2 \)), the random walk \( S_n \) is recurrent, its range is of order \( n^{1/\alpha} \), its local times are of order \( n^{1-1/\alpha} \), so that \( Z_n \) is of order \( n^{1-\frac{1}{\alpha} + \frac{1}{\beta}} \). In this situation, [3] and [11] proved a functional limit theorem for \( n^{-1-\frac{1}{\alpha} + \frac{1}{\beta}} (Z_{[nt]})_{t \geq 0} \) in the space \( C([0, \infty)) \) of continuous functions endowed with the uniform topology, the limiting process being a self-similar process, but not a stable one;
- when \( \alpha = d \) (i.e. \( \alpha = d = 1 \), or \( \alpha = d = 2 \)), \( S_n \) is recurrent, its range is of order \( n/\log(n) \), its local times are of order \( \log(n) \) so that \( Z_n \) is of order \( n^{1/2} \log(n)^{\frac{d-1}{d}} \). In this situation, a functional limit theorem in the space of continuous functions was proved in [2] for \( d = \alpha = \beta = 2 \), and in [8] for \( d = \alpha = 1 \) and \( \beta = 2 \).

Our first result gives a limit theorem for \( \alpha = d \) (and so \( d \in \{1, 2\} \)) and for any value of \( \beta \in (0, 2) \) in the finite distributional sense.

**Theorem 1.** Let us assume that \( \beta \in (0, 2) \) and that

(a) either \( d = 2 \) and \( X_1 \) is centered, square integrable with invertible variance matrix \( \Sigma \) and then we define \( A := 2\sqrt{\det \Sigma} \);
(b) or \( d = 1 \) and \( \left( \frac{\xi_0}{n} \right) \) converges in distribution to a random variable with characteristic function given by \( t \mapsto \exp(-a|t|) \) with \( a > 0 \) and then we define \( A := a \).
Then, the finite-dimensional distributions of the sequence of random variables
\[
\left( \frac{Z_{[nt]}}{(n^{1/\beta} \log(n))^{(\beta-1)/\beta}} \right)_{t \geq 0}, n \geq 2
\]
converges to the finite-dimensional distributions of the process
\[
\left( \frac{\Gamma(\beta + 1)}{(\pi A)^{\beta-1}} Y(t) \right)_{t \geq 0}.
\]
Moreover, if \( \beta < 2 \), the sequence
\[
\left( \frac{Z_{[nt]}}{(n^{1/\beta} \log(n))^{(\beta-1)/\beta}} \right)_{t \geq 0}, n \geq 2
\]
is not tight in \( \mathcal{D}([0, \infty)) \) endowed with the \( J_1 \)-topology.

Local limit theorem.

Our next results concern a local limit theorem for \( (Z_n)_n \). The \( d = 1 \) case was treated in [5] for \( \alpha \in (0; 2] \setminus \{1\} \) and all values of \( \beta \in (0; 2] \). Here, we complete this study by proving a local limit theorem for \( \alpha = d = 1 \) (and \( \beta \in (0; 2] \)). By a direct adaptation of the proof of this result, we also establish a local limit theorem for \( \alpha = d = 2 \) (we just adapt the definition of "peaks", see section 3.5). Let us notice that the same adaptation can be done from [5] (case \( \alpha < 1 \)) to get local limit theorems for \( d \geq 2, \alpha < d \) and \( \beta \in (0; 2] \).

We give two results corresponding respectively to the case when \( \xi_0 \) is lattice and to the case when it is strongly non-lattice. We denote by \( \varphi_\xi \) the characteristic function of \( \xi_0 \).

**Theorem 2.** Assume that \( \xi_0 \) takes its values in \( \mathbb{Z} \). Let \( d_0 \geq 1 \) be the integer such that \( \{u : |\varphi_\xi(u)| = 1\} = \frac{\pi \sigma}{d_0} \mathbb{Z} \). Let \( b_n := n^{1/\beta}(\log(n))^{(\beta-1)/\beta} \). Under the previous assumptions on the random walk and on the scenery, for \( \alpha = d \in \{1, 2\} \), for every \( \beta \in (0, 2] \), and for every \( x \in \mathbb{R} \),

- if \( \mathbb{P}(n \xi_0 - |b_n x| \notin d_0 \mathbb{Z}) = 1 \), then \( \mathbb{P}(Z_n = |b_n x|) = 0 \);
- if \( \mathbb{P}(n \xi_0 - |b_n x| \in d_0 \mathbb{Z}) = 1 \), then
\[
\mathbb{P}(Z_n = |b_n x|) = d_0 \frac{C(x)}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}} + o(n^{-1/\beta}(\log(n))^{-(\beta-1)/\beta})
\]
uniformly in \( x \in \mathbb{R} \), where \( C(\cdot) \) is the density function of \( \tilde{Y}_1 \).

**Theorem 3.** Assume now that \( \xi_0 \) is strongly non-lattice which means that
\[
\limsup_{|u| \to +\infty} |\varphi_\xi(u)| < 1.
\]
We still assume that \( \alpha = d \in \{1, 2\} \) and \( \beta \in (0; 2] \). Then, for every \( x, a, b \in \mathbb{R} \) such that \( a < b \), we have
\[
\lim_{n \to +\infty} b_n \mathbb{P}(Z_n \in [b_n x + a; b_n x + b]) = C(x)(b - a),
\]
with \( b_n := n^{1/\beta}(\log(n))^{(\beta-1)/\beta} \) and where \( C(\cdot) \) is the density function of \( \tilde{Y}_1 \).
Before proving the theorem, we prove some technical lemmas. For any real number $\gamma > 0$, any integer $m \geq 1$, any $\theta_1, \ldots, \theta_m \in \mathbb{R}$, any $t_0 = 0 < t_1 < \ldots < t_m$, we consider the sequences of random variables $(L_n(\gamma))_{n \geq 2}$ and $(L'_n(\gamma))_{n \geq 2}$ defined by

$$L_n(\gamma) := \frac{1}{n(\log n)^{\gamma - 1}} \sum_{x \in \mathbb{Z}^d} \left| \sum_{i=1}^m \theta_i (N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)) \right|^\gamma$$

and

$$L'_n(\gamma) := \frac{1}{n(\log n)^{\gamma - 1}} \sum_{x \in \mathbb{Z}^d} \left| \sum_{i=1}^m \theta_i (N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)) \right|^\gamma \text{sgn} \left( \sum_{i=1}^m \theta_i (N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)) \right).$$

Lemma 4. For any real number $\gamma > 0$, any integer $m \geq 1$, any $\theta_1, \ldots, \theta_m \in \mathbb{R}$, any $t_0 = 0 < t_1 < \ldots < t_m$, the following convergences hold $\mathbb{P}$-almost surely

$$\lim_{n \to +\infty} L_n(\gamma) = \frac{\Gamma(\gamma + 1)}{(\pi A)^{\gamma - 1}} \sum_{i=1}^m |\theta_i|^\gamma (t_i - t_{i-1})$$

and

$$\lim_{n \to +\infty} L'_n(\gamma) = \frac{\Gamma(\gamma + 1)}{(\pi A)^{\gamma - 1}} \sum_{i=1}^m |\theta_i|^\gamma \text{sgn}(\theta_i)(t_i - t_{i-1}).$$

