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SCROLLS AND HYPERBOLICITY

C. CILIBERTO, M. ZAIDENBERG

Abstract. Using degeneration to scrolls, we give an easy proof of non-existence of curves of low genera on general surfaces in $\mathbb{P}^3$ of degree $d \geq 5$. We show, along the same lines, boundedness of families of curves of small enough genera on general surfaces in $\mathbb{P}^3$. We also show that there exist Kobayashi hyperbolic surfaces in $\mathbb{P}^3$ of degree $d = 7$ (a result so far unknown), and give a new construction of such surfaces of degree $d = 6$. Finally we provide some new lower bounds for geometric genera of surfaces lying on general hypersurfaces of degree $3d \geq 15$ in $\mathbb{P}^3$.

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Introduction

What is the lowest geometric genus $\eta(n, d)$ of a reduced, irreducible curve on a very general hypersurface of degree $d$ in $\mathbb{P}^n$? The case $n = 2$ is trivial. For $n = 3$ one has

$$\eta(3, d) = 0 \text{ if } d \leq 4 \text{ while } \eta(3, d) = \left(\frac{d-1}{2}\right) - 3 \text{ if } d \geq 5$$

(1)

and for any $d \geq 6$ this bound is achieved by tritangent plane sections, and only by these [49]. Similarly, $\eta(4, d) = 0$ if $d \leq 5$, while $\eta(4, 6) \geq 2$ [14]. More generally, for $n \geq 4$ one has

$$\eta(n, d) = 0 \quad \forall d \leq 2n - 3 \quad \text{and} \quad \eta(n, d) \geq 1 \quad \forall d \geq 2n - 2$$

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see [46] (in the case $d = 2n - 3$ see also [29] [39] [45]). Presumably, $\eta(n, d) \to \infty$ as $d \to \infty$, however, the asymptotic of $\eta(n, d)$ is unknown. One is equally interested in bounds for the geometric genus or other numerical invariants of higher dimensional subvarieties in general hypersurfaces, see e.g. [12] [18] [19] [37] [47] [48] [50].

A projective variety $X$ is algebraically hyperbolic if it does not admit a non–constant morphism from an abelian variety. If there is an algebraically hyperbolic hypersurface of degree $d$ in $\mathbb{P}^n$, then a very general hypersurface of degree $d$ is algebraically hyperbolic as well. For instance, a very general surface $X$ of degree $d \geq 5$ in $\mathbb{P}^3$ is algebraically hyperbolic. Indeed, $X$ does not contain rational or elliptic curves since $\nu(3, d) \geq 3$ for $d \geq 5$ by [1]. This also follows from Proposition 2.1 below if $d \geq 6$, while Corollary 2.2 offers a short proof of Xu’s and Voisin’s result about non–existence of rational curves on a very general quintic in $\mathbb{P}^3$. Since $X$ is of general type it cannot be dominated by an abelian variety.

Similarly, a general sextic threefold $X$ in $\mathbb{P}^4$ is algebraically hyperbolic. Indeed, $X$ does not contain rational or elliptic curves since $\nu(4, 6) \geq 2$. By [50] Theorem 1 it also does not contain surfaces with desingularization of geometric genus at most 2. Therefore, every map from an abelian variety to $X$ is constant.

A variety $X$ is Kobayashi hyperbolic, or simply hyperbolic, if it does not admit any non–constant entire curve $\mathbb{C} \to X$. Hyperbolicity implies algebraic hyperbolicity, and it is stable under small deformations.

Given one of the two above hyperbolicity notions, one can ask what is the lowest degree $d = d(n)$ such that a very general projective hypersurface in $\mathbb{P}^n$ of degree $d$ possesses this property. For instance, the classical Kobayashi problem suggests that a very general hypersurface of degree $d \geq 2n - 1$ in $\mathbb{P}^n$ is hyperbolic.

It is known that, indeed, a very general surface of degree $d \geq 18$ in $\mathbb{P}^3$ is hyperbolic [38] (see also [15] [31]). The existence of hyperbolic surfaces in $\mathbb{P}^3$ of degree $d$ for all $d \geq 8$ was established with a degeneration argument in [41] (see the references in [41] for other constructions), and for $d = 6$ in [17]. In [43] below (see, in particular, Theorem 3.3) we give an alternative proof for the case $d = 6$, which works also in the (so far unknown) case $d = 7$. The case $d = 5$ in the Kobayashi problem for $\mathbb{P}^3$ remains open.

Our method consist in degenerating a general hypersurface to a certain special one, following the limits in the degeneration of entire curves or of algebraic curves or surfaces, according to the hyperbolicity notion we are dealing with. In this framework the concept of Brody curves and their limits is very useful, cf. e.g. [43] [44] [51] [52]. We recall a minimum of basics on this subject in [41]. Our preferable degenerations here are to scrolls, and we recall their main properties in [41]. In subsection 2.1 we give an easy proof of non–existence of curves of low genera on very general surfaces in $\mathbb{P}^3$ of a given degree. In [33] we treat the higher dimensional case. In particular, in Theorem 3.3 we provide a lower bound for the geometric genus of surfaces contained in very general hypersurfaces of degree $3d \geq 15$ in $\mathbb{P}^4$.

By a well-known theorem of Bogomolov [8], on a smooth surface $S$ of general type with $c_2^2(S) > c_2(S)$, the curves of a fixed geometric genus vary in a bounded family. This was partially extended in [30] to any smooth surface $S$ of general type by showing that there are only a finite number of rational and elliptic curves on $S$ with a fixed number of nodes and ordinary triple points and no other singularities. In subsection 2.2 we address the question whether curves of a given geometric genus have bounded degree on a general surface of degree $d \geq 5$ in $\mathbb{P}^3$. We give an affirmative answer for all genera $g \leq d^2 + O(d)$.

Finally in subsection 4.3 we prove the aforementioned Theorem 4.6.

1. Scrolls

1.1. Generalities on scrolls. By a scroll in $\mathbb{P}^n$ we mean the image $\Sigma = \varphi(S)$ of a smooth, proper $\mathbb{P}^1$-bundle $\pi : S \to E$ under a birational morphism $\varphi : S \to \Sigma \subset \mathbb{P}^n$ which sends the rulings of $S$ (i.e. the fibres of $\pi$) to projective lines, called rulings of $\Sigma$. The variety $E$ is called the base of the scroll. We will denote by $H$ and $F$ a hyperplane section and a ruling of $\Sigma$, respectively. We may abuse notation denoting by $H$ and $F$ also their proper transforms on $S$.

The induced morphism $\mu : E \to \text{Gr}(1, n)$ to the Grassmanian of lines in $\mathbb{P}^n$ is birational onto its image. Any such morphism $\mu$ appears in this way, where $\pi : S \to E$ is induced via $\mu$ by the tautological $\mathbb{P}^1$–bundle
over the Grassmanian. Furthermore, $d = \deg(\Sigma)$ is equal to the degree of the subvariety $\mu(E)$ under the Plücker embedding of the Grassmanian \cite{12.4}, [15 11.4.1].

We will suppose form now on that $\varphi : S \to \Sigma$ coincides with the normalization morphism. We denote by $\text{br}(\Sigma)$ the set of multibranch points of $\Sigma$, i.e. the set of points $x \in \Sigma$ such that $\varphi^{-1}(x)$ consists of more than one point. If $x \notin \text{br}(\Sigma)$, e.g. $x$ is a smooth point of $\Sigma$, then there is just one ruling passing through $x$. Since $\varphi$ is finite, there is no point on $\Sigma$ which belongs to infinitely many rulings.

We let $\Delta_{\Sigma} = \text{br}(\Sigma) \subseteq \Sigma$ and $\Delta_{S} = \varphi^{-1}(\Delta_{\Sigma}) \subseteq S$. We will assume that the following conditions hold:

(C1) $\dim(\Sigma) = n - 1$;
(C2) $\Delta_{\Sigma}$ coincides with $\text{Sing}(\Sigma)$;
(C3) $\Delta_{\Sigma}$ and $\Delta_{S}$ are both irreducible of dimension $n - 2$;
(C4) a general point $x \in \Delta_{\Sigma}$ is a normal crossing double point of $\Sigma$. In particular, $\varphi^{-1}(x)$ has cardinality 2, and $x$ sits on two different rulings;
(C5) $\Delta_{\Sigma}$ contains no ruling, i.e. $\mu : E \to \text{Gr}(1, n)$ is injective.

In this situation $\Delta_{\Sigma}$ and $\Delta_{S}$ both have natural scheme structures, and $\Delta_{S}$ is a reduced divisor on $S$.

Conditions (C1)–(C4) are verified if $S \subseteq \mathbb{P}^{n+k}$ is a smooth scroll of dimension $n - 1$ and $\varphi : S \to \Sigma$ is induced by a general linear projection $\mathbb{P}^{n+k} \to \mathbb{P}^{n}$; see [23]. The last condition (C5) can be easily checked by induction; we leave the details to the reader.

**Lemma 1.1.** In the above setting, a general ruling of $\Sigma$ meets the double locus $\Delta_{\Sigma}$ in $d - n + 1$ points. In particular, $\Sigma$ is swept out by an $(n - 2)$-dimensional family of $(d - n + 1)$-secant lines of $\Delta_{\Sigma}$.

**Proof.** By the Ramification Formula [21 9.3.7(b)] there is a linear equivalence relation on $S$

$$\Delta_{S} \sim (d - n - 1)H - K_{S}. \quad (2)$$

Since $F \cdot H = 1$ and $F \cdot K_{S} = -2$, we have $F \cdot \Delta_{S} = d - n + 1$. Since $\Delta_{S}$ is reduced, the general ruling of $S$ meets $\Delta_{S}$ in $d - n + 1$ distinct points. The assertions follow because $\varphi$ induces an isomorphism of each ruling of $S$ to its image. $\Box$

### 1.2. Surface Scrolls with Ordinary Singularities.

We restrict here to the case $n = 3$. So $E$ is a smooth curve of genus $g$ and $S \subseteq \mathbb{P}^{3+k}$ and $\Sigma \subseteq \mathbb{P}^{3}$ are surfaces, called *scrolls of genus* $g$; here $g$ is the *sectional genus* of the scroll.

