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On the automorphism group of the first Weyl algebra

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Abstract

Let $A_1 := \mathbb{k}[t, \partial]$ be the first algebra over a field $\mathbb{k}$ of characteristic zero. One can associate to each right ideal $I$ of $A_1$ its Stafford subgroup, which is a subgroup of $\text{Aut}_\mathbb{k}(A_1)$, the automorphism group of the ring $A_1$. In this article we show that each Stafford subgroup is equal to its normalizer. For that, we study when the Stafford subgroup of a right ideal of $A_1$ contains a given Stafford subgroup.

Introduction

Let $\mathbb{k}$ be a commutative field of characteristic zero. We note $A_1$ the first Weyl algebra over $\mathbb{k}$ i.e. :

$$A_1 := A_1(\mathbb{k}) = \mathbb{k}[t, \partial]$$
where \( \partial, t \) are related by \( \partial t - t \partial = 1 \).

**Definition 1** For a right ideal \( I \) of \( A_1 \), the Stafford subgroup associated to \( I \) is :

\[
H(I) := \{ \sigma \in \text{Aut}_\kappa(A_1) : \sigma(I) \simeq I \}
\]

(where the symbol “\( \simeq \)” means “\( \sigma(I) \) is isomorphic to \( I \) as a right-\( A_1 \)-module”).

By [5], it is known that each subgroup \( H(I) \) is isomorphic to an automorphism group \( \text{Aut}_\kappa(\mathcal{D}(X)) \), where \( \mathcal{D}(X) \) is the \( \kappa \)-algebra of differential operators over an algebraic affine curve \( X \).

A natural question is :
“are the Stafford subgroups normal in \( \text{Aut}_\kappa(A_1) \)”?

The answer is no.

Stafford showed that if \( X_2 \) is the famous algebraic plane curve defined by the equation :

\[
x^2 = y^3
\]

and if \( I_2 \) is the right ideal of \( A_1 \) :

\[
I_2 := \{ d \in A_1 : d(\kappa[t]) \subseteq \kappa[t^2, t^3] \}
\]

then the subgroup \( H(I_2) \) is isomorphic to \( \text{Aut}_\kappa(\mathcal{D}(X_2)) \) and is equal to its own normalizer in \( \text{Aut}_\kappa(A_1) \).

We will show in this paper that the subgroup \( H(I) \) is equal to its own normalizer for all right ideal \( I \) of \( A_1 \).

We begin by giving some definitions and by fixing some notations that will be used in this paper.

## 1 Definitions and some properties

The ring \( A_1 \) contains the subrings \( R := \kappa[t] \) and \( S := \kappa[\partial] \). It is well known that \( A_1 \) is a two-sided noetherian integral domain. Since the characteristic of \( \kappa \) is zero, \( A_1 \) is also hereditary (cf [4]) i.e. every non zero right ideal of \( A_1 \) is a projective right-\( A_1 \)-module.

The ring \( A_1 \) has a quotient division ring, denoted by \( Q_1 \). For any finitely generated right-submodule \( M \) of \( Q_1 \), the dual \( M^* \), as a left-\( A_1 \)-module will be identified with the set \( \{ u \in Q_1 : uM \subseteq A_1 \} \), and \( \text{End}_{A_1}(M) \) with the set \( \{ d \in Q_1 : dM \subseteq M \} \) (cf [3]).
The division ring \( Q_1 \) contains the subrings \( D := \mathbb{k}(t)[\partial] \) and \( E := \mathbb{k}(\partial)[t] \). The elements of \( D \) are \( \mathbb{k} \)-linear endomorphisms of \( \mathbb{k}(t) \). More precisely, if 
\[
d := a_n\partial^n + \ldots + a_1\partial + a_0 \quad \text{for some } a_i \in \mathbb{k}(t) \quad \text{and if } h \in \mathbb{k}(t), \text{ then :}
\]
\[
d(h) := a_n h^{(n)} + \ldots + a_1 h^{(1)} + a_0 h,
\]
where \( h^{(i)} \) denotes the \( i \)-th derivative of \( h \) and \( a_i h^{(i)} \) is a product in \( \mathbb{k}(t) \).

We note that:
\[
\forall d, d' \in \mathbb{k}(t)[\partial], \forall h \in \mathbb{k}(t), (dd')(h) = d(d'(h)).
\]

For \( V \) and \( W \) two vector subspaces of \( \mathbb{k}(t) \), we set :
\[
\mathcal{D}(V, W) := \{ d \in \mathbb{k}(t)[\partial] : d(V) \subseteq W \}.
\]

Notice that \( \mathcal{D}(R, V) \) is a right \( A_1 \)-submodule of \( Q_1 \) and \( \mathcal{D}(V, R) \) is a left \( A_1 \)-submodule of \( Q_1 \). If moreover \( V \subseteq R \), then \( \mathcal{D}(R, V) \) is a right ideal of \( A_1 \). When \( V = R \), one has \( \mathcal{D}(R, R) = \mathcal{A}_1 \).

