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LIMIT THEOREMS FOR ONE AND TWO-DIMENSIONAL RANDOM WALKS IN RANDOM SCENERY

FABIENNE CASTELL, NADINE GUILLOTIN-PLANTARD, AND FRANÇOISE PÈNE

Abstract. Random walks in random scenery are processes defined by
\[ Z_n := \sum_{k=1}^{n} \xi X_k + \ldots + X_k, \]
where \((X_k, k \geq 1)\) and \((\xi_y, y \in \mathbb{Z}^d)\) are two independent sequences of i.i.d. random variables with
values in \(\mathbb{Z}^d\) and \(\mathbb{R}\) respectively. We suppose that the distributions of \(X_1\) and \(\xi_0\) belong to the
normal basin of attraction of stable distribution of index \(\alpha \in (0, 2]\) and \(\beta \in (0, 2].\) When \(d = 1\)
and \(\alpha \neq 1,\) a functional limit theorem has been established in [11] and a local limit theorem
in [5]. In this paper, we establish the convergence of the finite-dimensional distributions and a
local limit theorem when \(\alpha = d\) (i.e. \(\alpha = d = 1\) or \(\alpha = d = 2\)) and \(\beta \in (0, 2].\) Let us mention
that functional limit theorems have been established in [2] and recently in [8] in the particular
case where \(\beta = 2\) (respectively for \(\alpha = d = 2\) and \(\alpha = d = 1\)).

1. Introduction

Random walks in random scenery (RWRS) are simple models of processes in disordered media
with long-range correlations. They have been used in a wide variety of models in physics to
study anomalous dispersion in layered random flows [14], diffusion with random sources, or spin
depolarization in random fields (we refer the reader to Le Doussal’s review paper [12] for a
discussion of these models).

On the mathematical side, motivated by the construction of new self-similar processes with
stationary increments, Kesten and Spitzer [11] and Borodin [3, 4] introduced RWRS in dimension
one and proved functional limit theorems. This study has been completed in many works, in
particular in [2] and [8]. These processes are defined as follows. Let \(\xi := (\xi_y, y \in \mathbb{Z}^d)\) and
\(X := (X_k, k \geq 1)\) be two independent sequences of independent identically distributed random
variables taking values in \(\mathbb{R}\) and \(\mathbb{Z}^d\) respectively. The sequence \(\xi\) is called the random scenery.
The sequence \(X\) is the sequence of increments of the random walk \((S_n, n \geq 0)\) defined by \(S_0 := 0\)
and \(S_n := \sum_{i=1}^{n} X_i,\) for \(n \geq 1.\) The random walk in random scenery \(Z\) is then defined by
\[ Z_0 := 0 \text{ and } \forall n \geq 1, \ Z_n := \sum_{k=0}^{n-1} \xi S_k. \]
Denoting by \(N_n(y)\) the local time of the random walk \(S:\)
\[ N_n(y) := \# \{ k = 0, \ldots, n - 1 : S_k = y \}, \]
it is straightforward to see that \(Z_n\) can be rewritten as \(Z_n = \sum_{y} \xi_y N_n(y).\)

As in [11], the distribution of \(\xi_0\) is assumed to belong to the normal domain of attraction of
a strictly stable distribution \(S_\beta\) of index \(\beta \in (0, 2],\) with characteristic function \(\phi\) given by
\[ \phi(u) = e^{-|u|^\beta(A_1 + iA_2 \text{sgn}(u))} \quad u \in \mathbb{R}, \]

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Our first result is concerned with a functional limit theorem for a functional limit theorem. Our first result gives a limit theorem for the distribution of a stable distribution. Then the following weak convergences hold in the space of càdlàg real-valued functions defined on the real line when

\[ \sup_{t>0} |\mathbb{E} [\xi_0 \mathbb{1}_{\{\xi_0 \leq t\}}]| < +\infty. \]  

(1)

Under these conditions (for \( \beta \in (0;2] \)), there exists \( C_\xi > 0 \) such that we have

\[ \forall t > 0, \ \mathbb{P} (|\xi_t| \geq t) \leq C_\xi t^{-\beta}. \]  

(2)

Concerning the random walk, the distribution of \( X_1 \) is assumed to belong to the normal basin of attraction of a stable distribution \( S_\alpha \) with index \( \alpha \in (0,2] \).

Then the following weak convergences hold in the space of càdlàg real-valued functions defined on \([0,\infty)\) and on \( \mathbb{R} \) respectively, endowed with the Skorohod \( J_1 \)-topology (see [1, chapter 3]):

\[
\left( n^{-1/\alpha} S_{[nt]} \right)_{t \geq 0} \xrightarrow{\mathcal{L}} (U(t))_{t \geq 0}
\]

and

\[
\left( n^{-\beta} \sum_{k=0}^{\lfloor nx \rfloor} \xi_{ke_1} \right)_{x \in \mathbb{R}} \xrightarrow{\mathcal{L}} (Y(x))_{x \in \mathbb{R}}, \text{ with } e_1 = (1,0,\ldots,0) \in \mathbb{Z}^d,
\]

where \( U \) and \( Y \) are two independent Lévy processes such that \( U(0) = 0, Y(0) = 0, U(1) \) has distribution \( S_\alpha \), \( Y(1) \) and \( Y(-1) \) have distribution \( S_\beta \).

**Functional limit theorem.**

Our first result is concerned with a functional limit theorem for \( (Z_{[nt]})_{t \geq 0} \). Intuitively speaking,

- when \( \alpha < d \), the random walk \( S_n \) is transient, its range is of order \( n \), and \( Z_n \) has the same behaviour as a sum of about \( n \) independent random variables with the same distribution as the variables \( \xi_x \). Therefore, \( n^{-1/\beta} (Z_{[nt]})_{t \geq 0} \) weakly converges in the space \( D([0,\infty)) \) of càdlàg functions endowed with the Skorohod \( J_1 \)-topology, to a multiple of the process \( (Y_t) \), as proved in [4];
- when \( \alpha > d \) (i.e. \( d = 1 \) and \( 1 < \alpha \leq 2 \)), the random walk \( S_n \) is recurrent, its range is of order \( n^{1/\alpha} \), its local times are of order \( n^{1-1/\alpha} \), so that \( Z_n \) is of order \( n^{1-1/\alpha + 1/\beta} \). In this situation, [3] and [11] proved a functional limit theorem for \( n^{-(1-1/\alpha + 1/\beta)} (Z_{[nt]})_{t \geq 0} \) in the space \( C([0,\infty)) \) of continuous functions endowed with the uniform topology, the limiting process being a self-similar process, but not a stable one;
- when \( \alpha = d \) (i.e. \( \alpha = d = 1 \), or \( \alpha = d = 2 \)), \( S_n \) is recurrent, its range is of order \( n/\log(n) \), its local times are of order \( \log(n) \) so that \( Z_n \) is of order \( n^{1/\alpha \log(n)} \). In this situation, a functional limit theorem in the space of continuous functions was proved in [2] for \( d = \alpha = \beta = 2 \), and in [8] for \( d = \alpha = 1 \) and \( \beta = 2 \).

Our first result gives a limit theorem for \( \alpha = d \) (and so \( d \in \{1,2\} \)) and for any value of \( \beta \in (0;2) \) in the finite distributional sense.

**Theorem 1.** Let us assume that \( \beta \in (0;2] \) and that

(a) either \( d = 2 \) and \( X_1 \) is centered, square integrable with invertible variance matrix \( \Sigma \) and then we define \( A := 2\sqrt{\det \Sigma} \);

(b) or \( d = 1 \) and \( (\xi_{nt})_n \) converges in distribution to a random variable with characteristic function given by \( t \mapsto \exp(-a|t|) \) with \( a > 0 \) and then we define \( A := a \).
Then, the finite-dimensional distributions of the sequence of random variables
\[
\left( \frac{Z_{[nt]}}{n^{1/\beta} \log(n)^{(\beta-1)/\beta}} \right)_{t \geq 0} \bigg|_{n \geq 2}
\]
converges to the finite-dimensional distributions of the process
\[
\left( \tilde{Y}_t := \left( \frac{\Gamma(\beta + 1)}{\pi A^{\beta-1}} \right)^{1/\beta} Y(t) \right)_{t \geq 0}.
\]
Moreover, if \( \beta < 2 \), the sequence
\[
\left( \frac{Z_{[nt]}}{n^{1/\beta} \log(n)^{(\beta-1)/\beta}} \right)_{t \geq 0} \bigg|_{n \geq 2}
\]
is not tight in \( D([0, \infty)) \) endowed with the \( J_1 \)-topology.

