Families of quasi-rational solutions of the NLS equation as an extension of higher order Peregrine breathers.

Pierre Gaillard

Families of quasi-rational solutions of the NLS equation as an extension of higher order Peregrine breathers.

[+]Pierre Gaillard, [+] Université de Bourgogne, Dijon, France :
e-mail: Pierre.Gaillard@u-bourgogne.fr,

Mars 06, 2011

Abstract

We construct a multi-parametric family of solutions of the focusing nonlinear Schrödinger equation (NLS) equation from the known result describing the multi phase almost-periodic elementary solutions given in terms of Riemann theta functions. We give a new representation of their solutions in terms of Wronskians determinants of order $2N$ composed of elementary trigonometric functions. When we perform a special passage to the limit when all the periods tend to infinity, we obtain a family of quasi-rational solutions. This leads to efficient representations for the Peregrine breathers of orders $N = 1, 2, 3$ first constructed by Akhmediev and his co-workers and also allows us to obtain a simpler derivation of the generic formulas corresponding the three or six rogue-waves formation in frame of the NLS model first explained in 2010. Our formulation allows us to isolate easily the second or third order Peregrine breather from "generic" solutions, and also to compute the Peregrine breathers of order 2 and 3 easier with respect to other approaches. In the cases $N = 2, 3$ we get the comfortable formulas to study the deformation of higher Peregrine breather of order 2 to the three rogue-waves or order 3 to the six rogue-waves solution via variation of the free parameters of our construction.

1 Introduction

The nonlinear Schrödinger equation (NLS) was first derived by Zakharov [19] in 1968. It was solved in 1972 by the inverse scattering method by Zakharov and Shabat [20] in which in particular the amplitude of N-solitons solutions to the focusing NLS equation was explicitly calculated.
The periodic and almost periodic algebro-geometric solutions to the focusing NLS equation were first found in 1976 by Its and Kotlyarov [14].
The study of quasi-rational solutions was preceded by the works by Kuznetsov, Ma, Kawata and Akhmediev who constructed some special periodic solutions to the NLS equation. In 1983, performing an appropriate passage to the limit in one of this solutions, Peregrine discovered a quasi-rational solution to the NLS equation nowadays called worldwide Peregrine breather. In 1986 Eleonski, Akhmediev and Kulagin obtained the two-phase almost periodic solution to the NLS equation and by taking an appropriate limit obtained the first higher order analogue of the Peregrine breather [3]. A few families of higher order were constructed in a series of articles by Akhmediev et al. [1, 2] using Darboux transformations. Other solutions were found for reduced self-induced transparency (SIT) integrable systems by Matveev, Rybin and Salle [15]. In [13], the N-phase quasi-periodic modulations of the plane waves solutions were constructed via appropriate degeneration of the finite gap periodic solutions of the NLS equation.
Recently, it has been shown in [8] that rational solutions of NLS equation can be written as a quotient of two wronskians using modified version of [10]; moreover, it has been established the link between quasi-rational solutions of the focusing NLS equation and the rational solution of the KP-I equation. Also with this formulation we recover as particular case, Akhmediev's quasi-rational solutions of NLS equation.
In [6], Calini and Schober have studied solutions of NLS equation using the method of Hirota [12], in particular for the orders 2 and 3 (corresponding to our notations) and obtained multi-rogue waves whose pictures were very similar to the rational case obtained in [8]. It was clear from this remark that rational solutions could be obtained from the solutions in terms of Riemann theta functions given by A. Its [13] by a specific passage to the limit.

In this paper, we construct a representation of the solutions of the NLS equation in terms of a ratio of two wronskians determinants of even order $2N$ composed of elementary functions; we will call these related solutions,

solutions of NLS of order N. When we perform the passage to the limit when some parameter tends to 0, we get families of multi-rogue wave solutions of the focusing NLS equation depending on a certain number of parameters. It allows to recognize the famous Peregrine breather [17] and also higher order Peregrine breathers constructed by Akhmediev [1, 4].
Conversely, in the approach of [8], it is very difficult from the general formula given therein, to isolate higher order Peregrine breathers.
As a particular case, we obtain for $N = 1$, the well known Peregrine's solution [17] of the focusing NLS equation.
For $N = 2$, we get Akhmediev's breathers with certain choices of the parameters. Surprisingly, we recover after reductions, exactly the same analytical expression of the solutions given in [8]. We get for an arbitrary choice of the parameters the shape of Akhmediev's breathers; we can also get easily, for particular parameters, the apparition of the three peaks for the modulus of the solution v in the $(x; y)$ coordinates (three sisters).
For $N = 3$, we get Akhmediev's breathers for an arbitrary choice of parameters. Choosing particular parameters, we observe also the apparition of the six peaks for the modulus of the solution in the (x, t) coordinates.
For N=4, we give only the analytical expression of Akhmediev's breather in the case $t = 0$ and the corresponding graphic in the (x, t) plane.

In this approach, we get an alternative way to get quasi-rational solutions of the focusing NLS equation depending on a certain number of parameters, in particular, higher order Peregrine breathers and multi-rogue waves, different from all previous works.

2 Expression of solutions of NLS equation in terms of Fredholm determinant

2.1 Solutions of NLS equation in terms of θ functions

We use here a general formulation of the solution of the NLS equation given in [13], different from that used in [11]. We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0, \tag{1}$$

The solution is given in terms of truncated theta function by

$$v(x,t) = \frac{\theta_3(x,t)}{\theta_1(x,t)} \exp(2it - i\varphi). \tag{2}$$

The functions $\theta_r(x,t)$ are the functions defined by

$$\theta_r(x,t) = \sum_{k \in \{0;1\}^{2N}} g_{r,k}, \quad r = 1,\, 3 \tag{3}$$

with $g_{r,k}$ given by

$$g_{r,k} = \exp\left\{ \sum_{\mu>\nu,\, \mu,\nu=1}^{2N} \ln\left(\frac{\gamma_\nu - \gamma_\mu}{\gamma_\nu + \gamma_\mu}\right)^2 k_\mu k_\nu \right. \tag{4}$$

$$\left. + \left(\sum_{\nu=1}^{2N} i\kappa_\nu x - 2\delta_\nu t + (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i} + \sum_{\mu=1,\, \mu\neq\nu}^{2N} \ln\left|\frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu}\right| + \pi i \epsilon_\nu + e_\nu \right) k_\nu \right\}.$$

The solutions depend on a certain number of parameters :
φ;
N parameters λ_j, satisfying the relations

$$0 < \lambda_j < 1, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \le j \le N; \tag{5}$$

$2N$ parameters e_ν, $1 \le \nu \le 2N$ satisfying the relations

$$e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \quad 1 \le j \le N. \tag{6}$$

The terms ϵ_ν, $1 \le \nu \le 2N$ are arbitrary numbers equal to 0 or 1.
In the preceding formula, the terms κ_ν, δ_ν, γ_ν are functions of the parameters λ_ν, $\nu = 1, \ldots, 2N$, and they are given by the following equations,

$$\kappa_\nu = 2\sqrt{1-\lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}. \tag{7}$$

We also note that

$$\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1 \ldots N. \tag{8}$$

2.2 Relation between θ and Fredholm determinant

We know from [13] that the function θ_r defined in (3) can be written as a Fredholm determinant. The expression given in [13] is different from which we need in the following. We need different choices of ϵ_ν :

$$\begin{aligned} \epsilon_\nu = 0, \quad 1 \leq \nu \leq N \\ \epsilon_\nu = 1, \quad N+1 \leq \nu \leq 2N. \end{aligned} \tag{9}$$