**Remark 1.2.** For an irreducible curve $C$ on $S$ of genus $g'$ such that $C \cdot F = \nu$, the Riemann–Hurwitz Formula implies the inequalities $g' \geq \nu(g - 1) + 1 \geq g$. In particular, for $g \geq 1$ the only irreducible curves on $S$ of geometric genus $g' < g$ are the rulings, and for $g' = g \geq 2$ the curve $C$ is a *unisecant* i.e., the intersection number $\nu$ of $C$ with rulings is 1. The same holds on $\Sigma$.

We say that $\Sigma$ has *ordinary singularities* if, in addition to conditions (C1)–(C5), the following hold:

(C6) the singularities of the double curve $\Delta_{\Sigma}$ consist of finitely many triple points, which are also ordinary triple points of the surface $\Sigma$ (these are locally analytically isomorphic to the surface singularity $xyz = 0$ in $\mathbb{C}^{3}$ at the origin);
(C7) the non–normal crossings singularities of $\Sigma$ are finitely many *pinch points*. These are the points in $\Delta_{\Sigma} \setminus \text{br}(\Sigma)$, and there is just one ruling through each of them. A pinch point has just one preimage on $S$, which, abusing terminology, we will also call a pinch point;
(C8) the only singularities of $\Delta_{S}$ are ordinary double points, three of them over each triple point of $\Delta_{\Sigma}$.

Furthermore, the degree two map $\varphi : \Delta_{S} \to \Delta_{\Sigma}$ is ramified exactly over the pinch points of $\Sigma$.

These are the singularities of a general projection to $\mathbb{P}^{3}$ of a smooth surface in $\mathbb{P}^{4}$, or even of a surface in $\mathbb{P}^{4}$ with finitely many nodes, i.e. double points with tangent cone formed by two planes spanning $\mathbb{P}^{4}$. In this case the curves $\Delta_{\Sigma}$ and $\Delta_{S}$ are irreducible, except for the projection in $\mathbb{P}^{3}$ of the Veronese surface of degree 4 in $\mathbb{P}^{5}$ (cf. [21 22 32 33]). Note that a general projection to $\mathbb{P}^{4}$ of any smooth surface in $\mathbb{P}^{r}$ (with $r > 4$) has only nodes as singularities and the Veronese surface of degree 4 in $\mathbb{P}^{5}$ is the only one whose general projection to $\mathbb{P}^{4}$ is smooth (see 31 53).
The basic invariants of $S$ are

$$c_2 = K_S^2 = 8(1-g), \quad c_2 = e(S) = 4(1-g), \quad \text{and} \quad \chi(O_S) = \frac{c_1^2 + c_2}{12} = 1 - g$$

(see [20], [26] Ch. 5, §2). The following projective invariants are also important

$$\delta_\Sigma = \deg(\Delta_\Sigma)$$
$$\gamma_\Sigma = \text{the geometric genus of } \Delta_\Sigma$$
$$t_\Sigma = \text{the number of triple points of } \Delta_\Sigma$$
$$\pi_\Sigma = \text{the number of pinch points of } \Sigma$$
$$\tilde{\gamma}_\Sigma = \text{the geometric genus of } \Delta_S$$

(in the sequel we suppress the index $\Sigma$ when unnecessary).

For the proof of the following formulas see e.g. [8], [16, §11.5], [20, p. 176], [39], [42, (1)-(10)], and references therein.

**Proposition 1.3.** Let $\Sigma$ stands as before for a scroll in $\mathbb{P}^3$ of degree $d$ and genus $g$ with ordinary singularities. Then the projective invariants of $\Sigma$ are given by the Bonnesen’s formulas

$$\delta = \left(\frac{d-1}{2}\right) - g,$$  \hspace{1cm} (4)
$$\gamma = \left(\frac{d-3}{2}\right) + (d-5)g,$$  \hspace{1cm} (5)
$$t = \left(\frac{d-2}{3}\right) - (d-4)g,$$  \hspace{1cm} (6)
$$p = 2d + 4(g-1),$$  \hspace{1cm} (7)
$$\tilde{\gamma} = 2(\gamma + g) + d - 3.$$  \hspace{1cm} (8)

**Remark 1.4.** Due to (6), for $d \geq 5$ the inequality $t \geq 0$ reads $g \leq \frac{1}{2}(d-2)(d-3)$. This implies

$$g \leq d - 4, \quad \text{if} \quad d = 5, 6, 7.$$  \hspace{1cm} (9)

In the sequel we also need the inequality

$$\gamma > 3(g-1) \quad \text{for all} \quad g \geq 1 \quad \text{and} \quad d \geq 5.$$  \hspace{1cm} (10)

This follows from [4] for $d \geq 8$ and from [4] and [9] for $d = 5, 6, 7$ (actually, $\gamma > 3g$ for all $g \geq 1$ and $d \geq 5$ except for $g = 2, d = 6$).

1.3. **Surface scrolls with general moduli.** We recall a result from [4] (cf. also [10] Theorem 1.2]).

**Theorem 1.5.** Let $g \geq 0$ be an integer and let $k = \min\{1, g-1\}$. If $d \geq 2g+3+k$, then there exists a unique irreducible component $\mathcal{H}_{d,g}$ of the Hilbert scheme of scrolls of degree $d$ and sectional genus $g$ in $\mathbb{P}^r$, where $r = d - 2g + 1$, such that the general point $[S] \in \mathcal{H}_{d,g}$ represents a smooth scroll $S$ with $h^1(S, O_S(1)) = 0$, i.e. $S$ is non-special. Furthermore $\mathcal{H}_{d,g}$ dominates the moduli space $\mathcal{M}_g$ of smooth curves of genus $g$ via the map sending a scroll to its base.
Remarks 1.6. (i) Assuming that \( d \geq 2g + 3 + k \) (as in the above theorem), we have \( r \geq 3 \) if \( g = 0, r \geq 4 \) if \( g = 1 \), and \( r \geq 5 \) if \( g \geq 2 \), and we can project smooth scrolls \( S \) with \([S] \in \mathcal{H}_{d,g}\) thus obtaining scrolls \( \Sigma \) in \( \mathbb{P}^3 \) with ordinary singularities and irreducible double curve.

(ii) The assumption of Theorem 1.5 gives \( d \geq 2g + 4 \) for \( g \geq 2 \) and \( d \geq 2g + 3 = 5 \) for \( g = 1 \). In fact, similar results hold also for \( g \geq 2 \) and \( d = 2g + 3 \) or \( d = 2g + 2 \), while the corresponding scrolls are no longer smooth.

More precisely, let \( g \geq 2 \) and \( d = 2g + 3 \) (i.e., \( r = 4 \)). Then \( \mathcal{H}_{d,g} \) is a component of the Hilbert scheme, whose general point \([S'] \in \mathcal{H}_{d,g}\) represents a scroll \( S' \subset \mathbb{P}^4 \) with only nodes as singularities and with a smooth normalization \( S \) such that \( h^0(S, O_S(1)) = 5 \) and \( h^1(S, O_S(1)) = 0 \) (this can be shown with the same analysis as in [10]). Once again, \( \mathcal{H}_{d,g} \) dominates the moduli space \( \mathcal{M}_g \).

If \( g \geq 2 \) and \( d = 2g + 2 \) (i.e., \( r = 3 \)), a similar assertion holds. However, now \( \mathcal{H}_{d,g} \) is no longer a component of the Hilbert scheme, but a locally closed subset of the projective space \( \mathcal{L}_d = |O_{\mathbb{P}^3}(d)| \) of all surfaces of degree \( d \) in \( \mathbb{P}^3 \). It is reasonable to expect that a general point \([\Sigma] \in \mathcal{H}_{d,g}\) represents a scroll \( \Sigma \subset \mathbb{P}^3 \) with ordinary singularities. This would follow by going deeper into the analysis performed in [10], but we do not use this here in the full generality. We investigate below in more detail various examples (see especially Example 1.9).

Example 1.7. Elliptic quartic scrolls. Let \( E \) be a smooth curve of type \((a, b)\) on \( \mathbb{P}^1 \times \mathbb{P}^1 \), identified with a smooth quadric in \( \mathbb{P}^3 \). The genus of \( E \) is \( g = ab - a - b + 1 \). Consider a pair of skew lines \( R_1, R_2 \) in \( \mathbb{P}^3 \). Identifying these lines with the factors of \( \mathbb{P}^3 \times \mathbb{P}^1 \), we can interpret the canonical projections of \( E \) to the factors as maps \( \varphi_i : E \to R_i, i = 1, 2 \), of degree \( a \) and \( b \), respectively. For each \( x \in E \) we consider the line \( L_x \) joining the points \( \varphi_i(x), i = 1, 2 \). This yields the map \( \mu : x \in E \to L_x \in \text{Gr}(1, 3) \). Its image is a smooth curve on \( \text{Gr}(1, 3) \) under the Plücker embedding of the Grassmanian \( \text{Gr}(1, 3) \) as a quadric in \( \mathbb{P}^5 \). The associated scroll

\[
\Sigma = \Sigma_{a,b} = \bigcup_{x \in E} L_x
\]

in \( \mathbb{P}^3 \) with base \( E \) has degree \( a + b \). Indeed, it has singularities of multiplicities \( a \) along \( R_1 \) and \( b \) along \( R_2 \). So a line \((A, B)\), where \( A \in R_1 \) and \( B \in R_2 \), meets \( \Sigma \) only in \( A \) and \( B \).

In particular, for \( a = b = 2 \) we obtain a quartic scroll in \( \mathbb{P}^3 \) of genus 1 with two skew double lines, and for \( a = 3, b = 2 \) a quintic scroll of genus 2 with a double line and a triple line.