**Local limit theorem.**

Our next results concern a local limit theorem for \((Z_n)_n\). The \( d = 1 \) case was treated in [5] for \( \alpha \in (0; 2] \setminus \{1\} \) and all values of \( \beta \in (0; 2] \). Here, we complete this study by proving a local limit theorem for \( \alpha = d = 1 \) (and \( \beta \in (0; 2] \)). By a direct adaptation of the proof of this result, we also establish a local limit theorem for \( \alpha = d = 2 \) (we just adapt the definition of "peaks", see section 3.5). Let us notice that the same adaptation can be done from [5] (case \( \alpha < 1 \)) to get local limit theorems for \( d \geq 2 \), \( \alpha < d \) and \( \beta \in (0; 2] \).

We give two results corresponding respectively to the case when \( \xi_0 \) is lattice and to the case when it is strongly non-lattice. We denote by \( \varphi_\xi \) the characteristic function of \( \xi_0 \).

**Theorem 2.** Assume that \( \xi_0 \) takes its values in \( \mathbb{Z} \). Let \( d_0 \geq 1 \) be the integer such that \( \{ u : |\varphi_\xi(u)| = 1 \} = \frac{2\pi}{d_0} \mathbb{Z} \). Let \( b_n := n^{1/\beta}(\log(n))^{(\beta-1)/\beta} \). Under the previous assumptions on the random walk and on the scenery, for \( \alpha = d \in \{1, 2\} \), for every \( \beta \in (0, 2] \), and for every \( x \in \mathbb{R} \),

- if \( \mathbb{P}(n\xi_0 - |b_n x| \notin d_0 \mathbb{Z}) = 1 \), then \( \mathbb{P}(Z_n = |b_n x|) = 0 \);
- if \( \mathbb{P}(n\xi_0 - |b_n x| \notin d_0 \mathbb{Z}) = 1 \), then
  \[
  \mathbb{P}(Z_n = |b_n x|) = d_0 n^{1/\beta}(\log(n))^{(\beta-1)/\beta} + o(n^{-1/\beta}(\log(n))^{-(\beta-1)/\beta})
  \]
uniformly in \( x \in \mathbb{R} \), where \( C(\cdot) \) is the density function of \( \tilde{Y}_1 \).

**Theorem 3.** Assume now that \( \xi_0 \) is strongly non-lattice which means that
\[
\limsup_{|u| \to +\infty} |\varphi_\xi(u)| < 1.
\]
We still assume that \( \alpha = d \in \{1, 2\} \) and \( \beta \in (0; 2] \). Then, for every \( x, a, b \in \mathbb{R} \) such that \( a < b \), we have
\[
\lim_{n \to +\infty} b_n \mathbb{P}(Z_n \in [b_n x + a; b_n x + b]) = C(x)(b - a),
\]
with \( b_n := n^{1/\beta}(\log(n))^{(\beta-1)/\beta} \) and where \( C(\cdot) \) is the density function of \( \tilde{Y}_1 \).
Before proving the theorem, we prove some technical lemmas. For any real number $\gamma > 0$, any integer $m \geq 1$, any $\theta_1, \ldots, \theta_m \in \mathbb{R}$, any $t_0 = 0 < t_1 < \ldots < t_m$, we consider the sequences of random variables $(L_n(\gamma))_{n \geq 2}$ and $(L'_n(\gamma))_{n \geq 2}$ defined by

$$L_n(\gamma) := \frac{1}{n(\log n)^{\gamma-1}} \sum_{x \in \mathbb{Z}^d} \left| \sum_{i=1}^{m} \theta_i N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x) \right|^\gamma$$

and

$$L'_n(\gamma) := \frac{1}{n(\log n)^{\gamma-1}} \sum_{x \in \mathbb{Z}^d} \left| \sum_{i=1}^{m} \theta_i (N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)) \right|^\gamma \text{sgn} \left( \sum_{i=1}^{m} \theta_i (N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)) \right).$$

**Lemma 4.** For any real number $\gamma > 0$, any integer $m \geq 1$, any $\theta_1, \ldots, \theta_m \in \mathbb{R}$, any $t_0 = 0 < t_1 < \ldots < t_m$, the following convergences hold $\mathbb{P}$-almost surely

$$\lim_{n \to +\infty} L_n(\gamma) = \frac{\Gamma(\gamma+1)}{(\pi A)^{\gamma-1}} \sum_{i=1}^{m} |\theta_i|^\gamma (t_i - t_{i-1})$$

and

$$\lim_{n \to +\infty} L'_n(\gamma) = \frac{\Gamma(\gamma+1)}{(\pi A)^{\gamma-1}} \sum_{i=1}^{m} |\theta_i|^\gamma \text{sgn}(\theta_i)(t_i - t_{i-1}).$$

**Proof.** We fix an integer $m \geq 1$ and 2$m$ real numbers $\theta_1, \ldots, \theta_m, t_1, \ldots, t_m$ such that $0 < t_1 < \ldots < t_m$ and we set $t_0 := 0$. To simplify notations, we write $b_{i,n}(x) := N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)$. Following the techniques developed in [6], we first have to prove (3) and (4) for integer $k$: for every integer $k \geq 1$, $\mathbb{P}$-almost surely, as $n$ goes to infinity, we have

$$\frac{1}{n(\log n)^{k-1}} \sum_{x \in \mathbb{Z}^d} \left( \sum_{i=1}^{m} \theta_i b_{i,n}(x) \right)^k \to \frac{\Gamma(k+1)}{(\pi A)^{k-1}} \sum_{i=1}^{m} \theta_i^k (t_i - t_{i-1}).$$

Let us assume (5) for a while, and let us end the proof of (3) and (4) for any positive real $\gamma$. Given the random walk $S := (S_n)_{n}$, let $(U_n)_{n \geq 1}$ be a sequence of random variables with values in $\mathbb{Z}^d$, such that for all $n$, $U_n$ is a point chosen uniformly in the range of the random walk up to time $[nt_m]$, that is

$$\mathbb{P}(U_n = x|S) = R_{[nt_m]}^{-1}(x) 1_{\{N_{[nt_m]}(x) \geq 1\}},$$

with $R_k := \#\{y : N_k(y) > 0\}$. Moreover, let $U'$ be a random variable with values in $\{1, \ldots, m\}$ and distribution

$$\mathbb{P}(U' = i) = (t_i - t_{i-1})/t_m$$

and let $T$ be a random variable with exponential distribution with parameter one and independent of $U'$. Then, for $\mathbb{P}$—almost every realization of the random walk $S$, the sequence of random variables

$$W_n := \frac{\pi A}{\log(n)} \sum_{i=1}^{m} \theta_i b_{i,n}(U_n)$$

converges in distribution to the random variable $W := \theta_U T$. Indeed, the moment of order $k$ of $W_n$ given $S$ is

$$\mathbb{E}(W_n^k|S) = \frac{(\pi A)^k}{n(\log n)^{k-1}} \sum_{x \in \mathbb{Z}^d} \left( \sum_{i=1}^{m} \theta_i b_{i,n}(x) \right)^k \frac{n}{\log(n)R([nt_m])}.$$
Using (5) and the fact that \(((\log n) R_n/n)_n\) converges almost surely to \(\pi A\) (see [9, 13]), the moments \(\mathbb{E}(W^n_k|S)\) converges a.s. to \(\mathbb{E}(W^k) = \Gamma(k + 1) \sum_{i=1}^{m} \theta_i^k (t_i - t_{i-1})/t_m\), which proves the convergence in distribution of \((W_n)_n\) (given \(S\)) to \(W\). This ensures, in particular, the convergence in distribution of \((|W_n|^\gamma)_n\) and \((|W_n|^\gamma \text{sgn}(W_n))_n\) (given \(S\)) to \(|W|^\gamma\) and \(|W|^\gamma \text{sgn}(W)\) respectively (for every real number \(\gamma \geq 0\) and for \(P-\) almost every realization of the random walk \(S\)). Since any moment of \(|W_n|\) can be bounded from above by an integer moment, we deduce that, for any \(\gamma \geq 0\), we have \(P\)-almost surely

\[
\lim_{n \to +\infty} \mathbb{E}(|W_n|^\gamma |S) = \mathbb{E}(|W|^\gamma) \quad \text{and} \quad \lim_{n \to +\infty} \mathbb{E}(|W_n|^\gamma \text{sgn}(W_n) |S) = \mathbb{E}(|W|^\gamma \text{sgn}(W)),
\]

which proves lemma 4. Let us prove (5). Let \(k \geq 1\). According to Theorem 1 in [6] (proved for \(\alpha = d = 2\), but also valid for \(\alpha = d = 1\)), we have

\[
\forall i \in \{1, \ldots, m\}, \quad \lim_{n \to +\infty} \frac{1}{n(\log n)^{k-1}} \sum_{x \in \mathbb{Z}^d} (b_i, n(x))^k = \frac{\Gamma(k + 1)}{(\pi A)^{k-1}} (t_i - t_{i-1})^k, \quad P-\text{a.s.} \quad (6)
\]

