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# Strict Lyapunov functionals for nonlinear parabolic partial differential equations

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**Abstract:** For families of partial differential equations (PDEs) with particular boundary conditions, strict Lyapunov functionals are constructed. The PDEs under consideration are parabolic and, in addition to the diffusion term, may contain a nonlinear source term plus a convection term. The boundary conditions may be either the classical Dirichlet conditions, or the Neumann boundary conditions or a periodic one. The constructions rely on the knowledge of weak Lyapunov functionals for the nonlinear source term. The strict Lyapunov functionals are used to prove asymptotic stability in the framework of an appropriate topology. Moreover, when an uncertainty is considered, our construction of a strict Lyapunov functional makes it possible to establish some robustness properties of Input-to-State Stability (ISS) type.

*Keywords:* Strictification, Lyapunov functional, partial differential equation

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## 1. INTRODUCTION

Lyapunov functional based techniques are central in the study of partial differential equations (PDEs). The techniques are useful for the stability analysis of systems of many different families (although other approaches can be used too, especially when parabolic PDEs are studied; see in particular the contributions Matignon and Prieur (2005); Iftime and Demetriou (2009); Drame et al. (2008)).

Amongst the remarkable results for PDEs which extensively use Lyapunov functionals, it is worth mentioning the following. In Cazenave and Haraux (1998) a Lyapunov functional is used to establish the existence of a global solution to the celebrated heat equation. In Krstic and Smyshlyaev (2008), Lyapunov functionals are designed for the heat equation with unknown destabilizing parameters (see also Smyshlyaev and Krstic (2007a,b) for further results on the design of output stabilizers). Lyapunov functionals have been also used to establish controllability results for semilinear heat equations. For example in Coron and Trélat (2004) the computation of a Lyapunov functional, in combination with the quasi-static deformation method, is a key ingredient of the proof of the global controllability of this equation. In all the preceding papers, parabolic PDEs are considered, but Lyapunov functionals can also be useful for other kinds of dynamics. For instance, in Coron and d'Andréa Novel (1998) the stabilization of a linear dynamic equation modeling a rotating beam is achieved through a control design relying on weak Lyapunov functional, i.e. a Lyapunov functional whose derivative along the trajectories of the system which is considered is non-positive (but not necessarily negative definite), and in Coron and Trélat (2006) the controllability of the wave equation is demonstrated via a Lyapunov func-

tional. Besides, the knowledge of Lyapunov functionals can be useful for the stability analysis of nonlinear hyperbolic systems (see the recent work Coron et al. (2008)) or even for designing boundary controls which stabilize a system of conservation laws (see Coron et al. (2007))

To demonstrate asymptotic stability through the knowledge of a weak Lyapunov functional, the celebrated LaSalle invariance principle has to be invoked (see e.g. Cazenave and Haraux (1998); Slemrod (1974); Luo et al. (1999)). It requires to demonstrate a precompactness property for the solutions, which may be difficult to prove (and is not even always satisfied, as illustrated by the hyperbolic systems considered in Coron et al. (2007)). This technical step is not needed when is available a strict Lyapunov functional i.e. a Lyapunov functional whose derivative along the trajectories of a system which is considered is negative definite. Thus designing such a Lyapunov functional is a way to overcome this technical difficulty. This is not the unique motivation for designing strict Lyapunov functionals. From the knowledge of the explicit expression of strict Lyapunov functionals one can estimate the robustness of the stability of a system with respect to the presence of uncertainties and one can analyze the sensitivity of the solutions with respect to external disturbances.

The present paper is devoted to new techniques of constructions of strict Lyapunov functionals for parabolic PDEs. For particular families of PDEs with diffusion and convection terms and specific boundary conditions, we modify weak Lyapunov functionals, which are readily available, to obtain strict Lyapunov functionals given by explicit formulas. The resulting functionals have rather simple explicit expressions. The underlying concept of strictification used in our paper is the same as the one exposed in Malisoff and Mazenc (2009), (see also Mazenc

and Nescic (2007), Mazenc et al. (2009)). However, due to the specificity of PDEs, the techniques of construction that we shall present are by no means a direct application of any constructions available for ordinary differential equations.

In a second part of our work, we design strict Lyapunov functionals to establish robustness properties of Input-to-State (ISS) type for a family of globally asymptotically stable PDEs with disturbances. Although the ISS notion is very popular in the area of the dynamical systems of finite dimension (see e.g. the recent survey Sontag (2007)) and, for a few years, begins to be used in the domain of the systems with delay (see for instance Mazenc et al. (2008), Pepe (2009), Pepe and Ito (2010), Karafyllis et al. (2008)), the present work is, to the best of our knowledge, the first one which uses it to characterize a robustness property of a PDE.

Our paper is organized as follows. Basic definitions and notations are introduced in Section 2. Constructions of Lyapunov functionals under various sets of assumptions are performed in Section 3. In Section 4 the analysis of the robustness of a family PDEs with uncertainties is carried out by means of the design of a so called ISS Lyapunov functional. An example in Section 5 illustrates the main result of Section 4. Concluding remarks in Section 7 end the work. Due to space limitation, some of the proofs are omitted.