The function θ_r defined in (3) can be rewritten with a summation in terms of subsets of $[1, .., 2N]$

$$\theta_r(x,t) = \sum_{J \subset \{1,..,2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J,\, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right|$$

$$\times \exp\{\sum_{\nu \in J} i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu\},$$

with

$$x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N, \tag{10}$$

in particular

$$\begin{aligned} x_{r,j} = (r-1) \ln \frac{\gamma_j - i}{\gamma_j + i}, \quad 1 \leq j \leq N, \\ x_{r,N+j} = -(r-1) \ln \frac{\gamma_j - i}{\gamma_j + i} - (r-1) i\pi, \quad 1 \leq j \leq N. \end{aligned} \tag{11}$$

We consider $A_r = (a_{\nu\mu})_{1 \leq \nu, \mu \leq 2N}$ the matrix defined by

$$a_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\lambda \neq \mu} \left| \frac{\gamma_\lambda + \gamma_\nu}{\gamma_\lambda - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \tag{12}$$

Then $\det(I + A_r)$ has the following form

$$\begin{aligned} \det(I + A_r) = \sum_{J \subset \{1,...,2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J\, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(i\kappa_\nu x \\ -2\delta_\nu t + x_{r,\nu} + e_\nu). \end{aligned} \tag{13}$$

From the beginning of this section, $\tilde{\theta}$ has the same expression as in (13) so, we have clearly the equality

$$\theta_r = \det(I + A_r). \tag{14}$$

Then the solution of NLS equation takes the form

$$v(x,t) = \frac{\det(I + A_3(x,t))}{\det(I + A_1(x,t))} \exp(2it - i\varphi). \tag{15}$$

3 Expression of solutions of NLS equation in terms of wronkian determinant

3.1 Link between Fredholm determinants and wronskians

We use here the same ideas as those exposed in [11]. The proofs are the same. We don't reproduce it in this text. The reader can see the aforementioned paper.
We consider the following functions

$$\begin{array}{ll} \phi^r_\nu(y) = \sin(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), & 1 \le \nu \le N, \\ \phi^r_\nu(y) = \cos(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), & N+1 \le \nu \le 2N. \end{array} \tag{16}$$

For simplicity, in this section we denote them $\phi_\nu(y)$.
We use the following notations :
$\Theta_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2, \quad 1 \le \nu \le 2N$.
$W_r(y) = W(\phi_1, \ldots, \phi_{2N})$ is the wronskian

$$W_r(y) = \det[(\partial_y^{\mu-1}\phi_\nu)_{\nu,\,\mu\in[1,\ldots,2N]}]. \tag{17}$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu,\,\mu\in[1,\ldots,2N]}$ defined by

$$\begin{array}{l} d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\lambda\neq\mu} \left|\frac{\gamma_\lambda+\gamma_\nu}{\gamma_\lambda-\gamma_\mu}\right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu), \\ \quad 1 \le \nu \le 2N, \quad 1 \le \mu \le 2N, \end{array}$$

with

$$x_{r,\nu} = (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i}.$$

Then we have the following statement

Theorem 3.1

$$\det(I + D_r) = k_r(0) \times W_r(\phi_1, \ldots, \phi_{2N})(0), \tag{18}$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_\nu)}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

Proof : The proof is the same as this given in [11]. We don't reproduce it here to avoid to have a too long text.

3.2 Wronskian representation of solutions of NLS equation

From the previous section, we get the following result :

Theorem 3.2 *The function v defined by*

$$v(x,t) = \frac{W_3(0)}{W_1(0)} \exp(2it - i\varphi). \tag{19}$$

is solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

Remark 3.1 *In formula (19), $W_r(y)$ is the wronskian defined in (17)with the functions ϕ_ν^r given by (16); κ_ν, δ_ν, γ_ν are defined by (7); λ_ν are arbitrary parameters given by (5); e_ν are defined by (6).*

4 Construction of quasi-rational solutions of NLS equation

4.1 Taking the limit when the parameters $\lambda_j \to 1$ for $1 \le j \le N$ and $\lambda_j \to -1$ for $N+1 \le j \le 2N$

In the following, we show how we can obtain quasi-rational solutions of NLS equation by a simple limiting procedure.

For simplicity, we denote d_j the term $\frac{c_j}{\sqrt{2}}$.
We consider the parameter λ_j written in the form

$$\lambda_j = 1 - 2\epsilon^2 d_j^2, \quad 1 \leq j \leq N. \tag{20}$$

When ϵ goes to 0, we realize limited expansions at order p, for $1 \leq j \leq N$, of the terms
$\kappa_j = 4d_j\epsilon(1-\epsilon^2 d_j^2)^{1/2}$, $\delta_j = 4d_j\epsilon(1-2\epsilon^2 d_j^2)(1-\epsilon^2 d_j^2)^{1/2}$,
$\gamma_j = d_j\epsilon(1-\epsilon^2 d_j^2)^{-1/2}$, $x_{r,j} = (r-1)\ln\frac{1+i\epsilon d_j(1-\epsilon^2 d_j^2)^{-1/2}}{1-i\epsilon d_j(1-\epsilon^2 d_j^2)^{-1/2}}$,
$\kappa_{N+j} = 4d_j\epsilon(1-\epsilon^2 d_j^2)^{1/2}$, $\delta_{N+j} = -4d_j\epsilon(1-2\epsilon^2 d_j^2)(1-\epsilon^2 d_j^2)^{1/2}$,
$\gamma_{N+j} = 1/(d_j\epsilon)(1-\epsilon^2 d_j^2)^{1/2}$, $x_{r,N+j} = (r-1)\ln\frac{1-i\epsilon d_j(1-\epsilon^2 d_j^2)^{-1/2}}{1+i\epsilon d_j(1-\epsilon^2 d_j^2)^{-1/2}}$.
For example, the expansions at order 1 gives :

$$\begin{array}{l}
\kappa_j = 4d_j\epsilon + O(\epsilon^2), \quad \gamma_j = d_j\epsilon + O(\epsilon^2), \quad \delta_j = 4d_j\epsilon + O(\epsilon^2), \\
x_{r,j} = (r-1)(2id_j\epsilon + O(\epsilon^2)), \\
\kappa_{N+j} = 4d_j\epsilon + O(\epsilon^2), \quad \gamma_{N+j} = 1/(d_j\epsilon) - (d_j\epsilon)/2 + O(\epsilon^2), \quad \delta_{N+j} = -4d_j\epsilon + O(\epsilon^2), \\
x_{r,N+j} = -(r-1)(2id_j\epsilon + O(\epsilon^2)), \\
1 \leq j \leq N.
\end{array}$$

Then, we realize limited expansions at order p in ϵ of the functions $\phi_j^r(0)$ and $\phi_{N+j}^r(0)$, for $1 \leq j \leq N$:

$$\begin{array}{l}
\phi_j^1(0) = P_j + O(\epsilon^{p+1}), \\
\phi_j^3(0) = Q_j + O(\epsilon^{p+1}), \\
\phi_{N+j}^1(0) = P_j' + O(\epsilon^{p+1}), \\
\phi_{N+j}^3(0) = Q_j' + O(\epsilon^{p+1}).
\end{array}$$

Here, it is the important point to get non trivial rational solution depending on the whole parameters : we choose λ_j as (20), for $1 \leq N$. The parameters a_j and b_j, for $1 \leq N$ must be carrefully chosen. They must depend on ϵ and are expressed in the form

$$a_j = \tilde{a_j}\epsilon^{M-1}, \quad b_j = \tilde{b_j}\epsilon^{M-1}, \quad 1 \leq j \leq N, \quad M = 2N. \tag{21}$$

Theorem 4.1 *With the parameters λ_j defined by (20), a_j and b_j chosen as in (21), for $1 \leq j \leq N$, the function v defined by*

$$v(x,t) = \exp(2it - i\varphi)\lim_{\epsilon\to 0}\frac{W_3(0)}{W_1(0)}, \tag{22}$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

depending on $3N$ *parameters* d_j, $\tilde{a}_j$, $\tilde{b}_j$, $1 \leq j \leq N$.