Proof. We fix an integer $m \geq 1$ and $2m$ real numbers $\theta_1, \ldots, \theta_m, t_1, \ldots, t_m$ such that $0 < t_1 < \ldots < t_m$ and we set $t_0 := 0$. To simplify notations, we write $b_{i,n}(x) := N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)$. Following the techniques developed in [6], we first have to prove (3) and (4) for integer $\gamma$: for every integer $k \geq 1$, $\mathbb{P}$-almost surely, as $n$ goes to infinity, we have

$$\frac{1}{n(\log n)^{k-1}} \sum_{x \in \mathbb{Z}^d} \left( \sum_{i=1}^m \theta_i b_{i,n}(x) \right)^k \to \frac{\Gamma(k + 1)}{(\pi A)^{k-1}} \sum_{i=1}^m |\theta_i|^k (t_i - t_{i-1}).$$

(5)

Let us assume (5) for a while, and let us end the proof of (3) and (4) for any positive real $\gamma$. Given the random walk $S := (S_n)_n$, let $(U_n)_{n \geq 1}$ be a sequence of random variables with values in $\mathbb{Z}^d$, such that for all $n$, $U_n$ is a point chosen uniformly in the range of the random walk up to time $[nt_m]$, that is

$$\mathbb{P}(U_n = x | S) = R_{[nt_m]}^{-1}(x) \mathbb{1}_{\{N_{[nt_m]}(x) \geq 1\}},$$

with $R_k := \#\{y : N_k(y) > 0\}$. Moreover, let $U'$ be a random variable with values in $\{1, \ldots, m\}$ and distribution

$$\mathbb{P}(U' = i) = (t_i - t_{i-1})/t_m$$

and let $T$ be a random variable with exponential distribution with parameter one and independent of $U'$. 

Then, for $\mathbb{P}$—almost every realization of the random walk $S$, the sequence of random variables

$$\left( W_n := \frac{\pi A}{\log(n)} \sum_{i=1}^m \theta_i b_{i,n}(U_n) \right)_n$$

converges in distribution to the random variable $W := \theta_{U'}T$. Indeed, the moment of order $k$ of $W_n$ given $S$ is

$$\mathbb{E}(W^n_k | S) = \frac{(\pi A)^k}{n(\log n)^{k-1}} \sum_{x \in \mathbb{Z}^d} \left( \sum_{i=1}^m \theta_i b_{i,n}(x) \right)^k \frac{n}{\log(n) R([nt_m])}.$$
Using (5) and the fact that \(((\log n) R_n/n) n\) converges almost surely to \(\pi A\) (see [9, 13]), the moments \(E(W_n^k|S)\) converges a.s. to \(E(W^k) = \Gamma(k + 1) \sum_{i=1}^{\infty} \theta_i^k (t_i - t_{i-1})/t_m\), which proves the convergence in distribution of \(W_n\) to \(W\). This ensures, in particular, the convergence in distribution of \(|W_n|\gamma\) and of \(|W_n|\gamma \sgn(W_n)\) (given \(S\)) to \(|W|\gamma\) and \(|W|\gamma \sgn(W)\) respectively (for every real number \(\gamma \geq 0\) and for \(P\) almost every realization of the random walk \(S\)). Since any moment of \(|W_n|\) can be bounded from above by an integer moment, we deduce that, for any \(\gamma \geq 0\), we have \(P\)-almost surely
\[
\lim_{n \to +\infty} E(|W_n|\gamma |S) = E(|W|\gamma) \quad \text{and} \quad \lim_{n \to +\infty} E(|W_n|\gamma \sgn(W_n) |S) = E(|W|\gamma \sgn(W)),
\]
which proves lemma 4.

Let us prove (5). Let \(k \geq 1\). According to Theorem 1 in [6] (proved for \(\alpha = d = 2\), but also valid for \(\alpha = d = 1\)), we have
\[
\forall i \in \{1, \ldots, m\}, \quad \lim_{n \to +\infty} \frac{1}{n (\log n)^{k-1}} \sum_{x \in \mathbb{Z}^d} (b_{i,n}(x))^k = \frac{\Gamma(k + 1)}{(\pi A)^{k-1}} (t_i - t_{i-1}), \quad P - a.s. \quad (6)
\]

We define
\[
\Sigma_n(\theta_1, \ldots, \theta_m) := \sum_{x \in \mathbb{Z}^d} \left( \sum_{i=1}^{m} \theta_i b_{i,n}(x) \right)^k - \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{m} (\theta_i)^k (b_{i,n}(x))^k. \quad (7)
\]

According to (6), it is enough to prove that \(P\)-a.s., \(\Sigma_n(\theta_1, \ldots, \theta_m) = o(n (\log n)^{k-1})\). We observe that \(\Sigma_n(\theta_1, \ldots, \theta_m)\) is the sum of the following terms
\[
\sum_{x \in \mathbb{Z}^d} \prod_{j=1}^{k} (\theta_{i_j} b_{i_j,n}(x)). \quad (8)
\]

over all the \(k\)-tuple \((i_1, \ldots, i_k) \in \{1, \ldots, m\}^k\), with at least two distinct indices. We observe that
\[
|\Sigma_n(\theta_1, \ldots, \theta_m)| \leq \max(|\theta_1|, \ldots, |\theta_m|)^k \Sigma_n(1, \ldots, 1).
\]

But, we have
\[
\Sigma_n(1, \ldots, 1) = \sum_{x \in \mathbb{Z}^d} (N_{[nt_m]}(x))^k - \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{m} (b_{i,n}(x))^k
\]
\[
= \sum_{x \in \mathbb{Z}^d} (N_{[nt_m]}(x))^k - \sum_{i=1}^{m} \sum_{x \in \mathbb{Z}^d} (b_{i,n}(x))^k = o(n \log(n)^{k-1}),
\]
according to (6). \(\square\)

**Lemma 5.** For any \(\rho > 0\),
\[
\sup_{x \in \mathbb{Z}^d} N_n(x) = o(n^\rho) \quad a.s.
\]

**Proof.** See Lemma 2.5 in [2]. \(\square\)

**Proof of Theorem 1.** Let an integer \(m \geq 1\) and \(2m\) real numbers \(\theta_1, \ldots, \theta_m, t_1, \ldots, t_m\) such that
\(0 < t_1 < \ldots < t_m\). We set \(t_0 := 0\). Again, we use the notation \(b_{i,n}(x) := N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)\).

Let us write \(Z_n := \frac{1}{n^{1/\beta}(\log n)^{(\beta - 1)/\beta}} \sum_{i=1}^{m} \theta_i (Z_{[nt_i]} - Z_{[nt_{i-1}]}).\) We have to prove that
\[
E[e^{iZ_n}] \to \prod_{i=1}^{m} \phi \left( \theta_i (t_i - t_{i-1})^{1/\beta} \left( \frac{\Gamma(\beta + 1)}{(\pi A)^{\beta - 1}} \right)^{1/\beta} \right), \quad (9)
\]
as $n$ goes to infinity. We observe that $\bar{Z}_n = \frac{1}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}} \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^m \theta_i b_{i,n}(x) \xi_x$. Hence we have

$$E[e^{i\bar{Z}_n}|S] = \prod_{x \in \mathbb{Z}^d} \varphi_x \left( \frac{\sum_{i=1}^m \theta_i b_{i,n}(x)}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}} \right).$$

