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Marie-Claude Arnaud

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Lower and upper bounds for the Lyapunov exponents of twisting dynamics: a relationship between the exponents and the angle of the Oseledet's splitting

M.-C. ARNAUD *†‡

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Abstract

We consider locally minimizing measures for the conservative twist maps of the d-dimensional annulus or for the Tonelli Hamiltonian flows defined on a cotangent bundle T^*M. For weakly hyperbolic such measures (i.e. measures with no zero Lyapunov exponents), we prove that the mean distance/angle between the stable and the unstable Oseledet's bundles gives an upper bound of the sum of the positive Lyapunov exponents and a lower bound of the smallest positive Lyapunov exponent. Some more precise results are proved too.

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†ANR DynNonHyp

‡Université d'Avignon et des Pays de Vaucluse, Laboratoire d'Analyse non linéaire et Géométrie (EA 2151), F-84 018Avignon, France. e-mail: Marie-Claude.Arnaud@univ-avignon.fr

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1 Introduction

The purpose of this article is to give some relationships between the positive Lyapunov exponents and the angles of the Oseledet's bundles for the invariant minimizing Borel probability measures of the conservative twisting dynamics.

Conservative twisting dynamics are either what is called a Tonelli Hamiltonian defined on the cotangent bundle T^*M of a connected and compact manifold M of a twist map of the d-dimensional annulus $\mathbb{A}_d = \mathbb{T}^d \times \mathbb{R}^d$. Their two main properties are the following ones:

- they twist the verticals;
- they are symplectic.

A lot of famous dynamical systems are such conservative twisting dynamics. Let us mention at first all the geodesic flows and mechanical systems (sum of a kinetic energy and a potential one): they define Tonelli Hamiltonian flows. The twist maps of the two-dimensional annulus were introduced at the end of the nineteenth century by H. Poincaré in the study of the restricted planar circular three body problem (which is a kind of modeling of the system Sun-Earth-Moon). Let us mention too that billiard maps are conservative twist maps and that the Frenkel-Kontorova model can be represented by a conservative twist map (see e.g. [11]).

To such dynamics we can associate what is called an *action*[1], defined via a generating function or a Lagrangian functional. Two types of action can be defined: either it is a functional defined along the pieces of orbits or the action of every invariant probability measure is defined. The objects of our study are then the locally minimizing orbits or measures. In the case of Tonelli Hamiltonians, it is well-known that those orbits (resp. measures) are exactly those that have no conjugate points (see for example [7], [1]). Following [6], we will see in subsection 2.3 that the locally minimizing orbits of the twist maps of $\mathbb{A}_d$ also have no conjugate points.

The following fact is proved in [6] in the case of symplectic twist maps and in [1], [7] in the case of Tonelli Hamiltonians.
If an orbit is locally minimizing (this means that every piece of this orbit minimizes locally the action among the segments that have same ends), then there exist along the orbit two Lagrangian sub-bundles, invariant under the linearized dynamics and transverse to the vertical bundle, called the Green bundles. These Green bundles enjoy a lot of nice properties that we will precisely describe later. Following [1] and [2], we will denote them by G_- and G_+.

[1] It will be precisely defined later.

Let us assume now that there exists along such a locally minimizing orbit either an Oseledet's splitting or a partially hyperbolic splitting. We denote the stable, unstable and center bundles corresponding to this splitting by E^s, E^u an E^c. It is proved in [4] that:

$$E^s \subset G_- \subset E^s \oplus E^c \quad \text{and} \quad E^u \subset G_+ \subset E^u \oplus E^c.$$

Hence for a minimizing Borel probability μ, the whole information concerning the positive Lyapunov exponents is contained in the linearized dynamics restricted to the positive Green bundle G_+ above the support of μ. Moreover, the angle/distance between the stable and unstable bundles is related to angle/distance between the two Green bundles.
Let us recall too that we proved in [4] that for an ergodic locally minimizing measure of a Tonelli Hamiltonian flow, two times the almost everywhere dimension of $G_- \cap G_+$ is equal to the number of zero Lyapunov exponents. The result is valid for twist maps too.

For general dynamical systems, one "inequality" between the angles of the Oseledet's splitting and the Lyapunov exponents is well-known; roughly speaking, the smaller the angle/distance between E^s and E^u is, the closer to zero the Lyapunov exponents are. This will be recalled in section 3. In this section too, we will prove two exact formulas linking the distance between the two Green bundles and the Lyapunov exponents of the minimizing measures of the conservative dynamics. They are contained in the following theorems.

Theorem 1. *Let μ be a Borel probability measure with no conjugate points that is ergodic for a Tonelli Hamiltonian flow. If G_+ is the graph of $\mathbb{U}$ and G_- the graph of $\mathbb{S}$, the sum of the positive Lyapunov exponents of μ is equal to:*

$$\Lambda_+(\mu) = \frac{1}{2}\int \mathrm{tr}(\frac{\partial^2 H}{\partial p^2}(\mathbb{U} - \mathbb{S}))d\mu.$$

Theorem 1 is a slight improvement of a theorem of A. Freire and R. Mané concerning the geodesic flows that is contained in [10] (see [9] and[7] too). A similar statement was given in the (non published) thesis of G. Kniepper.
Theorem 2 gives a similar statement for the twist maps. In this statement, $G_k(x) = Df^k(f^{-1}(x))V(x)$ is some image of the vertical $V(x)$ that will be precisely defined in section 2.