If \( I \) is a right ideal of \( A_1 \), we set :
\[
I \star 1 := \{ d(1) : d \in I \}.
\]

It is clear \( I \star 1 \) is a \( \mathbb{k} \)-vector subspace of \( \mathbb{k}[t] \) and that :
\[
I \subseteq \mathcal{D}(R, I \star 1).
\]

The inclusions \( A_1 \subset k(\partial)[t] \) and \( A_1 \subset k(t)[\partial] \) show that, at least, two notions of degree can be defined on \( A_1 \): the degree in “\( t \)” or \( t \)-degree and the degree in “\( \partial \)” or \( \partial \)-degree. Naturally, those degree notions extend to \( Q_1 \). We will note them, respectively, \( \text{deg}_t \) and \( \text{deg}_\partial \).

2 Primary decomposable subspaces

In order to describe the right ideals of \( A_1 \), it is convenient to use the notion of primary decomposable subspaces of \( \mathbb{k}[t] \).

Recall that \( \mathbb{k} \) is not necessarily algebraically closed.

Let \( b, h \in R = \mathbb{k}[t] \) and \( V \) a \( k \)-subspace of \( R \). We set:
\[
\mathcal{O}(b) := \{ a \in R : a' \in bR \},
\]

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where $a'$ denotes the formal derivative of $a$.

E.g.: one has $\mathcal{O}(t^{n-1}) = k + t^n k[t]$.

We set also:

$$S(V) := \{a \in R : aV \subseteq V\} \text{ and } \mathcal{C}(R, V) := \{a \in R : aR \subseteq V\}.$$  

Clearly $\mathcal{O}(b)$ and $S(V)$ are $k$-subalgebras of $R$. If $b \neq 0$, the Krull dimension of $\mathcal{O}(b)$ is $\dim_k(\mathcal{O}(b)) = 1$.

The set $\mathcal{C}(R, V)$ is an ideal of $R$ contained in both $S(V)$ and $V$. Moreover, if $\mathcal{O}(b) \subseteq S(V)$ then $b^2 R \subseteq V$ i.e. $b^2 \in \mathcal{C}(R, V)$.

**Definition 2** A non-zero $k$-vector subspace $V$ of $k[t]$ is said to be primary decomposable (p.d. for short) if $S(V)$ contains a $k$-subalgebra $\mathcal{O}(b)$, with $b \neq 0$. In this case $\mathcal{C}(R, V)$ is a non zero ideal of $R$. A p.d. subspace $V$ of $k[t]$ is said irreducible (p.d.i.) if $V$ is not contained in a proper ideal of $k[t]$.

In [1], R.C. Cannings and M.P. Holland have shown that for p.d. $V$ of $R$, there is the equality

$$\mathcal{D}(V, V) = \text{End}_{A_1}(\mathcal{D}(R, V)).$$

It is shown in [4] that for any non zero right ideal $I$ of $A_1$, there exists $x \in Q_1$ and $\sigma \in \text{Aut}_k(A_1)$ such that:

$$x\sigma(I) = \mathcal{D}(R, k[X_n]),$$

where $n \in \mathbb{N}$ and $k[X_n] := k + t^n k[t]$ is the ring of regular functions on an affine algebraic affine curve $X_n$.

We will show that the inclusion:

$$H(\mathcal{D}(R, k[X_n])) \subseteq H(\mathcal{D}(R, V))$$

where $V$ is a proper p.d.i. subspace of $R$, implies:

$$k[X_n] = V.$$  

That result will lead us to the conclusion that the subgroup $H(\mathcal{D}(R, k[X_n]))$ is equal to its own normalizer in $\text{Aut}_k(A_1)$. 

4
3 The characteristic elements of a right ideal

The first step in the classification of right ideals of the first Weyl algebra $A_1$ is the following:

**Theorem 3.1 (Stafford [8, lemma 4.2])** If $I$ is a non-zero right ideal of $A_1$, then there exist $e, e' \in Q_1$ such that:

(i) $eI \subseteq A_1$ and $eI \cap k[t] \neq \{0\}$;

(ii) $e'I \subseteq A_1$ and $e'I \cap k[\partial] \neq \{0\}$.

With (i) we see that any non-zero right ideal $I$ of $A_1$ is isomorphic to another ideal $I'$ such that $I' \cap k[t] \neq \{0\}$.

**Remark:** the element $e$ (resp. $e'$) of the theorem is a minimal $\partial$–degree element of $I^*$ (resp. a minimal $t$–degree element of $I^*$).

**Corollary 3.2** There exists an unique element (modulo the multiplicative group $k^*$) $f \in I$ such that the full set of elements of $I$ with minimum $t$–degree be exactly:

$$f k[\partial].$$

In the same way, there exists an unique element (modulo the multiplicative group $k^*$) $e^* \in I^*$ such that the full set of elements of $I^*$ with minimum $t$–degree be exactly:

$$k[\partial] e^*.$$

**Proof:** For example, for $f$ : let $e' \in I^*$ such that $e'I \cap k[\partial] \neq \{0\}$. Let $s \in k[\partial]$ such that $e'I \cap k[\partial] = sk[\partial]$. We can take : $f := e'^{-1}s$. q.e.d.