Remark 4.1 *In (22),* $W_r(y)$ *is the wronskian defined in (17) with the functions* ϕ_ν^r *given by (16).*

Proof : The idea is similar as this given in [11]. We postpone the details of the proof to a further publication.

Remark 4.2 *If we replace the parameters defined in (21) by* $a_j = \tilde{a}_j \epsilon^{p(M)}$, $b_j = \tilde{b}_j \epsilon^{p(M)}$, $1 \leq j \leq N$ *with* $p(M) \neq M-1$, *the parameters* $\tilde{a}_j$ *and* $\tilde{b}_j$ *disappear in the limit when* ϵ *goes ta* 0 *and we get particular cases of solutions.*
If $p(M) < M-1$, *we get trivial solution (i. e.* $v(x,t) = \exp(2it - i\varphi)$*).*
If we take $p(M) > M-1$, *we recover in this case higher order Peregrine's breathers.*

4.2 Quasi-rational solutions of order N

To get solutions of NLS equation written in the context of fiber optics

$$iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0, \tag{23}$$

from these of (1), we can make the following changes of variables

$$\begin{aligned} t &\rightarrow X/2 \\ x &\rightarrow T. \end{aligned} \tag{24}$$

In the following, we give all the solutions for (1).

4.2.1 Case N=1

From (22), we realize an expansion at order 1 of W_3 and W_1 in ϵ. The solution of NLS equation can be written as

$$v(x,t) = \frac{-16d_1^2t^2 + 16id_1^2t - 4id_1\tilde{b}_1 - 4d_1x\tilde{a}_1 - \tilde{a}_1^2 + 3d_1^2 + 8d_1t\tilde{b}_1 - \tilde{b}_1^2 - 4d_1^2x^2}{4d_1^2x^2 + 4d_1x\tilde{a}_1 + 16d_1^2t^2 - 8d_1t\tilde{b}_1 + \tilde{a}_1^2 + \tilde{b}_1^2 + d_1^2}\exp(2it - i\varphi).$$

Apparently, it depends on $3N+1=4$ parameters.
But in fact it can be written in the form

$$v(x,t)=\frac{(4(x+\frac{\tilde{a}_1}{2d_1})^2+16(t-\frac{\tilde{b}_1}{4d_1})^2-16i(t-\frac{\tilde{b}_1}{4d_1})-3)}{(4(x+\frac{\tilde{a}_1}{2d_1})^2+16(t-\frac{\tilde{b}_1}{4d_1})^2+1))}\exp(2it-i\varphi).$$

We note that the parameter d_1 disappears, and the remaining parameters are only translation parameters. By denoting $X=x+\frac{\tilde{a}_1}{2d_1}$ and $T=t-\frac{\tilde{b}_1}{4d_1}$, it can be rewritten as

$$v(x,t)=\frac{(4X^2+16T^2-16iT-3)}{(4X^2+16T^2+1))}\exp(2it-i\varphi).$$

We recover the well known Peregrine breather.
Thus, in this case $N=1$, the parameters can be reduced to only 2 parameters of translation and φ. The changes of these parameters don't affect the aspect of the form of the representation of $|v(x,t)|$ in the (x,t) variables.
Moreover, if we make the preceding change of variable (24), and take $\tilde{a}_1=\tilde{b}_1=0$, we get exactly Peregrine's solution (see [17]).
We represent in the figure 1, the modulus of v in function of $x\in[-5;5]$ and $t\in[-5;5]$, for $\tilde{a}_1=\tilde{b}_1=1$ and $d_1=1$.

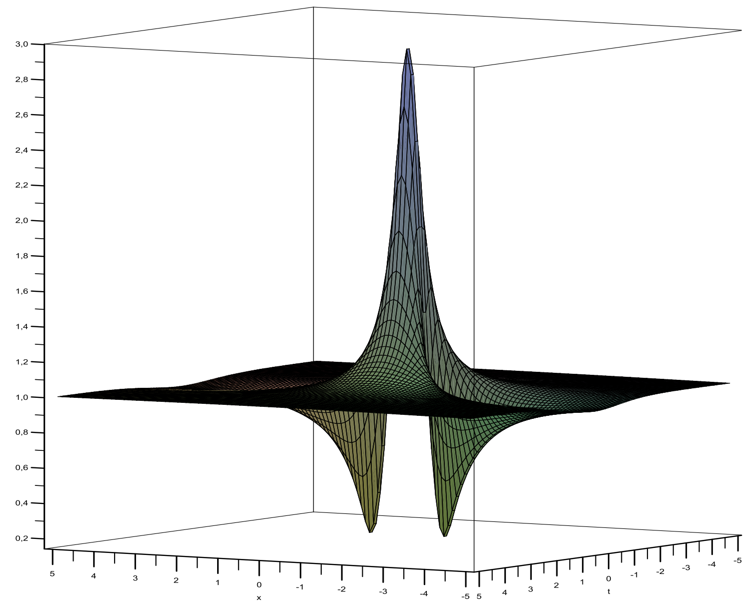

Figure 1: Solution to the NLS equation for N=1 with $\tilde{a}_1 = \tilde{b}_1 = 1$, $d_1 = 1$.

4.2.2 Case N=2

In the case $N = 2$, we realize an expansion at order 3 in ϵ. From (22), the solution of NLS equation can be written as

$$v(x,t) = \frac{n(x,t)}{d(x,t)} \exp(2it - i\varphi),$$

with

$$n(x,t) = (128 d_1^4 d_2^4 - 64 d_1^6 d_2^2 - 64 d_2^6 d_1^2) x^6$$

$$((-768 d_1^6 d_2^2 + 1536 d_1^4 d_2^4 - 768 d_2^6 d_1^2) t^2 + ((768i) d_1^6 d_2^2 + (768i) d_1^2 d_2^6 - (1536i) d_1^4 d_2^4) t - 288 d_1^4 d_2^4 + 144 d_2^6 d_1^2 + 144 d_1^6 d_2^2) x^4$$

$$(-48 d_1^2 d_2^3 \tilde{a}_2 + 48 \tilde{a}_1 d_1 d_2^4 + 48 d_1^4 d_2 \tilde{a}_2 - 48 d_1^3 d_2^2 \tilde{a}_1) x^3$$

$$((-3072 d_1^6 d_2^2 + 6144 d_1^4 d_2^4 - 3072 d_2^6 d_1^2) t^4 + ((6144i) d_1^6 d_2^2 - (12288i) d_1^4 d_2^4 + (6144i) d_1^2 d_2^6) t^3 + (-11520 d_1^4 d_2^4$$
$$+ 5760 d_1^6 d_2^2 + 5760 d_2^6 d_1^2) t^2 + (-(1152i) d_1^6 d_2^2 + 288 \tilde{b}_2 d_1^2 d_2^3 - (1152i) d_1^2 d_2^6 + 288 d_1^3 d_2^2 \tilde{b}_1 - 288 d_1^4 d_2 \tilde{b}_2 + (2304i) d_1^4 d_2^4$$
$$- 288 \tilde{b}_1 d_1 d_2^4) t + (144i) d_1^4 d_2 \tilde{b}_2 + 180 d_1^6 d_2^2 - 360 d_1^4 d_2^4 - (144i) d_1^3 d_2^2 \tilde{b}_1 - (144i) d_2^3 d_1^2 \tilde{b}_2 + (144i) \tilde{b}_1 d_1 d_2^4 + 180 d_2^6 d_1^2) x^2$$