*Notation.* Throughout the paper, the argument of the functions will be omitted or simplified when no confusion can arise from the context. Given a matrix  $A$ , its induced matrix,  $\text{Sym}(A) = \frac{1}{2}(A + A^\top)$  stands for the symmetric part of  $A$ . The norm  $\|\bullet\|_{L^2(0,L)}$  is defined by:  $\|\phi\|_{L^2(0,L)} = \sqrt{\int_0^L |\phi(z)|^2 dz}$ . Finally, we denote  $C_L = C^2([0, L], \mathbb{R}^n)$ , the set of all twice-differentiable  $\mathbb{R}^n$ -valued functions defined on a given interval  $[0, L]$ .

## 2. BASIC DEFINITIONS AND NOTIONS

Throughout our work, we will consider partial differential equations of the form

$$\frac{\partial X}{\partial t}(z, t) = \frac{\partial^2 X}{\partial z^2}(z, t) + \Delta(z, t) \frac{\partial X}{\partial z}(z, t) + f(X(z, t)) + u(z, t), \quad (1)$$

with  $z \in [0, L]$  and  $X(\cdot, t) \in C_L$  for all  $t \geq 0$ , where  $\Delta$  is continuous and bounded in norm,  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuously differentiable and where  $u$  is a continuous function (which typically is unknown and represents disturbances).

Let us introduce the notions of weak and strict Lyapunov functionals that we will consider in this paper (see e.g. (Luo et al., 1999, Def. 3.62)).

**Definition 2.1.** Let  $\mu: C_L \rightarrow \mathbb{R}$  be a continuously differentiable function. The functional  $\mu$  is said to be a weak Lyapunov functional for (1), if there are two functions  $\kappa_S$  and  $\kappa_M$  of class  $\mathcal{K}_\infty$  such that, for all functions  $\phi \in C_L$ ,

$$\kappa_S(\|\phi\|_{L^2(0,L)}) \leq \mu(\phi) \leq \int_0^L \kappa_M(|\phi(z)|) dz \quad (2)$$

and, in the absence of  $u$ , for all solutions of (1), for all  $t \geq 0$ ,

$$\frac{d\mu(X(\cdot, t))}{dt} \leq 0.$$

The functional  $\mu$  is said to be a strict Lyapunov functional for (1) if, additionally, in the absence of  $u$ , there exists  $\lambda_1 > 0$  such that, for all solutions of (1), for all  $t \geq 0$ ,

$$\frac{d\mu(X(\cdot, t))}{dt} \leq -\lambda_1 \mu(X(\cdot, t)).$$

The functional  $\mu$  is said to be an ISS Lyapunov functional for (1) if, additionally, there exist  $\lambda_1 > 0$  and a function  $\lambda_2$  of class  $\mathcal{K}$  such that, for all continuous functions  $u$ , for all solutions of (1), and for all  $t \geq 0$ ,

$$\frac{d\mu(X(\cdot, t))}{dt} \leq -\lambda_1 \mu(X(\cdot, t)) + \int_0^L \lambda_2(|u(z, t)|) dz.$$

**Remark 1.** 1. For conciseness, we will often use the notation  $\dot{\mu}$  instead of  $\frac{d\mu(X(\cdot, t))}{dt}$ .

2. When a strict Lyapunov functional exists and  $u$  is not present, then the value of a strict Lyapunov functional for (1) along the solutions of (1) exponentially decays to zero and therefore each solution  $X(z, t)$  satisfies  $\lim_{t \rightarrow +\infty} \|X(\cdot, t)\|_{L^2(0,L)} = 0$ . When in addition, there exists a function  $\kappa_L$  of class  $\mathcal{K}_\infty$ , such that, for all functions  $\phi \in C_L$ ,

$$\mu(\phi) \leq \kappa_L(\|\phi\|_{L^2(0,L)}), \quad (3)$$

then the system (1) is globally asymptotically stable for the topology of the norm  $L^2$ .

3. When the system (1) admits an ISS Lyapunov functional  $\mu$ , then, one can check through elementary calculations that, for all solutions of (1) and for all instants  $t \geq t_0$ , the inequality

$$\|X(\cdot, t)\|_{L^2(0,L)} \leq \kappa_S^{-1} \left( 2e^{-\lambda_1(t-t_0)} \int_0^L \kappa_M(|X(z, t_0)|) dz \right) + \kappa_S^{-1} \left( \frac{2}{\lambda_1} \sup_{\ell \in [t_0, t]} \left( \int_0^L \lambda_2(|u(z, \ell)|) dz \right) \right)$$

holds. This inequality is the analogue for the PDE (1) of the ISS inequalities for ordinary differential equations. It gives an estimation with the  $L_2$  norm of the influence of the disturbance  $u$  on the solutions of the system (1). Of course this inequality does not imply that a similar inequality holds when another norm is selected.  $\circ$

## 3. CONSTRUCTIONS OF LYAPUNOV FUNCTIONALS

In this section, we give several constructions of Lyapunov functionals for the system

$$\frac{\partial X}{\partial t}(z, t) = \frac{\partial^2 X}{\partial z^2}(z, t) + f(X(z, t)) \quad (4)$$

with  $z \in [0, L]$ ,  $X(z, t) \in \mathbb{R}^n$  and where  $f$  is a nonlinear function of class  $C^1$ .