Observe next that

$$|\varphi_x(t) - \exp \left( -|t|^\beta (A_1 + iA_{2\text{sgn}}(t)) \right) | \leq |t|^\beta h(|t|) \quad \text{for all } t \in \mathbb{R},$$

with $h$ a continuous and monotone function on $[0, +\infty)$ vanishing in $0$. This implies in particular the existence of $\varepsilon_0 > 0$ and $\sigma > 0$ such that $\max(|\varphi_x(t)|, \exp(-A_1|t|^\beta)) \leq e^{-\sigma|t|^\beta}$ for any $t \in [-\varepsilon_0, \varepsilon_0]$. According to lemma 5, $\mathbb{P}$-almost surely, for every $n$ large enough, we have

$$b_n := \sup_x \left| \frac{\sum_{i=1}^m \theta_i b_{i,n}(x)}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}} \right| \leq \varepsilon_0$$

and so

$$E[e^{i\bar{Z}_n}|S] - \prod_{x \in \mathbb{Z}^d} e^{-\sum_{i=1}^m \theta_i b_{i,n}(x)|^\beta/h(b_n)e^{-\sigma \sum_{y \in \mathbb{Z}^d} \sum_{i=1}^m \theta_i b_{i,n}(y)|^\beta/b_n^\beta} \right) \right|$$

is less than $\sum_{x \in \mathbb{Z}^d} \sum_{i=1}^m \theta_i b_{i,n}(x)|^\beta/h(b_n)e^{-\sigma \sum_{y \in \mathbb{Z}^d} \sum_{i=1}^m \theta_i b_{i,n}(y)|^\beta/b_n^\beta} \right) \right|$. Hence, according to lemmas 4 and 5, $\mathbb{P}$-almost surely, we have

$$\lim_{n \to +\infty} E[e^{i\bar{Z}_n}|S] = e^{-\frac{\Gamma(\beta+1)}{\Gamma(\beta)} \sum_{i=1}^m |\theta_i|^\beta(t_i-t_{i-1})(A_1+iA_{2\text{sgn}}(\theta_i))}$$

which gives (9) thanks to the Lebesgue dominated convergence theorem.

Finally we prove that the sequence

$$\left( \frac{Z_{[nt]}}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}} \right)_{t \in [0;1]}$$

is not tight in $D([0,\infty))$. It is enough to prove that it is not tight in $D([0,1])$. To this aim, let $b_n = n^{1/\beta}(\log(n))^{(\beta-1)/\beta}$, and $(Z_n(t), t \in [0,1])$ denote the linear interpolation of $(Z_{[nt]}, t \in [0,1])$, i.e.

$$Z_n(t) = Z_{[nt]} + (nt-[nt])\xi_{S_{[nt]}}.$$ 

Then, $\forall \varepsilon > 0$,

$$\mathbb{P} \left[ \sup_{t \in [0,1]} |Z_n(t) - Z_{[nt]}| \geq \varepsilon b_n \right] = \mathbb{P} \left[ \max_{i=0}^{n-1} |\xi_{S_i}| \geq \varepsilon b_n \right]$$

$$= \mathbb{P} \left[ \exists x \in \{S_0, \ldots, S_{n-1}\} \text{ s.t. } |\xi_x| \geq \varepsilon b_n \right]$$

$$\leq \mathbb{E}(\# \{S_0, \ldots, S_{n-1}\}) \mathbb{P} [ |\xi_0| \geq \varepsilon b_n ]$$

$$\leq C \frac{n}{\log(n)} \varepsilon^{-\beta} b_n^{-\beta} = C \varepsilon^{-\beta} \log(n)^{-\beta},$$

where the last inequality comes from (2) and Theorem 6.9 of [13]. Therefore, if $\left( \frac{Z_{[nt]}}{b_n} \right)_{t \in [0;1]}$ converges weakly to $\left( \tilde{Y}_t \right)_{t \in [0,1]}$, the same is true for $\left( \frac{Z_n(t)}{b_n} \right)_{t \in [0;1]}$. Using the fact that
the sequence $\left( \frac{Z_{n(t)}}{b_n^2} \right)_{t \in [0;1]}$ is a sequence in the space $\mathbb{C}([0,1])$ and that the Skorohod $J_1$-topology coincides with the uniform one when restricted to $\mathbb{C}([0,1])$, one deduces that $\left( \frac{Z_{n(t)}}{b_n^2} \right)_{t \in [0;1]}$ converges weakly in $\mathbb{C}([0,1])$, and that the limiting process $\left( \tilde{Y}_t \right)_{t \in [0,1]}$ is therefore continuous, which is false as soon as $\beta < 2$.

\section{Proof of the local limit theorem in the lattice case}

\subsection{The event $\Omega_n$}

Set $N^*_n := \sup_y N_n(y)$ and $R_n := \# \left\{ y : N_n(y) > 0 \right\}$.

\begin{lemma}
For every $n \geq 1$ and $1 > \gamma > 0$, set

$$\Omega_n = \Omega_n(\gamma) := \left\{ R_n \leq \frac{n}{(\log \log(n))^{1/4}} \text{ and } N^*_n \leq n^\gamma \right\}.$$ \hspace{1cm} (10)

Then, $\mathbb{P}(\Omega_n) = 1 - o(b_n^{-1})$. Moreover, the following also holds on $\Omega_n$:

$$(\log \log(n))^{1/4} \leq N^*_n \quad \text{and} \quad V_n \geq n^{1-\gamma(1-\beta)}.$$ \hspace{1cm} (11)

\end{lemma}

\begin{proof}
We first prove that

$$\mathbb{P} \left( R_n \geq n(\log \log(n))^{-1/4} \right) = o(b_n^{-1}).$$ \hspace{1cm} (12)

Let us recall that for every $a, b \in \mathbb{N}$, we have

$$\mathbb{P}(R_n \geq a + b) \leq \mathbb{P}(R_n \geq a)\mathbb{P}(R_n \geq b).$$ \hspace{1cm} (13)

The proof is given for instance in [7]. We will moreover use the fact that $\mathbb{E}[R_n] \sim cn(\log(n))^{-1}$ and $Var(R_n) = O \left( n^2 \log^{-4}(n) \right)$ (see [13]). Hence, for $n$ large enough, there exists $C > 0$ such that we have

$$\mathbb{P} \left( R_n \geq \frac{n}{(\log \log(n))^{1/4}} \right) \leq \mathbb{P} \left( R_n \geq \left[ \frac{n(\log \log(n))^{1/4}}{\log(n)} \right] \right)^{[\log(n)(\log \log(n))^{-1/2}]}$$

$$\leq \mathbb{P} \left( |R_n - \mathbb{E}[R_n]| \geq \frac{1}{2} \left[ \frac{n(\log \log(n))^{1/4}}{\log(n)} \right] \right)^{[\log(n)(\log \log(n))^{-1/2}]}$$

$$\leq \left( \frac{5\text{Var}(R_n) \log^2(n)}{n^2(\log \log(n))^{1/2}} \right)^{[\log(n)(\log \log(n))^{-1/2}]}$$

$$\leq \left( \frac{Cn^2 \log^2(n) / \log^4(n)}{n^2 \log \log(n)} \right)^{[\log(n)(\log \log(n))^{-1/2}]}$$

$$\leq \left( \frac{C}{(\log(n))^2} \right)^{[\log(n)(\log \log(n))^{-1/2}]} = \exp \left( - \log(n) \sqrt{\log \log(n)} \left( 1 - \frac{\log(C)}{2 \log \log(n)} \right) \right).$$

This ends the proof of (11).

Let us now prove that

$$\mathbb{P} [N^*_n \geq n^{\gamma}] = o(b_n^{-1}).$$ \hspace{1cm} (14)
We have
\[
\mathbb{P}(N_n^* \geq n^\gamma) \leq \sum_x \mathbb{P}(N_n(x) \geq n^\gamma) \\
= \sum_x \mathbb{P}(T_x \leq n; N_n(x) \geq n^\gamma), \text{ where } T_x := \inf \{n > 1, \text{ s.t. } S_n = x\} \\
\leq \sum_x \mathbb{P}(T_x \leq n) \mathbb{P}(N_n(0) \geq n^\gamma) \\
\leq \mathbb{E}[R_n] \mathbb{P}(T_0 \leq n) n^\gamma.
\]

Hence, (13) follows now from \(\mathbb{E}[R_n] \sim cn(\log(n))^{-1}\), and from \(\mathbb{P}(T_0 > n) \sim C/\log(n)\).