$((-576\tilde{a}_1d_1d_2^4+576d_1^3d_2^2\tilde{a}_1-576d_1^4d_2\tilde{a}_2+576d_1^2d_2^3\tilde{a}_2)t^2+(-(576i)\tilde{a}_1d_1^3d_2^2+(576i)d_1^4d_2\tilde{a}_2+(576i)\tilde{a}_1d_1d_2^4$
$-(576i)d_1^2d_2^3\tilde{a}_2)t-108d_1^2d_2^3\tilde{a}_2+108d_1^4d_2\tilde{a}_2-108d_1^3d_2^2\tilde{a}_1+108\tilde{a}_1d_1d_2^4)x$
$(-4096d_2^6d_1^2-4096d_1^6d_2^2+8192d_1^4d_2^4)t^6+(-(24576i)d_1^4d_2^4+(12288i)d_1^2d_2^6+(12288i)d_1^6d_2^2)t^5+(-16896d_1^4d_2^4$
$+8448d_2^6d_1^2+8448d_1^6d_2^2)t^4+((1536i)d_1^2d_2^6+(1536i)d_1^6d_2^2-384d_1^3d_2^2\tilde{b}_1-384\tilde{b}_2d_1^2d_2^3+384d_1^4d_2\tilde{b}_2+384\tilde{b}_1d_1d_2^4$
$-(3072i)d_1^4d_2^4)t^3+(-(576i)d_1^4d_2\tilde{b}_2+(576i)d_2^3d_1^2\tilde{b}_2-3744d_1^4d_2^4+1872d_2^6d_1^2-(576i)\tilde{b}_1d_1d_2^4+1872d_1^6d_2^2$
$+(576i)d_1^3d_2^2\tilde{b}_1)t^2+((1440i)d_1^4d_2^4-72d_1^4d_2\tilde{b}_2-72\tilde{b}_1d_1d_2^4-(720i)d_1^2d_2^6-(720i)d_1^6d_2^2+72d_1^3d_2^2\tilde{b}_1+72\tilde{b}_2d_1^2d_2^3)t$
$-9\tilde{b}_2^2d_1^2-(36i)d_1^3d_2^2\tilde{b}_1+18\tilde{a}_1d_2d_1\tilde{a}_2+(36i)\tilde{b}_1d_1d_2^4-9\tilde{b}_1^2d_2^2+90d_1^4d_2^4-45d_1^6d_2^2+(36i)d_1^4d_2\tilde{b}_2+18d_1\tilde{b}_2d_2\tilde{b}_1$
$-9\tilde{a}_1^2d_2^2-(36i)d_2^3d_1^2\tilde{b}_2-9\tilde{a}_2^2d_1^2-45d_2^6d_1^2$

and

$$d(x,t)=(64d_2^6d_1^2+64d_1^6d_2^2-128d_1^4d_2^4)x^6$$
$$((768d_2^6d_1^2-1536d_1^4d_2^4+768d_1^6d_2^2)t^2+48d_2^6d_1^2-96d_1^4d_2^4+48d_1^6d_2^2)x^4$$
$$(-48\tilde{a}_1d_1d_2^4-48d_1^4d_2\tilde{a}_2+48d_1^3d_2^2\tilde{a}_1+48d_1^2d_2^3\tilde{a}_2)x^3$$
$((3072d_2^6d_1^2-6144d_1^4d_2^4+3072d_1^6d_2^2)t^4+(-1152d_2^6d_1^2+2304d_1^4d_2^4-1152d_1^6d_2^2)t^2+(288\tilde{b}_1d_1d_2^4-288\tilde{b}_2d_1^2d_2^3$
$-288d_1^3d_2^2\tilde{b}_1+288d_1^4d_2\tilde{b}_2)t+108d_1^6d_2^2+108d_2^6d_1^2-216d_1^4d_2^4)x^2$
$((576\tilde{a}_1d_1d_2^4-576d_1^3d_2^2\tilde{a}_1+576d_1^4d_2\tilde{a}_2-576d_1^2d_2^3\tilde{a}_2)t^2+36\tilde{a}_1d_1d_2^4-36d_1^2d_2^3\tilde{a}_2+36d_1^4d_2\tilde{a}_2-36d_1^3d_2^2\tilde{a}_1)x$
$(-8192d_1^4d_2^4+4096d_2^6d_1^2+4096d_1^6d_2^2)t^6+(6912d_2^6d_1^2-13824d_1^4d_2^4+6912d_1^6d_2^2)t^4+(384d_1^3d_2^2\tilde{b}_1-384\tilde{b}_1d_1d_2^4$
$-384d_1^4d_2\tilde{b}_2+384\tilde{b}_2d_1^2d_2^3)t^3+(1584d_2^6d_1^2-3168d_1^4d_2^4+1584d_1^6d_2^2)t^2+(-216\tilde{b}_1d_1d_2^4+216d_1^3d_2^2\tilde{b}_1-216d_1^4d_2\tilde{b}_2$
$+216\tilde{b}_2d_1^2d_2^3)t+9d_1^6d_2^2+9d_2^6d_1^2-18\tilde{a}_1d_2d_1\tilde{a}_2+9\tilde{a}_1^2d_2^2-18d_1^4d_2^4+9\tilde{b}_1^2d_2^2+9\tilde{b}_2^2d_1^2$
$+9\tilde{a}_2^2d_1^2-18d_1\tilde{b}_2d_2\tilde{b}_1$

Remark 4.3 *This solution depends on $3N+1=7$ parameters. In fact, like in the case $N=1$, it can be reduced and the final expression depends only on two parameters (φ being not taking into account).*
If we denote

$$\alpha=\frac{3(\tilde{b}_2d_1-\tilde{b}_1d_2)}{2d_1d_2(d_1^2-d_2^2)},$$

and

$$\beta=\frac{3(\tilde{a}_2d_1-\tilde{a}_1d_2)}{d_1d_2(d_1^2-d_2^2)},$$

the preceding solution $v(x,t)$ can be written as

$$v(x,t)=\frac{n_1(x,t)}{d_1(x,t)}\exp(2it-i\varphi), \tag{25}$$

with

$$n_1(x,t) = 64x^6+(768t^2-144-(768i)t)x^4-16\beta x^3+(3072t^4+192\alpha t-(96i)\alpha-5760t^2-(6144i)t^3$$

$$+(1152i)t-180)x^2+(192\beta t^2-36\beta-(192i)\beta t)x+45-(1536i)t^3+\beta^2+4\alpha^2-1872t^2$$

$$-8448t^4-256\alpha t^3+4096t^6+48\alpha t+(720i)t-(12288i)t^5+(384i)\alpha t^2-(24i)\alpha$$

$$d_1(x,t) = 64x^6+(768t^2+48)x^4-16\beta x^3+(-1152t^2+192\alpha t+108+3072t^4)x^2+(12\beta$$

$$+192\beta t^2)x+4096t^6+6912t^4-256\alpha t^3+1584t^2-144\alpha t+\beta^2+4\alpha^2+9.$$

In [8], we have constructed the solution v_2 of (1). Choosing the parameters as follows $B=1$, $\varphi_1=3\varphi_3$, $\varphi_2=2\varphi_4+\frac{3+\sqrt{5}}{16}\sqrt{10-2\sqrt{5}}$ (as specified in [9]), the solution can be written exactly in the form (25).
The two different methods give well the same analytical expression $v(x,t)$ as solution of NLS (1), but the choices of parameters φ_i in the method [8] are difficult to isolate Akhmediev's breathers as to identify with the solutions given in this paper.