### 3.1 Weak Lyapunov functional for the system (4)

To prepare the construction of strict Lyapunov functionals of the forthcoming sections, we recall how a weak Lyapunov functional can be constructed for the system (4) under the following assumptions:

**Assumption 1.** *There is a symmetric positive definite matrix  $Q$  such that the function*

$$W_1(\Xi) := -\frac{\partial V}{\partial \Xi}(\Xi)f(\Xi), \quad (5)$$

with  $V(\Xi) = \frac{1}{2}\Xi^\top Q \Xi$ , is nonnegative.

**Assumption 2.** *The boundary conditions are such that, for all  $t \geq 0$ ,*

$$\begin{aligned} \text{either } & |X(L, t)| \left| \frac{\partial X}{\partial z}(L, t) \right| = |X(0, t)| \left| \frac{\partial X}{\partial z}(0, t) \right| = 0, \\ \text{or } & X(L, t) = X(0, t) \text{ and } \frac{\partial X}{\partial z}(L, t) = \frac{\partial X}{\partial z}(0, t). \end{aligned} \quad (6)$$

Some comments on Assumptions 1 and 2 follow.

**Remark 2.** 1. Assumption 1 is equivalent to claiming that  $V$  is a weak Lyapunov function for the ordinary differential equation

$$\dot{\Xi} = f(\Xi) \quad (7)$$

with  $\Xi \in \mathbb{R}^n$ . Therefore it implies that this system is globally stable.

2. Assumption 2 is satisfied in particular if the Dirichlet or Neumann conditions or the periodic conditions, i.e.  $X(0, t) = X(L, t)$  and  $\frac{\partial X}{\partial z}(0, t) = \frac{\partial X}{\partial z}(L, t)$  for all  $t$  (see Chen and Matano (1989)), are satisfied.

3. Since  $Q$  is positive definite, there exist two positive real values  $q_1$  and  $q_2$  such that, for all  $\Xi \in \mathbb{R}^n$ ,

$$q_1|\Xi|^2 \leq V(\Xi) \leq q_2|\Xi|^2. \quad (8)$$

The constants  $q_1$  and  $q_2$  will be used in the constructions of strict Lyapunov functionals we shall perform later.  $\circ$

The construction we perform below is given for instance in Krstic and Smyshlyaev (2008); Coron and Trélat (2004).

**Lemma 3.1.** *Under Assumptions 1 and 2, the functional*

$$U(\phi) = \int_0^L V(\phi(z))dz \quad (9)$$

is a weak Lyapunov functional whose derivative along the solutions of (4) satisfies

$$\begin{aligned} \dot{U} = & - \int_0^L \frac{\partial X}{\partial z}(z, t)^\top Q \frac{\partial X}{\partial z}(z, t) dz \\ & - \int_0^L W_1(X(z, t)) dz. \end{aligned} \quad (10)$$

**Proof.** The proof is omitted.

*3.2 Strict Lyapunov functional for the system (4): first result*

In this section, we show that the functional  $U$  given in (9) is a strict Lyapunov functional for (4) when this system is associated with special families of boundary conditions or when  $W_1$  is larger than a positive definite quadratic function. We state and prove the following result:

**Theorem 1.** *Assume that the system (4) satisfies Assumptions 1 and 2 and that one of the following property is satisfied*

(i) *there exists a constant  $\alpha > 0$  such that, for all  $\Xi$  in  $\mathbb{R}^n$ ,*

$$W_1(\Xi) \geq \alpha|\Xi|^2,$$

(ii)  *$X(0, t) = 0$  for all  $t \geq 0$ ,*

(iii)  *$X(L, t) = 0$  for all  $t \geq 0$ .*

*Then the functional  $U$  given in (9) is a strict Lyapunov functional for (4).*

**Proof.** Let us assume that the property (i) holds. Then it follows straightforwardly from (8) and (10) that

$$\dot{U} \leq -\frac{\alpha}{q_2} \int_0^L V(X(z, t)) dz$$

and  $U$  is a strict Lyapunov functional (see Definition 2.1).

We consider now the cases (ii) and (iii). We consider now the cases (ii) and (iii). Let us recall the Poincaré inequality.