**Definition 3** The elements $e^* \in I^*$ and $f \in I$ are called the characteristic elements of the ideal $I$.

If $V$ is a p.d. subspace and if $I = \mathcal{D}(R, V)$, we will also say that $e^*$ and $f$ are the characteristic elements of the p.d.i. subspace $V$.

**Remark:** if $e^*, f$ are the characteristic elements of a right ideal $I$, then $e^* f \in k[\partial]$ and:

$$e^* I \cap k[\partial] = e^* f k[\partial].$$
E.g.: the characteristic elements of $\mathbb{k}[X_n] = \mathbb{k} + t^n\mathbb{k}[t]$ are:

$$e_n^* := t^{-n}(t\partial) \in \mathcal{D}(\mathbb{k}[X_n], R),$$

$$f_n := (t\partial - 1)...(t\partial - (n - 1)) \in \mathcal{D}(R, \mathbb{k}[X_n])$$

and $e_n^* f_n = \partial^n$.

We now recall some important properties of the p.d.i. subspaces $V$ and of the associated right ideals $\mathcal{D}(R, V)$.

**Lemma 3.3** Let $I$ be a right ideal of $A_1$ such that $I \cap \mathbb{k}[t] \neq \{0\}$.

— If $V := I \ast 1 := \{d(1) : d \in I\}$, then $V$ is a p.d. subspace of $R$ and $I = \mathcal{D}(R, V)$.

— For any p.d. subspace $W$ of $R$, one has

$$\mathcal{D}(R, W) \ast 1 = W \text{ and } \mathcal{C}(R, W) = \mathcal{D}(R, W) \cap \mathbb{k}[t].$$

For a proof cf [1, theorem §3.2].

Henceforth, $\theta$ will denote the $\mathbb{k}$-automorphism of $A_1$ such that:

$$\theta(\partial) = t \text{ and } \theta(t) = -\partial.$$

According to [1], the above lemma has the following consequence:

**Corollary 3.4** For any non zero right ideal $I$ of $A_1$,

i) there is a unique $x \in Q_1$ (modulo the multiplicative group $\mathbb{k}^*$) and a unique p.d.i. subspace $V$ of $R$ such that $xI = \mathcal{D}(R, V)$;

ii) there is a unique $y \in Q_1$ (modulo the multiplicative group $\mathbb{k}^*$) and a unique p.d.i. subspace $W$ of $R$ such that: $\theta(yI) = \mathcal{D}(R, W)$.

Now let us give some properties which characterize a p.d.i. subspace $V$ of $R$.

**Proposition 3.5** Let $V$ be a p.d.i. subspace of $R$ and $m := \dim_{\mathbb{k}} R/V$.

i) For any $0 \neq d \in \mathcal{D}(R, V)$, $d(R) \subseteq V$ and $\deg_t(d) \geq \dim_{\mathbb{k}} R/V$.

ii) If $0 \neq f \in \mathcal{D}(R, V)$ has minimal $t$-degree, then $f(R) = V.$
iii) Let $e^*$ and $f$ be the characteristic elements of $V$. As $e^* \in \mathcal{D}(R, V)^*$ and as $\mathcal{D}(R, V) \cap \mathbb{k}[t] \neq \{0\}$, we have $e^* \in \mathbb{k}(t)[\partial]$. Moreover if:

$$f = t^m c_m(\partial) + t^{m-1} c_{m-1}(\partial) + ... + c_0(\partial)$$

for some $c_i(\partial) \in \mathbb{k}[\partial]$, and if

$$e^* = b_m(\partial)t^{-m} + u$$

where $b_m(\partial) \in \mathbb{k}[\partial]$, $u \in \mathbb{k}(t)[\partial]$ and $\deg(u) < -m$, then:

$$e^* f = b_m(\partial)c_m(\partial).$$

Those properties have all been proved in [3, remarques 1,2,3].

Remarks:

— Note that the $\mathbb{k}$-vector space $R/V$ has finite dimension since $\{0\} \neq \mathcal{C}(R, V) \subseteq V$.

— How to calculate $f'$? We take any $f' \in \mathcal{D}(R, V)$ with minimal $t$–degree $m$, and we expand $f'$ as polynomial in $t$:

$$f' = t^m a_m(\partial) + t^{m-1} a_{m-1}(\partial) + ... + a_0(\partial)$$

where $a_i(\partial) \in \mathbb{k}[\partial]$ for all $i$.

If $p(\partial) := \text{hcf}(a_m(\partial), a_{m-1}(\partial), ..., a_0(\partial))$ then we get $f' = f p(\partial)$ (modulo $\mathbb{k}^*$).