Since \(n = \sum_y N_n(y) \leq R_n N_n^*\), we get that \(N_n^* \geq \frac{n}{R_n} \geq (\log \log(n))^{1/4}\) on \(\Omega_n\).

To prove the lower bound for \(V_n\), note that for \(\beta \geq 1\), \(V_n = \sum_y N_n(y)^{\beta} \geq \sum_y N_n(y) = n\). For \(\beta < 1\,\text{ on }\Omega_n\),
\[
n = \sum_y N_n(y) = \sum_y N_n(y)^\beta N_n(y)^{1-\beta} \leq V_n(N_n^*)^{1-\beta} \leq V_n n^{\gamma(1-\beta)}.
\]

\(\square\)

3.2. Scheme of the proof. It is easy to see (cf the proof of lemma 5 in [5]) that \(\mathbb{P}(Z_n = [b_n x]) = 0\) if \(\mathbb{P}(n \xi_0 - [b_n x] \notin d_0 \mathbb{Z}) = 1\), and that if \(\mathbb{P}(n \xi_0 - [b_n x] \in d_0 \mathbb{Z}) = 1\),
\[
\mathbb{P}(Z_n = [b_n x]) = \frac{d_0}{2\pi} \int_{-\frac{x_0}{d_0}}^{\frac{x_0}{d_0}} e^{-it[b_n x]} \mathbb{E}\left[\prod_y \varphi(t N_n(y))\right] dt.
\]

In view of lemma 6, we have to estimate
\[
\frac{d_0}{2\pi} \int_{-\frac{x_0}{d_0}}^{\frac{x_0}{d_0}} e^{-it[b_n x]} \mathbb{E}\left[\prod_y \varphi(t N_n(y))\mathbb{1}_{\Omega_n}\right] dt.
\]

This is done in several steps presented in the following propositions.

**Proposition 7.** Let \(\gamma \in (0, 1/(\beta + 1))\) and \(\delta \in (0, 1/(2\beta))\) s.t. \(\gamma^{(1-\beta)+}\frac{1}{\beta} < \delta < 1/\beta - \gamma\). Then, we have
\[
\frac{d_0}{2\pi} \int_{\{t| t \leq \delta / b_n\}} e^{-it[b_n x]} \mathbb{E}\left[\prod_y \varphi(t N_n(y))\mathbb{1}_{\Omega_n}\right] dt = \frac{d_0}{b_n} \frac{C(x)}{b_n} + o(b_n^{-1}),
\]
uniformly in \(x \in \mathbb{R}\).

Recall next that the characteristic function \(\phi\) of the limit distribution of \((n^{-1/\beta} \sum_{k=1}^n \xi_k)\) has the following form:
\[
\phi(u) = e^{-|u|^\beta (A_1 + i A_2 \text{sgn}(u))},
\]
with \(0 < A_1 < \infty\) and \(|A_1^{-1}A_2| \leq |\tan(\pi \beta/2)|\). It follows that the characteristic function \(\varphi\xi\) of \(\xi_0\) satisfies:
\[
1 - \varphi\xi(u) \sim |u|^\beta (A_1 + i A_2 \text{sgn}(u)) \quad \text{ when } u \to 0.
\]

Therefore there exist constants \(\varepsilon_0 > 0\) and \(\sigma > 0\) such that
\[
\max(|\phi(u)|, |\varphi\xi(u)|) \leq \exp\left(-\sigma |u|^\beta\right) \quad \text{for all } u \in [-\varepsilon_0, \varepsilon_0].
\]

Since \(\varphi\xi(t) = \varphi\xi(-t)\) for every \(t \geq 0\), the following propositions achieve the proof of Theorem 2:
Proposition 8. Let $\delta$ and $\gamma$ be as in Proposition 7. Then there exists $c > 0$ such that
\[
\int_{n^{\delta}-b_n}^{n^{\delta}} \mathbb{E} \left[ \prod_y \varphi(tN_n(y))|1_{\Omega_n} \right] dt = o(e^{-cn}).
\]

Proposition 9. There exists $c > 0$ such that
\[
\int_{\omega_{n-\gamma}}^{\omega_n} \mathbb{E} \left[ \prod_y \varphi(tN_n(y))|1_{\Omega_n} \right] dt = o(e^{-cn}).
\]

3.3. Proof of Proposition 7. Remember that $V_n = \sum_{z \in \mathbb{Z}^d} N_n^\beta(z)$. We start by a preliminary lemma.

Lemma 10.  
1. If $\beta > 1$, $\sup_n \mathbb{E} \left[ \frac{(n \log(n)^{\beta-1})}{V_n} \right]^{1/(\beta-1)} \leq \frac{\log(n)R_n}{n}$

2. If $\beta \leq 1$, $\forall p \in \mathbb{N}$, $\sup_n \mathbb{E} \left[ (\frac{n \log(n)^{\beta-1}}{V_n})^p \right] < +\infty$.

Proof. For $\beta > 1$, using Hölder’s inequality with $p = \beta$, we get
\[
n = \sum_x N_n(x) \leq V_n \frac{\beta-1}{\beta} R_n \frac{1}{\beta-1}
\]
which means that
\[
\left( \frac{n \log(n)^{\beta-1}}{V_n} \right)^{1/(\beta-1)} \leq \frac{\log(n)R_n}{n}
\]
But it is proved in [13] Equation (7.a) that $\mathbb{E}[R_n] = O(n/\log(n))$. The result follows.

The result is obvious for $\beta = 1$. For $\beta < 1$, Hölder’s inequality with $p = 2 - \beta$ yields
\[
n = \sum_x N_n^\beta(x) N_n^{2(1-\beta)} \leq V_n^{1-\beta} \left( \sum_x N_n^2(x) \right)^{\frac{1-\beta}{2-\beta}}
\]
and so
\[
\frac{n \log(n)^{\beta-1}}{V_n} \leq \left( \frac{\sum_x N_n^2(x)}{n \log(n)} \right)^{1-\beta}.
\]
It is therefore enough to prove that there exists $c > 0$ such that
\[
\sup_n \mathbb{E} \left[ \exp \left( \frac{c \sum_x N_n^2(x)}{n \log(n)} \right) \right] < \infty.
\]

Note that $\sum_x N_n^2(x) = \sum_{k=0}^{n-1} N_n(S_k)$. By Jensen’s inequality, we get thus
\[
\mathbb{E} \left[ \exp \left( \frac{c \sum_x N_n^2(x)}{n \log(n)} \right) \right] \leq \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E} \left[ \exp \left( \frac{N_n(S_k)}{\log(n)} \right) \right].
\]
Observe now that $N_n(S_k) = \sum_{j=0}^k 1_{\{S_j - S_{j-1} = 0\}} + \sum_{j=k+1}^{n-1} 1_{\{S_j - S_{k} = 0\}} = N_{k+1}(0) + N_{n-k}(0) - 1$, where $(N_n'(x), n \in \mathbb{N}, x \in \mathbb{Z}^d)$ is an independent copy of $(N_n(x), n \in \mathbb{N}, x \in \mathbb{Z}^d)$. Hence,
\[
\mathbb{E} \left[ \exp \left( \frac{c \sum_x N_n^2(x)}{n \log(n)} \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{N_n(0)}{\log(n)} \right)^2 \right].
\]
But, $\forall t > 0$,
\[
\mathbb{P}(N_n(0) \geq t \log(n)) \leq \mathbb{P}(T_0 \leq n)^{\lceil t \log(n) \rceil},
\]

where $T_0$ is the stopping time.
and
\[ \mathbb{E} \left[ \exp \left( c \frac{N_n(0)}{\log(n)} \right) \right] \leq 1 + \int_0^\infty c \exp(ct) \exp \left( - \left\lfloor t \log(n) \right\rfloor \mathbb{P}(T_0 > n) \right) \, dt. \]

Now (16) follows then from the fact that \( \exists C > 0 \) such that \( \mathbb{P}(T_0 > n) \sim C/\log(n) \) for any integer \( n \geq 1 \).