**Lemma 3.2.** *For any function  $w$ , continuously differentiable on  $[0, 1]$ , and for  $c = 0$  or  $c = 1$ ,*

$$\int_0^1 |w(z)|^2 dz \leq 2w^2(c) + 4 \int_0^1 \left| \frac{\partial w}{\partial z}(z) \right|^2 dz. \quad (11)$$

From this lemma, we deduce that for all  $L \geq 0$  and  $c = 0$  or  $L$ , and for any function  $w$ , continuously differentiable on  $[0, L]$ , the inequality

$$\int_0^L |w(z)|^2 dz \leq 2Lw^2(c) + 4L^2 \int_0^L \left| \frac{\partial w}{\partial z}(z) \right|^2 dz, \quad (12)$$

is valid. We deduce that when  $X(0, t) = 0$  for all  $t \geq 0$  or  $X(L, t) = 0$  for all  $t \geq 0$  then, for all  $t \geq 0$ , the inequality

$$\int_0^L |X(z, t)|^2 dz \leq \frac{4L^2}{q_1} \int_0^L \frac{\partial X}{\partial z}(z, t)^\top Q \frac{\partial X}{\partial z}(z, t) dz$$

where  $q_1$  is the constant in (8), is satisfied. Combining this inequality with (10) yields

$$\dot{U} \leq -\frac{q_1}{2L^2} \int_0^L |X(z, t)|^2 dz.$$

Using (8) again, we can conclude that  $U$  is a strict Lyapunov functional for the system (4).  $\bullet$

*3.3 Strict Lyapunov functional for the system (4): second result*

One can check easily that Assumptions 1 and 2 alone do not ensure that the system (4) admits the zero solution as an asymptotically stable solution.<sup>1</sup> Therefore an extra assumption must be introduced to guarantee that a strict Lyapunov functional exists. In Section 3.2 we have exhibited simple conditions which ensure that  $U$  is a strict Lyapunov functional. In this section, we introduce a new assumption, less restrictive than the condition (i) of Theorem 1, which ensures that a strict Lyapunov functional different from  $U$  can be constructed.

**Assumption 3.** *There exist a nonnegative function  $M : \mathbb{R}^n \rightarrow \mathbb{R}$  of class  $C^2$ , and a continuous function  $W_2 : \mathbb{R}^n \rightarrow \mathbb{R}$  such that*

$$M(0) = 0, \quad \frac{\partial M}{\partial \Xi}(0) = 0, \quad (13)$$

$$\frac{\partial M}{\partial \Xi}(\Xi)f(\Xi) \leq -W_2(\Xi), \quad \forall \Xi \in \mathbb{R}^n, \quad (14)$$

$$\left| \frac{\partial^2 M}{\partial \Xi^2}(\Xi) \right| \leq \frac{q_1}{2}, \quad \forall \Xi \in \mathbb{R}^n, \quad (15)$$

<sup>1</sup> More precisely, we can construct examples of systems (4) which are not asymptotically stable when Assumption 1 is satisfied and Assumption 2 holds with the Neumann boundary conditions.

and there exists a constant  $q_3 > 0$  such that  $W_1 + W_2$  is positive definite and

$$W_1(\Xi) + W_2(\Xi) \geq q_3 |\Xi|^2, \quad \forall \Xi \in \mathbb{R}^n : |\Xi| \leq 1 \quad (16)$$

where  $W_1$  is the function defined in (5).

We are ready to state and prove the following result:

**Theorem 2.** *Under Assumptions 1 to 3, there exists a function  $k$  of class  $\mathcal{K}_\infty$ , of class  $C^2$  such that  $k'$  is positive,  $k''$  is nonnegative and the functional*

$$\bar{U}(\phi) = \int_0^L k(V(\phi(z)) + M(\phi(z))) dz \quad (17)$$

is a strict Lyapunov functional for (4).

**Remark 3.** Assumption 3 seems to be restrictive. In fact, it can be significantly relaxed. Indeed, if the system

$$\dot{\Xi} = f(\Xi) \quad (18)$$

is locally exponentially stable and satisfies one of Matrosov's conditions which ensure that a strict Lyapunov function can be constructed then one can construct a function  $M$  which satisfies Assumption 3. For constructions of strict Lyapunov functions under Matrosov's conditions, the reader is referred to Malisoff and Mazenc (2009).  $\square$

**Proof.** Let us consider the functional  $\bar{U}$  defined in (17). Since we impose a priori on  $k$  to be of class  $\mathcal{K}_\infty$  and (15) holds, we deduce easily that inequalities of the type (2) are satisfied.

Next, let us evaluate what is the time derivative of  $\bar{U}$  along the solutions of (4). With the notation  $S = V + M$ , we have

$$\begin{aligned} \dot{\bar{U}} &= \int_0^L k'(S(X(z, t))) \frac{\partial S}{\partial \Xi}(X(z, t)) \frac{\partial X}{\partial t}(z, t) dz \\ &= \mathcal{T}_1(X(\cdot, t)) + \mathcal{T}_2(X(\cdot, t)) \end{aligned} \quad (19)$$

with

$$\begin{aligned} \mathcal{T}_1(\phi) &= \int_0^L k'(S(\phi(z))) \frac{\partial S}{\partial \Xi}(\phi(z)) f(\phi(z)) dz, \\ \mathcal{T}_2(\phi) &= \int_0^L k'(S(\phi(z))) \frac{\partial S}{\partial \Xi}(\phi(z)) \frac{\partial^2 \phi}{\partial z^2}(z) dz. \end{aligned} \quad (20)$$