From now on, we concentrate on an elliptic quartic scroll \( \Sigma = \Sigma_{2,2} \). The preimage \( \Delta_S \) of \( \Delta_{\Sigma} \) on \( S \) consists of two disjoint copies \( E_1, E_2 \) of \( E \) with \( \varphi_i : E_i \to R_i, i = 1, 2 \), corresponding to two distinct \( g_{1,1} \)'s on \( E \). There are in total 8 pinch points of \( \Sigma \), 4 on each of the lines \( R_1, R_2 \). These are the branch points of the maps \( \varphi_i, i = 1, 2 \). If these maps are sufficiently general, also the pinch points are generically localized along \( R_1, R_2 \) and the ruling passing through a pinch point does not contain any other pinch point.

Let us illustrate on this example our degeneration method. Any smooth elliptic quartic curve is a complete intersection of two quadrics in \( \mathbb{P}^3 \). Hence it embeds as well to the Grassmanian \( \text{Gr}(1, 3) \). By virtue of Remark 1.6 to Theorem 1.5 (the case \( r = 3 \)) these curves fill in a unique irreducible component \( \mathcal{H}_{4,1} \) of the Hilbert scheme of curves of degree 4 in \( \text{Gr}(1, 3) \), which dominates the moduli space \( \mathcal{M}_1 \). The component \( \mathcal{H}_{4,1} \) contains all limit curves, e.g. all reduced, nodal curves of degree 4 and arithmetic genus 1 spanning a \( \mathbb{P}^3 \). For instance, the union \( E_0 \) of two conics \( \Gamma_1, \Gamma_2 \) meeting transversally at two distinct points \( f_1, f_2 \) is such a limit curve. The curve \( E_0 \) corresponds to the union \( \Sigma_0 \) of two quadrics surfaces \( Q_1, Q_2 \) in \( \mathbb{P}^3 \) associated to the conics \( \Gamma_1, \Gamma_2 \) on the Grassmanian \( \text{Gr}(1, 3) \). We may assume these quadrics to be smooth. They intersect along the quadrilateral \( F_1 \cup F_2 \cup G_1 \cup G_2 \), where the lines \( F_1, F_2 \) correspond to \( f_1, f_2 \) and belong to the same ruling on each quadric, and \( G_1, G_2 \) are distinct lines belonging to the other ruling. We let \( p_{ij} = F_i \cap G_j, i, j = 1, 2 \).

The surface \( \Sigma_0 \) can be seen as a flat limit of surfaces of type \( \Sigma \), since it corresponds to a point in \( \mathcal{H}_{4,1} \). The limit of the ruling of \( \Sigma \) is the union of the two rulings of \( Q_1 \) and \( Q_2 \) containing \( F_1, F_2 \). The limits of the double lines \( R_1, R_2 \) are the lines \( G_1, G_2 \). The limit of each of the components \( E_i \) of the curve \( \Delta_S \) on \( S \) consists of two copies of \( G_i \), glued at \( p_{11}, p_{21} \). Each of these points is the limit of two pinch points of \( \Sigma \).
Conversely, when we deform $\Sigma_0$ to $\Sigma$, the two double lines $F_1$ and $F_2$ of $\Sigma_0$ disappear, because we are smoothing the two nodes of $E_0$. Each of the points $p_{ij}$ ($i, j = 1, 2$) gives rise to two pinch points generically located along the double line of $\Sigma$, which deforms $G_j$.

**Example 1.8.** Elliptic quintic scrolls. Consider now the case where $d = 5$ and $g = 1$. By Theorem [5], a general point $[S] \in H_{5,1}$ represents a smooth scroll $S$ in $\mathbb{P}^4$, whose general projection $\Sigma$ to $\mathbb{P}^3$ has ordinary singularities. According to Bonnesen’s formulas (1.1–8), the double curve $C = \Delta \Sigma$ is an irreducible, smooth, elliptic quintic curve, which contains the 10 pinch points of $S$. Its preimage $\tilde{C} = \Delta S$ is a smooth, irreducible curve on $S$ of genus 6. By Lemma [1.1] the rulings of $\Sigma$ are trisecant lines to $C$.

Conversely, for any smooth elliptic quintic curve $C$ in $\mathbb{P}^3$, the trisecant lines to $C$ sweep out a quintic scroll $\Sigma$, which is singular exactly along $C$ (cf. Berzolari’s Formula, Proposition 1 and Corollary 2 in [6]). Such a surface $\Sigma$ is an elliptic scroll, and by the Riemann–Roch Theorem it comes as a projection of a surface represented by a point in $H_{5,1}$ as above.

Any such scroll $\Sigma$ corresponds to an embedding of an elliptic quintic curve $E$ in $\text{Gr}(1,3)$ via the map $\mu$ as in (1.1). The image of $E$ is a quintic elliptic normal curve, contained in a hyperplane section of $\text{Gr}(1,3)$. Indeed, any normal, elliptic quintic curve lies on some smooth quadric in $\mathbb{P}^4$, hence on a hyperplane section of $\text{Gr}(1,3)$.

There is another interpretation of these elliptic quintic scrolls. Let $E$ be an elliptic curve. Consider its symmetric product $E(2)$, formed by all degree 2 effective divisors on $E$. The class of the diagonal $D = \{2p, p \in E\}$ is divisible by 2 in $\text{Pic}(E(2))$; we denote by $\vartheta$ the class of its half. One has $K_{E(2)} \sim -\vartheta$.

The Abel–Jacobi map $\alpha : E(2) \to \text{Pic}^{(2)}(E) \cong E$ makes $E(2)$ a $\mathbb{P}^1$–bundle with base $E$. The rulings are the $g_1^2$’s on $E$. The coordinate curves $E_p = \{x + p, x \in E\} \cong E$ are unisecant curves of the rulings and form a one–dimensional family parametrized by the point $p$ varying on $E$. We have $E_p^2 = 1$. If $F_1, F_2$ are rulings, then the divisor class of the curve $E_p + F_1 + F_2$ is ample on $E(2)$ and maps isomorphically the surface $E(2)$ onto a quintic scroll $S$ in $\mathbb{P}^4$. Each coordinate curve $E_p$ is mapped to a smooth plane cubic on $S$ which is the residual intersection of $S$ with a hyperplane containing two rulings. Conversely any smooth plane cubic on $S$ is a coordinate curve: indeed, it sits on a 1–dimensional family of hyperplane sections of $S$ and their residual intersections with $S$ is a pair of lines.

Let as before $\Sigma$ denote the image of $S$ under a general projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$. Any coordinate curve on $S$ is isomorphically mapped to a smooth plane cubic and the images on $\Sigma$ of two distinct coordinate cubics on $S$ are distinct. This provides a complete, one–parameter family of smooth plane cubic curves on $\Sigma$ which are the only plane cubics on $\Sigma$. Let $L$ be the plane containing one of them $\tilde{E}$. The residual intersection on $L \cap \Sigma$ must be a union of two rulings, which meet on $C$. The corresponding rulings on $S$ span a hyperplane which cuts out on $S$ a coordinate cubic $\tilde{E}$ plus the two rulings. Hence $\tilde{E}$ is the image of $\tilde{E}$ on $\Sigma$.

Let $x \in C$ be a general point and $F_1, F_2$ the two rulings through $x$. The plane $\pi$ spanned by them cuts $\Sigma$ in the union of $F_1, F_2$ and a smooth cubic $\tilde{E}$, which is the projection of a unique coordinate curve. When $x$ varies, we obtain in this way all projections of coordinate curves. This shows that $C$ is isomorphic to $E$, since it parametrizes the family of coordinate curves.

When the center of projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ varies we obtain a monodromy action. The following argument shows that this monodromy is irreducible on appropriately chosen objects.

The cubic curve $\tilde{E}$ as above does not pass through $x$, and cuts the ruling $F_1$ in three (generically distinct) points $p_i, q_{1i}, q_{2i}$, $i = 1, 2$, such that $q_{1i}, q_{2i} \in C$. Indeed, $p_i, q_{1i}, q_{2i}$, $i = 1, 2$, are the five intersection points of $\pi$ with $C$. By moving the centre of projection, we may assume that the pair of rulings $(F_1, F_2)$ corresponds to a general divisor of a given $g_1^2$ on $E$, and that $q_{11} + q_{12}$ ($q_{21} + q_{22}$, respectively) is a general divisor in the $g_2^3$ cut out on $\tilde{E}$ by the lines through $p_1$ (through $p_2$, respectively). In conclusion, by moving the centre of projection the monodromy interchanges the pairs $q_{11} + q_{12}$ and $q_{21} + q_{22}$ and also interchanges the points in each pair separately.

**Example 1.9.** Sextic scroll of genus two. By the case $g \geq 2$, $r = 3$ of Remark [1.6], there exist sextic scrolls $\Sigma$ of genus two in $\mathbb{P}^3$. They correspond to genus 2 curves of degree 6 on the Grassmanian $\text{Gr}(1,3)$. In fact, by the Riemann–Roch Theorem, any smooth curve of genus 2 embeds in $\mathbb{P}^4$ as a sextic. This sextic
spans \( \mathbb{P}^4 \) and lies on a smooth quadric in \( \mathbb{P}^4 \), hence on a hyperplane section of the Grassmanian \( \text{Gr}(1,3) \). These curves fill in a unique component \( \mathcal{H}_{6,2} \) of the Hilbert scheme of curves of degree 6 and genus 2 in \( \text{Gr}(1,3) \), which dominates \( \mathcal{M}_2 \) via the natural map. As in Example 1.7.2, \( \mathcal{H}_{6,2} \) contains limit curves, and in particular all reduced, nodal curves of degree 6 and arithmetic genus 2 spanning a \( \mathbb{P}^4 \).

Assuming that a general such scroll \( \Sigma \) has ordinary singularities, Lemma 1.1 and Proposition 1.3 say that the rulings of \( \Sigma \) are four–secant lines to the double curve \( C = \Delta_\Sigma \), which is a smooth, irreducible curve in \( \mathbb{P}^3 \) of degree 8 and genus 5, passing through all 16 pinch points of \( \Sigma \). The preimage \( \bar{C} = \Delta_\Sigma \) of \( C \) on \( S \) is a smooth curve of degree 16 and of genus 17.