If we make the preceding changes of variables defined by (24), and take $\tilde{a}_1=\tilde{a}_2=\tilde{b}_1=\tilde{b}_2=0$, it can be reduced exactly at the second order Akhmediev's solution (see [1]).

Contrary to the case $N=1$, in this case there are two important parameters different from parameters of translation which play a central role in the deformation of solution. It is the crucial point. With these parameters the shape of the curve of $|v|$ change radically as we prove it in the following. We recover second order Peregrine breather as well the three sisters of the same amplitude.
We represent the modulus of v in function of $x\in[-5;5]$ and $t\in[-5;5]$ in two cases.
If we take $d_1=1$, $d_2=2$, $\tilde{a}_1=\tilde{a}_2=\tilde{b}_1=\tilde{b}_2=1$, we get the well known Peregrine breather of order 2 described in the figure 2 :

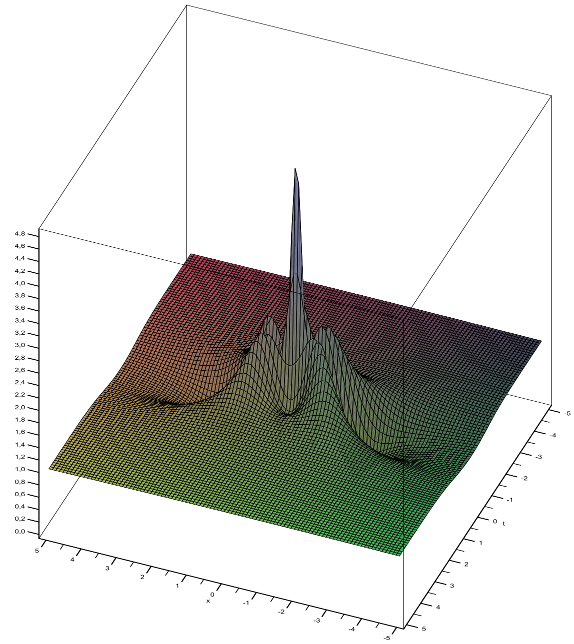

Figure 2: Solution to the NLS equation for N=2 with $d_1 = 1$, $d_2 = 2$, $\tilde{a}_1 = \tilde{a}_2 = \tilde{b}_1 = \tilde{b}_2 = 1$.

If we take $d_1 = 1$, $d_2 = 2$, $\tilde{a}_1 = \tilde{a}_2 = 0$, $\tilde{b}_1 = \tilde{b}_2 = 1000$ we get the case of the three sisters described by the figure 3 :

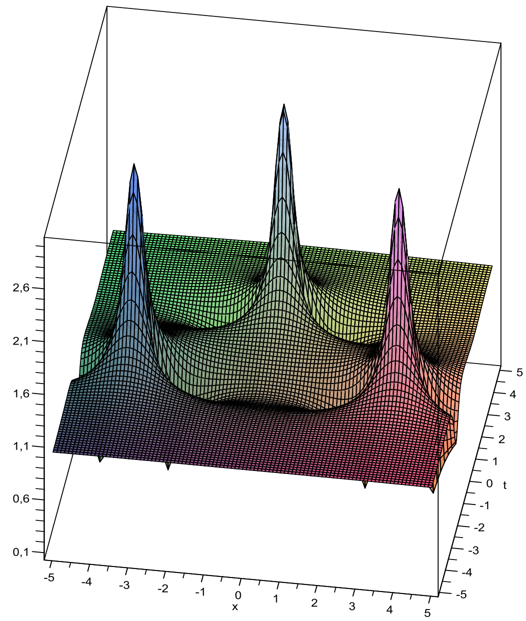

Figure 3: Solution to the NLS equation for N=2 with $d_1 = 1$, $d_2 = 2$, $\tilde{a}_1 = \tilde{a}_2 = 0$, $\tilde{b}_1 = \tilde{b}_2 = 1000$.

We presented here an example of deformation of solutions of the NLS equation according to the parameters a_j and b_j giving various known shapes of the modulus of these one, to illustrate the power of the method. The study of the zones of appearance of these various types of solutions according to a_j and b_j, or of α and β is in progress and will be the subject of a next publication not to weigh down the text of this article.

4.2.3 Case N=3

In the case $N = 3$, we realize an expansion at order 5 in ϵ. We get from (22), the solution of NLS equation (23) in the form

$$v(x,t) = \frac{n(x,t)}{d(x,t)} \exp(2it - i\varphi).$$

In this case, the analytical expression takes about 36 pages of usual format. We can't reproduce it in this text.
We give the following graphics for the modulus of v in function of $x \in [-5;5]$ and $t \in [-5;5]$ in three cases.
If we take the following parameters : $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = \tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = 1$, we get the Peregrine breather of order 3 given by the figure 4 :

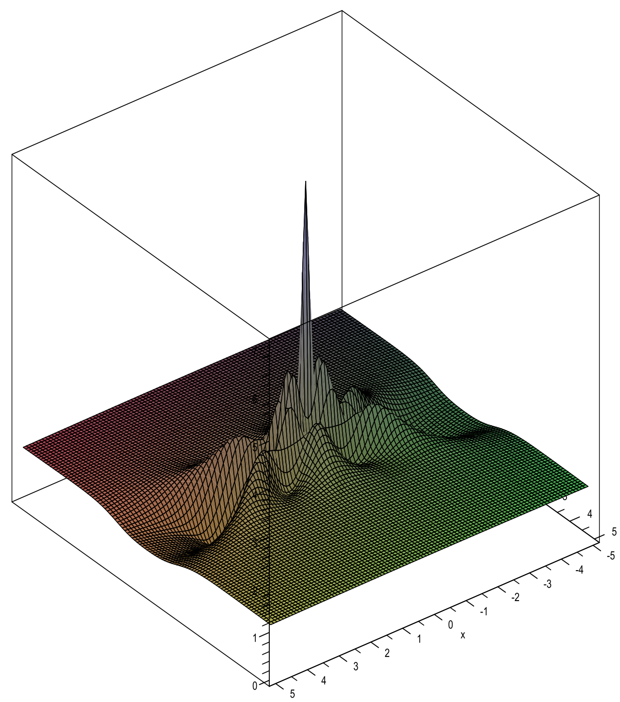

Figure 4: Solution to the NLS equation for N=3 with $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = \tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = 1$.

If we take the following parameters $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $\tilde{a}_1 = 10000$, $\tilde{a}_2 = \tilde{a}_3 = 0$, $\tilde{b}_1 = 10000$, $\tilde{b}_2 = \tilde{b}_3 = 0$, the shape of the modulus of v in the (x, t) coordinates change to get 6 peaks as described in the approach of Matveev et al. (see [8]) and is given by the figure 5.