Since

$$\frac{\partial S}{\partial \Xi}(\Xi) f(\Xi) = \frac{\partial V}{\partial \Xi}(\Xi) f(\Xi) + \frac{\partial M}{\partial \Xi}(\Xi) f(\Xi),$$

we deduce from Assumptions 1 and 3 that

$$\mathcal{T}_1(\phi) \leq - \int_0^L k'(S(\phi(z))) [W_1(\phi(z)) + W_2(\phi(z))] dz. \quad (21)$$

Now, we consider  $\mathcal{T}_2$ . By integrating by part, we obtain

$$\mathcal{T}_2(\phi) = \mathcal{T}_3(\phi) - \int_0^L \frac{\partial H(\phi(z))}{\partial z} \frac{\partial \phi}{\partial z}(z) dz \quad (22)$$

with

$$\begin{aligned} \mathcal{T}_3(\phi) &= k'(S(\phi(L))) \frac{\partial S}{\partial \Xi}(\phi(L)) \frac{\partial \phi}{\partial z}(L) \\ &\quad - k'(S(\phi(0))) \frac{\partial S}{\partial \Xi}(\phi(0)) \frac{\partial \phi}{\partial z}(0), \end{aligned} \quad (23)$$

and

$$H(\Xi) = k'(S(\Xi)) \frac{\partial S}{\partial \Xi}(\Xi). \quad (24)$$

Since

$$\begin{aligned} \frac{\partial H(\phi(z))}{\partial z} &= k''(S(\phi(z))) \frac{\partial S}{\partial \Xi}(\phi(z)) \frac{\partial \phi}{\partial z}(z) \frac{\partial S}{\partial \Xi}(\phi(z)) \\ &\quad + k'(S(\phi(z))) \frac{\partial \phi}{\partial z}(z) \frac{\partial^2 S}{\partial \Xi^2}(\phi(z)), \end{aligned} \quad (25)$$

we deduce from (22) that

$$\mathcal{T}_2(\phi) = \mathcal{T}_3(\phi) - \mathcal{T}_4(\phi) - \mathcal{T}_5(\phi) \quad (26)$$

with

$$\begin{aligned} \mathcal{T}_4(\phi) &= \int_0^L k''(S(\phi(z))) \left( \frac{\partial S}{\partial \Xi}(\phi(z)) \frac{\partial \phi}{\partial z}(z) \right)^2 dz, \\ \mathcal{T}_5(\phi) &= \int_0^L k'(S(\phi(z))) \frac{\partial \phi}{\partial z}(z) \frac{\partial^2 S}{\partial \Xi^2}(\phi(z)) \frac{\partial \phi}{\partial z}(z) dz. \end{aligned}$$

Since we impose on  $k$  to be such that  $k''$  is nonnegative, we immediately deduce that

$$\mathcal{T}_2(\phi) \leq \mathcal{T}_3(\phi) - \mathcal{T}_5(\phi). \quad (27)$$

Now, observe that

$$\frac{\partial^2 S}{\partial \Xi^2}(\phi(z)) = Q + \frac{\partial^2 M}{\partial \Xi^2}(\phi(z)). \quad (28)$$

This equality, inequalities (8), Assumption 3 and the fact that we impose on  $k$  to be such that  $k' > 0$  ensure that

$$\begin{aligned} \mathcal{T}_5(\phi) &\geq 2q_1 \int_0^L k'(S(\phi(z))) \left| \frac{\partial \phi}{\partial z}(z) \right|^2 dz \\ &\quad - \frac{q_1}{2} \int_0^L k'(S(\phi(z))) \left| \frac{\partial \phi}{\partial z}(z) \right|^2 dz \\ &= \frac{3q_1}{2} \int_0^L k'(S(\phi(z))) \left| \frac{\partial \phi}{\partial z}(z) \right|^2 dz. \end{aligned}$$

It follows that

$$\mathcal{T}_2(\phi) \leq \mathcal{T}_3(\phi) - \frac{3q_1}{2} \int_0^L k'(S(\phi(z))) \left| \frac{\partial \phi}{\partial z}(z) \right|^2 dz. \quad (29)$$

Hence (19), (21), and (29) give

$$\begin{aligned} \dot{\bar{U}} &\leq - \int_0^L k'(S(X(z, t))) W_3(X(z, t)) dz \\ &\quad + \mathcal{T}_3(X(\cdot, t)) - \frac{3q_1}{2} \int_0^L k'(S(X(z, t))) \left| \frac{\partial X}{\partial z}(z, t) \right|^2 dz \end{aligned}$$

with  $W_3 = W_1 + W_2$ . Assumption 2 ensures that, for all  $t \geq 0$ ,  $\mathcal{T}_3(X(\cdot, t)) = 0$ . We deduce that

$$\dot{\bar{U}} \leq - \int_0^L k'(S(X(z, t))) W_3(X(z, t)) dz.$$

By (16) and the inequality  $S(\Xi) \geq q_1 |\Xi|^2$ , we can construct, through simple but lengthy calculations, a function  $k$  of class  $C^2$  of class  $\mathcal{K}_\infty$  such that  $k'$  is positive,  $k''$  is nonnegative and

$$k'(S(\Xi)) W_3(\Xi) \geq C k(S(\Xi)), \quad \forall \Xi \in \mathbb{R}^n.$$

where  $C$  is a positive constant. Therefore, selecting this function  $k$ , we obtain