Theorem 2. *Let $f : \mathbb{A}_d \to \mathbb{A}_d$ be a twist map and let μ be a locally minimizing ergodic measure with compact support . Then, if $\Lambda(\mu)$ is the sum of the non-negative exponents of μ, if S_-, S_+ designate the symmetric matrices whose graphs are the two Green bundles G_- and G_+ and S_k designates the symmetric matrix whose graph is G_k, we have:*

$$\Lambda(\mu) = \frac{1}{2}\int \log\left(\frac{\det\left(S_+(x) - S_{-1}(x)\right)}{\det\left(S_-(x) - S_{-1}(x)\right)}\right) d\mu(x).$$

For general dynamics, there is no inequality in the other sense. More precisely, the distance between the stable and unstable bundles can be big for measures having Lyapunov exponents that are close to zero. We will see that this phenomenon cannot happen for conservative twisting dynamics. In the following theorems, we denote by $q_+(S)$ the smallest positive eigenvalue of a semi-positive non-zero matrix S and we use the same notation as in theorem 1 for $\mathbb{U}$ and $\mathbb{S}$.

Theorem 3. *Let μ be an ergodic measure with no conjugate points and with at least one non zero Lyapunov exponent for the Tonelli Hamiltonian flow of $H : T^*M \to \mathbb{R}$; then its smallest positive Lyapunov exponent $\lambda(\mu)$ satisfies: $\lambda(\mu) \geq \frac{1}{2}\int q_+(\frac{\partial^2 H}{\partial p^2}).q_+(\mathbb{U}-\mathbb{S})d\mu$.*

Hence, the gap between the two Green bundles gives a lower bound of the smallest positive Lyapunov exponent.
For the conservative twist maps, we obtain a similar inequality when all the Lyapunov exponents are non-zero. In this case, the two Green bundles are nothing else but the stable and unstable bundles.

Theorem 4. *Let $f : \mathbb{A}_d \to \mathbb{A}_d$ be a symplectic twist map and let μ be a locally minimizing ergodic measure with no zero Lyapunov exponents. We denote the smallest positive Lyapunov exponent of μ by $\lambda(\mu)$ and an upper bound for $\|s_1 - s_{-1}\|$ above* suppμ *by C. Then we have:*

$$\lambda(\mu) \geq \frac{1}{2}\int \log\left(1 + \frac{1}{C}q_+(\mathbb{U}(x) - \mathbb{S}(x))\right) d\mu(x).$$

J.-C. Yoccoz pointed to me the following illustration of this last result. Let us consider a minimizing fixed point x_0 of a two-dimensional twist map $f : \mathbb{A}_1 \to \mathbb{A}_1$. At such a minimizing fixed point, Df has necessarily two positive eigenvalues denoted by λ and $\frac{1}{\lambda}$. Let us denote the matrix of $Df(x_0)$ in the usual coordinates by : $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The twist condition gives a constant $\alpha > 0$ such that $b \geq \alpha$. If $Df(x_0)$ is bounded by a constant C, this implies that N cannot be to close to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence N cannot have simultaneously two different eigenvalues close to 1 and a big distance between its eigenspaces. Hence we understand for this example the result contained in the last theorem.

Let us comment forward about related results. In [3], we proved some results concerning the invariant probability measures of a 2-dimensional twist map whose

support is an irrational Aubry-Mather set $\mathcal{A}$. We defined at each point x of such an Aubry-Mather set its Bouligand's paratingent cone $C_x\mathcal{A}$, that is a kind of generalized tangent bundle for sets that are not manifolds. We can identify $C_x\mathcal{A}$ with the set $\mathcal{S}_x\mathcal{A}$ of the slopes of its vectors. Then, if g_- and g_+ designate the slopes of the two Green bundles, we proved the following inequality : $g_-(x) \leq \mathcal{S}_x \leq g_+(x)$. From that and from theorem 4, we deduce :
The more irregular the Aubry-Mather set is, i.e. the bigger its paratingent cone is, the bigger the Lyapunov exponents are.

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2 Some results about the Green bundles

NOTATIONS. We assume that M is a compact and connected d-dimensional manifold endowed with a fixed Riemannian metric (the associated scalar product is denoted by $(.|.)$). We denote a point of its cotangent bundle T^*M by (q,p) where $p \in T_q^*M$. If q are some (local) coordinates on M, then p designate the dual coordinates. This means that if $\eta \in T^*M$ and $\eta = \sum \eta_i dq_i$, then $p_i = \eta_i$.
Let us recall that T^*M can be endowed with a 1-form λ called the Liouville 1-form, whose expression in all dual coordinates is $\lambda(q,p) = \sum p_i dq_i$. Then the canonical symplectic form ω is defined on M by $\omega = -d\lambda$. All the dual coordinates are symplectic for ω.
We will denote the usual projection from T^*M to M by $\pi : T^*M \to M$. For every $x = (q,p) \in T^*M$, we will denote the vertical by $V(x) = \ker(D\pi(x))$. It is a Lagrangian linear subspace of $T_x(T^*M)$.
When $M = \mathbb{T}^d$ we will use the global coordinates of $\mathbb{A}_d = \mathbb{T}^d \times \mathbb{R}^d$.

2.1 Comparison of two Lagrangian subspaces that are transverse to the vertical

Let us recall that a d-dimensional subspace G of $T_x(T^*M)$ that is transverse to the vertical $V(x)$ is Lagrangian if and only for every dual linear coordinates $(\delta q, \delta p)$ of $T_x(T^*M)$, then G is the graph of a symmetric matrix in these coordinates.