We define

\[
\Sigma_n(\theta_1, \ldots, \theta_m) := \sum_{x \in \mathbb{Z}^d} \left( \sum_{i=1}^{m} \theta_i b_i, n(x) \right)^k - \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{m} (\theta_i)^k (b_i, n(x))^k.
\]

According to (6), it is enough to prove that \(P-\text{a.s.}, \Sigma_n(\theta_1, \ldots, \theta_m) = o(n(\log n)^{k-1})\). We observe that \(\Sigma_n(\theta_1, \ldots, \theta_m)\) is the sum of the following terms

\[
\sum_{x \in \mathbb{Z}^d} \prod_{j=1}^{k} \left( \theta_{i_j} b_{i_j, n}(x) \right).
\]

over all the \(k\)-tuple \((i_1, \ldots, i_k) \in \{1, \ldots, m\}^k\), with at least two distinct indices. We observe that

\[
|\Sigma_n(\theta_1, \ldots, \theta_m)| \leq \max(|\theta_1|, \ldots, |\theta_m|)^k \Sigma_n(1, \ldots, 1).
\]

But, we have

\[
\Sigma_n(1, \ldots, 1) = \sum_{x \in \mathbb{Z}^d} \left( N_{[nt_m]}(x) \right)^k - \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{m} (b_i, n(x))^k
\]

\[
= \sum_{x \in \mathbb{Z}^d} \left( N_{[nt_m]}(x) \right)^k - \sum_{i=1}^{m} \sum_{x \in \mathbb{Z}^d} (b_i, n(x))^k = o(n \log(n)^{k-1}),
\]

according to (6). \(\square\)

**Lemma 5.** For any \(\rho > 0\),

\[
\sup_{x \in \mathbb{Z}^d} N_n(x) = o(n^\rho) \quad \text{a.s.}
\]

**Proof.** See Lemma 2.5 in [2]. \(\square\)

**Proof of Theorem 1.** Let an integer \(m \geq 1\) and \(2m\) real numbers \(\theta_1, \ldots, \theta_m, t_1, \ldots, t_m\) such that \(0 < t_1 < \ldots < t_m\). We set \(t_0 := 0\). Again, we use the notation \(b_i, n(x) := N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)\). Let us write \(Z_n := \frac{1}{n^\beta(\log(n))^{\beta-1/\beta}} \sum_{i=1}^{m} \theta_i \left( Z_{[nt_i]} - Z_{[nt_{i-1}]} \right)\). We have to prove that

\[
\mathbb{E}[e^{iZ_n}] \to \prod_{i=1}^{m} \phi \left( \theta_i (t_i - t_{i-1})^{1/\beta} \left( \frac{\Gamma(\beta + 1)}{(\pi A)^{\beta-1/\beta}} \right)^{1/\beta} \right), \quad (9)
\]

\[
\lim_{n \to +\infty} \mathbb{E}(|W_n|^\gamma |S) = \mathbb{E}(|W|^\gamma) \quad \text{and} \quad \lim_{n \to +\infty} \mathbb{E}(|W_n|^\gamma \text{sgn}(W_n) |S) = \mathbb{E}(|W|^\gamma \text{sgn}(W)),
\]
as \( n \) goes to infinity. We observe that \( \tilde{Z}_n = \frac{1}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}} \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^m \theta_i b_{i,n}(x) \xi_x \). Hence we have

\[
\mathbb{E}[e^{i\tilde{Z}_n|S}] = \prod_{x \in \mathbb{Z}^d} \varphi_x \left( \frac{\sum_{i=1}^m \theta_i b_{i,n}(x)}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}} \right).
\]

Observe next that

\[
|\varphi_x(t) - \exp\left(-|t|^\beta (A_1 + i A_2 \text{sgn}(t))\right| \leq |t|^\beta h(|t|) \quad \text{for all } t \in \mathbb{R},
\]

with \( h \) a continuous and monotone function on \([0, +\infty)\) vanishing in 0. This implies in particular the existence of \( \varepsilon_0 > 0 \) and \( \sigma > 0 \) such that \( \max(|\varphi_x(t)|, \exp(-A_1|t|^\beta)) \leq e^{-\sigma|t|^\beta} \) for any \( t \in [-\varepsilon_0, \varepsilon_0] \). According to lemma 5, \( \mathbb{P} \)-almost surely, for every \( n \) large enough, we have

\[
b_n := \sup_x \frac{|\sum_{i=1}^m \theta_i b_{i,n}(x)|}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}} \leq \varepsilon_0
\]

and so

\[
\mathbb{E}[e^{i\tilde{Z}_n|S}] - \prod_{x \in \mathbb{Z}^d} e^{\frac{|\sum_{i=1}^m \theta_i b_{i,n}(x)|^\beta}{n(\log(n))^{\beta-1}} (A_1 + i A_2 \text{sgn}(\sum_{i=1}^m \theta_i b_{i,n}(x)))} \leq \sum_{x \in \mathbb{Z}^d} |\sum_{i=1}^m \theta_i b_{i,n}(x)|^\beta h(b_n) e^{-\sigma \left( \frac{\sum_{x \in \mathbb{Z}^d} |\sum_{i=1}^m \theta_i b_{i,n}(x)|^\beta}{n(\log(n))^{\beta-1}} - b_n^\beta \right)}.
\]

Hence, according to lemmas 4 and 5, \( \mathbb{P} \)-almost surely, we have

\[
\lim_{n \to +\infty} \mathbb{E}[e^{i\tilde{Z}_n|S}] = e^{-\frac{r(\beta+1)}{(\pi A)^{\beta+1}}} \sum_{i=1}^m |\theta_i|^\beta (t_i - t_{i-1})(A_1 + i A_2 \text{sgn}(\theta_i))
\]

which gives (9) thanks to the Lebesgue dominated convergence theorem.

Finally we prove that the sequence

\[
\left( \frac{Z_{[nt]}}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}} \right)_{t \in [0,1]}
\]

is not tight in \( \mathcal{D}([0, \infty)) \). It is enough to prove that it is not tight in \( \mathcal{D}([0, 1]) \). To this aim, let \( b_n = n^{1/\beta}(\log(n))^{(\beta-1)/\beta} \), and \( (Z_n(t), t \in [0, 1]) \) denote the linear interpolation of \( (Z_{[nt]}, t \in [0, 1]) \), i.e.

\[
Z_n(t) = Z_{[nt]} + (nt - [nt])\xi_{S_{[nt]}}.
\]

Then, \( \forall \varepsilon > 0 \),

\[
\mathbb{P} \left[ \sup_{t \in [0,1]} \left| Z_n(t) - Z_{[nt]} \right| \geq \varepsilon b_n \right] = \mathbb{P} \left[ \max_{i=0}^{n-1} |\xi_{S_i}| \geq \varepsilon b_n \right] = \mathbb{P} \left[ \exists x \in \{S_0, \ldots, S_{n-1}\} \text{ s.t. } |\xi_x| \geq \varepsilon b_n \right] \leq \mathbb{E}(\# \{S_0, \ldots, S_{n-1}\}) \mathbb{P} \left( |\xi_0| \geq \varepsilon b_n \right) \leq C\frac{n}{\log(n)} \varepsilon^{-\beta} b_n^{-\beta} = C\varepsilon^{-\beta} \log(n)^{-\beta},
\]

where the last inequality comes from (2) and Theorem 6.9 of [13]. Therefore, if \( \left( \frac{Z_{[nt]}}{b_n} \right)_{t \in [0,1]} \) converges weakly to \( \left( \tilde{Y}_t \right)_{t \in [0,1]} \), the same is true for \( \left( \frac{Z_n(t)}{b_n} \right)_{t \in [0,1]} \). Using the fact that
the sequence \( \left( \frac{Z_n(t)}{b_n} \right)_{t \in [0;1]} \) is a sequence in the space \( \mathbb{C}([0,1]) \) and that the Skorohod \( J_1 \)-
topology coincides with the uniform one when restricted to \( \mathbb{C}([0,1]) \), one deduces that \( \left( \frac{Z_n(t)}{b_n} \right)_{t \in [0;1]} \)
converges weakly in \( \mathbb{C}([0,1]) \), and that the limiting process \( \left( \hat{Y}_t \right)_{t \in [0,1]} \) is therefore continuous, which is false as soon as \( \beta < 2 \).