$$\dot{\bar{U}} \leq -C \bar{U}(X(\cdot, t)).$$

It follows that  $\bar{U}$  is a strict Lyapunov functional for (4).  $\bullet$

#### 4. ISS PROPERTY FOR A FAMILY OF PDES

In the previous section, we have constructed Lyapunov functionals for PDEs without uncertainties and without convection term. In this section, we show how our technique of construction can be used to estimate the impact of uncertainties on the solutions of PDEs with a convection term and uncertainties of the form

$$\begin{aligned} \frac{\partial X}{\partial t}(z, t) &= \frac{\partial^2 X}{\partial z^2}(z, t) + [D_1 + v(z, t)] \frac{\partial X}{\partial z}(z, t) \\ &\quad + f(X(z, t)) + u(z, t) \end{aligned} \quad (30)$$

where  $D_1$  is a constant matrix,  $v$  is an unknown matrix function and  $u$  is an unknown continuous function.

**Remark 4.** For a linear finite dimensional system

$$\dot{x} = Ax + Bu \quad (31)$$

where  $A$  and  $B$  are constant matrices respectively in  $\mathbb{R}^{n \times n}$  and in  $\mathbb{R}^{n \times 1}$ , it is well-known that if the linear system (31) without input  $u$ , i.e.  $\dot{x} = Ax$ , is asymptotically stable then bounded inputs result in bounded solutions. However, for nonlinear finite dimensional systems, global asymptotic stability does not imply input/state stability of any sort. See (Sontag, 2007, Section 2.6) for a simple scalar example, which is globally asymptotically stable but which have solutions with a finite time explosion for a suitable constant input.

Also for linear infinite dimensional system, asymptotic stability does not imply input-to-state stability. More precisely, we exhibit in Section 6 below an example of linear system which is globally asymptotically stable without any input, but which may have unbounded solutions in the presence of a bounded input.  $\circ$

To cope with the presence of a convection term and the uncertainty  $v$ , we introduce the following assumption:

**Assumption 4.** *Considering the symmetric positive definite matrix  $Q$  given by Assumption 1, there exists a nonnegative real number  $\delta$  such that*

$$|v(z, t)| \leq \frac{\delta}{|Q|}, \quad \forall z \in [0, L], t \geq 0. \quad (32)$$

Moreover, the matrix  $QD_1$  is symmetric.

Moreover, we replace Assumption 3 by a more restrictive assumption:

**Assumption 5.** *There exists a nonnegative function  $M : \mathbb{R}^n \rightarrow \mathbb{R}$  such that, for all  $\Xi \in \mathbb{R}^n$ ,*

$$M(0) = 0, \quad \frac{\partial M}{\partial \Xi}(\Xi)f(\Xi) = -W_2(\Xi), \quad (33)$$

where  $W_2$  is a nonnegative function and there exist  $c_a > 0$ ,  $c_b > 0$  and  $c_c > 0$  such that, for all  $\Xi \in \mathbb{R}^n$ , the inequalities

$$\left| \frac{\partial M}{\partial \Xi}(\Xi) \right| \leq c_a |\Xi|, \quad \left| \frac{\partial^2 M}{\partial \Xi^2}(\Xi) \right| \leq c_b, \quad (34)$$

$$|\Xi|^2 \leq c_c [W_1(\Xi) + W_2(\Xi)], \quad (35)$$

where  $W_1$  is the function defined in (5), are satisfied.

**Remark 5.** If  $f$  is linear and  $\dot{\Xi} = f(\Xi)$  is exponentially stable, then Assumption 5 is satisfied with a positive definite quadratic function as function  $M$ .  $\circ$

We are ready to state and prove the main result of the section

**Theorem 3.** *Assume that the system (30) satisfies Assumptions 1, 4 and 5 and is associated with boundary conditions satisfying*

$$X(L, t) = X(0, t) \text{ and } \frac{\partial X}{\partial z}(L, t) = \frac{\partial X}{\partial z}(0, t), \quad \forall t \geq 0. \quad (36)$$

Then the functional

$$\bar{U}(\phi) = \int_0^L [KV(\phi(z)) + M(\phi(z))] dz \quad (37)$$

with

$$K = \max \left\{ 1, \frac{2c_b}{q_1}, \frac{8c_c c_a^2 (|D_1| + 1)^2}{q_1} \right\} \quad (38)$$

satisfies, along the trajectories of (30),

$$\dot{\bar{U}} \leq -\lambda_1 \bar{U}(X(z, t)) + \lambda_2 \int_0^L |u(z, t)|^2 dz \quad (39)$$

for some positive constants  $\lambda_1, \lambda_2$ , provided that  $\delta$  in Assumption 4 satisfies

$$\delta \leq \min \left\{ |Q|, \frac{\sqrt{q_1}}{2\sqrt{2c_c K}} \right\}. \quad (40)$$