4 About the automorphisms that stabilizes an ideal

Let $I$ be a right ideal of $A_1$ such that $I \cap \mathbb{k}[t] \neq \{0\}$ or $I \cap \mathbb{k}[\partial] \neq \{0\}$. In this paragraph we wish to determine the automorphisms $\sigma \in \text{Aut}_\mathbb{k}(A_1)$ such that $\sigma(I) = I$.

We introduce some particular automorphisms of $A_1$.

If $p \in R$ we define $\sigma := \text{exp(ad}(p))$ by:

$$\forall d \in A_1, \sigma(d) := d + [d, p] + \frac{1}{2!}[[d, p], p] + \frac{1}{3!} [[[d, p], p], p] + ...$$

where $[d, p] := dp - pd$ for all $d \in A_1$. 

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As the application:

\[ A_1 \to A_1, \quad d \mapsto [d, p] \]

is a locally nilpotent derivation, σ is a well defined automorphism of the ring \( A_1 \). Moreover \( \sigma^{-1} = \exp(\text{ad}(-p)) \).

The following theorem is fundamental in this paper.

**Theorem 4.1** Let \( V \) be a p.d. subspace of \( R \) and \( \sigma := \exp(\text{ad}(p)) \) where \( p \in k[t] \). Then \( \sigma(\mathcal{D}(R, V)) = \mathcal{D}(R, V) \) if and only if \( p \in S(V) \).

**Proof:**

**Suppose that** \( p \in S(V) \).

Let \( d \in \mathcal{D}(R, V) \).

Clearly \( dp \) and \( pd \) are both in \( \mathcal{D}(R, V) \), so \( [d, p] \in \mathcal{D}(R, V) \), and this implies the first inclusion: \( \sigma(\mathcal{D}(R, V)) \subseteq \mathcal{D}(R, V) \). In the same way, since \(-p \in S(V)\) we get the second inclusion \( \sigma^{-1}(\mathcal{D}(R, V)) \subseteq \mathcal{D}(R, V) \) and then the equality \( \sigma(\mathcal{D}(R, V)) = \mathcal{D}(R, V) \).

**Now suppose that** \( \sigma(\mathcal{D}(R, V)) \subseteq \mathcal{D}(R, V) \).

Let us take an element \( f \in \mathcal{D}(R, V) \).

In the formal power series ring \( \mathbb{k}[T] \), let

\[
\log(1 + T) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} T^k .
\]

In the ring \( \mathbb{k}[T] \), there is the equality:

\[
\log(1 + (e^T - 1)) = T .
\]

If we specialize in \( T = \text{ad}p \), we get:

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\sigma - \text{Id}_{A_1})^k(f) = [f, p]
\]

(the left hand side sum is finite because \( f \in A_1 \) and \( p \in R \)).

Now, all the terms \( f, \sigma(f), \sigma^2(f), ..., \sigma^n(f), ... \) belong to \( \mathcal{D}(R, V) \), so do the terms:

\[
(\sigma - \text{Id}_{A_1})^k(f)
\]
Therefore, \([f, p] \in \mathcal{D}(R, V)\) and we have :

\[
[f, p](1) = f(p) - pf(1) \in V
\]

\[
\implies pf(1) \in V.
\]

But we have :

\[
V = \mathcal{D}(R, V) \star 1
\]

\[
= \{f(1) : f \in \mathcal{D}(R, V)\}
\]

so \(pV \subseteq V\) i.e. \(p \in S(V)\).

A similar result holds with \(\partial\) instead of \(t\) :

**Corollary 4.2** Let \(I\) be a right ideal of \(A_1\) such that \(I \cap k[\partial] \neq \{0\}\). Let \(W\) be the p.d. subspace of \(R\) such that \(\theta(I) = \mathcal{D}(R, W)\). Let \(q(\partial) \in k[\partial]\) and \(\tau := \exp(\text{ad}(q(\partial)))\).

Then:

\[
\tau(I) = I \iff \theta(q(\partial)) \in S(W).
\]

## 5 The Stafford subgroups

Recall that if \(I\) is a right ideal of \(A_1\) the Stafford subgroup of \(I\) is noted \(H(I)\). In the definition, the notion of isomorphism of right \(A_1\) modules appears. Now, as the ring \(A_1\) is hereditary, if \(I, J\) are isomorphic right ideals of \(A_1\), then there exists \(x \in Q_1\) such that \(xI = J\). So, if \(I\) is a right ideal of \(A_1\), then we have :

i) \(\forall \sigma \in \text{Aut}_{k}(A_1), H(\sigma(I)) = \sigma H(I)\sigma^{-1}\);

ii) \(\forall 0 \neq z \in Q_1, H(zI) = H(I)\).

We will simply note \(H(V) := H(\mathcal{D}(R, V))\) for any p.d. subspace \(V\) of \(R\).

Following the above remark, a Stafford subgroup of \(\text{Aut}_{k}(A_1)\) is of the form \(H(V)\) for some p.d.i. subspace of \(R\).