}\[\square\]

3. Proof of the local limit theorem in the lattice case

3.1. The event \( \Omega_n \). Set

\[ N^*_n := \sup_{y} N_n(y) \quad \text{and} \quad R_n := \#\{y : N_n(y) > 0\}. \]

Lemma 6. For every \( n \geq 1 \) and \( 1 > \gamma > 0 \), set

\[ \Omega_n = \Omega_n(\gamma) := \left\{ R_n \leq \frac{n}{(\log \log(n))^{1/4}} \text{ and } N^*_n \leq n^\gamma \right\}. \]

Then, \( \mathbb{P}(\Omega_n) = 1 - o(b_n^{-1}) \). Moreover, the following also holds on \( \Omega_n \):

\[ (\log \log(n))^{1/4} \leq N^*_n \quad \text{and} \quad V_n \geq n^{1-\gamma(1-\beta)_+}. \]

Proof. We first prove that

\[ \mathbb{P} \left( R_n \geq n(\log \log(n))^{-1/4} \right) = o(b_n^{-1}). \]

Let us recall that for every \( a, b \in \mathbb{N} \), we have

\[ \mathbb{P}(R_n \geq a + b) \leq \mathbb{P}(R_n \geq a)\mathbb{P}(R_n \geq b). \]

The proof is given for instance in [7]. We will moreover use the fact that \( \mathbb{E}[R_n] \sim cn(\log(n))^{-1} \)
and \( \text{Var}(R_n) = O\left(\frac{n^2 \log^{-2}(n)}{\log \log(n)}\right) \) (see [13]). Hence, for \( n \) large enough, there exists \( C > 0 \) such that we have

\[ \mathbb{P} \left( R_n \geq \frac{n}{(\log \log(n))^{1/4}} \right) \leq \mathbb{P} \left( R_n \geq \left[ \frac{n(\log \log(n))^{1/4}}{\log(n)} \right] \right)^{\log(n)(\log \log(n))^{-1/2}} \]

\[ \leq \mathbb{P} \left( |R_n - \mathbb{E}[R_n]| \geq \frac{1}{2} \left[ \frac{n(\log \log(n))^{1/4}}{\log(n)} \right] \right)^{\log(n)(\log \log(n))^{-1/2}} \]

\[ \leq \left( \frac{5\text{Var}(R_n) \log^2(\log(n))}{n(\log \log(n))^{1/2}} \right)^{\log(n)(\log \log(n))^{-1/2}} \]

\[ \leq \left( \frac{Cn^2 \log^2(\log(n))}{n^2 \sqrt{\log \log(n)}} \right)^{\log(n)(\log \log(n))^{-1/2}} \]

\[ \leq \left( \frac{C}{(\log(n))^2} \right)^{\log(n)(\log \log(n))^{-1/2}} \cdot \exp \left( - \log(n) \sqrt{\log \log(n)} \left( 1 - \frac{\log(C)}{2 \log \log(n)} \right) \right). \]

This ends the proof of (11).

Let us now prove that

\[ \mathbb{P} \left[ N^*_n \geq n^\gamma \right] = o(b_n^{-1}). \]

(13)
We have
\[ \mathbb{P}(N_n^* \geq n^\gamma) \leq \sum_x \mathbb{P}(N_n(x) \geq n^\gamma) \]
\[ = \sum_x \mathbb{P}(T_x \leq n; N_n(x) \geq n^\gamma), \text{ where } T_x := \inf \{ n > 1, \text{ s.t. } S_n = x \}, \]
\[ \leq \sum_x \mathbb{P}(T_x \leq n) \mathbb{P}(N_n(0) \geq n^\gamma) \]
\[ \leq \mathbb{E}[R_n] \mathbb{P}(T_0 \leq n)^n. \]

Hence, (13) follows now from \( \mathbb{E}[R_n] \sim cn(\log(n))^{-1} \), and from \( \mathbb{P}(T_0 > n) \sim C/\log(n) \).

Since \( n = \sum_y N_n(y) \leq R_n N_n^* \), we get that \( N_n^* \geq \frac{n}{R_n} \geq (\log \log(n))^1/4 \) on \( \Omega_n \).

To prove the lower bound for \( V_n \), note that for \( \beta \geq 1 \), \( V_n = \sum_y N_n(y)^\beta \geq \sum_y N_n(y) = n \). For \( \beta < 1 \), on \( \Omega_n \),
\[ n = \sum_y N_n(y) = \sum_y N_n(y)^\beta N_n(y)^{1 - \beta} \leq V_n(N_n^*)^{1 - \beta} \leq V_n n^{\gamma(1 - \beta)}. \]

\( \square \)

3.2. Scheme of the proof. It is easy to see (cf the proof of lemma 5 in [5]) that \( \mathbb{P}(Z_n = [b_n x]) = 0 \) if \( \mathbb{P}(n \xi_0 - [b_n x] \notin d_0 \mathbb{Z}) = 1 \), and that \( \mathbb{P}(n \xi_0 - [b_n x] \in d_0 \mathbb{Z}) = 1 \),
\[ \mathbb{P}(Z_n = [b_n x]) = \frac{d_0}{2\pi} \int_{\frac{-\pi}{d_0}}^{\frac{\pi}{d_0}} e^{-i t [b_n x]} \mathbb{E} \left[ \prod_y \varphi(t N_n(y)) \right] dt. \]

In view of lemma 6, we have to estimate
\[ \frac{d_0}{2\pi} \int_{\frac{-\pi}{d_0}}^{\frac{\pi}{d_0}} e^{-i t [b_n x]} \mathbb{E} \left[ \prod_y \varphi(t N_n(y)) \mathbb{1}_{\Omega_n} \right] dt. \]

This is done in several steps presented in the following propositions.

**Proposition 7.** Let \( \gamma \in (0, 1/(\beta + 1)) \) and \( \delta \in (0, 1/(2\beta)) \) s.t. \( \frac{(1 - \beta) + \delta}{\beta} < \delta < 1/\beta - \gamma \). Then, we have
\[ \frac{d_0}{2\pi} \int_{\{ |t| \leq \delta b_n \}} e^{-i t [b_n x]} \mathbb{E} \left[ \prod_y \varphi(t N_n(y)) \mathbb{1}_{\Omega_n} \right] dt = \frac{d_0}{b_n} C(x) + o(b_n^{-1}), \]
uniformly in \( x \in \mathbb{R} \).

Recall next that the characteristic function \( \phi \) of the limit distribution of \( n^{-1/\beta} \sum_{k=1}^n \xi_k \) has the following form :
\[ \phi(u) = e^{-|u|^\beta (A_1 + iA_2 \text{sgn}(u))}, \]
with \( 0 < A_1 < \infty \) and \( |A_1^{-1}A_2| \leq |\tan(\pi\beta/2)| \). It follows that the characteristic function \( \varphi_\xi \) of \( \xi_0 \) satisfies:
\[ 1 - \varphi_\xi(u) \sim |u|^\beta (A_1 + iA_2 \text{sgn}(u)) \quad \text{when } u \to 0. \]
(14)
Therefore there exist constants \( \varepsilon_0 > 0 \) and \( \sigma > 0 \) such that
\[ \max(|\phi(u)|, |\varphi_\xi(u)|) \leq \exp \left( -\sigma |u|^\beta \right) \quad \text{for all } u \in [-\varepsilon_0, \varepsilon_0]. \]
(15)
Since \( \overline{\varphi_\xi(t)} = \varphi_\xi(-t) \) for every \( t \geq 0 \), the following propositions achieve the proof of Theorem 2:
Proposition 8. Let $\delta$ and $\gamma$ be as in Proposition 7. Then there exists $c > 0$ such that
\[
\int_{n^{\delta}/b_n}^{\infty} \mathbb{E} \left[ \prod_y |\varphi(tN_n(y))| 1_{\Omega_n} \right] dt = o(e^{-nc}).
\]

Proposition 9. There exists $c > 0$ such that
\[
\int_{n^{\delta}/b_n}^{\infty} \mathbb{E} \left[ \prod_y |\varphi(tN_n(y))| 1_{\Omega_n} \right] dt = o(e^{-nc}).
\]

3.3. Proof of Proposition 7. Remember that $V_n = \sum_{z \in \mathbb{Z}^d} N_n^\beta(z)$. We start by a preliminary lemma.

Lemma 10. (1) If $\beta > 1$, $\sup_n \mathbb{E} \left[ \left( \frac{n \log(n)^{\beta-1}}{V_n} \right)^{1/(\beta-1)} \right] < +\infty$.