We defined in [1] an order relation for such Lagrangian subspaces of $T_x(T^*M)$ that are transverse to the vertical. The definition is intrinsic and doesn't depend on the

chosen dual coordinates, but let us recall its interpretation in terms of symmetric matrices: we say that the graph of ℓ_1 is *above* (resp. *strictly above*) the graph of ℓ_2 if the symmetric matrix $\ell_1 - \ell_2$ is a positive semi-definite (resp. definite) matrix.
We need to be more precise. Let us recall what we did in [1]; we associated its *height* $Q(S,U)$ to each pair (S,U) of Lagrangian linear subspaces of $T_x(T^*M)$ that are transverse to the vertical. This height is a quadratic form defined on the quotient linear space $T_x(T^*M)/V(x)$. As this last space is canonically isomorphic to T_qM if $q = \pi(x)$, we modify slightly the set where this quadratic form is defined in comparison with [1][2]: if U, S are two Lagrangian linear subspaces of $T_x(T^*M)$ that are transverse to the vertical $V(x)$, the relative height between S and U is the quadratic form $q(S,U)$ defined on T_qM by the following way:

if $\delta q \in T_qM$, if $\delta x_U \in U$ (resp. $\delta x_S \in S$) is the vector of U (resp. S) such that $D\pi(\delta x_U) = \delta q$ (resp. $D\pi(\delta x_S) = \delta q$), then we have: $q(S,U)(\delta q) = \omega(\delta x_S, \delta x_U)$.

Of course, this definition doesn't depend on the dual coordinates that we choose. We associate to this bilinear form a unique symmetric operator $s(S,U) : T_qM \to T_qM$ defined by: $q(S,U)(\delta q_1, \delta q_2) = (s(S,U)\delta q_1 | \delta q_2)$. The operator $s(S,U)$ depends only on the Riemannian product $(.|.)$. Hence, the eigenvalues of $s(S,U)$ are intrinsically defined. We denote them by: $\lambda_1(S,U) \leq \cdots \leq \lambda_d(S,U)$.

DEFINITION. The quadratic form $q(S,U) : T_qM \to \mathbb{R}$ is called the *height* of U above S. The numbers $\lambda_1(S,U) \leq \cdots \leq \lambda_d(S,U)$ are the *characteristic numbers* of U above S.

Let us recall some properties that are proved in [1].

Proposition 5. *Let L_1, L_2 and L_3 be three Lagrangian subspaces of $T_x(T^*M)$ that are transverse to the vertical. Then:*

1. $\ker q(L_1, L_2) = D\pi(L_1 \cap L_2)$;
2. $q(L_1, L_2) = -q(L_2, L_1)$;
3. $q(L_1, L_2) + q(L_2, L_3) = q(L_1, L_3)$.

DEFINITION. The *distance* between S and U is then
$\Delta(S,U) = \|q(S,U)\| = \max_{\|\delta q_i\|=1, i=1,2} \omega(\delta x_S^1, \delta x_U^2)$ where δx_U^i (resp. δx_S^i) designates the element of U (resp. S) whose projection on T_qM is δq_i.

[2]We thank F. Laudenbach for this suggestion.

Let us notice that $\Delta(S,U)$ is not symplectically invariant.

REMARK.There is a relationship between the distance $\Delta(S,U)$ and the characteristic numbers: $\Delta(S,U) = \max\{|\lambda_1|, |\lambda_d|\}$.

2.2 Tonelli Hamiltonians

We recall some well-known facts concerning Hamiltonian and Lagrangian dynamics (see [5], [8]).
DEFINITION. A C^2 function $H : T^*M \to \mathbb{R}$ is called a *Tonelli Hamiltonian* if it is:

• superlinear in the fiber, i.e. $\forall A \in \mathbb{R}, \exists B \in \mathbb{R}, \forall (q,p) \in T^*M, \|p\| \geq B \Rightarrow H(q,p) \geq A\|p\|$;
• C^2-convex in the fiber i.e. for every $(q,p) \in T^*M$, the Hessian $\frac{\partial^2 H}{\partial p^2}$ of H in the fiber direction is positive definite as a quadratic form.
We denote the Hamiltonian flow of H by (φ_t) and the Hamiltonian vector-field by X_H.
A *Lagrangian function* $L : TM \to \mathbb{R}$ is associated with H. It is defined by
$L(q,v) = \max_{p \in T_q^*M} (p.v - H(q,p))$.

Then L is C^2-convex and superlinear in the fiber and has the same regularity as H. We denote its Euler-Lagrange flow by (f_t). Then (φ_t) and (f_t) are conjugated by the Legendre diffeomorphism $\mathcal{L} : (q,p) \in T^*M \to (q, \frac{\partial H}{\partial p}(q,p)) \in TM$; more precisely, we have $\mathcal{L} \circ \varphi_t = f_t \circ \mathcal{L}$.
Let us recall that the orbit of $x \in T^*M$ is $(x_t) = (\varphi_t x)_{t \in \mathbb{R}}$. An *infinitesimal orbit* along the orbit (x_t) is then $(D\varphi_t.\delta x)_{t \in \mathbb{R}}$ where $\delta x \in T_x(T^*M)$. Such an infinitesimal orbit is a solution of the linearized Hamilton equations along the orbit (x_t).