**Proposition 5.1** Let \(V\) and \(W\) be two p.d.i. subspaces of \(R\).

If \(H(V) \subseteq H(W)\) then :

i) \(S(V) \subseteq S(W)\) and ii) \(C(R, V) \subseteq C(R, W)\) .
Proof:

i) Let $p \in S(V)$. By the theorem 4.1, $\sigma := \exp(\text{ad}(p)) \in H(V)$, thus $\sigma \in H(W)$. So there exists $0 \neq a \in Q_1$ such that $\sigma(\mathcal{D}(R,W)) = a\mathcal{D}(R,W)$. So $a\mathcal{D}(R,W) \subseteq A_1$. In particular, $a \in \mathbb{k}(t)[\partial]$. We have also:

$$\deg_\partial a \leq \deg_\partial d$$

for all $d \in \mathcal{D}(R,W)$. Therefore, $\deg_\partial a = 0$ and $a \in \mathbb{k}(t)$. But:

$$\sigma(\mathcal{D}(R,W)) \star 1 = a\mathcal{D}(R,W) \star 1$$

$$= aW$$

thus $aW \subseteq R$ and $a \in R$ because $RW = R$. Since $\sigma(t) = t$, we have:

$$a^{-1}\mathcal{D}(R,W) = \sigma^{-1}(\mathcal{D}(R,W))$$

so $a^{-1} \in \mathbb{k}[t]$ too. Therefore, $a \in \mathbb{k}^*$ and

$$\sigma(\mathcal{D}(R,W)) = \mathcal{D}(R,W)$$

$$\implies p \in S(W)$$

by the theorem 4.1, again.

ii) If $a \in \mathcal{C}(R,V)$, then:

$$\mathcal{C}(R,V) = V$$

$$\implies aV \subseteq V$$

$$\implies aR \subseteq S(V)$$

$$\implies aR \subseteq S(W)$$

(by i))

$$\implies aRW \subseteq W$$

$$\implies aR \subseteq W$$

i.e. $a \in \mathcal{C}(R,W)$.

q.e.d.

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Proposition 5.2 Let $I$ and $J$ be two right ideals of $A_1$ such that: $\theta(I) = \mathcal{D}(R, V)$ and $\theta(J) = \mathcal{D}(R, W)$ with $V$ and $W$ two p.d.i. subspaces of $R$. Then:

$$H(I) \subseteq H(J) \implies I \cap k[\partial] \subseteq J \cap k[\partial].$$

Proof: We apply the proposition 5.1 to $\theta(I)$ and $\theta(J)$. q.e.d.

Now, we deduce the following for the characteristic elements of p.d. subspaces of $R$:

Corollary 5.3 Let $V$ and $W$ be two p.d.i. subspaces of $R$ such that $H(V) \subseteq H(W)$.

If $e^*_V \in \mathcal{D}(R, V)^*$ and $f_V \in \mathcal{D}(R, V)$ are the characteristic elements of $V$ and $e^*_W \in \mathcal{D}(R, W)^*$, $f_W \in \mathcal{D}(R, W)$ are those of $W$, then:

$$e^*_V f_V \in e^*_W f_W k[\partial].$$

Proof: We have $H(\mathcal{D}(R, V)) = H(e^*_V \mathcal{D}(R, V))$ and $H(\mathcal{D}(R, W)) = H(e^*_W \mathcal{D}(R, W))$, so we have the inclusion:

$$H(e^*_V \mathcal{D}(R, V)) \subseteq H(e^*_W \mathcal{D}(R, W)).$$

Now, we can show that $e^*_V$ is the unique element in $Q_1$ (modulo $k^*$) such that:

$$\theta(e^*_V \mathcal{D}(R, V)) = \mathcal{D}(R, V')$$

for some p.d.i. subspace $V'$ of $R$.

By the proposition 5.2 above, we have:

$$e^*_V \mathcal{D}(R, V) \cap k[\partial] \subseteq e^*_W \mathcal{D}(R, W) \cap k[\partial].$$

Since we have $e^*_V \mathcal{D}(R, V) \cap k[\partial] = e^*_V f_V k[\partial]$ and $e^*_W \mathcal{D}(R, W) \cap k[\partial] = e^*_W f_W k[\partial]$, we obtain:

$$e^*_V f_V \in e^*_W f_W k[\partial].$$

q.e.d.

Now we are ready to prove the main proposition.

We will say that a p.d. subspace of $R$ is monomial if it can be generated by monomials.
Proposition 5.4 Let $V \subset R$ be a proper p.d.i. subspace of $R$ such that $H(\mathbb{k}[X_n]) \subseteq H(V)$. Then:

i) $V$ is monomial;

ii) $\mathcal{C}(R, V) = t^n \mathbb{k}[t]$;

iii) $V = \mathbb{k}[X_n]$.