(2) If $\beta \leq 1$, $\forall p \in \mathbb{N}$, $\sup_n \mathbb{E} \left[ \left( \frac{n \log(n)^{\beta-1}}{V_n} \right)^p \right] < +\infty$.

Proof. For $\beta > 1$, using Hölder’s inequality with $p = \beta$, we get
\[
n = \sum_x N_n(x) \leq V_n^{\frac{1}{\beta}} R_n^{\frac{1}{\beta-1}}
\]
which means that
\[
\left( \frac{n \log(n)^{\beta-1}}{V_n} \right)^{1/(\beta-1)} \leq \frac{\log(n) R_n}{n}.
\]
But it is proved in [13] Equation (7.a) that $\mathbb{E}[R_n] = O(n/\log(n))$. The result follows.

The result is obvious for $\beta = 1$. For $\beta < 1$, Hölder’s inequality with $p = 2 - \beta$ yields
\[
n = \sum_x \frac{N_n(x)^{\beta}}{n^{\beta-1}} \cdot N_n^{2(2-\beta)/(2-\beta)}(x) \leq V_n^{\frac{1}{2-\beta}} \left( \sum_x N_n^2(x) \right)^{1-\frac{\beta}{2-\beta}}
\]
and so
\[
\frac{n \log(n)^{\beta-1}}{V_n} \leq \left( \frac{\sum_x N_n^2(x)}{n \log(n)} \right)^{1-\beta}\left( \frac{n \log(n)^{\beta-1}}{V_n} \right)^{1/(\beta-1)}.
\]
It is therefore enough to prove that there exists $c > 0$ such that
\[
\sup_n \mathbb{E} \left[ \exp \left( \frac{\sum_x N_n^2(x)}{n \log(n)} \right) \right] < \infty. \quad (16)
\]
Note that $\sum_x N_n^2(x) = \sum_{k=0}^{n-1} N_n(S_k)$. By Jensen’s inequality, we get thus
\[
\mathbb{E} \left[ \exp \left( \frac{\sum_x N_n^2(x)}{n \log(n)} \right) \right] \leq \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E} \left[ \exp \left( \frac{N_n(S_k)}{\log(n)} \right) \right].
\]
Observe now that $N_n(S_k) = \sum_{j=0}^{k} 1_{\{S_j-S_i=0\}} + \sum_{j=k+1}^{n-1} 1_{\{S_j-S_k=0\}} \Rightarrow_{d} N_{k+1}(0) + N_{n-k}(0) - 1$,
where $(N_n'(x), n \in \mathbb{N}, x \in \mathbb{Z}^d)$ is an independent copy of $(N_n(x), n \in \mathbb{N}, x \in \mathbb{Z}^d)$. Hence,
\[
\mathbb{E} \left[ \exp \left( \frac{\sum_x N_n^2(x)}{n \log(n)} \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{N_n(0)}{\log(n)} \right)^2 \right].
\]
But, $\forall t > 0$,
\[
\mathbb{P} \left[ N_n(0) \geq t \log(n) \right] \leq \mathbb{P} \left[ T_1 \leq n \right]^{t \log(n)},
\]
Proof. It suffices to prove that

\[ \mathbb{E} \left[ \exp \left( c \frac{N_n(0)}{\log(n)} \right) \right] \leq 1 + \int_0^{\infty} c \exp(\sqrt{t}) \exp(- \sqrt{t} \log(n)) \mathbb{P}(T_0 > n) \, dt. \]

Now (16) follows then from the fact that \( \mathbb{P}(T_0 > n) \sim C/\log(n) \) for any integer\( n \geq 1. \)

The next step is

**Lemma 11.** Under the hypotheses of Proposition 7, we have

\[ \int_{\{|t| \leq n^\delta / b_n\}} e^{-it|b_n x|} \mathbb{E} \left[ \left\{ \prod_y \varphi_\xi(tN_n(y)) - e^{-\sqrt{t}|N_n^\beta(A_1 + iA_2 \text{sgn}(t))V_n} \right\} 1_{\Omega_n} \right] \, dt = o(b_n^{-1}), \]

uniformly in \( x \in \mathbb{R} \).

**Proof.** It suffices to prove that

\[ \int_{\{|t| \leq n^\delta / b_n\}} \mathbb{E}[|E_n(t)| 1_{\Omega_n}] \, dt = o(b_n^{-1}) \]

with

\[ E_n(t) := \prod_y \varphi_\xi(tN_n(y)) - \prod_y \exp \left( -|t|^\beta N_n^\beta(y)(A_1 + iA_2 \text{sgn}(t)) \right). \]

Observe that

\[ E_n(t) = \sum_y \left( \prod_{z < y} \varphi_\xi(tN_n(z)) \right) \left( \varphi_\xi(tN_n(y)) - e^{-|t|^\beta N_n^\beta(y)(A_1 + iA_2 \text{sgn}(t))} \right) \times \left( \prod_{z > y} e^{-|t|^\beta N_n^\beta(z)(A_1 + iA_2 \text{sgn}(t))} \right), \]

where an arbitrary ordering of sites of \( \mathbb{Z}^d \) has been chosen. But on \( \Omega_n \), if \( |t| \leq n^\delta b_n^{-1} \), then

\[ |t| N_n(z) \leq n^{\gamma + \delta} b_n^{-1}. \]

Since \( \gamma + \delta < \beta^{-1} \), this implies in particular that \( |t| N_n(z) < \varepsilon_0 \) for \( n \) large enough. Thus, by using (15), we get

\[ |E_n(t)| \leq \sum_y |\varphi_\xi(tN_n(y)) - \exp \left( -|t|^\beta N_n^\beta(y)(A_1 + iA_2 \text{sgn}(t)) \right) \exp \left( -\sigma |t|^\beta \sum_{z \neq y} N_n^\beta(z) \right), \]

for \( n \) large enough. Observe next that (14) implies

\[ |\varphi_\xi(u) - \exp \left( -|u|^\beta (A_1 + iA_2 \text{sgn}(u)) \right) | \leq |u|^\beta h(|u|) \quad \text{for all} \; u \in \mathbb{R}, \]

with \( h \) a continuous and monotone function on \( [0, +\infty) \) vanishing in 0. Therefore by using (17) we get

\[ |E_n(t)| \leq |t|^\beta h(n^{\gamma + \delta} b_n^{-1}) \sum_y N_n^\beta(y) \exp \left( -\sigma |t|^\beta \sum_{z \neq y} N_n^\beta(z) \right). \]

Now, according to (10) and since \( \gamma < \frac{1}{\beta + 1} \leq \frac{1}{\beta + (1 - \beta)_+} \), if \( n \) is large enough, we have on \( \Omega_n \)

\[ \sum_{z \neq y} N_n^\beta(z) \geq V_n / 2 \quad \text{for all} \; y \in \mathbb{Z}. \]
By using this and the change of variables $v = tV_n^{1/\beta}$, we get
\[ \int_{\{|t| \leq n^\delta b_n^{-1}\}} \mathbb{E} [|E_n(t)| \cdot 1_{\Omega_n}] \, dt \leq h(n^\gamma + \delta b_n^{-1}) \mathbb{E}[V_n^{-1/\beta}] \int_{\mathbb{R}} |v|^\beta \exp \left( -\sigma |v|^3 / 2 \right) \, dv = o(\mathbb{E}[V_n^{-1/\beta}]), \]
which proves the result according to Lemma 10.

Finally Proposition 7 follows from the

**Lemma 12.** Under the hypotheses of Proposition 7, we have
\[
\frac{d_0}{2\pi} \int_{\{|t| \leq n^\delta b_n^{-1}\}} e^{-it[b_n x]} e^{-|t|^{\beta} V_n(A_1 + iA_2 \text{sgn}(t))} 1_{\Omega_n} \, dt = d_0 \frac{C(x)}{b_n} + o(b_n^{-1}),
\]
uniformly in $x \in \mathbb{R}$.