The Lagrangian action $A_L(\gamma)$ of a C^1 arc $\gamma : [a,b] \to M$ is defined by:

$$A_L(\gamma) = \int_a^b L(\gamma(s), \dot{\gamma}(s)) ds.$$

A C^1 arc $\gamma_0 : [a,b] \to M$ is *minimizing* (resp. *locally minimizing*) if for every C^1 arc $\gamma : [a,b] \to M$ that has the same endpoints as γ_0, i.e. such that $\gamma_0(a) = \gamma(a)$ and $\gamma_0(b) = \gamma(b)$ (resp. that has the same endpoints as γ_0, i.e. such that $\gamma_0(a) = \gamma(a)$ and $\gamma_0(b) = \gamma(b)$ and that is sufficiently close to γ_0 for the C^1-topology), we have: $A_L(\gamma_0) \leq A_L(\gamma)$. Such a minimizing (resp. locally minimizing) arc is the projection of a unique piece of orbit of the Hamiltonian flow (and then of the Lagrangian flow too). We will say that the corresponding piece of orbit $(\varphi_t(x))_{t \in [a,b]}$ is minimizing (resp. locally minimizing). We say that a complete orbit is minimizing (resp. locally minimizing) if all its restrictions to compact intervals are minimizing (resp. locally minimizing). J. Mather proved at the end of the 80's (see [15]) that there always exist

some minimizing orbits. More precisely, he proved the existence of minimizing measures, i.e. Borel probability invariant measures of T^*M whose support is filled with minimizing orbits.

It is well-known that an orbit $(x_t) = (\varphi_t x)$ is locally minimizing if and only if it has no conjugate points. This means that $\forall t \neq u, D\varphi_{t-u}V(x_u) \cap V(x_t) = \{0\}$. At every point y of such a minimizing orbit, the family $(G_t(y)) = (D\varphi_t.V(\varphi_{-t}y))_{t>0}$ (resp. $(G_{-t}(y)) = (D\varphi_{-t}.V(\varphi_t y))_{t>0}$ is a decreasing (resp. increasing) family of Lagrangian subspaces that are transverse to the vertical $V(y)$ (see [7], [13] or [1]) and for every $t > 0$, $G_{-t}(y)$ is strictly under $G_t(y)$. Then we define the two Green bundles by

$$G_-(y) = \lim_{t\to+\infty} G_{-t}(y) \quad \text{and} \quad G_+(y) = \lim_{t\to+\infty} G_t(y).$$

They are transverse to the vertical, between all the G_{-t} and G_t and G_+ is above G_- (see [1] for details).

As at the end of the introduction, let us assume now that there exists along a locally minimizing orbit either an Oseledet's splitting or a partially hyperbolic splitting. We denote the stable, unstable and center bundles corresponding to this splitting by E^s, E^u an E^c. It is proved in [4] and [3] that:

$$E^s \oplus \mathbb{R}X_H \subset G_- \subset E^s \oplus E^c \quad \text{and} \quad E^u \oplus \mathbb{R}X_H \subset G_+ \subset E^u \oplus E^c.$$

2.3 Twist maps

The main part of this subsection comes from [6] (see [11] too), even if we changed some proofs. All of what concerns the comparison between the two Green bundles is new. We consider a C^2-function $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that:

1. Φ is $\mathbb{Z}^d$-periodic, i.e: $\forall k \in \mathbb{Z}^d, \forall (q, Q) \in \mathbb{R}^d \times \mathbb{R}^d, \Phi(q + k, Q + k) = \Phi(q, Q)$;
2. Φ satisfies the uniform twist condition, i.e there exists $K > 0$ such that:

$$\forall \zeta \in \mathbb{R}^d, \sum_{i,j} \frac{\partial^2 \Phi(q, Q)}{\partial q_i \partial Q_j} \zeta_i \zeta_j \leq -K\|\zeta\|^2.$$

Then, if we denote the derivative with respect to the q_i, Q_j variables by Φ_1 and Φ_2 respectively, the following implicit formula defines a symplectic diffeomorphism $\tilde{f}$ of $\mathbb{R}^d$:

$$\tilde{f}(q, p) = (Q, P) \quad \text{where} \quad P = \Phi_2(q, Q) \quad \text{and} \quad p = -\Phi_1(q, Q).$$

We say then that Φ is a *generating function* for $\tilde{f}$. We associate a formal function defined on $(\mathbb{R}^d)^{\mathbb{Z}}$ to Φ:

$$\mathcal{A}((q_n)_{n\in\mathbb{Z}}) = \sum_{n=-\infty}^{+\infty} \Phi(q_n, q_{n+1}).$$

Even if this function is not well-defined, its critical points are well-defined, they satisfy the equations:

$$\forall n \in \mathbb{Z}, \Phi_2(q_{n-1}, q_n) + \Phi_1(q_n, q_{n+1}) = 0.$$

We can denote the partial actions $\mathcal{A}_{M,N}$ for $M \leq N$ by:

$$\mathcal{A}_{M,N}((q_n)_{M\leq n\leq N}) = \sum_{n=M}^{N-1} \Phi(q_n, q_{n+1}).$$

Then $(q_n)_{M\leq n\leq N}$ is a critical point of $\mathcal{A}_{M,N}$ restricted to the set of the finite sequences that have the same endpoints as $(q_n)_{M\leq n\leq N}$ if and only if it is a the projection of a finite piece of orbit $(q_n, p_n)_{M\leq n\leq N}$ for $\tilde{f}$. In this case, we have:

- $p_M = -\Phi_1(q_M, q_{M+1})$; $p_N = \Phi_2(q_{N-1}, q_N)$;
- $\forall n \in [M+1, N-1], p_n = \Phi_2(q_{n-1}, q_n) = -\Phi_1(q_n, q_{n+1})$.