Proof:

If $n = 1$, clearly $V$ would be equal to $R$, contrary to our hypothesis. So let $n \geq 2$. Let $e^* \in \mathcal{D}(R, V)^*$ and $f \in \mathcal{D}(R, V)$ be the characteristic elements of $V$.

i) We have $\mathbb{k}[X_n] = \mathbb{k} + t^n \mathbb{k}[t]$, $\mathcal{C}(R, \mathbb{k}[X_n]) = t^n \mathbb{k}[t]$ and $t^n \mathbb{k}[t] \subseteq \mathcal{C}(R, V)$. In particular, $t^n \in \mathcal{D}(R, V)$ and as $e^* \in \mathcal{D}(R, V)^*$, we have:

$$e^* = f \in \mathbb{k}[t, t^{-1}, \partial].$$

So, we can use the standard form to describe $e^*$ and $f$:

$$e^* = t^p a_p(t\partial) + \ldots + t^q a_q(t\partial)$$

$$f = t^r b_r(t\partial) + \ldots + t^s b_s(t\partial)$$

for some integers $p \leq q$ and $r \leq s$ and some polynomials $a_i(T), b_j(T) \in \mathbb{k}[T]$.

The characteristic elements of $\mathbb{k}[X_n]$ are:

$$e^n_n := t^{-n}(t\partial), \ f_n := (t\partial - 1)...(t\partial - (n - 1))$$

and:

$$e^n_n f_n = \partial^n.$$

According to the corollary 5.3, we have:

$$e^n_n f_n \in e^* f \mathbb{k}[\partial]$$

and so

$$e^* f = \partial^l$$

for some integer $0 \leq l \leq n$. That forces $p = q$ and $r = s$. Therefore $V = f(R)$ is spanned by its monomial terms $t^{p+i} f_p(i), i \geq 0$, and is monomial.

In fact, $1 \leq l \leq n$. Otherwise:

$$e^* f = 1 \Rightarrow e^* \in \mathbb{k}(t)$$
and:

\[ e^* V = e^* \mathcal{D}(R, V) \star 1 \subseteq R \]

\[ \Rightarrow e^* R = e^* RV \subseteq R \]

\[ \Rightarrow e^* \in \mathbb{k}[t] \]

\[ \Rightarrow e^* = f = 1 \]

\[ \Rightarrow V = R \]

which is impossible.

\[ * \]

**ii)** As \( V \) is monomial and irreducible, \( 1 \in V \) and so \( \mathbb{k}[X_n] \subseteq V \).

Suppose \( t^{n-1} \in V \) and let us consider the automorphism \( \sigma := \exp(\text{ad}(t^{n-1})) \). Clearly \( t^{n-1} \) would belong to \( S(V) \) since \( V \) is monomial and \( t^n \mathbb{k}[t] \subseteq V \). Then we would have \( \sigma \in H(V) \). By applying \( \sigma \) to \( H(\mathbb{k}[X_n]) \), we get a new inclusion:

\[ \sigma H(\mathbb{k}[X_n]) \sigma^{-1} \subseteq \sigma H(V) \sigma^{-1} \]

\[ \iff H(\sigma(\mathcal{D}(R, \mathbb{k}[X_n]))) \subseteq H(V) \] \hspace{1cm} (1)

But for all \( d \in A_1 \), for all \( j \geq 0 \):

\[ \sum_{j \geq 0} \frac{d(t^{n-1})^j r}{j!} - \sum_{j \geq 1} \frac{t^{n-1} d(t^{n-1})^{j-1} r}{(j-1)!} \mod t^n A_1. \]

So, for all \( d \in A_1 \), for all \( r \in R \):

\[ \sigma(d)(r) = \sum_{j \geq 0} \frac{d(t^{n-1})^j r}{j!} - \sum_{j \geq 1} \frac{t^{n-1} d(t^{n-1})^{j-1} r}{(j-1)!} \mod t^n \mathbb{k}[t] \]

(those are in fact finite sums because \( d(t^{n-1})^j r \in t^n \mathbb{k}[t] \) for \( j >> 0 \)).

Thus for all \( d \in \mathcal{D}(R, \mathbb{k}[X_n]) \), for all \( r \in R \), we get:

\[ \sigma(d)(r) = (1 - t^{n-1}) \sum_{j \geq 0} \frac{d(t^{n-1})^j r}{j!} \mod t^n \mathbb{k}[t] \]

\[ \in (1 - t^{n-1}) \mathbb{k}[X_n] + t^n \mathbb{k}[t] = \mathbb{k}(1 - t^{n-1}) + t^n \mathbb{k}[t]. \]

Let \( U_n := \mathbb{k}(1 - t^{n-1}) + t^n \mathbb{k}[t] \). We have just proved:

\[ \sigma(\mathcal{D}(R, \mathbb{k}[X_n])) \subseteq \mathcal{D}(R, U_n). \]

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In the same way, we can prove:

\[ \sigma^{-1}(\mathcal{D}(R, U_n)) \subseteq \mathcal{D}(R, \mathcal{K}[X_n]) \, . \]

Therefore, we have exactly:

\[ \sigma(\mathcal{D}(R, \mathcal{K}[X_n])) = \mathcal{D}(R, U_n) \, . \]