**Proof.** Set
\[
I_{n,x} := \int_{\{|t| \leq n^\delta b_n^{-1}\}} e^{-it[b_n x]} e^{-|t|^{\beta} V_n(A_1 + iA_2 \text{sgn}(t))} \, dt,
\]
which can be rewritten
\[
I_{n,x} = \int_{\{|t| \leq n^\delta b_n^{-1}\}} e^{-it[b_n x]} \phi(tV_n^{1/\beta}) \, dt.
\]
Since $|b_n x - b_n x| \leq 1$, for all $n$ and $x$, it is immediate that
\[
I_{n,x} = \int_{\{|t| \leq n^\delta b_n^{-1}\}} e^{-itb_n x} \phi(tV_n^{1/\beta}) \, dt + O(n^{2\delta b_n^{-2}}).
\]
But $\delta < (2\beta)^{-1}$ by hypothesis. So actually
\[
I_{n,x} = \int_{\{|t| \leq n^\delta b_n^{-1}\}} e^{-itb_n x} \phi(tV_n^{1/\beta}) \, dt + o(b_n^{-1}).
\]
Next, with the change of variable $v = tb_n$, we get:
\[
\int_{\{|t| \leq n^\delta b_n^{-1}\}} e^{-itb_n x} \phi(tV_n^{1/\beta}) \, dt = b_n^{-1} \left\{ V_n^{-1/\beta} b_n f(x V_n^{-1/\beta} b_n) - J_{n,x} \right\},
\]
where $f$ is the density function of the distribution with characteristic function $\phi$ and where
\[
J_{n,x} := \int |v| \geq n^\delta} e^{-iv x} \phi(vb_n^{-1} V_n^{1/\beta}) \, dv.
\]
By lemma 4 (applied with $m = 1$, $t_1 = \theta_1 = 1$, $\gamma = \beta$, $(W_n := b_n V_n^{-1/\beta})_n$ converges almost surely, as $n \to \infty$, to the constant $\Gamma(\beta + 1) - 1/\beta (\pi A)^{1-1/\beta}$. Moreover, Lemma 10 ensures that the sequence $(W_n, n \geq 1)$ is uniformly integrable, so actually the convergence holds in $\mathbb{L}^1$. Let us deduce that
\[
\mathbb{E}[g_x(W_n)] = \mathbb{E}[g_x(W)] + o(1),
\]
where $g_x : z \mapsto z f(xz)$ and the $o(1)$ is uniform in $x$. First
\[
|\mathbb{E}[g_x(W_n)] - \mathbb{E}[g_x(W)]| \leq \sup_{x, z \in \mathbb{R}} |(g_x)'(z)| \mathbb{E}[|W_n - W|] \leq \sup_{u} |f(u) + uf'(u)| \mathbb{E}[|W_n - W|].
\]
This proves (19). We observe that $\mathbb{E}[g_x(W)] = C(x)$.
In view of (18), it only remains to prove that $E[J_{n,x} 1_{\Omega_n}] = o(1)$ uniformly in $x$. But this follows from the basic inequality

$$E[|J_{n,x} 1_{\Omega_n}|] \leq \int_{|v| \geq \eta n} E \left[ e^{-A_1 |v|^{\beta} \frac{V_n}{b_n}} 1_{\Omega_n} \right] dv,$$

and from the lower bound for $V_n$ given in (10) and from the choice $\delta > \gamma (1 - \beta) + / \beta$. 

3.4. Proof of Proposition 8. Recall that on $\Omega_n$, $N_n(y) \leq n^\gamma$, for all $y \in \mathbb{Z}^d$. Hence by (15),

$$K_n := \int_{n^{\delta}/b_n}^{\varepsilon n^{-\gamma}} E \left[ \prod_y |\varphi_\xi(t N_n(y))| 1_{\Omega_n} \right] dt \leq \int_{n^{\delta}/b_n}^{\varepsilon n^{-\gamma}} E \left[ \exp \left( -\sigma t^\beta V_n \right) 1_{\Omega_n} \right] dt .$$

With the change of variable $s = t V_n^{1/\beta}$, we get

$$K_n \leq E \left[ V_n^{-1/\beta} \int_{n^{1/\beta} V_n^{1/\beta} b_n}^{\infty} \exp \left( -\sigma s^\beta \right) ds 1_{\Omega_n} \right] \leq \frac{1}{n^d} \int_{\delta^{-1} (1 - \beta) / \beta}^{\infty} \frac{1}{\log(n)} \frac{1}{s^\beta} \exp \left( -\sigma s^\beta \right) ds ,$$

which proves the proposition since $\delta > \gamma (1 - \beta) + / \beta$.

3.5. Proof of Proposition 9. We adapt the proof of [5, Proposition 10]. We will see that the argument of "peaks" still works here. We endow $\mathbb{Z}^d$ with the ordered structure given by the relation $<$ defined by

$$(\alpha_1, ..., \alpha_d) < (\beta_1, ..., \beta_d) \iff \exists i \in \{1, ..., d\}, \ \alpha_i < \beta_i, \ \forall j < i, \ \alpha_j = \beta_j .$$

We consider $C^+ = (x_1, ..., x_T) \in (\mathbb{Z}^d \setminus \{0\})^T$ for some positive integer $T$ such that:

- $x_1 + ... + x_T = 0$;
- for every $i = 1, ..., T$, $P(x_1 = x_i) > 0$;
- there exists $I_1 \in \{1, ..., T\}$ such that
  - for every $i = 1, ..., I_1$, $x_i > 0$,
  - for every $i = I_1 + 1, ..., T$, $x_i < 0$.

Let us write $C^- := (x_{T-i+1})_{i=1}^T$. We define $B := \sum_{i=1}^{I_1} x_i$. We observe that

$$p := P((X_1, ..., X_T) = C^+) = P((X_1, ..., X_T) = C^-) > 0 .$$

We notice that $(X_1, ..., X_T) = C^+$ corresponds to a trajectory visiting $B$ only once before going back to the origin at time $T$ (and without visiting $-B$). Analogously, $(X_1, ..., X_T) = C^-$ corresponds to a trajectory that goes down to $-B$ and comes back up to 0 (and without visiting $B$), and staying at a distance smaller than $\bar{d}$ from the origin with $\bar{d} := \sum_{i=1}^T |x_i|$ (where $|.|$ is the absolute value if $d = 1$ and $|[a, b]| = \max(|a|, |b|)$ if $d = 2$). We introduce now the event

$$D_n := \left\{ C_n > \frac{np}{2T} \right\} ,$$

where

$$C_n := \# \left\{ k = 0, ..., \left\lfloor \frac{n}{T} \right\rfloor - 1 : (X_{kT+1}, ..., X_{(k+1)T}) = C^+ \right\} .$$

Since the sequences $(X_{kT+1}, ..., X_{(k+1)T})$, for $k \geq 0$, are independent of each other, Chernoff's inequality implies that there exists $c > 0$ such that

$$P(D_n) = 1 - o(e^{-cn}) .$$
We introduce now the notion of "loop". We say that there is a loop based on \( y \) at time \( n \) if \( S_n = y \) and \( (X_{n+1}, \ldots, X_{n+T}) = C^\pm \). We will see (in Lemma 13 below) that, on \( \Omega_n \cap D_n \), there is a large number of \( y \in \mathbb{Z}^d \) on which are based a large number of loops. For any \( y \in \mathbb{Z}^d \), let
\[
C_n(y) := \# \{ k = 0, \ldots, \lfloor \frac{n}{T} \rfloor : S_{kT} = y \text{ and } (X_{kT+1}, \ldots, X_{(k+1)T}) = C^\pm \},
\]
be the number of loops based on \( y \) before time \( n \) (and at times which are multiple of \( T \)), and let
\[
p_n := \# \{ y \in \mathbb{Z} : C_n(y) \geq \frac{\log \log(n)^{1/4}}{4T} \},
\]
be the number of sites \( y \in \mathbb{Z} \) on which at least \( a_n := \lceil \frac{\log \log(n)^{1/4}}{4T} \rceil \) loops are based.

**Lemma 13.** On \( \Omega_n \cap D_n \), we have, \( p_n \geq c' n^{1-\gamma} \) with \( c' = p'/(4T) \).

**Proof.** Note that \( C_n(y) \leq N_n^* \) for all \( y \in \mathbb{Z}^d \). Thus on \( \Omega_n \cap D_n \), we have
\[
\frac{np}{2T} \leq \sum_{y \in \mathbb{Z}^d : C_n(y) < a_n} C_n(y) + \sum_{y \in \mathbb{Z}^d : C_n(y) \geq a_n} C_n(y)
\]
\[
\leq R_n a_n + N_n^* p_n \leq \frac{np}{4T} + p_n n^\gamma,
\]
according to lemma 6. This proves the lemma. \( \square \)

We have proved that, if \( n \) is large enough, the event \( \Omega_n \cap D_n \) is contained in the event
\[
\mathcal{E}_n := \{ p_n \geq c' n^{1-\gamma} \}.
\]
Now, on \( \mathcal{E}_n \), we consider \( (Y_i)_{i=1,\ldots,\lfloor c' n^{1-\gamma}\rfloor} \) (with \( c'' := c'/(2\tilde{d}) \) if \( d = 1 \) and with \( c'' := c'/2\tilde{d}^2 \) if \( d = 2 \)) such that
\[
\bullet \text{ on each } Y_i, \text{ at least } a_n \text{ loops are based,}
\]
\[
\bullet \text{ for every } i, j \text{ such that } i \neq j, \text{ we have } |Y_i - Y_j| > \tilde{d}/2.
\]
For every \( i = 1, \ldots, \lfloor c'' n^{1-\gamma}\rfloor \), let \( t_i^{(1)}, \ldots, t_i^{(a_n)} \) be the \( a_n \) first times (which are multiples of \( T \)) when a loop is based on the site \( Y_i \). We also define \( N_n^0(Y_i + B) \) as the number of visits of \( S \) before time \( n \) to \( Y_i + B \), which do not occur during the time intervals \( [t_i^{(j)}, t_i^{(j)} + T], \) for \( j \leq a_n \).