Let us show that a general sextic scroll \( \Sigma \) in \( \mathbb{P}^3 \) of genus 2 has ordinary singularities and an irreducible double curve \( C \). Consider a reducible sextic curve \( E_0 \subseteq \mathbb{P}^4 \) of arithmetic genus 2, which consists of a general smooth elliptic normal quintic curve \( E' \) and a line \( D \) meeting \( E' \) transversally in two distinct points. Such a curve \( E_0 \) corresponds to a point in \( \mathcal{H}_{6,2} \), hence to a reducible surface \( \Sigma_0 \), which is a limit of genus 2 sextic scrolls \( \Sigma \). On the other hand, \( \Sigma_0 \) is the union of a general quintic elliptic scroll \( \Sigma' \) in \( \mathbb{P}^3 \) arising from \( E' \) as in Example 1.8 plus a plane \( \pi \) through the two rulings \( F_1, F_2 \) of \( \Sigma_0 \), which correspond to the intersection points of \( E' \) and \( D \). These rulings meet at a point \( p \) of the double curve \( C' \) of \( \Sigma' \). The ruling on \( \pi \) is given by the pencil of lines passing through \( p \), which corresponds to the line \( D \). The plane \( \pi \) cuts out on \( \Sigma' \) the union of the rulings \( F_1, F_2 \) and a smooth plane cubic \( E \), as described in Example 1.8. The singularities of \( \Sigma' \) consist of \( C', F_1, F_2, \) and \( E \).

When we deform \( E_0 \) to a general smooth sextic \( E \) on \( \text{Gr}(1,3) \), the scroll \( \Sigma_0 \) is deformed to an irreducible sextic scroll \( \Sigma \). The double lines \( F_1 \) and \( F_2 \) of \( \Sigma_0 \) disappear, because we are smoothing the two nodes of \( E_0 \). This means that the flat limit on \( \Sigma_0 \) of the singular locus of \( \Sigma \) is the nodal curve \( C_0 = C' \cup \bar{E} \) of arithmetic genus 5. Hence \( \Sigma \) is singular only along a double curve \( C \), which has arithmetic genus 5. The latter curve is irreducible. Indeed, otherwise it would be still a union of the form \( C' \cup \bar{E} \), and so the four-secant lines to \( C \) would sweep out a union of an elliptic scroll and a plane. However, this is impossible since \( \Sigma \) is irreducible and swept out by the four-secants of the double curve \( \bar{C} = \Delta_\Sigma \).

Since \( C_0 \) is nodal so is \( C \). We claim that \( C \) is actually smooth. Indeed, we may restrict our family to a general irreducible curve germ in \( \mathcal{H}_{2,6} \) through \( \Sigma_0 \), and then normalize this germ. In this way we obtain a family of sextic scrolls over the disc \( \mathbb{D} \) with a family \( C \to \mathbb{D} \) of double curves. The central fibre of \( C \) is a reducible nodal curve \( C_0 = C' \cup \bar{E} \) with 4 nodes. Assuming that no one of these nodes is smoothed on a general fibre \( C \) of the family, \( C \) should also have 4 nodes. These nodes represent an étale four-sheeted cover over the disc. Now we can normalize the fibres of the family \( C \) simultaneously (see e.g., [40]), thus obtaining a smooth family with an irreducible general fibre and a disconnected central fibre. The latter contradicts the Connectedness Principle (see [27] Ch. III, Ex. 11.4, p. 281]).

Consequently, at least one of the four nodes of \( C_0 \) has to be smoothed in the deformation to \( C \). But then by the irreducibility of the monodromy (see the final part of Example 1.8) all nodes of \( C_0 \) have to be smoothed.

2. Bounding degrees of low genera curves on surfaces

2.1. Algebraic hyperbolicity. Scrolls can be used to establish algebraic hyperbolicity of very general surfaces of a given degree \( d \) in \( \mathbb{P}^3 \). For \( d \geq 6 \) this is done in Proposition 2.1 below. In the proof we use the Albanese inequality (see [1], 35) (see also [34] §4(b)), which says the following: if a reduced projective curve \( C \) of geometric genus \( g \) degenerates into an effective cycle \( C_0 = \sum_i m_i C_i \), where \( C_i \) is a reduced projective curve of geometric genus \( g_i \), then

\[
g \geq \sum_{g_i \geq 1} (m_i (g_i - 1) + 1) .
\]

In particular, \( m_i (g_i - 1) \leq g - 1 \) if \( g_i \geq 1 \). So \( g_i \leq g \) for all \( i \).

**Proposition 2.1.** Assume that there exists a scroll \( \Sigma \) of degree \( d \geq 5 \) and genus \( g \geq 1 \) in \( \mathbb{P}^3 \) with ordinary singularities. Then a very general surface \( X \) in \( \mathbb{P}^3 \) of degree \( d \) does not contain curves of geometric genus \( g' < g \).
Proof. Let $X$ be a very general surface in $\mathbb{P}^3$ of degree $d$. By the Noether-Lefschetz Theorem, the Picard group of $X$ is generated by $\mathcal{O}_X(1)$. Consider the pencil $\{X_t\}_{t \in \mathbb{D}}$ generated by $X_0 = \Sigma$ and $X_\infty = X$. This gives rise to a flat family of surfaces $f: \mathcal{X} \to \mathbb{D}$ over a disc $\mathbb{D}$, where the central fibre over 0 is $X_0$, all fibres $X_t$ with $t \in \mathbb{D} \setminus \{0\}$ are smooth and Pic$(X_t)$ is generated by $\mathcal{O}_{X_t}(1)$ for a very general such fibre. We claim that a very general surface of this family does not contain any curve of geometric genus $g' < g$. We argue by contradiction and assume that this is not the case for some $g' < g$.

For each positive integer $n$ we may consider the locally closed subset $\mathcal{H}_{n,g'}$ of the relative Hilbert scheme of $f: \mathcal{X} \setminus X_0 \to \mathbb{D} \setminus \{0\}$, whose points correspond, for each $t \neq 0$, to the irreducible curves of geometric genus $g'$ in $|\mathcal{O}_{X_t}(n)|$. By our assumption, there is a component of $\mathcal{H}_{n,g'}$ which dominates $\mathbb{D} \setminus \{0\}$. Let $\mathcal{H}$ be the closure of this component in the relative Hilbert scheme of $f: \mathcal{X} \to \mathbb{D}$. By the properness of the relative Hilbert scheme, $\mathcal{H}$ surjects onto $\mathbb{D}$. Hence there is a curve $C_0 \in \mathcal{O}_{X_0}(n)$ on $X_0$, which corresponds to a point in $\mathcal{H}$. By Albanese’s inequality (11), every component of $C_0$ has geometric genus $g'' \leq g' < g$. By (5) (for $g \geq 2$) and Example 1.8 (for $g = 1$) we have $\gamma \geq g > g''$, where $\gamma$ stands as before for the geometric genus of the double curve $\Delta_{\Sigma}$ of $X_0 = \Sigma$. Hence no component of $C_0$ coincides with $\Delta_{\Sigma}$. Now the pull-back $\Gamma$ of $C_0$ on the normalization $\varphi: S \to \Sigma$ belongs to the linear system $|\varphi^*(\mathcal{O}_{\Sigma}(n))|$ and maps birationally to $C_0$ by the finite map $\varphi$. Since the only curves of genus smaller than $g$ on $S$ are rulings, $\Gamma$ consists of rulings. In particular, $\Gamma^2 = 0$. On the other hand, since $\Gamma \in |\varphi^*(\mathcal{O}_{\Sigma}(n))| = |\mathcal{O}_S(n)|$ we have $\Gamma^2 = n^2d > 0$, a contradiction. \qed

In Proposition 2.10 below we slightly strengthen Proposition 2.1 using Proposition 2.8 and Corollary 2.9.

Keeping in mind Example 1.8 Proposition 2.1 provides an alternative quick proof of the following result originally established by Xu [49] and Voisin [46] [47].

**Corollary 2.2.** On a very general surface of degree $d \geq 5$ in $\mathbb{P}^3$ there is no rational curve.

Very general in Corollary 2.2 can be replaced by general provided the following question is answered in negative.

**Question 2.3.** Does there exist a sequence of smooth quintic surfaces $X_n$ in $\mathbb{P}^3$ such that $X_n$ contains a rational curve of degree $d_n$ and not smaller, with $d_n \to \infty$?

**Remark 2.4.** Notice that for any integers $n \geq 3$, $d > 0$ and $0 \leq \delta \leq d^2(n-1)+1$, the linear system $|\mathcal{O}_S(d)|$ on a general K3 surface $S$ of degree $2n-2$ in $\mathbb{P}^n$ with Picard group generated by $\mathcal{O}_S(1)$, contains a $(d^2(n-1)-\delta+1)$-dimensional family of irreducible $\delta$-nodal curves, whose geometric genus equals $d^2(n-1)-\delta+1$ (see [11]). So $S$ contains nodal curves of every geometric genus $g \geq 0$. This applies in particular to general quartic surfaces in $\mathbb{P}^3$.

### 2.2. Bounding degrees of curves of low genera on general surfaces in $\mathbb{P}^3$.

In this section we address the following boundedness question (cf. [8] [30] and the related discussion in the Introduction):

**Question 2.5.** Given integers $d \geq 5$ and $g \geq 0$, does there exist a bound $n_{d,g}$ such that every irreducible curve of geometric genus $g$ on a very general surface of degree $d$ in $\mathbb{P}^3$ has degree $n \leq n_{d,g}$?

If $d = 4$ the answer is negative (see [11] [25] and Remark 2.4). The argument in the proof of Propositions 2.1 and 2.10 can be used to give an affirmative answer for $d \geq 6$ and small enough $g$.

**Proposition 2.6.** Suppose there exists a scroll $\Sigma$ of degree $d \geq 6$ and genus $g \geq 2$ with ordinary singularities. Then the answer to Question 2.5 is affirmative for all genera $g' < \gamma$, where $\gamma$ is defined in (3).