We say that $(q_n)_{M\leq n\leq N}$ is *minimizing* (resp. *locally minimizing*) if it is minimizing (resp. locally minimizing) among all the segments that have the same endpoints. Then the corresponding piece of orbit $(q_n, p_n)_{M\leq n\leq N}$ is said to be minimizing (resp. locally minimizing) too. We say that $(q_n)_{n\in\mathbb{Z}}$ or $(q_n, p_n)_{n\in\mathbb{Z}}$ is minimizing (resp. locally minimizing) if all its restrictions to segments are minimizing (resp. locally minimizing).

If now $(x_n) = (q_n, p_n) \in (\mathbb{A}_d)^{\mathbb{Z}}$ is an orbit for f, we say that it is *minimizing* (resp. *locally minimizing*) if its lifted orbit $(\tilde{q}_n, p_n)$ for $\tilde{f}$ is minimizing. Moreover, we will denote the partial action of the lift by:

$$\Phi_{N,M}((q_n)) = \mathcal{A}_{N,M}((\tilde{q}_n)).$$

Let us now fix an orbit $(x_n) = (q_n, p_n)$ for f. We call an *infinitesimal orbit* along (x_n) a sequence $(Df^n(x_0)\delta x)_{n\in\mathbb{Z}}$, i.e. an infinitesimal orbit is an orbit for the derivative of f. The projection of an infinitesimal orbit is called a *Jacobi field*. Then (ζ_n) is a Jacobi field if and only if we have:

$$\forall n \in \mathbb{Z}, {}^t b_{n-1}\zeta_{n-1} + a_n\zeta_n + b_n\zeta_{n+1} = 0;$$

where $b_n = \Phi_{12}(q_n, q_{n+1})$ and $a_n = \Phi_{11}(q_n, q_{n+1}) + \Phi_{22}(q_{n-1}, q_n)$.
The Hessian of $\Phi_{M,N}$ is:

$$D^2\Phi_{M,N}((x_n)) = \begin{pmatrix} a_M & b_M & 0 & \dots & \dots & \dots & 0 \\ {}^t b_M & a_{M+1} & b_{M+1} & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & a_{N-1} & b_{N-1} \\ 0 & \dots & \dots & \dots & 0 & {}^t b_{N-1} & a_N \end{pmatrix}.$$

The kernel of this Hessian is made with the Jacobi fields $(\zeta_n)_{M\leq n\leq N}$ such that $\zeta_{M-1} = \zeta_{N+1} = 0$.
If we assume that (x_n) is locally minimizing, then all the Hessians $D^2\Phi_{M,N}((x_n))$ are a priori positive semi-definite. Following [6], let us prove that these Hessians are in fact positive definite.

Proposition 6. (Bialy-MacKay, [6]) *If the orbit (x_n) of f is locally minimizing, then all the Hessians $D^2\Phi_{M,N}((x_n))$ are positive definite and then the orbit has no conjugate vectors.*

PROOF If not, there exist $M \leq N$ and a Jacobi field $(\zeta_n)_{n\in\mathbb{Z}}$ that is different from (0) but such that $\zeta_{M-1} = \zeta_{N+1} = 0$. In other words, this Jacobi field has what is usually called *conjugate vectors*. In this case, $(0, 0, \zeta_M, \zeta_{M+1}, \dots, \zeta_{N-1}, \zeta_N, 0, 0)$ is in the isotropic cone of $D^2\Phi_{M-2,N+2}((x_n))$ but not in its kernel (because it is not a Jacobi field); this contradicts the fact that the kernel is equal to the isotropic cone (because this Hessian is positive semi-definite). □

Hence the Jacobi fields along any locally minimizing orbit have no conjugate vectors. This implies that for any $k \in \mathbb{Z}^*$ and any $n \in \mathbb{Z}$, $G_k(x_{n+k}) = Df^k(x_n).V(x_n)$ is transverse to $V(x_{n+k}) = V(f^k x_n)$.

Proposition 7. (Bialy-MacKay, [6]) *Let (x_k) be a locally minimizing orbit. Then, for all $k \geq 1$, we have along this orbit:*

- *G_{-1} is strictly under G_k and G_{-k} is strictly under G_1;*
- *G_{k+1} is strictly under G_k and G_{-k} is strictly under $G_{-(k+1)}$.*

This result is proved in [6], but we give a slightly different proof.

We deduce that $(G_k)_{k\geq 1}$ is a decreasing sequence of Lagrangian subspaces that are all above G_{-1}, hence we can define $G_- = \lim_{k\to+\infty} G_k$. Similarly, $(G_{-k})_{k\geq 1}$ is an increasing sequence of Lagrangian subspaces that are all under G_1, hence we can define G_+ by taking the limit.

DEFINITION. If the orbit of x is locally minimizing, the two *Green bundles* at x are the two Lagrangian subspaces of $T_x(T^*M)$ that are transverse to the vertical and defined by:

$$G_-(x) = \lim_{k\to+\infty} G_{-k}(x) \quad \text{and} \quad G_+(x) = \lim_{k\to+\infty} G_k(x).$$

PROOF We denote the symmetric matrix whose graph is $G_k(x_{n+k})$ by $S_k(x_{n+k})$. Let us notice that:

$$Df(x_n) = \begin{pmatrix} -b_n^{-1}\Phi_{11}(q_n, q_{n+1}) & -b_n^{-1} \\ {}^t b_n - \Phi_{22}(q_n, q_{n+1})b_n^{-1}\Phi_{11}(q_n, q_{n+1}) & -\Phi_{22}(q_n, q_{n+1})b_n^{-1} \end{pmatrix}.$$

We deduce that: $G_1(x_{n+1}) = \text{graph}(\Phi_{22}(q_n, q_{n+1}))$, $G_{-1}(x_n) = \text{graph}(-\Phi_{11}(q_n, q_{n+1}))$ and then $S_1(x_n) = \Phi_{22}(q_{n-1}, q_n)$, $S_{-1}(x_n) = -\Phi_{11}(q_n, q_{n+1})$. Hence: $a_n = S_1(x_n) - S_{-1}(x_n)$ is the matrix of the relative height between $G_{-1}(x_n)$ and $G_1(x_n)$ (see subsection 2.1 for definition). Hence G_1 is strictly above G_{-1}.