The next step is

**Lemma 11.** Under the hypotheses of Proposition 7, we have
\[ \int_{\{|t| \leq n^{\delta}/b_n\}} e^{-it|\phi_n|} \mathbb{E} \left[ \left\{ \prod_y \varphi(tN_n(y)) - e^{-|t|\beta(A_1 + iA_2 \text{sgn}(t))V_n} \right\} 1_{\Omega_n} \right] \, dt = o(b_n^{-1}), \]
uniformly in \( x \in \mathbb{R} \).

**Proof.** It suffices to prove that
\[ \int_{\{|t| \leq n^{\delta}/b_n\}} \mathbb{E}[|E_n(t)|1_{\Omega_n}] \, dt = o(b_n^{-1}) \]
with
\[ E_n(t) := \prod_y \varphi(tN_n(y)) - \prod_y \exp \left( -|t|\beta N_n^\beta(y)(A_1 + iA_2 \text{sgn}(t)) \right). \]

Observe that
\[ E_n(t) = \sum_y \left( \prod_{z < y} \varphi(tN_n(z)) \right) \left( \varphi(tN_n(y)) - e^{-|t|\beta N_n^\beta(y)(A_1 + iA_2 \text{sgn}(t))} \right) \times \left( \prod_{z > y} e^{-|t|\beta N_n^\beta(z)(A_1 + iA_2 \text{sgn}(t))} \right), \]
where an arbitrary ordering of sites of \( \mathbb{Z}^d \) has been chosen. But on \( \Omega_n \), if \( |t| \leq n^{\delta}b_n^{-1} \), then
\[ |t|N_n(z) \leq n^{\gamma+\delta}b_n^{-1}. \]
Since \( \gamma + \delta < \beta^{-1} \), this implies in particular that \( |t|N_n(z) < \varepsilon_0 \) for \( n \) large enough. Thus, by using (15), we get
\[ |E_n(t)| \leq \sum_y \left| \varphi(tN_n(y)) - \exp \left( -|t|\beta N_n^\beta(y)(A_1 + iA_2 \text{sgn}(t)) \right) \right| \exp \left( -\sigma|t|\beta \sum_{z \neq y} N_n^\beta(z) \right), \]
for \( n \) large enough. Observe next that (14) implies
\[ \left| \varphi(u) - \exp \left( -|u|\beta(A_1 + iA_2 \text{sgn}(u)) \right) \right| \leq |u|\beta h(|u|) \quad \text{for all } u \in \mathbb{R}, \]
with \( h \) a continuous and monotone function on \([0, +\infty)\) vanishing in 0. Therefore by using (17) we get
\[ |E_n(t)| \leq |t|\beta h(n^{\gamma+\delta}b_n^{-1}) \sum_y N_n^\beta(y) \exp \left( -\sigma|t|\beta \sum_{z \neq y} N_n^\beta(z) \right). \]

Now, according to (10) and since \( \gamma < \frac{1}{\beta+1} \leq \frac{1}{\beta+(1-\beta)_+} \), if \( n \) is large enough, we have on \( \Omega_n \)
\[ \sum_{z \neq y} N_n^\beta(z) \geq V_n/2 \quad \text{for all } y \in \mathbb{Z}. \]
Lemma 12. Under the hypotheses of Proposition 7, we have

\[
\int_{\{t \leq n^\delta b_n^{-1}\}} e^{-|t| b_n x} e^{-|t|^\beta V_n(A_1+iA_2\text{sgn}(t))} \, dt = d_0 \frac{C(x)}{b_n} + o(b_n^{-1}),
\]

uniformly in \( x \in \mathbb{R} \).

Proof. Set

\[
I_{n,x} := \int_{\{t \leq n^\delta b_n^{-1}\}} e^{-it[b_n x]} e^{-|t|^\beta V_n(A_1+iA_2\text{sgn}(t))} \, dt,
\]

which can be rewritten

\[
I_{n,x} = \int_{\{t \leq n^\delta b_n^{-1}\}} e^{-it[b_n x]} \phi(t V_n^{1/\beta}) \, dt.
\]

Since \(|b_n x| - b_n x| \leq 1\), for all \( n \) and \( x \), it is immediate that

\[
I_{n,x} = \int_{\{t \leq n^\delta b_n^{-1}\}} e^{-it b_n x} \phi(t V_n^{1/\beta}) \, dt + O(n^2 \delta b_n^{-2}).
\]

But \( \delta < (2\beta)^{-1} \) by hypothesis. So actually

\[
I_{n,x} = \int_{\{t \leq n^\delta b_n^{-1}\}} e^{-it b_n x} \phi(t V_n^{1/\beta}) \, dt + o(b_n^{-1}).
\]

Next, with the change of variable \( v = tb_n \), we get:

\[
\int_{\{t \leq n^\delta b_n^{-1}\}} e^{-it b_n x} \phi(t V_n^{1/\beta}) \, dt = b_n^{-1} \left\{ V_n^{-1/\beta} b_n f(x V_n^{-1/\beta} b_n) - J_{n,x} \right\},
\]

(18)

where \( f \) is the density function of the distribution with characteristic function \( \phi \) and where

\[
J_{n,x} := \int_{\{|v| \leq n^\delta\}} e^{-ivx} \phi(v b_n^{-1} V_n^{1/\beta}) \, dv.
\]

By lemma 4 (applied with \( m = 1, t_1 = \theta_1 = 1, \gamma = \beta \), \( W_n := b_n V_n^{-1/\beta} \) converges almost surely, as \( n \to \infty \), to the constant \( \Gamma(\beta + 1)^{-1/\beta}(\pi A)^{1-1/\beta} \). Moreover, Lemma 10 ensures that the sequence \( (W_n, n \geq 1) \) is uniformly integrable, so actually the convergence holds in \( L^1 \). Let us deduce that

\[
\mathbb{E}[g_x(W_n)] = \mathbb{E}[g_x(W)] + o(1),
\]

(19)

where \( g_x : z \mapsto z f(xz) \) and the \( o(1) \) is uniform in \( x \). First

\[
|\mathbb{E}[g_x(W_n)] - \mathbb{E}[g_x(W)]| \leq \sup_{x, z \in \mathbb{R}} |(g_x)'(z)| \mathbb{E}[|W_n - W|]
\]

\[
\leq \sup_u |f(u) + uf'(u)| \mathbb{E}[|W_n - W|].
\]

This proves (19). We observe that \( \mathbb{E}[g_x(W)] = C(x) \).
In view of (18), it only remains to prove that $E[J_n x 1_{\Omega_n}] = o(1)$ uniformly in $x$. But this follows from the basic inequality

$$ E[|J_n x 1_{\Omega_n}|] \leq \int_{|v| \geq \eta} E \left[ e^{-A_1 |v|^\delta \frac{\Delta t}{b_n}} 1_{\Omega_n} \right] dv, $$

and from the lower bound for $V_n$ given in (10) and from the choice $\delta > (1 - \beta)_+/\beta$. □