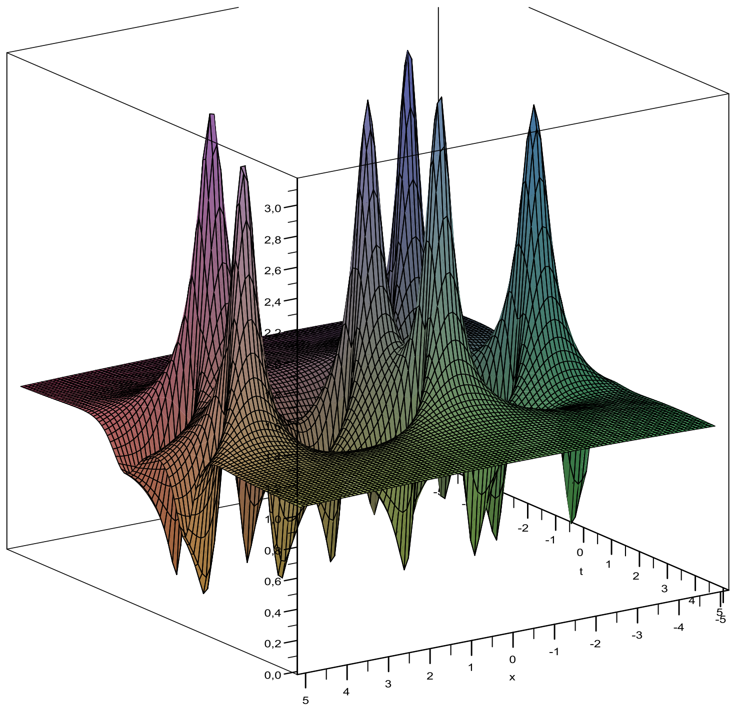

Figure 5: Solution to the NLS equation for N=3 with $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $\tilde{a}_1 = 10000$, $\tilde{a}_2 = \tilde{a}_3 = 0$, $\tilde{b}_1 = 10000$, $\tilde{b}_2 = \tilde{b}_3 = 0$.

If we take $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = \tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = 0$ and make the preceding changes of variables defined by (24), we recover the solution given recently by Akhmediev [1]. For example, if we take all parameters a_j and b_j equal to zero, we obtain the Peregrine breather of order 4. If we choose the following representation of the NLS equation

$$v_N(x,t) = \frac{n(x,t)}{d(x,t)} \exp(2it - i\varphi) = (1 - \alpha_N \frac{G_N(2x,4t) + iH_N(2x,4t)}{Q_N(2x,4t)})e^{2it-i\varphi}$$

with

$$\begin{array}{rcl} G_N(X,T) & = & \sum_{k=0}^{N(N+1)} \mathbf{g}_k(T)X^k \\ H_N(X,T) & = & \sum_{k=0}^{N(N+1)} \mathbf{h}_k(T)X^k \\ Q_N(X,T) & = & \sum_{k=0}^{N(N+1)} \mathbf{q}_k(T)X^k \end{array}$$

We get

$$\alpha_3 = 4, \quad \mathbf{g}_{12} = 0, \quad \mathbf{g}_{11} = 0, \quad \mathbf{g}_{10} = 6, \quad \mathbf{g}_9 = 0, \quad \mathbf{g}_8 = 90\,T^2 + 90, \quad \mathbf{g}_7 = 0,$$
$$\mathbf{g}_6 = 300\,T^4 - 360\,T^2 + 1260,$$
$$\mathbf{g}_5 = 0,$$
$$\mathbf{g}_4 = 420\,T^6 - 900\,T^4 + 2700\,T^2 - 2700,$$
$$\mathbf{g}_3 = 0,$$
$$\mathbf{g}_2 = 270\,T^8 + 2520\,T^6 + 40500\,T^4 - 81000\,T^2 + 180\,Tb - 4050,$$
$$\mathbf{g}_1 = 0,$$
$$\mathbf{g}_0 = 66\,T^{10} + 2970\,T^8 + 13140\,T^6 - 45900\,T^4 - 12150\,T^2 + 4050$$

$$\mathbf{h}_{12} = 0, \quad \mathbf{h}_{11} = 0, \quad \mathbf{h}_{10} = 6\,T, \quad \mathbf{h}_9 = 0, \quad \mathbf{h}_8 = 30\,T^3 - 90\,T, \quad \mathbf{h}_7 = 0,$$
$$\mathbf{h}_6 = 60\,T^5 - 840\,T^3 - 900\,T,$$
$$\mathbf{h}_5 = 0,$$
$$\mathbf{h}_4 = 60\,T^7 - 1260\,T^5 - 2700\,T^3 - 8100\,T,$$
$$\mathbf{h}_3 = 0,$$
$$\mathbf{h}_2 = 30\,T^9 - 360\,T^7 + 10260\,T^5 - 37800\,T^3 + 28350\,T,$$
$$\mathbf{h}_1 = 0,$$
$$\mathbf{h}_0 = 6\,T^{11} + 150\,T^9 - 5220\,T^7 - 57780\,T^5 - 14850\,T^3 + 28350T$$

$$\mathbf{q}_{12} = 1, \quad \mathbf{q}_{11} = 0, \quad \mathbf{q}_{10} = 6\,T^2 + 6, \quad \mathbf{q}_9 = 0, \quad \mathbf{q}_8 = 15\,T^4 - 90\,T^2 + 135, \quad \mathbf{q}_7 = 0,$$
$$\mathbf{q}_6 = 20\,T^6 - 180\,T^4 + 540\,T^2 + 2340,$$
$$\mathbf{q}_5 = 0,$$
$$\mathbf{q}_4 = 15\,T^8 + 60\,T^6 - 1350\,T^4 + 13500\,T^2 + 3375,$$
$$\mathbf{q}_3 = 0,$$
$$\mathbf{q}_2 = 6\,T^{10} + 270\,T^8 + 13500\,T^6 + 78300\,T^4 - 36450\,T^2 + 12150,$$
$$\mathbf{q}_1 = 0,$$
$$\mathbf{q}_0 = T^{12} + 126\,T^{10} + 3735\,T^8 + 15300\,T^6 + 143775\,T^4 + 93150T^2 + 2025$$

It can be notified that even in this case, the choices of the parameters for the method given in [8] to get Akhmediev's breathers are not yet found.

As in the previous section, we presented here deformations of solutions of the equation NLS according to the parameters a_j and b_j giving various known solutions. In order to make the text of this paper not too long, we postpone the study of the zones of appearance of these various types of solutions according to a_j and b_j to a next publication.