**Proof.** We apply the same argument as in the proof of Proposition 2.1. Keeping the notation of this proposition, we let again $C_0 \in |\mathcal{O}_C(n)|$ denote a curve which is a limit of a flat family of irreducible curves $\{C_t\}_{t \in \mathbb{D} \setminus \{0\}}$, $C_t \in |\mathcal{O}_{X_t}(n)|$, of genus $g'$, where $g' \geq g \geq 2$ by Proposition 2.1. Write $C_0 = m_1C_1 + \ldots + m_hC_h + C'$ as a cycle, where for every $i = 1, \ldots, h$ the curve $C_i$ is irreducible of geometric genus $g_i \geq 1$ and its transform on $S$ has positive intersections $n_i$ with the rulings, whereas $C'$ consists of rulings. Note that
\[ n = \sum_{i=1}^{h} m_i n_i. \] By Albanese’s inequality \((11)\) and our hypothesis \(g' < \gamma\), none of the components of \(C_0\) coincides with \(\Delta_{\Sigma}\), and
\[ g' \geq h + \sum_{i=1}^{h} m_i (g_i - 1). \]
The Riemann–Hurwitz formula yields: \(g_i - 1 \geq n_i (g - 1)\) for all \(i = 1, \ldots, h\), so that \(\gamma > g' \geq h + n(g - 1)\). This provides a bound \(n < (\gamma - 1)/(g - 1)\) (we remind that \(g \geq 2\)).

**Corollary 2.7.** Question \((2.3)\) has an affirmative answer for
\[ d = 6, \quad g \leq 5, \]
\[ d \geq 7 \quad \text{even}, \quad g < (d - 4)^2, \]
\[ d \geq 7 \quad \text{odd}, \quad g < \frac{(d - 3)(d - 9)}{2}. \]

**Proof.** For \(d = 6\) we use the sextic scroll of genus 2 as in Example \((1.3)\) For \(d \geq 7\) even we write \(d = 2m + 4\) and we consider in \(\mathbb{P}^3\) general projections of smooth scrolls of genus \(m\) and degree \(d\) in \(\mathbb{P}^5\) as in Theorem \((1.5)\). For \(d \geq 7\) odd we write \(d = 2m + 3\) and we consider general projections of scrolls of genus \(m\) and degree \(d\) in \(\mathbb{P}^4\) as in Remark \((1.6)\). Applying Proposition \((2.6)\) and taking into account \((1)\), the assertion follows. \(\square\)

### 2.3. Families of low degree curves of a given genus on general surfaces in \(\mathbb{P}^3\).

Proposition \((2.8)\) below extends a similar result by Arbarello–Cornalba \([2\) Theorem 3.1], \([3]\) and Zariski \([54]\); cf. also Knutsen \([28\) Lemma 4.4].

Let \(S\) be a smooth projective surface, \(\text{Hilb}_1(S)\) the Hilbert scheme of curves on \(S\), and \(\mathcal{V}_g(S)\) the locally closed subset of \(\text{Hilb}_1(S)\) formed by irreducible curves of geometric genus \(g\).

**Proposition 2.8.** In the setting as before, for an irreducible component \(\mathcal{V}\) of \(\mathcal{V}_g(S)\) we let \(v = \dim(\mathcal{V})\) and \(\kappa = K_S \cdot \Gamma\), where a curve \(\Gamma\) on \(S\) corresponds to a general point in \(\mathcal{V}\). Then \(v \leq \max\{g, g - 1 - \kappa\}\). Furthermore, if \(v > g\) then \(v = g - 1 - \kappa\), and the general curve \(\Gamma\) of \(\mathcal{V}\) has only nodes as singularities.

**Proof.** Let \(f : C \to \Gamma\) be the normalization. The exact sequence
\[ 0 \to T_C \to f^*(T_S) \to N_f \to 0 \]
defines the normal sheaf \(N_f\) to the map \(f : C \to S\). It can be included into an exact sequence
\[ 0 \to \tau \to N_f \to N' \to 0, \]
where \(\tau\) is the torsion subsheaf of \(N_f\) supported at the points, where the rank of the differential of \(f\) drops, and \(N'\) is an invertible sheaf. Due to the Horikawa inclusion \(T_{[\Gamma]}(\mathcal{V}) \subseteq H^0(C, N')\) (see \([2\) (1.3)] or \([3\) Lemma 1.4]) we have \(v \leq h^0(C, N')\). By Riemann-Roch,
\[ h^0(C, N') = \deg(N') - g + 1 + h^1(C, N'), \quad \text{where} \quad \deg(N') \leq \deg(N_f) = 2g - 2 - \kappa. \]
If \(h^1(C, N') = 0\) this gives \(v \leq g - 1 - \kappa\). Otherwise \(N'\) is special, so \(h^0(C, N') \leq g\). In any case, \(v \leq \max\{g, g - 1 - \kappa\}\), as stated.

If \(v > g\) then \(h^1(C, N') = 0\). Since \(H^1(C, \tau) = 0\) this yields \(H^1(C, N_f) = 0\). As in \([2\) proof of (1.5) and p. 96] this implies \(\tau = 0\), hence \(\Gamma\) is immersed (i.e., has no cuspidal singularities). One ends the proof as in \([2\) pp. 96–98]. \(\square\)

For \(\mathcal{L}_d = |O_{\mathbb{P}^3}(d)|\) we let
\[ N_d = \dim(\mathcal{L}_d) = \binom{d + 3}{3} - 1. \tag{12} \]

Given a smooth surface \(X\) of degree \(d\) in \(\mathbb{P}^3\) and non–negative integers \(n, g\), we let \(\mathcal{V}_{n,g} = \mathcal{V}_{n,g}(X)\) denote the locally closed subset of \(\mathcal{L}_{X,n} = |O_X(n)|\) formed by irreducible curves on \(X\) of geometric genus \(g\). We also let
\[ g_{d,n} = \frac{dn(d + n - 4)}{2} + 1. \]
denote the arithmetic genus of the curves in \( \mathcal{L}_{X,n} \). Notice that \( g_{d,n} = g + \nu \) if a general member of \( \mathcal{V}_{n,g} \) is nodal with \( \nu \) nodes.

**Corollary 2.9.** Let \( X \) be a general surface of degree \( d \geq 3 \) in \( \mathbb{P}^3 \). If \( g \geq 0 \) and \( n \in \{1, 2\} \) are such that \( \mathcal{V}_{n,g} \) is nonempty, then

\[
g_{d,1} - 3 \leq g \leq g_{d,1} \quad \text{if} \quad n = 1 \quad \text{and} \quad g_{d,2} - 9 \leq g \leq g_{d,2} \quad \text{if} \quad n = 2.
\]

Furthermore, for every irreducible component \( \mathcal{V} \) of \( \mathcal{V}_{n,g} \), its general curve has exactly \( \nu \) nodes as singularities and its dimension is

\[
3 - \nu = g - g_{d,1} + 3 \quad \text{if} \quad n = 1 \quad \text{and} \quad 9 - \nu = g - g_{d,2} + 9 \quad \text{if} \quad n = 2.
\]

**Proof.** Let us show the assertion in the case \( n = 2 \), the case \( n = 1 \) being similar. Consider the incidence relation \( I \subseteq \mathcal{L}_d \times \mathcal{L}_2 \) consisting of all pairs \((X, Q)\) such that \( X \) is smooth and \( Q \) and \( X \) intersect in an irreducible curve \( C \) of geometric genus \( g \). Then \( I \) is locally closed and comes equipped with the natural projections \( p : I \rightarrow \mathcal{L}_d \) and \( q : I \rightarrow \mathcal{L}_2 \).

Note that if \((X, Q) \in I \) and \( C \) is the intersection of \( X \) and \( Q \), then we have a family of dimension \( \dim(\mathcal{L}_{d-2}) + 1 \) of pairs \((X', Q) \in I \) such that intersection of \( X' \) and \( Q \) is \( C \): indeed we can take \( X' \) general in the span of \( X \) and of all surfaces of degree \( d \) containing \( Q \).

By our assumption \( p \) is dominant. Let \( I' \) be an irreducible component of \( I \) which dominates \( \mathcal{L}_d \) via \( p \), so that \( \dim(I') \geq N_d \). We assume that \( q(I') \) contains a smooth quadric \( Q \) (the argument is similar otherwise, the details are left to the reader). Then \( I' \) dominates \( \mathcal{L}_2 \) via \( q \) and we may assume \( Q \) to be a general quadric. All components of \( q^{-1}(Q) \) have dimension \( \dim(I') - \dim(\mathcal{L}_2) \). Any such component can be identified with a family of surfaces of degree \( d \). By the above discussion, the family of curves \( \mathcal{V} \) they cut out on \( Q \) has dimension

\[
v = \dim(I') - \dim(\mathcal{L}_{d-2}) - \dim(\mathcal{L}_2) - 1.
\]

Moreover, \( \mathcal{V} \) is an irreducible component of \( \mathcal{V}_{d,g}(Q) \). We have

\[
v \geq N_d - N_{d-2} - N_2 - 1 = g_{d,2} + 4d - 10 > g_{d,2} \geq g.
\]

By Proposition 2.8 one has \( v = g - 1 + 4d \), which yields \( g_{d,2} - g \leq 9 \). Furthermore, by Proposition 2.8 the general curve in \( \mathcal{V} \) has at most nodes as singularities, which implies (13). \( \Box \)

Corollary 2.9 could be extended to handle also the case \( n = 3 \). This requires however to analyze a number of cases, which we avoid here.

Now we can strengthen Proposition 2.1 as follows.

**Proposition 2.10.** Assume that there exists a scroll \( \Sigma \) of degree \( d \geq 5 \) and genus \( g \geq 1 \) in \( \mathbb{P}^3 \) with ordinary singularities. Then a very general surface \( X \) of degree \( d \) in \( \mathbb{P}^3 \) does not contain curves of geometric genus \( g' \leq 3(g - 1) \).