3.4. Proof of Proposition 8. Recall that on $\Omega_n$, $N_n(y) \leq n^\gamma$, for all $y \in \mathbb{Z}^d$. Hence by (15),

$$ K_n := \int_{n^{\delta}/b_n}^{e \gamma n^{-\gamma}} E \left[ \prod_{y} (\xi(t N_n(y)) 1_{\Omega_n}) \right] dt \leq \int_{n^{\delta}/b_n}^{e \gamma n^{-\gamma}} E \left[ \exp \left( -\sigma t^\beta V_n \right) 1_{\Omega_n} \right] dt. $$

With the change of variable $s = tV_n^{1/\beta}$, we get

$$ K_n \leq \mathbb{E} \left[ V_n^{-1/\beta} \int_{n^{\delta}/b_n}^{e \gamma n^{-\gamma} V_n^{1/\beta}} \exp \left( -\sigma s^\beta \right) ds 1_{\Omega_n} \right], $$

$$ \leq \frac{1}{n^{\beta} \gamma (1 - \beta)_+} \int_{\delta - (1 - \beta)_+}^{+\infty} \exp \left( -\sigma s^\beta \right) ds, $$

which proves the proposition since $\delta > (1 - \beta)_+/\beta$.

3.5. Proof of Proposition 9. We adapt the proof of [5, Proposition 10]. We will see that the argument of "peaks" still works here. We endow $\mathbb{Z}^d$ with the ordered structure given by the relation $<$ defined by

$$(\alpha_1, ..., \alpha_d) < (\beta_1, ..., \beta_d) \iff \exists i \in \{1, ..., d\}, \ \alpha_i < \beta_i, \ \forall j < i, \ \alpha_j = \beta_j. $$

We consider $C^+ = (x_1, ..., x_T) \in (\mathbb{Z}^d \setminus \{0\})^T$ for some positive integer $T$ such that:

- $x_1 + ... + x_T = 0$;
- for every $i = 1, ..., T$, $\mathbb{P}(X_1 = x_i) > 0$;
- there exists $I_1 \in \{1, ..., T\}$ such that
  - for every $i = 1, ..., I_1$, $x_i > 0$,
  - for every $i = I_1 + 1, ..., T$, $x_i < 0$.  

Let us write $C^- = (x_{T-i+1})_{i=1}^T$. We define $B := \sum_{i=1}^I x_i$. We observe that

$$ p := \mathbb{P}((X_1, ..., X_T) = C^+) = \mathbb{P}((X_1, ..., X_T) = C^-) > 0. $$

We notice that $(X_1, ..., X_T) = C^+$ corresponds to a trajectory visiting $B$ only once before going back to the origin at time $T$ (and without visiting $-B$). Analogously, $(X_1, ..., X_T) = C^-$ corresponds to a trajectory that goes down to $-B$ and comes back up to 0 (and without visiting $B$), and staying at a distance smaller than $\tilde{d}/2$ of the origin with $\tilde{d} := \sum_{i=1}^T |x_i|$ (where $|\cdot|$ is the absolute value if $d = 1$ and $|(a, b)| = \max(|a|, |b|)$ if $d = 2$). We introduce now the event

$$ D_n := \left\{ C_n > \frac{np}{2T} \right\}, $$

where

$$ C_n := \# \left\{ k = 0, ..., \left\lfloor \frac{n}{T} \right\rfloor - 1 : (X_{kT+1}, ..., X_{(k+1)T}) = C^\pm \right\}. $$

Since the sequences $(X_{kT+1}, ..., X_{(k+1)T})$, for $k \geq 0$, are independent of each other, Chernoff's inequality implies that there exists $c > 0$ such that

$$ \mathbb{P}(D_n) = 1 - o(e^{-cn}). $$
We introduce now the notion of "loop". We say that there is a loop based on \( y \) at time \( n \) if \( S_n = y \) and \( (X_{n+1}, \ldots, X_{n+T}) = C^\pm \). We will see (in Lemma 13 below) that, on \( \Omega_n \cap D_n \), there is a large number of \( y \in \mathbb{Z}^d \) on which are based a large number of loops. For any \( y \in \mathbb{Z}^d \), let

\[
C_n(y) := \# \left\{ k = 0, \ldots, \left\lfloor \frac{n}{T} \right\rfloor - 1 : S_{kT} = y \text{ and } (X_{kT+1}, \ldots, X_{(k+1)T}) = C^\pm \right\},
\]

be the number of loops based on \( y \) before time \( n \) (and at times which are multiple of \( T \)), and let

\[
p_n := \# \left\{ y \in \mathbb{Z} : C_n(y) \geq \frac{\log \log(n)^{1/4} p}{4T} \right\},
\]

be the number of sites \( y \in \mathbb{Z} \) on which at least \( a_n := \left\lfloor \frac{\log \log(n)^{1/4} p}{4T} \right\rfloor \) loops are based.

**Lemma 13.** On \( \Omega_n \cap D_n \), we have, \( p_n \geq c'n^{1-\gamma} \) with \( c' = p/(4T) \).

**Proof.** Note that \( C_n(y) \leq N_n^* \) for all \( y \in \mathbb{Z}^d \). Thus on \( \Omega_n \cap D_n \), we have

\[
\frac{np}{2T} \leq \sum_{y \in \mathbb{Z}^d : C_n(y) < a_n} C_n(y) + \sum_{y \in \mathbb{Z}^d : C_n(y) \geq a_n} C_n(y)
\]

\[
\leq R_n a_n + N_n^* p_n \leq \frac{np}{4T} + p_n n^\gamma,
\]

according to lemma 6. This proves the lemma.

We have proved that, if \( n \) is large enough, the event \( \Omega_n \cap D_n \) is contained in the event

\[
\mathcal{E}_n := \{ p_n \geq c'n^{1-\gamma} \}.
\]

Now, on \( \mathcal{E}_n \), we consider \( (Y_i)_{i=1, \ldots, [c'n^{1-\gamma}]} \) (with \( c'' := c'/2\bar{d} \) if \( d = 1 \) and with \( c'' := c'/2\bar{d}^2 \) if \( d = 2 \)) such that

- on each \( Y_i \), at least \( a_n \) loops are based,
- for every \( i, j \) such that \( i \neq j \), we have \( |Y_i - Y_j| > \bar{d}/2 \).

For every \( i = 1, \ldots, [c''n^{1-\gamma}] \), let \( t_i^{(1)}, \ldots, t_i^{(a_n)} \) be the \( a_n \) first times (which are multiples of \( T \)) when a loop is based on the site \( Y_i \). We also define \( N_n^0(Y_i + B) \) as the number of visits of \( S \) before time \( n \) to \( Y_i + B \), which do not occur during the time intervals \( [t_i^{(j)}, t_i^{(j)} + T], \) for \( j \leq a_n \).

Since our construction is basically the same as in [5, section 2.8], the proof of the following lemma is exactly the same as the proof of [5, Lemma 16] and we do not prove it again.

**Lemma 14.** Conditionally to the event \( \mathcal{E}_n \), \( (N_n(Y_i + B) - N_n^0(Y_i + B))_{i \geq 1} \) is a sequence of independent identically distributed random variables with binomial distribution \( \mathcal{B}(a_n; \frac{1}{2}) \). Moreover this sequence is independent of \( (N_n^0(Y_i + B))_{i \geq 1} \).