**Remark 6.** Using Assumption 5, one can prove easily that the ISS Lyapunov functional  $\bar{U}$  defined in (37) is upper and lower bounded by a positive definite quadratic functional. We deduce easily that (39) leads to an ISS inequality of the type

$$\begin{aligned} |X(\cdot, t)|_{L^2(0, L)} &\leq \Lambda_1 e^{-\frac{\lambda_1}{2}(t-t_0)} |X(\cdot, t_0)|_{L^2(0, L)} \\ &\quad + \Lambda_2 \sup_{m \in [t_0, t]} \sqrt{\int_0^L |u(z, m)|^2 dz}, \end{aligned} \quad (41)$$

where  $\Lambda_1, \Lambda_2$  are positive real numbers.  $\circ$

**Proof.** The proof is omitted.

## 5. EXAMPLE

In this section, we illustrate Theorem 3 through the nonlinear system

$$\begin{cases} \frac{\partial x_1}{\partial t}(z, t) = \frac{\partial^2 x_1}{\partial z^2}(z, t) - \frac{\partial x_1}{\partial z}(z, t) \\ \quad + x_2(z, t)[1 + x_1(z, t)^2] + u_1(z, t) \\ \frac{\partial x_2}{\partial t}(z, t) = \frac{\partial^2 x_2}{\partial z^2}(z, t) - x_1(z, t)[1 + x_1(z, t)^2] \\ \quad - x_2(z, t)[2 + x_1(z, t)^2] + u_2(z, t) \end{cases} \quad (42)$$

with  $X = (x_1, x_2)^\top \in \mathbb{R}^2$  and where  $u_1$  and  $u_2$  are continuous real-valued functions. Equation (42) is a system of two heat equations with a convection term in the first. Let us check that Theorem 3 applies. One can check readily that Assumption 1, 4 and 5 are satisfied with the positive definite quadratic functions  $V(\Xi) = \frac{1}{2}[\xi_1^2 + \xi_2^2]$

and  $M(\Xi) = \xi_1^2 + \xi_2^2 + \xi_1 \xi_2$ , and the matrix  $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

Indeed, with  $f(\Xi) = (\xi_2[1 + \xi_1^2], -\xi_1[1 + \xi_1^2] - \xi_2[2 + \xi_1^2])^\top$ , we have, for all  $\Xi$  in  $\mathbb{R}^2$ ,

$$\begin{aligned} \frac{\partial V}{\partial \Xi}(\Xi)f(\Xi) &= -\xi_2^2[2 + \xi_1^2], \\ \frac{\partial M}{\partial \Xi}(\Xi)f(\Xi) &= -(3 + \xi_1^2)\xi_2^2 - (1 + \xi_1^2)\xi_1^2 - \xi_1 \xi_2(2 + \xi_1^2) \\ &\leq -\xi_2^2 - \left( \frac{1}{2} + \frac{3}{4}\xi_1^2 \right) \xi_1^2. \end{aligned}$$

Moreover, through elementary calculations, one obtains the following values:  $q_1 = \frac{1}{2}$ ,  $|D_1| = 1$ ,  $c_a = c_b = 3$ ,  $c_c = 2$  for the constants in (38). Therefore Theorem 3 guarantees that, if (36) is satisfied, then the functional

$$\begin{aligned} \bar{U}(\phi) &= 1153 \int_0^L [\phi_1(z)^2 + \phi_2(z)^2] dz \\ &\quad + \int_0^L \phi_1(z)\phi_2(z) dz \end{aligned}$$

is an ISS Lyapunov functional for the system (42).

## 6. APPENDIX: ILLUSTRATION OF REMARK 4

In this section, we introduce an example of a linear infinite dimensional system which is globally asymptotically

stable without any input, but which may have unbounded solutions in presence of a bounded input.

To do that, let us consider the following system composed by an infinite number of scalar ordinary differential equations written by, for each  $n$  in  $\mathbb{N}$ ,

$$\dot{X}_n = -\frac{1}{n+1}X_n + u_n. \quad (43)$$

Given an initial condition in  $l^2(\mathbb{N})$ , there exists a solution of (43) defined for all time  $t \geq 0$  for  $u_n \equiv 0$ , for all  $n$  in  $\mathbb{N}$ . Moreover system (43) is globally asymptotic stable, without any input, i.e. when  $u_n \equiv 0$ , for all  $n$  in  $\mathbb{N}$ .

Let us consider the input satisfying, for each  $n$  in  $\mathbb{N}$ , and for all  $t \geq 0$ ,

$$u_n(t) = \frac{1}{(n+1)^2} e^{-\frac{1}{(n+1)^2}t}.$$

Note that, denoting by  $\|\bullet\|_{l^2(\mathbb{N})}$  the usual norm in  $l^2(\mathbb{N})$ , we have  $t \rightarrow \|(u_n(t))_{n \in \mathbb{N}}\|_{l^2(\mathbb{N})}$  is bounded. We even have  $\|(u_n(t))_{n \in \mathbb{N}}\|_{l^2(\mathbb{N})} \leq \frac{\pi^2}{6} e^{-t} \rightarrow 0$ , as  $t \rightarrow \infty$ . Moreover the solution of (43) with, for all  $n$  in  $\mathbb{N}$ ,  $X_n(0) = 0$  is given by, for all  $t \geq 0$ ,

$$X_n(t) = \frac{1}{n} \left[ e^{-\frac{1}{(n+1)^2}t} - e^{-\frac{1}{n+1}t} \right]$$

for  $n \geq 1$ . Therefore  $X_n(t)$  is not in  $l^2(\mathbb{N})$  for  $t > 0$ , and the system (43) is not asymptotically stable. System (43) is an example of asymptotic stable systems which is not input-to-state stable, as considered in Remark 4.