**Proof.** By Proposition 2.1 we may suppose that \( d \geq 6 \) and \( g' \geq g \geq 2 \). We proceed as in the proof of this proposition, using the same notation. We argue by contradiction and assume that there is a positive \( g' \leq 3(g - 1) \), a positive integer \( n \) and a component of \( \mathcal{H}_{n,g'} \) which dominates \( \mathbb{D} \setminus \{0\} \). Consider a curve \( C_0 \in \mathcal{O}(n) \) as in the proof of Proposition 2.1. As shown in this proof, \( C_0 \) cannot be composed of rulings. Hence it contains a component \( C_i \) of geometric genus \( g_i > 0 \), appearing in \( C_0 \) with multiplicity \( m_i \). By Albanese's inequality \( 14 \) one has \( g' - 1 \geq m_i(g_i - 1) \). By \( 10 \) and our assumption \( g' \leq 3(g - 1) < g \), hence \( C_i \neq \Delta \mathcal{C} \). Therefore \( C_i \) lifts birationally to the normalization \( S \) of \( \Sigma \) yielding a \( \nu_i \)-secant of the ruling on \( S \). Combining the inequalities above, by Hurwitz Formula (see Remark 1.2) we obtain

\[
3(g - 1) - 1 \geq g' - 1 \geq m_i(g_i - 1) \geq \nu_i m_i(g - 1).
\]

Hence \( \nu_i m_i \leq 2 \) and so the only possibilities are

\[
\nu_i = m_i = 1, \quad \nu_i = 1, \quad m_i = 2, \quad \text{and} \quad \nu_i = 2, \quad m_i = 1.
\]
In the former case by (11) there can be at most two such components, while in the latter two cases at most one. We have \( n = \sum \nu_i m_i \), the sum over all components \( C_i \) of \( C_0 \) of positive genus. It follows that \( 1 \leq n \leq 2 \). Then Corollary (2.3) yields \( g' \geq g_{d,1} - 3 \), since \( g_{d,1} - 3 < g_{d,2} - 9 \) for \( d \geq 6 \). Thus we must have

\[
\frac{(d-1)(d-2)}{2} - 3 = g_{d,1} - 3 \leq g' \leq 3(g-1) \leq \frac{d(d-5)}{2},
\]

the last inequality coming from (9) for \( d \geq 6 \). But (11) gives a contradiction. \( \square \)

3. Bounding geometric genera of divisors on general 3-folds in \( \mathbb{P}^4 \)

A simple way of constructing higher dimensional scrolls consists in starting with the trivial \( \mathbb{P}^1 \)-bundle \( \pi: S = E \times \mathbb{P}^1 \to E \) over a smooth projective variety \( E \subseteq \mathbb{P}^m \) of degree \( d \) and dimension \( n \). Let \( \text{Seg}_{a,b} \) denote the image of \( \mathbb{P}^a \times \mathbb{P}^b \) via the Segre embedding. Then

\[
S \hookrightarrow \mathbb{P}^m \times \mathbb{P}^1 \xrightarrow{\text{Seg}_{m,1}} \mathbb{P}^{2m+1}
\]
yields an embedding of \( S \) as a smooth scroll of dimension \( n+1 \) and degree \( (n+1)d \) in \( \mathbb{P}^{2m+1} \). A general linear projection of \( S \) to \( \mathbb{P}^{n+2} \) gives a hypersurface scroll \( \Sigma \subseteq \mathbb{P}^{n+2} \) of degree \( (n+1)d \).

Consider, for instance, a surface \( E_d \) in \( \mathbb{P}^3 \) of degree \( d \), which we suppose to be very general. The above construction gives

\[
S_d := E_d \times \mathbb{P}^1 \hookrightarrow \text{Seg}_{3,1} \hookrightarrow \mathbb{P}^7,
\]
and \( S_d \) is a threefold of degree \( 3d \) in \( \mathbb{P}^7 \). A general linear projection of \( S_d \) to \( \mathbb{P}^4 \) yields a threefold scroll \( \Sigma_d \) of degree \( 3d \) in \( \mathbb{P}^4 \). It is swept out by a two-dimensional family of \( (3d-3) \)-secant lines to the double surface \( \Delta_{\Sigma} \) (see Lemma 1.1).

The following version of the Albanese inequality follows immediately from the Semistable Reduction Theorem \([31] \S 1\) and the Geometric Genus Criterion (see formula (1) on p. 119 in \([34] \S 6\) or, in the surface case, formula (8) in \([29] \Ch. 5, \S 5\)).

**Lemma 3.1.** Let \( X \) be a flat limit of a one-parameter family of smooth, irreducible, projective varieties of geometric genus \( \rho \). Let \( X_i \) be irreducible components of \( X_0 \) with geometric genera \( \rho_i \), \( i = 1, \ldots, h \). Then

\[
\rho \geq \sum_{i=1}^{h} \rho_i.
\]

Recall that, in the notation as in (12), the geometric genus \( \rho(E_d) \) of a smooth surface \( E_d \) of degree \( d \) in \( \mathbb{P}^3 \) is equal to \( \rho(E_d) = \binom{d-3}{3} = N_{d-4} + 1 \) (see e.g. \([29] \Ch. 4, (5.12.2)\)). The following lower bound on the geometric genus of the double surface is an analog of (10) in the case of surface scrolls.

**Lemma 3.2.** Let \( \Sigma_d \subseteq \mathbb{P}^4 \) be a threefold scroll of degree \( 3d \), constructed as before over a very general surface \( E_d \) in \( \mathbb{P}^3 \) of degree \( d \geq 5 \) as a base. Then for the geometric genus \( \rho_d \) of the double surface \( \Delta_{\Sigma_d} \) we have a lower bound

\[
\forall d \geq 5.
\]

**Proof.** Degenerate \( E_d \) to \( E_{d-1} \cup E_1 \), where \( E_{d-1} \) and \( E_1 \) are general. Then \( S_d \) degenerates to the union of \( S_{d-1} \) and \( S_1 = \text{Seg}_{1,2} \), meeting along the Segre image \( X \) of \( C \times \mathbb{P}^1 \), where \( C = E_1 \cap E_{d-1} \). Accordingly, \( \Sigma_d \) degenerates in \( \mathbb{P}^4 \) to the union of \( \Sigma_{d-1} \) and \( \Sigma_1 \), the latter being a hypersurface of degree 3 with a double plane. These threefolds intersect along the general projection \( Y \) of \( X \), plus another surface \( Z \). The limit of the double locus \( \Delta_{\Sigma_d} \) consists of the union of \( \Delta_{\Sigma_{d-1}} \), of the plane \( \Delta_{\Sigma_1} \), and of \( Z \). The ruling determines a dominant rational map \( Z \dashrightarrow E_{d-1} \). So there is at least one component \( Z' \) of \( Z \) with geometric genus \( \rho' \geq \rho(E_{d-1}) \). Now the first inequality in (15) follows from Lemma 3.1. In particular, \( \rho_5 \geq \rho' \geq 1 \). By induction for every \( d \geq 5 \) we obtain

\[
\rho_d \geq \rho(E_{d-1}) + \rho_{d-1} \geq \sum_{k=5}^{d-1} \rho(E_k) + \rho_5 \geq \sum_{k=0}^{d-2} \binom{k}{3} = \binom{d-1}{4},
\]

as required. \( \square \)
It would be interesting to find the precise value of \( \rho_d \).

**Theorem 3.3.** Any irreducible surface contained in a very general hypersurface of degree \( 3d \geq 15 \) in \( \mathbb{P}^4 \) has geometric genus \( \rho \geq \min \{ \rho_d, N_{d-4}+1 \} \). In particular, \( \rho \geq \rho(E_d) \) if \( d \geq 8 \).

**Proof.** The argument is similar to that in the proof of Proposition 2.1 so we will be brief. We let \( X_0 \) be the scroll \( \Sigma_d \) and \( X \) be a general hypersurface in \( \mathbb{P}^4 \) of degree \( 3d \). The pencil generated by \( X_0 \) and \( X \) gives rise as usual to a flat family \( f : \mathcal{X} \to \mathbb{D} \). Suppose that the general fibre of this family contains an irreducible surface \( Y \) of geometric genus \( \rho < \min \{ \rho_d, N_{d-4}+1 \} \). By Lemma 3.2 the limit \( Y_0 \) of such a surface in the central fibre does not contain \( \Delta_{\Sigma_d} \). By Lemma 3.1 all of its components have geometric genus \( \rho' \leq \rho < \min \{ \rho_d, N_{d-4}+1 \} \leq N_{d-4}+1 \). Hence they cannot dominate \( E_d \), which has geometric genus \( N_{d-4}+1 \). Thus all components of \( Y_0 \) pull–back to \( S_d \) to surfaces with zero intersection with the ruling. This yields a contradiction as in the proof of Proposition 2.1.

**Remark 3.4.** G. Xu gave in [31] Theorem 2] a sharp lower bound for the geometric genus of an irreducible divisor on a very general hypersurface of degree \( d \geq n+2 \) in \( \mathbb{P}^n \), with \( n \geq 4 \). Of course Theorem 3.3 above is weaker than Xu’s result. However, the method of proof is simple and it may possibly have further applications. Hence it would be interesting to extend Theorem 3.3 to other degrees (non–divisible by 3), as well as to higher dimensions. We wonder also whether in higher dimensions an analog of Proposition 2.6 holds. For instance, one can suggest by analogy that on a very general threefold in \( \mathbb{P}^4 \) of degree \( \geq 6 \), the divisors of geometric genera \( \rho' < \rho_d \) form bounded families.

4. DEGENERATION TO SCROLLS AND KOBAYASHI HYPERBOLICITY

4.1. Limiting Brody curves and Hurwitz Theorem. Let \( V \) be a subvariety of a hermitian complex manifold. A **Brody curve** in \( V \) is a holomorphic map \( f : \mathbb{C} \to V \) satisfying

\[
\sup_{z \in \mathbb{C}} ||df(z)|| = ||df(0)|| = 1.
\]

By Brody’s reparametrization lemma ([1]), if \( V \) is proper and non–hyperbolic then it contains a Brody curve. Furthermore, from any sequence of Brody curves in \( V \) one can extract a subsequence converging to a Brody curve, which is called a **limiting Brody curve**.