Let \( \eta \) be a real number such that \( \gamma < \eta < (1 - \gamma)/\beta \) (this is possible since \( \gamma < 1/(\beta + 1) \)). We define

\[
\forall n \geq 1, \quad d_n := n^{-\eta}.
\]

Let now \( \rho := \sup \{|\varphi_{\mathcal{E}}(u)| : d \left(u, \frac{2\pi}{d_n} \mathbb{Z}\right) \geq c_0\} \). According to Formula (15) and since \( \lim_{n \to \infty} d_n = 0 \), for \( n \) large enough, we have

\[
|\varphi_{\mathcal{E}}(u)| \leq \rho \mathbf{1}_{d(u, 2\pi Z) \geq c_0} + \exp \left(-\sigma d \left(u, \frac{2\pi}{d_n} \mathbb{Z}\right)^\beta\right) \mathbf{1}_{d(u, 2\pi Z) < c_0}.
\]

\[
\leq \exp \left(-\sigma d_n^\beta\right),
\]
as soon as \( d \left( u, \frac{2\pi Z}{d_0} \right) \geq d_n. \) Therefore, for \( n \) large enough,
\[
\prod_z |\varphi_z(tN_n(z))| \leq \exp \left( -\sigma \frac{d_n}{d_0} \# \left\{ z : d \left( tN_n(z), \frac{2\pi Z}{d_0} \right) \geq d_n \right\} \right).
\]
(20)

Then notice that
\[
d \left( tN_n(z), \frac{2\pi Z}{d_0} \right) \geq d_n \iff N_n(z) \in I := \bigcup_{k \in \mathbb{Z}} I_k,
\]
where for all \( k \in \mathbb{Z},
\[
I_k := \left[ \frac{2k\pi}{d_0} + \frac{d_n}{t}, \frac{2(k+1)\pi}{d_0} - \frac{d_n}{t} \right].
\]
In particular \( \mathbb{R} \setminus I = \bigcup_{k \in \mathbb{Z}} J_k \), where for all \( k \in \mathbb{Z},
\[
J_k := \left( \frac{2k\pi}{d_0} - \frac{d_n}{t}, \frac{2k\pi}{d_0} + \frac{d_n}{t} \right).
\]

**Lemma 15.** Under the hypotheses of Proposition 9, for every \( i \leq \left\lfloor c^n n^{1-\gamma} \right\rfloor \), \( t \in (\varepsilon_0 n^{-\gamma}, \pi/d_0) \) and \( n \) large enough,
\[
\mathbb{P} \left( N_n(Y_i + B) \in I \mid E_n, \ N_n^0(Y_i + B) \right) \geq \frac{1}{3} \quad \text{almost surely.}
\]

Assume for a moment that this lemma holds true and let us finish the proof of Proposition 9. Lemmas 14 and 15 ensure that conditionally to \( E_n \) and \( (N_n^0(Y_i + B), i \geq 1) \), the events \( \{N_n(Y_i + B) \in I\}, i \geq 1, \) are independent of each other, and all happen with probability at least \( 1/3 \). Therefore, since \( \Omega_n \cap D_n \subseteq E_n \), there exists \( c > 0 \), such that
\[
\mathbb{P} \left( \Omega_n \cap D_n, \ \# \left\{ i : N_n(Y_i + B) \in I \right\} \leq \frac{c^n n^{1-\gamma}}{4} \right) \leq \mathbb{P} \left( B_n \leq \frac{c^n n^{1-\gamma}}{4} \right) = o(\exp(-c n^{1-\gamma})),
\]
where for all \( n \geq 1 \), \( B_n \) has binomial distribution \( B \left( \left\lfloor c^n n^{1-\gamma} \right\rfloor ; \frac{1}{3} \right) \).

But if \( \# \left\{ z : N_n(z) \in I \right\} \geq \frac{c^n n^{1-\gamma}}{4} \), then by (20) and (21) there exists a constant \( c > 0 \), such that
\[
\prod_z |\varphi_z(tN_n(z))| \leq \exp \left( -c n^{1-\gamma} d_n^2 \right),
\]
which proves Proposition 9 since \( 1 - \gamma - \beta \eta > 0 \).

**Proof of Lemma 15.** First notice that by Lemma 14, for any \( H \geq 0, \)
\[
\mathbb{P} \left( N_n(Y_i + B) \in I \mid E_n, \ N_n^0(Y_i + B) = H \right) = \mathbb{P} \left( H + b_n \in I \right),
\]
where \( b_n \) is a random variable with binomial distribution \( B \left( a_n; \frac{1}{3} \right) \). We will use the following result whose proof is postponed.

**Lemma 16.** Under the hypotheses of Proposition 9, for every \( t \in (\varepsilon_0 n^{-\gamma}, \pi/d_0) \) and \( n \) large enough, the following holds:

(i) For any integer \( k \) such that all the elements of \( I_k - H \) are smaller than \( \frac{a_n}{2} \),
\[
\mathbb{P}(b_n \in (I_k - H)) \geq \mathbb{P}(b_n \in (J_k - H)).
\]

(ii) For any integer \( k \) such that all the elements of \( I_k - H \) are larger than \( \frac{a_n}{2} \),
\[
\mathbb{P}(b_n \in (I_k - H)) \geq \mathbb{P}(b_n \in (J_{k+1} - H)).
\]
Now call $k_0$ the largest integer satisfying the condition appearing in (i) and $k_1$ the smallest integer satisfying the condition appearing in (ii). We have $k_1 = k_0 + 1$ or $k_1 = k_0 + 2$. According to Lemma 16, we have

$$\mathbb{P}(H + b_n \in I_k) \geq \sum_{k \leq k_0} \mathbb{P}(H + b_n \in I_k) + \sum_{k \geq k_1} \mathbb{P}(H + b_n \in I_k)$$

$$\geq \sum_{k \leq k_0} \mathbb{P}(H + b_n \in J_k) + \sum_{k \geq k_1} \mathbb{P}(H + b_n \in J_{k+1})$$

$$= \mathbb{P}(H + b_n \not\in I) - \mathbb{P}(H + b_n \in J_{k_0+1} \cup J_1).$$

Hence,

$$\mathbb{P}(H + b_n \in I) \geq \frac{1}{2} [1 - \mathbb{P}(H + b_n \in J_{k_0+1} \cup J_1)].$$

Let $\bar{b}_n := 2 \left( b_n - \frac{a_n}{2} \right) \sqrt{a_n}$. Since $\lim_{n \to +\infty} a_n = +\infty$, $(\bar{b}_n)_n$ converges in distribution to a standard normal variable, whose distribution function is denoted by $\Phi$. The interval $J_{k_1}$ being of length $2d_n/t$,

$$\mathbb{P}(H + b_n \in J_{k_1}) = \mathbb{P}(\bar{b}_n \in [m_n, M_n]), \text{ with } M_n - m_n = 4 \frac{d_n}{t \sqrt{a_n}}$$

$$\leq \Phi(M_n) - \Phi(m_n) + C \frac{M_n - m_n}{\sqrt{a_n}} \quad \text{(by the Berry–Esseen inequality)}$$

$$\leq \frac{M_n - m_n}{\sqrt{2\pi}} + C \frac{d_n}{\sqrt{a_n}}$$

$$\leq C' \frac{d_n}{\varepsilon_0 n^{-\gamma} \sqrt{a_n}} + C \frac{d_n}{\sqrt{a_n}},$$

for $t \geq \varepsilon_0 n^{-\gamma}$, and some constants $C > 0$ and $C' > 0$. Since $\lim_{n \to +\infty} a_n = +\infty$ and $\lim_{n \to +\infty} d_n n^\eta (a_n)^{-1/2} = 0$ (since $\eta > \gamma$), we conclude that $\mathbb{P}(H + b_n \in J_{k_1}) = o(1)$. The same holds for $\mathbb{P}(H + b_n \in J_{k_0+1})$, so that for $n$ large enough,

$$\mathbb{P}(H + b_n \in I) \geq \frac{1}{2} [1 - o(1)] \geq \frac{1}{3}.$$

Together with (22), this concludes the proof of Lemma 15.