Since our construction is basically the same as in [5, section 2.8], the proof of the following lemma is exactly the same as the proof of [5, Lemma 16] and we do not prove it again.

**Lemma 14.** Conditionally to the event \( \mathcal{E}_n \), \( (N_n(Y_i + B) - N_n^0(Y_i + B))_{i \geq 1} \) is a sequence of independent identically distributed random variables with binomial distribution \( B (a_n; 1/2) \). Moreover this sequence is independent of \( (N_n^0(Y_i + B))_{i \geq 1} \).

Let \( \eta \) be a real number such that \( \gamma < \eta < (1 - \gamma)/\beta \) (this is possible since \( \gamma < 1/(\beta + 1) \)). We define
\[
\forall n \geq 1, \quad d_n := n^{-\eta}.
\]
Let now \( \rho := \sup \{|\varphi_z(u)| : d \left( u, \frac{2\pi}{d_n} \mathbb{Z} \right) \geq c_0 \} \). According to Formula (15) and since \( \lim_{n \to \infty} d_n = 0 \), for \( n \) large enough, we have
\[
|\varphi_z(u)| \leq \rho 1_{d(u, \frac{2\pi}{d_n} \mathbb{Z}) \geq c_0} + \exp \left( -\sigma d \left( u, \frac{2\pi}{d_0} \mathbb{Z} \right)^\beta \right) 1_{d(u, \frac{2\pi}{d_0} \mathbb{Z}) < c_0}
\]
\[
\leq \exp \left( -\sigma d_n^\beta \right),
\]
as soon as \( d \left( u, \frac{2\pi}{d_0} \mathbb{Z} \right) \geq d_n \). Therefore, for \( n \) large enough,
\[
\prod_z |\varphi_z(tN_n(z))| \leq \exp \left( -\sigma d_n^3 \# \left\{ z : d \left( tN_n(z), \frac{2\pi}{d_0} \mathbb{Z} \right) \geq d_n \right\} \right).
\] (20)

Then notice that
\[
d \left( tN_n(z), \frac{2\pi}{d_0} \mathbb{Z} \right) \geq d_n \iff N_n(z) \in \mathcal{I} := \bigcup_{k \in \mathbb{Z}} I_k,
\] (21)
where for all \( k \in \mathbb{Z} \),
\[
I_k := \left[ \frac{2k\pi}{d_0} + \frac{d_n}{t}, \frac{2(k+1)\pi}{d_0} - \frac{d_n}{t} \right].
\]
In particular \( \mathbb{R} \setminus \mathcal{I} = \bigcup_{k \in \mathbb{Z}} J_k \), where for all \( k \in \mathbb{Z} \),
\[
J_k := \left( \frac{2k\pi}{d_0} - \frac{d_n}{t}, \frac{2k\pi}{d_0} + \frac{d_n}{t} \right).
\]

**Lemma 15.** Under the hypotheses of Proposition 9, for every \( i \leq \lceil c'n^{1-\gamma} \rceil \), \( t \in (\varepsilon_0 n^{-\gamma}, \pi/d_0) \) and \( n \) large enough,
\[
\mathbb{P} \left( N_n(Y_i + B) \in \mathcal{I} \mid \mathcal{E}_n, \ N_n^0(Y_i + B) \right) \geq \frac{1}{3} \text{ almost surely.}
\]

Assume for a moment that this lemma holds true and let us finish the proof of Proposition 9. Lemmas 14 and 15 ensure that conditionally to \( \mathcal{E}_n \) and \( (N_n^0(Y_i + B), \ i \geq 1) \), the events \( \{N_n(Y_i + B) \in \mathcal{I}\}, \ i \geq 1 \), are independent of each other, and all happen with probability at least 1/3. Therefore, since \( \Omega_n \cap \mathcal{D}_n \subseteq \mathcal{E}_n \), there exists \( c > 0 \), such that
\[
\mathbb{P} \left( \Omega_n \cap \mathcal{D}_n, \ #\{i : N_n(Y_i + B) \in \mathcal{I}\} \leq \frac{c'n^{1-\gamma}}{4} \right) \leq \mathbb{P} \left( B_n \leq \frac{c'n^{1-\gamma}}{4} \right) = o(\exp(-cn^{1-\gamma})),
\]
where for all \( n \geq 1 \), \( B_n \) has binomial distribution \( \mathcal{B} \left( \left\lceil c'n^{1-\gamma} \right\rceil \ ; \ \frac{1}{3} \right) \).

But if \( \#\{z : N_n(z) \in \mathcal{I}\} \geq \frac{c'n^{1-\gamma}}{4} \), then by (20) and (21) there exists a constant \( c > 0 \), such that
\[
\prod_z |\varphi_z(tN_n(z))| \leq \exp \left( -cn^{1-\gamma}a_n^3 \right),
\]
which proves Proposition 9 since \( 1 - \gamma - \beta \eta > 0 \).

**Proof of Lemma 15.** First notice that by Lemma 14, for any \( H \geq 0 \),
\[
\mathbb{P} \left( N_n(Y_i + B) \in \mathcal{I} \mid \mathcal{E}_n, \ N_n^0(Y_i + B) = H \right) = \mathbb{P} \left( H + b_n \in \mathcal{I} \right),
\] (22)
where \( b_n \) is a random variable with binomial distribution \( \mathcal{B} \left( a_n ; \frac{1}{3} \right) \). We will use the following result whose proof is postponed.

**Lemma 16.** Under the hypotheses of Proposition 9, for every \( t \in (\varepsilon_0 n^{-\gamma}, \pi/d_0) \) and \( n \) large enough, the following holds:

(i) For any integer \( k \) such that all the elements of \( I_k - H \) are smaller than \( \frac{a_n}{2} \),
\[
\mathbb{P}(b_n \in (I_k - H)) \geq \mathbb{P}(b_n \in (J_k - H)).
\]
(ii) For any integer \( k \) such that all the elements of \( I_k - H \) are larger than \( \frac{a_n}{2} \),
\[
\mathbb{P}(b_n \in (I_k - H)) \geq \mathbb{P}(b_n \in (J_{k+1} - H)).
\]
Now call $k_0$ the largest integer satisfying the condition appearing in (i) and $k_1$ the smallest integer satisfying the condition appearing in (ii). We have $k_1 = k_0 + 1$ or $k_1 = k_0 + 2$. According to Lemma 16, we have

\[
\mathbb{P}(H + b_n \in \mathcal{I}) \geq \sum_{k \leq k_0} \mathbb{P}(H + b_n \in I_k) + \sum_{k \geq k_1} \mathbb{P}(H + b_n \in I_k) \\
\geq \sum_{k \leq k_0} \mathbb{P}(H + b_n \in J_k) + \sum_{k \geq k_1} \mathbb{P}(H + b_n \in J_{k+1}) \\
= \mathbb{P}(H + b_n \notin \mathcal{I}) - \mathbb{P}(H + b_n \in J_{k_0+1} \cup J_1).
\]

Hence,

\[
\mathbb{P}(H + b_n \in \mathcal{I}) \geq \frac{1}{2} [1 - \mathbb{P}(H + b_n \in J_{k_0+1} \cup J_1)].
\]