Let us prove that: $\forall k \geq 1, S_k(x_n) - S_{-1}(x_n) > 0$. If not, there exists $k \geq 2$ and $\eta \neq 0$ such that: ${}^t\eta(S_k(x_n) - S_{-1}(x_n))\eta \leq 0$. Then we consider the piece of infinitesimal orbit $\left(Df^{-j}\begin{pmatrix} \eta \\ S_k(x_n)\eta \end{pmatrix}\right)_{0 \leq j \leq k-1}$ and the Jacobi field that is the projection of this infinitesimal orbit: $\zeta_i = D\pi \circ Df^{i-n}\begin{pmatrix} \eta \\ S_k(x_n)\eta \end{pmatrix}$ for $n-k+1 \leq i \leq n$. Let us compute $D^2\Phi_{n-k+1,n}(x)\zeta = \Delta$.

1. as $Df^{-k}G_k(x_n) = V(x_{n-k})$, we have: $\Delta_{n-k+1} = a_{n-k+1}\zeta_{n-k+1} + b_{n-k+1}\zeta_{n-k+2} = -{}^t b_{n-k}\zeta_{n-k} = 0$;
2. as we have a Jacobi field, for $n-k+1 \leq i \leq n-2$, we have: $\Delta_{i+1} = {}^t b_i\zeta_i + a_{i+1}\zeta_{i+1} + b_{i+1}\zeta_{i+2} = 0$;
3. $\Delta_n = {}^t b_{n-1}\zeta_{n-1} + a_n\zeta_n = -b_n D\pi \circ Df\begin{pmatrix} \eta \\ S_k(x_n)\eta \end{pmatrix}$
$= -b_n\left(-b_n^{-1}(\Phi_{11}(q_n, q_{n+1}) - S_k(x_n))\right)\eta = -(S_{-1}(x_n) - S_k(x_n))\eta.$

We deduce that $D^2\Phi_{n-k+1,n}(x)(\zeta,\zeta) = {}^t\Delta.\zeta = {}^t\eta(S_k(x_n) - S_{-1}(x_n))\eta \leq 0$. This contradicts the fact that the Hessian is positive definite. Hence we have proved that for all positive k, G_k is strictly above G_{-1}.

Moreover, $G_{k+1}(x_{n+1})$ is represented by:

$$Df(x_n)\begin{pmatrix} \mathbf{1} \\ S_k(x_n) \end{pmatrix} = \begin{pmatrix} -b_n^{-1}(\Phi_{11}(q_n, q_{n+1}) + S_k(x_n)) \\ {}^t b_n - \Phi_{22}(q_n, q_{n+1})b_n^{-1}(\Phi_{11}(q_n, q_{n+1}) + S_k(x_n)) \end{pmatrix}.$$

This means: $S_{k+1}(x_{n+1}) = -{}^t b_n(\Phi_{11}(q_n, q_{n+1}) + S_k(x_n))^{-1}b_n + \Phi_{22}(q_n, q_{n+1})$ and then:
$(S_{k+1} - S_{-1})(x_{n+1}) = a_{n+1} - {}^t b_n((S_k - S_{-1})(x_n))^{-1}b_n$ i.e.:
$(S_{k+1} - S_{-1})(x_{n+1}) = (S_1 - S_{-1})(x_{n+1}) - {}^t b_n((S_k - S_{-1})(x_n))^{-1}b_n$.
In particular, we have: $(S_2 - S_{-1})(x_{n+1}) = (S_1 - S_{-1})(x_{n+1}) - {}^t b_n a_n^{-1}b_n$ then $S_2 < S_1$. We can subtract for any $k \geq 2$:

$$(S_{k+1} - S_k)(x_{n+1}) = {}^t b_n\left((S_{k-1} - S_{-1})(x_n))^{-1} - (S_k - S_{-1})(x_n)^{-1}\right)b_n.$$

We have proved that for all positive k, G_k is strictly above G_{-1}. We deduce that $(G_k(x_n))_{k\geq 1}$ is a strictly decreasing sequence of Lagrangians subspaces. Because all

these subspaces are above $G_{-1}(x_n)$, they converge to a Lagrangian subspace G_+ that is transverse to the vertical. In the same way, we obtain that $(G_{-k}(x_n)_{k\geq 0}$ is an increasing sequence of Lagrangian subspaces that are bounded from above by G_1, hence they converge to a Lagrangian subspace G_- that is transverse to the vertical. □

Proposition 8. *Let $x \in T^*M$ whose orbit is locally minimizing. Then for all $n, k \geq 1$, $G_{-k}(x)$ is strictly under $G_n(x)$.*
Hence G_- is under G_+.