Proof of Lemma 16. We only prove (i), since (ii) is similar. So let $k$ be an integer such that all the elements of $I_k - H$ are smaller than $\frac{a_n}{2}$. Assume that $(J_k - H) \cap \mathbb{Z}$ contains at least one nonnegative integer (otherwise $\mathbb{P}(b_n \in (J_k - H)) = 0$ and there is nothing to prove). Let $z_k$ denote the greatest integer in $J_k - H$, so that by our assumption $\mathbb{P}(b_n = z_k) > 0$ (remind that $0 \leq z_k < \frac{a_n}{2}$). By monotonicity of the function $z \mapsto \mathbb{P}(b_n = z)$, for $z \leq \frac{a_n}{2}$, we get

$$\mathbb{P}(b_n \in J_k - H) \leq \mathbb{P}(b_n = z_k) \#((J_k - H) \cap \mathbb{Z}) \leq \mathbb{P}(b_n = z_k) \left[ \frac{2d_n}{t} \right].$$

In the same way,

$$\mathbb{P}(b_n \in I_k - H) \geq \mathbb{P}(b_n = z_k) \#((I_k - H) \cap \mathbb{Z}) \geq \mathbb{P}(b_n = z_k) \left[ \frac{2d_n}{t} \right].$$

Hence

$$\mathbb{P}(b_n \in I_k - H) \geq \frac{2d_n}{t} \left[ \frac{2d_n}{t} \right] \mathbb{P}(b_n \in J_k - H).$$
But $\pi/(d_0 t) \geq 1$ and $\lim_{n \to +\infty} d_n = 0$ by hypothesis. It follows immediately that for $n$ large enough, we have $2d_n < \pi/(2d_0 t)$, and so

$$\frac{2\pi}{d_0 t} - \frac{2d_n}{t} \geq \frac{3\pi}{2d_0 t} \geq 1 + \frac{\pi}{2d_0 t} \geq \left\lceil \frac{\pi}{2d_0 t} \right\rceil \geq \left\lceil \frac{2d_n}{t} \right\rceil .$$

This concludes the proof of the lemma.

4. Proof of the Local Limit Theorem in the Strongly Nonlattice Case

As in [5], the proof in the strongly nonlattice case is closely related to the proof in the lattice case.

We assume here that $\xi$ is strongly nonlattice. In that case, there exist $\varepsilon_0 > 0$, $\sigma > 0$ and $\rho < 1$ such that $|\varphi_\xi(u)| \leq \rho$ if $|u| \geq \varepsilon_0$ and $|\varphi_\xi(u)| \leq \exp(-\sigma|u|^\beta)$ if $|u| < \varepsilon_0$.

We use here the notations of Section 3 with the hypotheses on $\gamma$, and $\delta$ of Proposition 7. Let $h_0$ be the density of Polya's distribution: $h_0(y) = \frac{1}{\pi} \frac{1-\cos(y)}{y^2}$, with Fourier transform $\hat{h}_0(t) = (1 - |t|)_+$. For $\theta \in \mathbb{R}$, let $h_{\theta}(y) = \exp(i\theta y) h_0(y)$ with Fourier transform $\hat{h}_\theta(t) = \hat{h}_0(t + \theta)$. As in [10, thm 5.4], it is enough to show that for all $\theta \in \mathbb{R}$,

$$\lim_{n \to \infty} b_n \mathbb{E}[h_{\theta}(Z_n - b_n x)] = C(x) \hat{h}_\theta(0). \quad (23)$$

By Fourier inverse transform, we have

$$b_n \mathbb{E}[h_{\theta}(Z_n - b_n x)] = \frac{b_n}{2\pi} \int_{\mathbb{R}} e^{-iub_n x} \mathbb{E} \left[ \prod_{x \in \mathbb{Z}^d} \varphi_\xi(uN_n(x)) \right] \hat{h}_\theta(u) \, du .$$

Since $\hat{h}_\theta \in L^1$, we can restrict our study to the event $\Omega_n$ of Lemma 6. The part of the integral corresponding to $|u| \leq n^\beta b_n^{-1}$ is treated exactly as in Proposition 7. The only change is that we have to check that

$$\lim_{n \to \infty} b_n \int_{\{|u| \leq n^\beta b_n^{-1}\}} \mathbb{E} \left[ e^{-|u|^\beta V_n(A_1 + iA_2 \text{sgn}(u))} 1_{\Omega_n} \right] (\hat{h}_\theta(u) - \hat{h}_\theta(0)) \, du = 0 ,$$

which is obviously true since $V_n \geq n^{1-\gamma(1-\beta)}_+$ and since $2\gamma(1 - \beta)_+ < 2\delta \beta < 1$, using the fact that $\hat{h}_\theta$ is a Lipschitz function.

Now, since $\hat{h}_\theta$ is bounded, the part corresponding to $n^\beta b_n^{-1} \leq |u| \leq \varepsilon_0 n^{-\gamma}$ is treated as in the proof of Proposition 8 (since it only uses the behavior of $\varphi_\xi$ around 0, which is the same).

Finally, it remains to prove that

$$\lim_{n \to \infty} b_n \int_{\{|u| \geq \varepsilon_0 n^{-\gamma}\}} e^{-iub_n x} \mathbb{E} \left[ \prod_{x \in \mathbb{Z}^d} \varphi_\xi(uN_n(x)) 1_{\Omega_n} \right] \hat{h}_\theta(u) \, du = 0 . \quad (24)$$

We note that, if $|u| \geq \varepsilon_0 n^{-\gamma}$ and $x \in \mathbb{Z}^d$, we have

$$|\varphi_\xi(uN_n(x))| \leq \exp(-\sigma|u|^\beta N_n^\beta(x)) 1_{\{\|uN_n(x)\| \leq \varepsilon_0\}} + \rho 1_{\{\|uN_n(x)\| \geq \varepsilon_0\}} \leq \exp(-\sigma \varepsilon_0^\beta n^{-\gamma \beta} N_n^\beta(x)) 1_{\{\|uN_n(x)\| \leq \varepsilon_0\}} + \rho 1_{\{\|uN_n(x)\| \geq \varepsilon_0\}} .$$

For $n$ large enough, $\rho \leq \exp(-\sigma \varepsilon_0^\beta n^{-\gamma \beta})$. Therefore, if $n$ is large enough, then for all $x$ and $u$ such that $N_n(x) \geq 1$ and $|u| \geq \varepsilon_0 n^{-\gamma}$, we have

$$|\varphi_\xi(uN_n(x))| \leq \exp(-\sigma \varepsilon_0^\beta n^{-\gamma \beta}) .$$
Hence,
\[
|E \left[ \prod_x \phi_\xi(uN_n(x)) 1_{\Omega_n} \right]| \leq E \left[ \exp(-\sigma \varepsilon_0^\beta n^{-\gamma(1+\beta)} R_n) 1_{\Omega_n} \right] \leq \exp(-\sigma \varepsilon_0^\beta n^{1-\gamma(1+\beta)}) .
\]
Therefore, since \(\gamma(1+\beta) < 1\), we have
\[
\lim_{n \to \infty} b_n \int_{\{|u| \geq \varepsilon_0 n^{-\gamma}\}} e^{-iub_n x} E \left[ \prod_x \phi_\xi(uN_n(x)) 1_{\Omega_n} \right] \hat{h}_\theta(u) \, du = 0 .
\]
This concludes the proof of Theorem 3. □

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**References**


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