Now, the characteristic elements of \( U_n \) are:

\[ e^*_U := \partial - 2t - n(\partial) + (-1)^n(n-1) + (n-1) \in \mathcal{D}(U_n, R) \]

\[ f_{U_n} = (t\partial - 1)\cdots(t\partial - (n-1)) + (-1)^n(n-1) \in \mathcal{D}(R, U_n) \, . \]

We have:

\[ e^*_U f_{U_n} = (\partial - 1)^2 \]

hence:

\[ e^*_U f_{U_n} \notin \partial^* \mathcal{K}[\partial] = e^* f \mathcal{K}[\partial] \, . \]

By the corollary 5.3, we deduce:

\[ H(\sigma(\mathcal{D}(R, \mathcal{K}[X_n]))) = H(U_n) \nsubseteq H(V) \]

contrary to (1).

So \( t^{n-1} \notin V \) and we have exactly:

\[ C(R, V) = t^n \mathcal{K}[t] \, . \]

\[ * \]

iii) Now, \( C(R, V) = t^n \mathcal{K}[t] \), and \( t^{n-1} \notin V \). For \( n = 2 \), we have already \( V = \mathcal{K}[X_2] \). So we will suppose \( n \geq 3 \).

Let \( 1 \leq n_1 < n_2 < ... < n_s < n - 1 \) be the integers such that \( t^{n_i} \notin V \).

We will show that \( s = n - 2 \) and thus \( V = \mathcal{K}[X_n] \).

We use again the automorphism \( \sigma = \exp(\text{ad}(t^{n-1})) \). We find:

\[ \sigma(\mathcal{D}(R, V)) = \mathcal{D}(R, V_\sigma) \]

where \( V_\sigma := \mathcal{K}(1-t^{n-1}) + V \cap t \mathcal{K}[t] \). We set:

\[ h(T) := (T - n_1)\cdots(T - n_s) \]
and
\[ \lambda := (n - 1)! \frac{h(0)}{h(n - 1)}. \]

We check that the element
\[ g_\sigma := h(t^\partial)(t^\partial - (n - 1))\partial^{n-2} + \lambda t h(t^\partial + 1) \]
is an element of \( \mathcal{D}(R, V_\sigma) \). We see that
\[ \deg_t(g_\sigma) = s + 1 \]
\[ = \dim_k R/V_\sigma \]
(for example because of the short exact sequence:
\[ 0 \to V_\sigma/(t \mathbb{k}[t] \cap V) \to R/(V \cap t \mathbb{k}[t]) \to R/V_\sigma \to 0 \].

So \( g_\sigma \) has minimum \( t \)-degree. The element \( g_\sigma \) can be expanded as:
\[ g_\sigma = t^{s+1}(\partial^{n-1} + \lambda)\partial^s + t^s b_s(\partial) + \ldots + tb_1(\partial) - (n - 1)h(0)\partial^{n-2} \]
for some polynomials \( b_i(T) \in \mathbb{k}[T] \).
Since \( \lambda \neq 0 \), the highest common factor of
\[ (\partial^{n-1} + \lambda)\partial^s, b_s(\partial), \ldots, b_1(\partial), \partial^{n-2} \]
must be some \( \partial^r \) where \( 0 \leq r \leq s \). So if \( e^*_\sigma, f_\sigma \) are the characteristic elements of \( V_\sigma \), we have \( g_\sigma = f_\sigma \partial^r \). Thus \( f_\sigma \) must be equal to:
\[ t^{s+1}(\partial^{n-1} + \lambda)\partial^{s-r} + t^s a_s(\partial) + \ldots + ta_1(\partial) - (n - 1)h(0)\partial^{n-2-r} \]
where \( a_i(\partial) := b_i(\partial)\partial^{-r} \in \mathbb{k}[\partial] \).
But:
\[ H(\mathbb{k}[X_n]) \subseteq H(V) \]
\[ \implies \sigma H(\mathbb{k}[X_n])\sigma^{-1} \subseteq \sigma H(V)\sigma^{-1} \]
\[ \implies H(U_n) \subseteq H(V_\sigma) \]
\[ \implies e^*_U f_{U_n} \in e^*_\sigma f_\sigma \mathbb{k}[\partial] \]
by the corollary [3.4]. Now, by the proposition [3.3, iii),
\[ e^*_\sigma f_\sigma \in (\partial^{n-1} + \lambda)\partial^{s-r}\mathbb{k}[\partial] \].
As a consequence:

\[
\left( \partial^{n-1} + (-1)^n(n-1)! \right)^2 \in (\partial^{n-1} + \lambda)\partial^{s-r}k[\partial]
\]

which implies that \( s = r \) and \( \lambda = (-1)^n(n-1)!^2 \).