Proof We denote $f^m(x)$ by x_m. Let us prove that for all $n, k \geq 1$, and all $m \in \mathbb{Z}$, then $G_n(x_m)$ is above $G_{-k}(x_m)$. We have proved this result for $k = 1$ or $n = 1$, then we assume that $n, k \geq 2$.
We recall that if F_1, F_2 are two transverse Lagrangian subspaces of a symplectic space whose dimension is denoted by $2d$, then the set $\mathcal{T}(F_1, F_2)$ of the Lagrangian subspaces that are transverse to both L_1 and L_2 has exactly $d + 1$ connected components: it depends on the signature of a certain quadratic form. Let us consider the connected component $\mathcal{C}$ of $\mathcal{T}(G_{k-1}(x_{k-1+m}), G_{k-1+n}(x_{k-1+m}))$ that contains $G_{k+n}(x_{k-1+m})$; we have proved that $G_{k+n}(x_{k-1+m})$ and $G_{-1}(x_{k-1+m})$ are under $G_{k-1}(x_{k-1+m})$ and $G_{k-1+n}(x_{k-1+m})$, hence they are in the same connected component $\mathcal{C}$ of $\mathcal{T}(G_{k-1}(x_{k-1+m}), G_{k-1+n}(x_{k-1+m}))$ and their images by $\left(Df^{k-1}(x_m)\right)^{-1}$, that are $G_{n+1}(x_m)$ and $G_{-k}(x_m)$, are in the same connected component of

$$\left(Df^{k-1}(x_m)\right)^{-1} \left(\mathcal{T}(G_{k-1}(x_{k-1+m}), G_{k-1+n}(x_{k-1+m}))\right).$$

This last set is equal to:
$\mathcal{T}(\left(Df^{k-1}(x_m)\right)^{-1} (G_{k-1}(x_{k-1+m}), \left(Df^{k-1}(x_m)\right)^{-1} (G_{k-1+n}(x_{k-1+m})))$
$= \mathcal{T}(V(x_m), G_n(x_m))$.
We have proved that $G_{n+1}(x_m)$ is under $G_n(x_m)$. As $G_{n+1}(x_m)$ and $G_{-k}(x_m)$ are in the same connected component of $\mathcal{T}(V(x_m), G_n(x_m))$, this implies that $G_{-k}(x_m)$ is under $G_n(x_m)$.
We deduce that G_- is under G_+. □

3 Sum of the positive Lyapunov exponents and upper bounds

Before explaining which results we obtain for the twisting dynamics, we have to explain that some results are true for general dynamics (not necessarily twisting) and explain the difference with our results.

3.1 Some general results

In this section, we review some more or less well-known results concerning the link between the Lyapunov exponents and the distance between the Oseledet's bundles. Because we didn't find any precise reference and because the proofs are rather short, we give here a proof a these results. A good reference for Lyapunov exponents is [14].

We work on a manifold N (not necessarily compact) and we ask ourselves the following question.

Question. If the Oseledet's splitting of an invariant measure of a C^1-diffeomorphism is such that E^s and E^u are close to each other (in a sense we have to specify), are the Lyapunov exponents all close to 0?

Let us explain that the answer is yes if E^s and E^u are 1-dimensional.

NOTATIONS. If E, F are two linear subspaces of T_xN that are d-dimensional with $d \geq 1$, the distance between E and F is:

$$\mathrm{dist}(E,F) = \inf_{(e_i),(f_i)} \max\{\|e_1 - f_1\|, \ldots, \|e_d - f_d\|\}$$

where the infimum is taken over all the orthonormal basis (e_i) of E, (f_i) of F.

Proposition 9. *Let K be a compact subset of N and let $C > 0$ be a real number. Then, for any $f \in \mathrm{Diff}^1(M)$ so that $\max\{\|Df_{|K}\|, \|Df^{-1}_{|K}\|\} \leq C$, if f has an invariant ergodic measure μ with support in K such that the Oseledet's stable and unstable bundles E^s and E^u of μ are one dimensional, if we denote by λ_u the positive Lyapunov exponent and by λ_s the negative one, then:*

$$0 < \lambda_u - \lambda_s \leq \log\left(1 + C^2 \int \mathrm{dist}(E^u, E^s) d\mu\right).$$

PROOF We denote $\mathrm{dist}(E^u(x), E^s(x))$ by $\alpha(x)$.We choose $x \in \mathrm{supp}\mu$ where E^s and E^u are defined and we choose $v \in E^u(x)\backslash\{0\}$. Then there exists $p_x(v) \in E^s(x)$ such that $\|p_x(v)\| = \|v\|$ and $\|p_x(v) - v\| \leq \alpha(x)\|v\|$. Then, we have:

$$\begin{aligned} \|Df_{|E^u(x)}\|.\|v\| = \|Df(x)v\| &\leq \|Df(x)p_x(v)\| + \|Df(x)\|.\|p_x(v) - v\| \\ &\leq \|Df_{|E^s}(x)\| \left(1 + \alpha(x)\tfrac{\|Df(x)\|}{\|Df_{|E^s}(x)\|}\right) \|v\|. \end{aligned}$$

We deduce:

$$\begin{aligned} \lambda_u - \lambda_s &= \int \log \|Df_{|E^u}\| d\mu - \int \log \|Df_{|E^s}\| d\mu \\ &\leq \int \log(1 + C^2\alpha(x)) d\mu(x) \leq \log\left(1 + C^2 \int \alpha(x) d\mu(x)\right) \end{aligned}$$

by Jensen inequality. □

In the higher dimension cases, we obtain a slightly less good estimation.