Assume there is a proper dominant map \( \pi : V \to C \) onto a smooth projective curve \( C \). If general fibres \( D_c = \pi^{-1}(c) \) \((c \in C)\) are non–hyperbolic, i.e., contain Brody curves, then every special fibre \( D_0 := \pi(c_0) \) is non–hyperbolic as well and contains limiting Brody curves. The Hurwitz Theorem imposes constrains on limiting Brody curve with respect to the singularities of \( D_0 \) (cf. e.g., [15] §1, [51] Theorem 2.1, and [52] Lemma 1.2). Let \( \Delta_0 = \text{br}(D_0) \) be the set of multi–branch points of \( D_0 \) such that locally the branches of \( D_0 \) are \( \mathbb{Q} \)-Cartier divisors on \( V \), and let \( \Delta \) be the Zariski closure of \( \Delta_0 \). Consider a limit \( f : \Omega \to D_0 \) of a sequence of holomorphic maps \( f_n : \Omega \to D_{c_n} \), with \( c_n \in C \setminus \{ c_0 \} \) such that \( c_n \to c_0 \), where \( \Omega \subseteq C \) is a connected domain. Hurwitz’ Theorem says that, if \( f(\Omega) \cap \Delta_0 \neq \emptyset \), then \( f(\Omega) \subseteq \Delta \). In particular, if \( \Delta \) is hyperbolic then any limiting Brody curve in \( D_0 \) is contained in \( D_0 \setminus \Delta_0 \). Hence if both \( \Delta \) and \( D_0 \setminus \Delta_0 \) are hyperbolic then all fibres \( D_c \) \((c \neq c_0)\) close enough to \( D_0 \) are hyperbolic as well (cf. [51]).

4.2. A hyperbolicity criterion for hypersurfaces in \( \mathbb{P}^n \). Let \( X_0, X_\infty \) be distinct hypersurfaces in \( \mathbb{P}^n \) of degree \( d \). Typically, \( X_\infty \) will be a general surface of degree \( d \) meeting \( \text{Sing}(X_0) \) in points, where locally \( X_0 \) is an union of two smooth branches intersecting transversally. Consider the associated linear pencil \( \{ X_t \}_{t \in \mathbb{P}^1} \).

Assume that for a general \( t \in \mathbb{P}^1 \) the hypersurface \( X_t \) is non–hyperbolic. Then there exists a sequence of Brody curves \( \varphi_n : \mathbb{C} \to X_{t_n} \) (with respect to the Fubini–Study metric on \( \mathbb{P}^n \)), where \( t_n \to 0 \), converging to a limiting (non–constant) Brody curve \( \varphi_0 : \mathbb{C} \to X_0 \).

**Proposition 4.1.** In the above setting, let \( B = X_\infty \cap \overline{\text{br}(X_0)} \). If \( \overline{\text{br}(X_0)} \) and \((X_0 \setminus \text{br}(X_0)) \cup B \) are both hyperbolic, then \( X_t \), for \( t \neq 0 \) close enough to 0, is hyperbolic as well.

**Proof.** By Hurwitz’ Theorem and the hypotheses, the image of \( \varphi \) cannot be contained in \( \text{br}(X_0) \), and it can meet \( \text{br}(X_0) \) only at \((\overline{\text{br}(X_0) \setminus \text{br}(X_0)}) \cup B \). But then it is contained in \((X_0 \setminus \text{br}(X_0)) \cup B \), a contradiction. \( \square \)
Remark 4.2. Hurwitz’ Theorem cannot be applied at points in \( \overline{\text{br}(X_0) \setminus \text{br}(X_0)} \) or at a pinch point of \( X_0 \subseteq \mathbb{P}^3 \), where \( X_0 \) is locally analytically isomorphic to the surface \( x^2 = y^2z \) in \( \mathbb{A}^3 = \mathbb{A}^3_C \) at the origin, or at a base point of the pencil situated on \( \text{br}(X_0) \).

Indeed, consider a linear pencil of surfaces given in an affine chart \( \mathbb{A}^3 \) of \( \mathbb{P}^3 \) as \( X_t = \{ x^2 - y^2z = t \} \). The origin \( 0 \in \mathbb{A}^3 \) is a pinch point of \( X_0 \) and is not a base point of the pencil. Consider also the family of entire curves
\[
\varphi_t : \mathbb{C} \to X_t, \quad u \mapsto (u^2 + \tau, u, u^2 + 2\tau), \quad \text{where} \quad \tau = t^2 \in \mathbb{C}.
\]
The limiting entire curve \( \varphi(\mathbb{C}) \subseteq X_0 \) passes through the pinch point \( 0 \in X_0 \) and is not contained in the singular locus \( \{ x = y = 0 \} = \text{br}(X_0) \cup \{ 0 \} \) of \( X_0 \).

Corollary 4.3. In the same setting as before, consider the normalization \( \nu : \tilde{X}_0 \to X_0 \). Suppose that \( \overline{\text{br}(X_0)} \) is hyperbolic and there is a morphism \( \pi : \tilde{X}_0 \to E \) onto a hyperbolic variety \( E \) such that for every \( x \in E \)
\[
\pi^{-1}(x) \setminus \nu^{-1}(\text{br}(X_0) \setminus (X_\infty \cap \text{br}(X_0)))
\]
is hyperbolic. Then any hypersurface \( X_t \neq X_0 \) for \( t \) close enough to 0 is hyperbolic. Consequently, a very general hypersurface of degree \( d \) in \( \mathbb{P}^n \) is algebraically hyperbolic.

Proof. We keep the notation introduced before. Suppose that for \( t \in \mathbb{P}^1 \) general, \( X_t \) is not hyperbolic. Let \( \varphi_0 : \mathbb{C} \to X_0 \) be a (non–constant) limiting Brody curve. Since its image cannot be contained in \( \overline{\text{br}(X_0)} \), there is a pullback \( \tilde{\varphi}_0 : \mathbb{C} \to \tilde{X}_0 \). Since \( E \) is hyperbolic, the composition \( \pi \circ \tilde{\varphi}_0 : \mathbb{C} \to E \) is constant. Hence \( \tilde{\varphi}_0(\mathbb{C}) \) is contained in a fibre \( \pi^{-1}(x) \) over a point \( x \in E \). Furthermore, it does not meet \( \nu^{-1}(\text{br}(X_0) \setminus (X_\infty \cap \text{br}(X_0))) \).

Indeed, otherwise \( \tilde{\varphi}_0(\mathbb{C}) \) would meet \( \text{br}(X_0) \setminus (X_\infty \cap \text{br}(X_0)) \) and, by Hurwitz’ Theorem, it would be contained in \( \overline{\text{br}(X_0)} \), which is impossible. Then \( \tilde{\varphi}_0(\mathbb{C}) \) lies in \( \{ 16 \} \), a contradiction. \( \square \)

4.3. Applying scrolls to Kobayashi hyperbolicity.

Proposition 4.4. We keep the notation as in Subsection 1.1. Let \( \Sigma \subseteq \mathbb{P}^n \) be a hypersurface scroll with ordinary singularities satisfying conditions (C1)-(C5). Suppose that:

(i) the base \( E \) of \( \Sigma \) and its double locus \( \Delta_\Sigma \) are both hyperbolic;

(ii) for a general hypersurface \( X \) in \( \mathbb{P}^n \) of degree \( d = \deg(\Sigma) \), every ruling \( F \) of \( \Sigma \) meets \( \text{br}(\Sigma) \) in at least three distinct points off \( X \cap F \).

Then a general hypersurface of degree \( d \) in \( \mathbb{P}^n \) is hyperbolic.

Proof. The assertion follows by applying Corollary 4.3 with \( X_\infty = X \), \( X_0 = \Sigma \), and \( \overline{\text{br}(X_0)} = \Delta_\Sigma \). \( \square \)

Consider a general sextic scroll of genus 2 as introduced in Example 1.9 and a general septic scroll, also of genus 2, with ordinary singularities in \( \mathbb{P}^3 \). The latter scroll exists according to Theorem 1.5 and Remark 1.6 (ii).

Lemma 4.5. For a general scroll \( \Sigma \subseteq \mathbb{P}^3 \) of genus 2 and degree either \( d = 6 \) or \( d = 7 \), the following hold:

(i) the projection \( \pi : \Delta_\Sigma \to E \) has only simple ramifications; in particular \( \Delta_\Sigma \) meets every ruling in at least three distinct points;

(ii) no pair of pinch points on \( S \) sit on the same ruling;

(iii) the rulings passing through the pinch points on \( S \) are not tangent to \( \Delta_\Sigma \).

Proof. We first treat the case \( d = 6 \).

The conditions (i)–(iii) are open in \( \mathcal{H} = \overline{\mathcal{H}_{6,2}} \). So it suffices to show that there is a surface in \( \mathcal{H} \) satisfying these conditions. The reducible surface \( \Sigma_0 \) in Example 1.9 could be used for this, once we know that the analogues of (i)–(iii) hold for a general elliptic quintic scroll. This is in fact the case, but we do not dwell on this here. We use instead a different degeneration of a general sextic scroll of genus 2. We keep the notation introduced in Example 1.9.

A smooth quadric \( Q \) in \( \mathbb{P}^4 \) can be viewed as a hyperplane section of the Grassmanian \( \text{Gr}(1,3) \) under the Plücker embedding of \( \text{Gr}(1,3) \) in \( \mathbb{P}^8 \). There exists a curve \( E_0 \) of degree 6 and arithmetic genus 2 on \( Q \),
which consists of three conics $\Gamma_0, \Gamma_1, \Gamma_2$ such that $\Gamma_1, \Gamma_2$ are disjoint and intersect both $\Gamma_0$ transversally at two points. Indeed, it is enough to take two general hyperplanes $H_1, H_2$ in $\mathbb{P}^4$ meeting in a plane $L_0$ and two other general planes $L_i \subseteq H_i$, $i = 1, 2$, and let $\Gamma_i = L_i \cap \tilde{Q}_i$, $i = 0, 1, 2$.