Let $\bar{b}_n := 2 (b_n - \frac{\alpha_n}{2}) \sqrt{a_n}$. Since $\lim_{n \to +\infty} a_n = +\infty$, $(\bar{b}_n)_n$ converges in distribution to a standard normal variable, whose distribution function is denoted by $\Phi$. The interval $J_{k_1}$ being of length $2d_n/t,$

\[
\mathbb{P}(H + b_n \in J_{k_1}) = \mathbb{P}(\bar{b}_n \in [m_n, M_n]), \text{ with } M_n - m_n = 4 \frac{d_n}{t \sqrt{a_n}} \\
\leq \Phi(M_n) - \Phi(m_n) + \frac{C}{\sqrt{a_n}} \quad \text{(by the Berry–Esseen inequality)} \\
\leq \frac{M_n - m_n}{\sqrt{2\pi}} + \frac{C}{\sqrt{a_n}} \\
\leq C' \frac{d_n}{\varepsilon_0 n^{-\gamma} \sqrt{a_n}} + \frac{C}{\sqrt{a_n}},
\]

for $t \geq \varepsilon_0 n^{-\gamma}$, and some constants $C > 0$ and $C' > 0$. Since $\lim_{n \to +\infty} a_n = +\infty$ and $\lim_{n \to +\infty} d_n n^{\gamma}(a_n)^{-1/2} = 0$ (since $\eta > \gamma$), we conclude that $\mathbb{P}(H + b_n \in J_{k_1}) = o(1)$. The same holds for $\mathbb{P}(H + b_n \in J_{k_0+1})$, so that for $n$ large enough,

\[
\mathbb{P}(H + b_n \in \mathcal{I}) \geq \frac{1}{2} [1 - o(1)] \geq \frac{1}{3}.
\]

Together with (22), this concludes the proof of Lemma 15. \hfill \Box

Proof of Lemma 16. We only prove (i), since (ii) is similar. So let $k$ be an integer such that all the elements of $I_k - H$ are smaller than $\frac{\alpha_n}{2}$. Assume that $(J_k - H) \cap \mathbb{Z}$ contains at least one nonnegative integer (otherwise $\mathbb{P}(b_n \in (J_k - H)) = 0$ and there is nothing to prove). Let $z_k$ denote the greatest integer in $J_k - H$, so that by our assumption $\mathbb{P}(b_n = z_k) > 0$ (remind that $0 \leq z_k < \frac{\alpha_n}{2}$). By monotonicity of the function $z \mapsto \mathbb{P}(b_n = z)$, for $z \leq \frac{\alpha_n}{2}$, we get

\[
\mathbb{P}(b_n \in J_k - H) \leq \mathbb{P}(b_n = z_k) \#((J_k - H) \cap \mathbb{Z}) \leq \mathbb{P}(b_n = z_k) \left\lfloor \frac{2d_n}{t} \right\rfloor.
\]

In the same way,

\[
\mathbb{P}(b_n \in I_k - H) \geq \mathbb{P}(b_n = z_k) \#((I_k - H) \cap \mathbb{Z}) \geq \mathbb{P}(b_n = z_k) \left\lfloor \frac{2\pi}{d_0} - \frac{2d_n}{t} \right\rfloor.
\]

Hence

\[
\mathbb{P}(b_n \in I_k - H) \geq \frac{2\pi}{d_0 t} - \frac{2d_n}{t} \mathbb{P}(b_n \in J_k - H).
\]
But $\pi/(d_0 t) \geq 1$ and $\lim_{n \to +\infty} d_n = 0$ by hypothesis. It follows immediately that for $n$ large enough, we have $2d_n < \pi/(2d_0)$, and so
\[
\left| \frac{2\pi}{d_0 t} - \frac{2d_n}{t} \right| \geq \left| \frac{3\pi}{2d_0 t} \right| \geq 1 + \left| \frac{\pi}{2d_0 t} \right| \geq \left| \frac{2d_n}{t} \right|.
\]
This concludes the proof of the lemma. \hfill \square

4. **Proof of the Local Limit Theorem in the Strongly Nonlattice Case**

As in [5], the proof in the strongly nonlattice case is closely related to the proof in the lattice case.

We assume here that $\xi$ is strongly nonlattice. In that case, there exist $\varepsilon_0 > 0$, $\sigma > 0$ and $\rho < 1$ such that $|\varphi_\xi(u)| \leq \rho$ if $|u| \geq \varepsilon_0$ and $|\varphi_\xi(u)| \leq \exp(-\sigma |u|^\beta)$ if $|u| < \varepsilon_0$.

We use here the notations of Section 3 with the hypotheses on $\gamma$, and $\delta$ of Proposition 7. Let $h_0$ be the density of Polya’s distribution: $h_0(y) = \frac{1}{\pi} \frac{1 - \cos(y)}{y^2}$, with Fourier transform $\hat{h}_0(t) = (1 - |t|)_+$. For $\theta \in \mathbb{R}$, let $h_\theta(y) = \exp(i\theta y)h_0(y)$ with Fourier transform $\hat{h}_\theta(t) = \hat{h}_0(t + \theta)$. As in [10, thm 5.4], it is enough to show that for all $\theta \in \mathbb{R}$,
\[
\lim_{n \to \infty} b_n \mathbb{E} [h_\theta(Z_n - b_n x)] = C(x) \hat{h}_\theta(0) .
\]

By Fourier inverse transform, we have
\[
b_n \mathbb{E} [h_\theta(Z_n - b_n x)] = \frac{b_n}{2\pi} \int_{\mathbb{R}} e^{-iub_n x} \mathbb{E} \left[ \prod_{x \in \mathbb{Z}^d} \varphi_\xi(uN_n(x)) \right] \hat{h}_\theta(u) \, du .
\]

Since $\hat{h}_\theta \in L^1$, we can restrict our study to the event $\Omega_n$ of Lemma 6. The part of the integral corresponding to $|u| \leq n^{\delta} b_n^{-1}$ is treated exactly as in Proposition 7. The only change is that we have to check that
\[
\lim_{n \to \infty} b_n \int_{\{|u| \leq n^{\delta} b_n^{-1}\}} \mathbb{E} \left[ e^{-|u|^\beta V_n(A_1 + iA_2 \text{sgn}(u))} \mathbf{1}_{\Omega_n} \right] (\hat{h}_\theta(u) - \hat{h}_\theta(0)) \, du = 0 ,
\]
which is obviously true since $V_n \geq n^{-1 - \gamma(1 - \beta)_+}$ and since $2\gamma(1 - \beta)_+ < 2\delta \beta < 1$, using the fact that $\hat{h}_\theta$ is a Lipschitz function.

Now, since $\hat{h}_\theta$ is bounded, the part corresponding to $n^{\delta} b_n^{-1} \leq |u| \leq \varepsilon_0 n^{-\gamma}$ is treated as in the proof of Proposition 8 (since it only uses the behavior of $\varphi_\xi$ around 0, which is the same).

Finally, it remains to prove that
\[
\lim_{n \to \infty} b_n \int_{\{|u| \geq \varepsilon_0 n^{-\gamma}\}} e^{-iub_n x} \mathbb{E} \left[ \prod_{x \in \mathbb{Z}^d} \varphi_\xi(uN_n(x)) \mathbf{1}_{\Omega_n} \right] \hat{h}_\theta(u) \, du = 0 .
\]

We note that, if $|u| \geq \varepsilon_0 n^{-\gamma}$ and $x \in \mathbb{Z}^d$, we have
\[
|\varphi_\xi(uN_n(x))| \leq \exp(-\sigma |u|^\beta N_n^\beta(x)) \mathbf{1}_{\{|uN_n(x)| \leq \varepsilon_0\}} + \rho \mathbf{1}_{\{|uN_n(x)| \geq \varepsilon_0\}} \leq \exp(-\sigma \varepsilon_0^\beta n^{-\gamma\beta} N_n^\beta(x)) \mathbf{1}_{\{|uN_n(x)| \leq \varepsilon_0\}} + \rho \mathbf{1}_{\{|uN_n(x)| \geq \varepsilon_0\}} .
\]

For $n$ large enough, $\rho \leq \exp(-\sigma \varepsilon_0^\beta n^{-\gamma\beta})$. Therefore, if $n$ is large enough, then for all $x$ and $u$ such that $N_n(x) \geq 1$ and $|u| \geq \varepsilon_0 n^{-\gamma}$, we have
\[
|\varphi_\xi(uN_n(x))| \leq \exp(-\sigma \varepsilon_0^\beta n^{-\gamma\beta}) .
\]
Hence,

\[
\left| \mathbb{E} \left[ \prod_x \varphi(\xi(uN_n(x))) \mathbf{1}_{\Omega_n} \right] \right| \leq \mathbb{E} \left[ \exp(-\sigma \varepsilon_0 n^{-\gamma} R_n) \mathbf{1}_{\Omega_n} \right] \leq \exp(-\sigma \varepsilon_0 n^{1-\gamma(1+\beta)}).
\]

Therefore, since \( \gamma(1+\beta) < 1 \), we have

\[
\lim_{n \to \infty} b_n \int \int_{\{|u| \geq \varepsilon_0 n^{-\gamma}\}} e^{-iu bn x} \mathbb{E} \left[ \prod_x \varphi(\xi(uN_n(x))) \mathbf{1}_{\Omega_n} \right] \hat{h}(u) \, du = 0.
\]

This concludes the proof of Theorem 3. \( \square \)

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**REFERENCES**


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