4.2.4 Cases of higher order

In the case of higher order $N \geq 4$, the work is actually in progress and we postpone to present the results in an other paper in order not to make this one too long. We just give a particular case with $N = 4$, to show the efficiency of the method. When we choose $a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4 = 0$, we get from (22), the solution of NLS equation (23) in the form

$$v(x,t) = \frac{n(x,t)}{d(x,t)} \exp(2it - i\varphi).$$

The expression of v is rather cumbersome. The polynomials N and D are polynomials in x and t of same degree $N(N+1) = 20$. It is too long to be published here. We postpone to give the explicit expression in a further publication.
In particular,

$$v(x,0) = \frac{n(x,0)}{d(x,0)},$$

$$n(x,0) = -200930625 + 2679075000x^2 + 9644670000x^4 - 11430720000x^6 - 9398592000x^8 - 6096384000x^{10} + 1354752000x^{12} + 324403200x^{14} + 44236800x^{16} + 7864320x^{18} - 1048576x^{20},$$

$$d(x,0) = 22325625 + 893025000x^2 + 1786050000x^4 + 8382528000x^6 + 4463424000x^8 + 1683763200x^{10} + 1741824000x^{12} + 265420800x^{14} + 26542080x^{16} + 2621440x^{18} + 1048576x^{20}.$$

For example, if we take all parameters a_j and b_j equal to zero, we obtain the Peregrine breather of order 4. The solutions of NLS equation take the form

$$v_N(x,t) = \frac{n(x,t)}{d(x,t)} \exp(2it - i\varphi) = (1 - \alpha_N \frac{G_N(2x,4t) + iH_N(2x,4t)}{Q_N(2x,4t)})e^{2it-i\varphi}$$

with

$$\begin{aligned} G_N(X,T) &= \textstyle\sum_{k=0}^{N(N+1)} \mathbf{g}_k(T) X^k \\ H_N(X,T) &= \textstyle\sum_{k=0}^{N(N+1)} \mathbf{h}_k(T) X^k \\ Q_N(X,T) &= \textstyle\sum_{k=0}^{N(N+1)} \mathbf{q}_k(T) X^k \end{aligned}$$

$\alpha_4 = 4$, $\mathbf{g}_{20} = 0$, $\mathbf{g}_{19} = 0$, $\mathbf{g}_{18} = 10$, $\mathbf{g}_{17} = 0$, $\mathbf{g}_{16} = 270\,T^2 + 270$, $\mathbf{g}_{15} = 0$, $\mathbf{g}_{14} = 1800\,T^4 -3600\,T^2 + 9000$, $\mathbf{g}_{13} = 0$, $\mathbf{g}_{12} = 5880\,T^6 - 54600\,T^4 - 12600\,T^2 + 189000$, $\mathbf{g}_{11} = 0$, $\mathbf{g}_{10} = 11340\,T^8 - 176400\,T^6 + 189000\,T^4 - 378000\,T^2 - 1077300$, $\mathbf{g}_9 = 0$, $\mathbf{g}_8 = 13860\,T^{10} - 207900\,T^8 + 2356200\,T^6 + 1701000\,T^4 - 56983500\,T^2 - 4819500$, $\mathbf{g}_7 = 0$, $\mathbf{g}_6 = 10920\,T^{12} - 18480\,T^{10} + 6967800\,T^8 + 56095200\,T^6 - 342657000\,T^4 +198450000\,T^2 - 11907000$, $\mathbf{g}_5 = 0$ $\mathbf{g}_4 = 5400\,T^{14} + 163800\,T^{12} + 9034200\,T^{10} +107919000\,T^8 - 615195000\,T^6 + 178605000\,T^4 + 654885000\,T^2 + 178605000$, $\mathbf{g}_3 = 0$, $\mathbf{g}_2 = 1530\,T^{16} + 133200\,T^{14} + 5506200\,T^{12} - 116802000\,T^{10} - 1731334500\,T^8 +2532222000\,T^6 - 893025000\,T^4 + 4643730000)T^2 + 223256250$, $\mathbf{g}_1 = 0$, $\mathbf{g}_0 = 190\,T^{18} + 33150\,T^{16} + 1294200\,T^{14} + 3288600\,T^{12} + 48629700\,T^{10} -2015401500\,T^8 - 1845585000\,T^6 + 14586075000)T^4 + 2098608750T^2 - 44651250$,

$\mathbf{h}_{20} = 0$, $\mathbf{h}_{19} = 0$, $\mathbf{h}_{18} = 10\,T$, $\mathbf{h}_{17} = 0$, $\mathbf{h}_{16} = 90\,T^3 - 270\,T$, $\mathbf{h}_{15} = 0$, $\mathbf{h}_{14} = 360\,T^5 -6000\,T^3 - 5400\,T$, $\mathbf{h}_{13} = 0$, $\mathbf{h}_{12} = 840\,T^7 - 29400\,T^5 + 12600\,T^3 - 138600\,T$, $\mathbf{h}_{11} = 0$, $\mathbf{h}_{10} = 1260\,T^9 - 65520\,T^7 + 259560\,T^5 - 529200\,T^3 - 1984500\,T$, $\mathbf{h}_9 = 0$, $\mathbf{h}_8 = 1260\,T^{11} - 77700\,T^9 + 718200\,T^7 - 5329800\,T^5 - 6142500\,T^3 + 29767500\,T$, $\mathbf{h}_7 = 0$, $\mathbf{h}_6 = 840\,T^{13} - 48720\,T^{11} + 718200\,T^9 + 2973600\,T^7 - 72765000\,T^5 +436590000\,T^3 + 146853000\,T, \mathbf{h}_5 = 0$, $\mathbf{h}_4 = 360\,T^{15} - 12600\,T^{13} + 138600\,T^{11} -5859000\,T^9 - 328293000\,T^7 + 1075599000\,T^5 + 773955000\,T^3 + 535815000T$, $\mathbf{h}_3 = 0$, $\mathbf{h}_2 = 90\,T^{17} + 1200\,T^{15} - 189000\,T^{13} - 40143600\,T^{11} -307786500\,T^9 + 2085426000\,T^7 - 4465125000\,T^5 + 4405590000T^3 - 1205583750T$, $\mathbf{h}_1 = 0$, $\mathbf{h}_0 = 10\,T^{19} + 930\,T^{17} - 86040\,T^{15} - 7018200\,T^{13} - 48100500\,T^{11} - 542902500\,T^9 +6039117000\,T^7 + 12942909000T^5 + 937676250T^3$, $\mathbf{q}_{20} = 1$, $\mathbf{q}_{19} = 0$, $\mathbf{q}_{18} = 10\,T^2 + 10$, $\mathbf{q}_{17} = 0$, $\mathbf{q}_{16} = 45\,T^4 - 270\,T^2 + 405$, $\mathbf{q}_{15} = 0$, $\mathbf{q}_{14} = 120\,T^6 - 1800\,T^4 + 1800\,T^2 + 16200$, $\mathbf{q}_{13} = 0$, $\mathbf{q}_{12} = 210\,T^8 - 4200\,T^6 + 6300\,T^4 +113400\,T^2 + 425250$, $\mathbf{q}_{11} = 0$, $\mathbf{q}_{10} = 252\,T^{10} - 3780\,T^8 + 63000\,T^6 +718200\,T^4 + 3005100\,T^2 + 1644300$, $\mathbf{q}_9 = 0$, $\mathbf{q}_8 = 210\,T^{12} + 1260\,T^{10} +255150\,T^8 - 567000\,T^6 + 23388750\,T^4 - 31468500\,T^2 + 17435250$, $\mathbf{q}_7 = 0$, $\mathbf{q}_6 = 120\,T^{14} + 5880\,T^{12} + 476280\,T^{10} + 16443000\,T^8 + 162729000\,T^6 -154791000\,T^4 + 130977000\,T^2 + 130977000$, $\mathbf{q}_5 = 0$, $\mathbf{q}_4 = 45\,T^{16} + 5400\,T^{14} +459900\,T^{12} + 19845000\,T^{10} + 153798750\,T^8 + 702513000\,T^6 - 89302500\,T^4 +1250235000T^2 + 111628125, \mathbf{q}_3 = 0$, $\mathbf{q}_2 = 10\,T^{18} +2250\,T^{16} + 225000\,T^{14} + 4422600\,T^{12} - 99508500\,T^{10} - 224248500\,T^8 +9704205000\,T^6 + 15181425000T^4 - 1920003750T^2 + 223256250$, $\mathbf{q}_1 = 0$, $\mathbf{q}_0 = T^{20} + 370\,T^{18} + 44325\,T^{16} + 2208600\,T^{14} + 62795250\,T^{12} + 693384300\,T^{10} +6641129250\,T^8 + 4346055000T^6 + 14042818125)T^4 + 2902331250T^2 + 22325625$

We recover a result of Akhmediev formulated in [1] in the case of initial condition $t = 0$. Here we give the complete solution in x and t.