## 7. CONCLUSIONS

For important families of PDEs, we have shown how weak Lyapunov functionals can be transformed into strict Lyapunov functionals. Robustness properties of the ISS type can be inferred from our constructions. Much remains to be done. Other types of equations can be studied and robustness with respect to other types of disturbances can be proved.

## REFERENCES

- Cazenave, T. and Haraux, A. (1998). *An introduction to semilinear evolution equations*. Oxford University Press.
- Chen, X.Y. and Matano, H. (1989). Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations. *Journal of Differential Equations*, 78(1), 160–190.
- Coron, J.M., Bastin, G., and d’Andréa Novel, B. (2008). Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. *SIAM Journal on Control and Optimization*, 47(3), 1460–1498.
- Coron, J.M. and d’Andréa Novel, B. (1998). Stabilization of a rotating body beam without damping. *IEEE Transactions on Automatic Control*, 43(5), 608–618.
- Coron, J.M., d’Andréa Novel, B., and Bastin, G. (2007). A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. *IEEE Transactions on Automatic Control*, 52(1), 2–11.
- Coron, J.M. and Trélat, E. (2004). Global steady-state controllability of one-dimensional semilinear heat equations. *SIAM Journal on Control and Optimization*, 43(2), 549–569.
- Coron, J.M. and Trélat, E. (2006). Global steady-state stabilization and controllability of 1D semilinear wave equations. *Commun. Contemp. Math.*, 8(4), 535–567.
- Drame, A., Dochain, D., and Winkin, J. (2008). Asymptotic behavior and stability for solutions of a biochemical reactor distributed parameter model. *IEEE Transactions on Automatic Control*, 53(1), 412–416.
- Iftime, O.V. and Demetriou, M.A. (2009). Optimal control of switched distributed parameter systems with spatially scheduled actuators. *Automatica*, 45(2), 312–323.
- Karafyllis, I., Pepe, P., and Jiang, Z.P. (2008). Input-to-output stability for systems described by retarded functional differential equations. *European Journal of Control*, 14(6), 539–555.
- Krstic, M. and Smyshlyaev, A. (2008). Adaptive boundary control for unstable parabolic PDEs—part I: Lyapunov design. *IEEE Transactions on Automatic Control*, 53(7), 1575–1591.
- Luo, Z.H., Guo, B.Z., and Morgul, O. (1999). *Stability and stabilization of infinite dimensional systems and applications*. Communications and Control Engineering. Springer-Verlag, New York.
- Malisoff, M. and Mazenc, F. (2009). *Constructions of Strict Lyapunov Functions*. Communications and Control Engineering Series. Springer, London, UK.
- Matignon, D. and Prieur, C. (2005). Asymptotic stability of linear conservative systems when coupled with diffusive systems. *ESAIM: Control, Optim. Cal. Var.*, 11(3), 487–507.
- Mazenc, F., Malisoff, M., and Bernard, O. (2009). A simplified design for strict Lyapunov functions under Matrosov conditions. *IEEE Transactions on Automatic Control*, 54(1), 177–183.
- Mazenc, F., Malisoff, M., and Lin, Z. (2008). Further results on input-to-state stability for nonlinear systems with delayed feedbacks. *Automatica*, 44(9), 2415–2421.
- Mazenc, F. and Nescic, D. (2007). Lyapunov functions for time-varying systems satisfying generalized conditions of Matrosov theorem. *Mathematics of Control, Signals, and Systems*, 19, 151–182.
- Pepe, P. (2009). Input-to-state stabilization of stabilizable, time-delay, control-affine, nonlinear systems. *IEEE Transactions on Automatic Control*, 54(7), 1688–1693.
- Pepe, P. and Ito, H. (2010). On saturation, discontinuities and time-delays in iISS and ISS feedback control redesign. In *Proc. of American Control Conference (ACC’10)*, 190–195.
- Slemrod, M. (1974). A note on complete controllability and stabilizability for linear control systems in Hilbert space. *SIAM Journal on Control*, 12, 500–508.
- Smyshlyaev, A. and Krstic, M. (2007a). Adaptive boundary control for unstable parabolic PDEs—part II: Estimation-based designs. *Automatica*, 43(9), 1543–1556.
- Smyshlyaev, A. and Krstic, M. (2007b). Adaptive boundary control for unstable parabolic PDEs—part III: Output feedback examples with swapping identifiers. *Automatica*, 43(9), 1557–1564.
- Sontag, E. (2007). Input to state stability: Basic concepts and results. In *Nonlinear and Optimal Control Theory*, 163–220. Springer-Verlag, Berlin.