Since \( s = r \), we have:

\[
g_\sigma = f_\sigma \partial^s
\]

and:

\[
g_\sigma(t^i) = 0
\]

for all \( 0 \leq i \leq s-1 \). Thus \( h(i+1) = 0 \) for \( i = 0, 1, ..., s-1 \). But \( n_1, ..., n_s \)
are the only roots of \( h \) so:

\[
\forall 0 \leq j \leq s, \ n_j = j .
\]

From the equality \( \lambda = (-1)^n(n-1)! \) we then deduce:

\[
\frac{(-1)^s n_1 \ldots n_s}{(n-2) \ldots (n-1-s)} = (-1)^n
\]

\[
\iff \frac{s!(n-2-s)!}{(n-2)!} = (-1)^{n+s}
\]

\[
\iff \binom{n-2}{s} = (-1)^{n+s}
\]

\[
\iff s = 0 \text{ and } n \text{ is even or } s = n-2 .
\]

If \( s = n-2 \), then \( V = k[X_n] \) and the proof is finished. If \( s = 0 \) and \( n \) is
even, then:

\[
V = k + kt + \ldots + kt^{n-2} + t^n k[t]
\]

with \( n \geq 4 \).

We set \( \sigma := \exp(\text{ad}(t^{n-2})) \). Then we have:

\[
H(k[X_n]) \subseteq H(V)
\]

\[
\iff \sigma H(k[X_n])\sigma^{-1} \subseteq \sigma H(V)\sigma^{-1}
\]

\[
\iff H(W_n) \subseteq H(V_\sigma)
\]

where:

\[
W_n = (1 - t^{n-2})k[X_n] + t^n k[t]
\]

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\[= \mathbb{k}(1 - t^{n-2}) + t^n \mathbb{k}[t]\]

and

\[V_{\sigma} = (1 - t^{n-2})V + t^n \mathbb{k}[t]\]

\[= \mathbb{k} + \mathbb{k}(t - t^{n-1}) + t^2 + ... + t^{n-2} + t^n \mathbb{k}[t].\]

Now, let \(e^*_W, f_W\) be the characteristic elements of \(W_n\) and \(e^*_\sigma, f_\sigma\) those of \(V_\sigma\). Because of the corollary [5.3] we should have :

\[e^*_W f_W \in e^*_\sigma f_\sigma \mathbb{k}[\partial].\]

But we can check that :

\[e^*_W = \left( \frac{\partial^{n-1}}{(n-1)!} + (-1)^{n-1}\partial \right) t^{1-n} + \left( \frac{\partial^{n-2}}{(n-2)!} + (-1)^{n-1} \right) t^{-n},\]

\[f_W = \frac{(t\partial - 1)...(t\partial - (n-1))}{(n-1)!} + (-1)^{n-1}t^{n-2}(t\partial - 1),\]

\[f_\sigma = t \left( \frac{\partial^{n-1}}{(n-1)!} + \partial \right) - \frac{\partial^{n-2}}{(n-2)!} - 1.\]

We deduce that :

\[e^*_W f_W = \left( \frac{\partial^{n-1}}{(n-1)!} + (-1)^{n-1}\partial \right)^2\]

\[= \left( \frac{\partial^{n-1}}{(n-1)!} - \partial \right)^2\]

because \(n\) is even and :

\[e^*_\sigma f_\sigma \mathbb{k}[\partial] \subseteq \left( \frac{\partial^{n-1}}{(n-1)!} + \partial \right) \mathbb{k}[\partial]\]

which contradicts [4].

Hence \(V = \mathbb{k}[X_n]\).

q.e.d.

Using the description of right ideals of \(A_1\) in [4], we deduce the following:
Corollary 5.5 For any non principal right ideals $I$ and $J$, the following equivalences are satisfied:

\[ H(I) \subset H(J) \]
\[ \iff H(I) = H(J) \]
\[ \iff \exists x \in \text{Frac}(A_1), \exists \sigma \in \text{Aut}_k(A_1), I = x\sigma(J). \]

We now obtain the announced result.

Proposition 5.6 Let $I$ be any right ideal of $A_1$. The subgroup $H(I)$ is equal to its own normalizer subgroup in $\text{Aut}_k(A_1)$.

Proof : By [3], we can suppose $I = \mathcal{D}(R, \mathbb{k}[X_n])$. Let $\gamma \in \text{Aut}(A_1)$ such that

\[ \gamma H(\mathbb{k}[X_n])\gamma^{-1} = H(\mathbb{k}[X_n]). \]

Then we have

\[ H(\gamma(\mathcal{D}(R, \mathbb{k}[X_n]))) = H(\mathbb{k}[X_n]). \]

We have also : $\gamma(\mathcal{D}(R, \mathbb{k}[X_n])) \simeq \mathcal{D}(R, V)$ for some p.d.i. subspace $V$ of $R$.

Thus, we get $H(\mathbb{k}[X_n]) = H(V)$, so $V = \mathbb{k}[X_n]$ by the proposition [5.4].

Finally, we have $\gamma(\mathcal{D}(R, \mathbb{k}[X_n])) \simeq \mathcal{D}(R, \mathbb{k}[X_n])$ and that means $\gamma \in H(I)$.

q.e.d.

References