We give the shape of the modulus of v in the (x, t) coordinates (corresponding in the general formulation to the elementary case $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = \tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = \tilde{b}_4 = 0$ in the figure 6 :

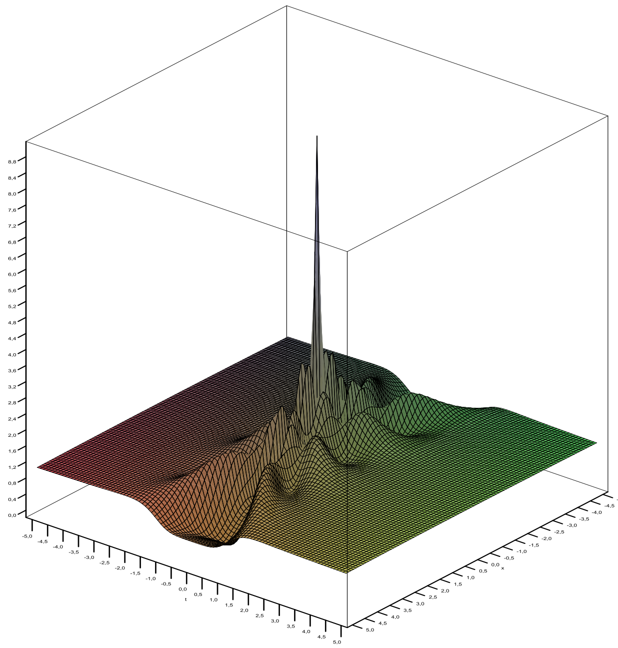

Figure 6: Solution to the NLS equation for N=4 with $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = \tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = \tilde{b}_4 = 0$.

Remark 4.4 *In this last case, $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = \tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = \tilde{b}_4 = 0$, the analytical expression of the solution of the NLS equation does not depend on the parameters d_1, d_2, d_3, d_4. It can be seen in the particular case $v(x, 0)$.*

5 Conclusion

We have given here an extension of a previous result exposed in [11] which gives with new parameters a family which recover a wide spectrum of solutions of the NLS equation. These solutions are written as a quotient of wronskians. An other approach has been given in [8].
The method described in the present paper provides a powerful tool to get explicitly solutions of the NLS equation.
This method with parameters gives higher Peregrine breathers of order N

as well solutions with peaks of similar amplitude. It is is reasonable to conjecture that in general there is $N(N+1)/2$ peaks for the modulus of any solution v in the (x,t) coordinates.
Because of the presence of a lot of redundant parameters $(3N+1)$, the present formulation give more flexibility to pass from Akhmediev's breathers to peaks of similar heights. This present method shows more adapted and efficient than this given in [8] to get all type of solutions.
This new formulation gives an infinite set of non singular (quasi-rational) solutions of NLS equation at any order and the results raise any scepticism about the use of determinants and theta functions.

Acknowledgements
I would like to thank sincerely and deeply V. B. Matveev to introduce me in the rogue wave community and particulary in this subject. I am very indebted for these precious remarks and advices on the writing of this article. I have to mention the central idea of A. Its at the origin of this work [13]. I am very grateful to him for showing interest about this work, of these various remarks, numerous fruitful discussions in particular during its last coming in Dijon to the university of Burgundy in June 2010, and for the second reading of this text.
This work has been supported partly by the grant ANR-09-BLAN-0117-01.

References

[1] N. Akhmediev, A. Ankiewicz, J.M. Soto-Crespo, Rogue waves and rational solutions of nonlinear Schrödinger equation, Physical Review E, V. **80**, N. 026601, (2009).

[2] N. Akhmediev, V. Eleonskii, N. Kulagin, Exact first order solutions of the nonlinear Schrödinger equation, Th. Math. Phys., V. **72**, N. 2, 183-196, (1987).

[3] N. Akhmediev, V. Eleonsky, N. Kulagin, Generation of periodic trains of picosecond pulses in an optical fiber : exact solutions, Sov. Phys. J.E.T.P., V. **62**, 894-899, (1985).

[4] N. Akhmediev, A. Ankiewicz, P.A. Clarkson, Rogue waves, rational solutions, the patterns of their zeros and integral relations, J. Phys. A : Math. Theor., V. **43**, 122002, 1-9, (2010).

[5] E.D. Belokolos, A.i. Bobenko, A.R. its, V.Z. Enolskij and V.B. Matveev, Algebro-geometric approach to nonlinear integrable equations, Springer series in nonlinear dynamics, Springer Verlag, 1-360, (1994).

[6] A. Calini, C. M. Schober, Homoclinic chaos increases the likelihood of rogue wave formation, Phy. Lett. A, 335-349, 2002

[7] A. Calini, C. M. Schober, Rogues waves in higher order nonlinear Scrödinger models, In : Extreme Waves, Eds. E. Pelinovsky, C. Kharif, Springer, 2008

[8] P. Dubard, P. Gaillard, C. Klein, V.B. Matveev, On multi-rogue waves solutions of the NLS equation and positon solutions of the KdV equation, Eur. Phys. J. Special Topics, V. **185**, 247-258, (2010).

[9] P. Dubard, Multi-rogue solutions to the focusing NLS equation, Thesis, Univ. Of Burgondy, p 24, (2011).

[10] V. Eleonskii, I. Krichever, N. Kulagin, Rational multisoliton solutions to the NLS equation, Soviet Doklady 1986 sect. Math. Phys., V. **287**, 606-610, (1986).

[11] P. Gaillard, Quasi-rational solutions of the NLS equation and rogue waves, halshs-00536287, 2011

[12] R. Hirota, Direct method of finding exact solutions of non linear evolution equations L.N.M., V. **515**, 40-68, (1976).

[13] A.R. Its, A.V. Rybin, M.A. Salle, Exact integration of nonlinear Schrödinger equation, Teore. i Mat. Fiz., V. **74**., N. 1, 29-45, (1988).

[14] A.R. Its, V.P. Kotlyarov, Explicit formulas for solutions of the nonlinear Schrödinger equation, Dockl. Akad. Nauk. Ukraine SSSR, S. A, V. **11**., 965-968, (1976).

[15] V.B. Matveev, M.A. Salle, Darboux transformations and solitons, Series in Nonlinear Dynamics, Springer Verlag, Berlin, (1991).

[16] V.B. Matveev, M.A. Salle, A.V. Rybin, Coherent Darboux transformations and interaction of the light pulsates with two-level media, Inv. Pro., V. **4**, 173-183, 1988.

[17] D. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, J. Austral. Math. Soc. Ser. B, V. **25**, 16-43, (1983).

[18] R.R.. Rosales, Exact solutions of some nonlinear evolution equations, Studies In Apll. Math., V. **59**, 117-151, (1978).

[19] V. E. Zakharov, Stability of periodic waves of finite amplitude on a surface of a deep fluid, J. Appl. Tech. Phys, V. **9**, 86-94, (1968)

[20] V. E. Zakharov, A.B. Shabat Exact theory of two dimensional self focusing and one dimensinal self modulation of waves in nonlinear media, Sov. Phys. JETP, V. **34**, 62-69, (1972)