The surface $\Sigma_0 \subseteq \mathbb{P}^3$, which corresponds to the curve $E_0$, is the union of the three quadrics $Q_0, Q_1, Q_2$, corresponding to $\Gamma_0, \Gamma_1, \Gamma_2$, respectively. We may suppose that these quadrics are smooth. The surface $\Sigma_0$ belongs to $H$. One has $Q_0 \cap Q_i = F_{ij} \cup G_{ij}, i, j = 1, 2$, where the lines $F_{ij}$'s correspond to the intersection points of $\Gamma_0$ with $\Gamma_i$ and belong to the same rulings of $Q_0$ and $Q_i$, and $G_{ij}$ are lines of the other rulings of $Q_0$ and $Q_i$. Furthermore $Q_i \cap Q_2 = \varrho$ is a smooth quartic curve of genus 1. By taking $Q_0, Q_1, Q_2$ sufficiently general, we may suppose that the lines $F_{ij}, G_{ij}$ are general in their rulings and $\varrho$ is also general. We denote by $p_{ij;hk}$ the intersection of $F_{ij}$ with $G_{hk}$, where $i, j, h, k \in \{1, 2\}$. We note that $\varrho$ meets $Q_0$ at the eight points $p_{ij;3-i,h}$, with $i, j, h \in \{1, 2\}$.

Regard now $\Sigma_0$ as a limit of a general sextic scroll $\Sigma$ of genus 2. The points of $\Gamma_0 \cap \Gamma_i$, $i = 1, 2$, are smoothed when deforming $E_0$ to $E$, hence also the lines $F_{ij}$ are. Therefore the limit of the smooth double curve $C = \Delta_\Sigma$ is the curve

$$C_0 = \Delta_{\Sigma_0} = \varrho \cup \bigcup_{i,j=1,2} G_{ij}$$

of degree 8 and arithmetic genus 5. The limit on $\Sigma_0$ of the ruling on $\Sigma$ is the union of the rulings of $Q_0, Q_1, Q_2$ containing the lines $F_{ij}, 1 \leq i \leq j$. By (7) there are 16 pinch points on $\Sigma$. Similarly as in Example [13] each of the eight points $p_{ij;ih}$, $i, j, h = 1, 2$ (not lying on $\varrho$) is the limit of two pinch points of $\Sigma$. We call them limit pinch points.

The smooth normalization $S$ of $\Sigma$ specializes to a partial normalization $S_0$ of $\Sigma_0$, ruled over the same nodal base curve $E_0$. The singular surface $S_0$ consists of three irreducible quadric surfaces $Q_0, \tilde{Q}_1, \tilde{Q}_2$ glued together along the common rulings $F_{ij}$ in the same way as before.

The limit $\tilde{C}_0 = \Delta_{S_0}$ of $\tilde{C} = \Delta_S$ is a nodal curve of arithmetic genus 17. It maps to $E_0$ with degree 4, and consists of ten components:

- two copies $g_i \subseteq \tilde{Q}_i$ of $\varrho$, each is mapped with degree 2 to $\Gamma_i$, $i = 1, 2$;
- two copies $G_{i;hk} \subseteq \tilde{Q}_i$ of $G_{hk}$, with $h, k = 1, 2$ and $i \in \{0, h\}$, eight curves in total. The curves $G_{0;hk}$ and $G_{h;hk}$ are glued at two points $p_{h1;hk}$ and $p_{h2;hk}$. Each of them is also glued to $g_h$ at two points, $h = 1, 2$. Hence the curves $g_1$ and $g_2$ meet in the eight points $p_{ij;3-i,h}$, with $i, j, h = 1, 2$. The four disjoint curves $G_{0;hk}$ on $\tilde{Q}_0$ are all mapped isomorphically to $\Gamma_0$, whereas for $h = 1, 2$ the two disjoint curves $G_{h;hk}$ on $\tilde{Q}_h$ ($k = 1, 2$) are mapped isomorphically to $\Gamma_h$.

Therefore, the limit of the 24 ramification points of the projection $\pi : \tilde{C} \to E$ are:

- (a) the ramification points of the degree 2 covers $g_i \to \Gamma_i$, $i = 1, 2$, in total 8 such points;
- (b) the connecting nodes of $g_h$ with $G_{h;hk}$, $k, h = 1, 2$, in total 8 distinct such points, each counted with multiplicity two.

We call these the limit ramification points.

Part (i) follows from this description, our generality assumption, and the observation that every limit ramification point of type (b) smooths to two ramification points on $\tilde{C}$ lying on different rulings.

As for (ii), the ruling $F_{ij}$ through $p_{ij;ih}$ misses all limit pinch points other than $p_{ij;3-i,h}$. Consider a partial deformation of $\Sigma_0$ to the union of a general elliptic quartic scroll $\Sigma_0'$ and a quadric $\tilde{Q}_1'$ containing two general rulings. This corresponds to a partial smoothing of $E_0$ to the union of an elliptic quartic curve $E'$, obtained by smoothing $\Gamma_0 + \Gamma_2$, plus a conic $\Gamma_1'$ (specializing to $\Gamma_1$) meeting $E'$ transversally at two points. In this way $\Sigma_0'$ has two double lines $R_1, R_2$ which respectively specialize to $G_{21}$ and $G_{22}$. For a fixed index $i \in \{0, 1\}$, the two limit pinch points $p_{2j;2i}$, $j = 1, 2$, deform to four pinch points of $\Sigma_0$ on $R_i$, and, as we saw in Example [13] they are general points on $R_1, R_2$ and are never pairwise on a ruling.

For the proof of (iii) note that, by generality assumptions, the rulings through the limit ramification points of type (a) do not contain any of the limit pinch points. In contrast, the rulings through limit ramification points of type (b) do contain limit pinch points. However, the same proof as for (ii) and generality assumptions imply that, in a general deformation of $\Sigma_0$ to $\Sigma$, this is no longer the case.
The case \( d = 7 \) is similar, hence we will be as brief as possible. The closure of \( \mathcal{H}_{7,2} \) contains points corresponding to a surface \( \Sigma_0 = \Sigma' \cup P \), where \( \Sigma_0 \) is a general sextic scroll of genus 2 and \( P \) is a general plane containing a general ruling \( F \). The intersection of \( P \) with \( \Sigma' \) consists of \( F \) plus a plane quintic curve \( D \) of genus 2, which has four nodes \( p_i, i = 1, \ldots, 4 \). The intersection of \( C' = \Delta_{\Sigma'} \) with \( P \) consists of the points \( q_i, i = 1, \ldots, 4 \), and four more points \( q_i \in F, i = 1, \ldots, 4 \). The intersection of \( D \) with \( F \) consists of the points \( q_i, i = 1, \ldots, 4 \), and of a further point \( q \) which is smooth on \( \Sigma' \), so that \( P \) is tangent to \( \Sigma' \) at \( q \).

The surface \( \Sigma_0 \) is the limit of a general scroll \( \Sigma \) of degree 7 and genus 2. If \( E \) is the base of \( \Sigma \) regarded as a curve in \( \text{Gr}(1,3) \), this corresponds to \( E \) degenerating to \( E_0 \), which is the union of a general sextic \( E' \) of genus 2 and a line \( L \) meeting \( E' \) transversally at one points \( f \), which corresponds to \( F \). The limit of the ruling of \( \Sigma \) is the ruling of \( \Sigma' \) plus the pencil in \( P \), corresponding to \( L \), with center a general point of \( F \).

The limit of \( C = \Delta_{\Sigma} \) is the curve \( C_0 = C' \cup D \) of degree 13. The points \( p_i, i = 1, \ldots, 4 \), are limits of the four triple points of \( C \). The geometric genus of a partial smoothing of \( C_0 \) at the points \( q_i, i = 1, \ldots, 4 \), is 10. All this agrees with \([5], [6], \) and \([7]\).

The usual analysis shows that the limit of the 18 pinch points of \( \Sigma \) (see \([8]\)) are the 16 pinch points of \( \Sigma' \) plus the point \( q \) counted with multiplicity 2.

The limit \( \tilde{C}_0 \) of \( \tilde{C} = \Delta_S \) maps with degree five to the curve \( E_0 = E' \cup L \). It consists of:

- a copy \( \tilde{C}' \) of \( \Delta_{S'} \) which maps to \( E' \) with degree four;
- a copy of the normalization \( \tilde{D} \) of \( D \), which maps isomorphically to \( E' \) and meets \( \tilde{C}' \) transversally at four points;
- a copy of \( D \) which maps to \( L \) with multiplicity five via the projection induced by the ruling on \( P \), and meets \( \tilde{C}' \) transversally at four points.

Hence the limit of the 44 ramification points of the projection \( \pi : \tilde{C} \to E \) are

- the 24 ramification points of the map \( \tilde{C}' \to E' \);
- the 12 ramification points of the map \( D \to L \);
- the 4 connecting nodes of \( \tilde{C}' \) with \( D \), each counted with multiplicity two.

With this in mind the proof proceeds similarly to the case \( d = 6 \). The details can be left to the reader. \( \square \)

**Theorem 4.6.** For every \( d \geq 6 \) there exists a hyperbolic surface in \( \mathbb{P}^3 \) of degree \( d \). Consequently, a very general surface in \( \mathbb{P}^3 \) of degree \( d \geq 6 \) is algebraically hyperbolic.

**Proof.** For \( d = 6, 7 \) this follows from Corollary \([4,3]\) and Lemma \([4,5]\). For \( d \geq 8 \) one can consider e.g. a general deformation of the union of two general cones in \( \mathbb{P}^3 \) of degrees \( d_1, d_2 \), where \( d_1 + d_2 = d \) and \( d_i \geq 4 \) (see \([44]\)). \( \square \)

**Remark 4.7.** Consider the union \( X_0 = X_1 \cup X_2 \) of projective cones with distinct vertices in \( \mathbb{P}^4 \) over two smooth hyperbolic surfaces in \( \mathbb{P}^3 \). According to \([44]\), \( X_0 \) can be deformed to a smooth hyperbolic threefold in \( \mathbb{P}^4 \) of degree \( \deg(X_1) + \deg(X_2) \). Thus there exist hyperbolic threefolds in \( \mathbb{P}^4 \) of any given degree \( d \geq 12 \). Consequently, a very general threefold in \( \mathbb{P}^4 \) of degree \( d \geq 12 \) is algebraically hyperbolic.

**References**


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