Proposition 10. *Let K be a compact subset of N, let $C > 0$ be a real number. Then, for any $f \in \mathrm{Diff}^1(M)$ so that $\max\{\|Df_{|K}\|, \|Df^{-1}_{|K}\|\} \leq C$, if f has an invariant ergodic measure μ with support in K such that the Oseledet's stable and unstable bundles E^s and E^u of μ have the same dimension d , if we denote by Λ_u the sum of the positive Lyapunov exponents and by Λ_s the sum of the negative Lyapunov exponents, then:*

$$0 < \Lambda_u - \Lambda_s \leq d \log\left(1 + (C^2+1)\int \mathrm{dist}(E^u, E^s) d\mu\right).$$

Proof We denote $\mathrm{dist}(E^u(x), E^s(x))$ by $\alpha(x)$. At all the points where E^s and E^u are defined, we choose an orthonormal basis $(e_1, \ldots, e_d)$ of E^s in a measurable way, and an orthogonal basis $(f_1, \ldots, f^d)$ of E^u that depends measurably on the considered point and is such that:

$$\mathrm{dist}(E^s, E^u) = \max\{\|e_1 - f_1\|, \ldots, \|e_d - f_d\|\}.$$

Then we denote by $P_x : E^u(x) \to E^s(x)$ the linear map such that $P_x(f_i(x)) = e_i(x)$; then, each P_x is an isometry. Moreover, because $\|P_x(f_i(x)) - f_i(x)\| = \|f_i(x) - e_i(x)\| \leq \alpha(x)$, we deduce: $\forall v \in E^u(x), \|P_x(v) - v\| \leq d\alpha(x)\|v\|$ and $\forall v \in E^s(x), \|(P_x)^{-1}(v) - v\| \leq d\alpha(x)\|v\|$. We have (we compute the determinant in the previous basis):

$$\begin{array}{rcl} \Lambda_u - \Lambda_s & = & \int \left(\log(|\det Df_{|E^u(x)}|) - \log(|\det Df_{|E^s(x)}|)\right) d\mu(x) \\ & = & \int \log\left|\det(Df_{|E^u(x)}(P_x)^{-1}(Df_{|E^s(x)})^{-1}P_{f(x)})\right| d\mu(x). \end{array}$$

Let us consider $v \in E^u(x)$. Then:

$$\begin{array}{lcl} \|Df(x)(P_x)^{-1} & (Df(x))^{-1}P_{f(x)}v - v\| & \\ & \leq & \left\|Df(x)\left((P_x)^{-1}(Df(x))^{-1}P_{f(x)}v - (Df(x))^{-1}P_{f(x)}v\right)\right\| + \left\|P_{f(x)}v - v\right\| \\ & \leq & Cd\alpha(x)\|(Df(x))^{-1}P_{f(x)}v\| + d\alpha(f(x))\|v\| \leq d(C^2\alpha(x) + \alpha(f(x)))\|v\|. \end{array}$$

Hence we have: $\|Df(x)(P_x)^{-1}(Df(x))^{-1}P_{f(x)} - \mathrm{Id}_{E^u(x)}\| \leq d(C^2\alpha(x) + \alpha(f(x)))$. We deduce:

$$\left|\det\left(Df_{|E^u(x)}(P_x)^{-1}(Df_{|E^s(x)})^{-1}P_{f(x)}\right)\right| \leq \left(1 + d(C^2\alpha(x) + \alpha(f(x)))\right)^d$$

and then:

$$\begin{array}{rcl} \Lambda_u - \Lambda_s & \leq & d\int \log\left(1 + d(C^2\alpha(x) + \alpha(f(x)))\right) d\mu(x) \\ & \leq & d\log\left(1 + (C^2+1)\int \alpha(x) d\mu(x)\right). \end{array}$$

□

Hence the fact that we will obtain a upper bound of the sum of the positive Lyapunov exponents that depends on the distance between the two Green bundles in the case of the twisting dynamics is not surprising. The results contained in section 4, that give lower bounds that are specific to the twisting dynamics, are more surprising.
What is more interesting in this section is that we obtain some exact formula for the sum of the positive Lyapunov exponents.

3.2 Tonelli Hamiltonians

Using a Riemannian metric on M, we define the horizontal subspace $\mathcal{H}$ as the kernel of the connection map. Then, for every Lagrangian subspace $\mathcal{G}$ of $T_x(T^*M)$, there exists a linear map $G : \mathcal{H}(x) \to V(x)$ whose graph is $\mathcal{G}$. That is the meaning of *graph* in the following theorem.
REMARK.If K is an invariant compact and locally minimizing subset of T^*M (for example the support of a locally minimizing ergodic measure), we have:

$$\forall x \in K, G_{-1}(x) \leq G_-(x) \leq G_+(x) \leq G_1(x)$$

where "$\leq$" designates the relation "to be below" for the Lagrangian subspaces that are transverse to the vertical.
Hence G_- and G_+ are uniformly bounded on K and $D\pi : G_\pm(x) \to T_xM$ is uniformly bilipschitz.

In the case of ergodic measures of a geodesic flow with support filled by locally minimizing orbits, i.e. in the case of measures with no conjugate points, A. Freire and R. Mané proved in [10] a nice formula for the sum of the positive exponents (see [9] and [7] too). A slight improvement of this formula gives:

Theorem. 1 *Let μ be a Borel probability measure with no conjugate points that is ergodic for a Tonelli Hamiltonian flow. If G_+ is the graph of $\mathbb{U}$ and G_- the graph of $\mathbb{S}$, the sum of the positive Lyapunov exponents of μ is equal to:*

$$\Lambda_+(\mu) = \frac{1}{2} \int \mathrm{tr}(\frac{\partial^2 H}{\partial p^2}(\mathbb{U} - \mathbb{S}))d\mu.$$

Hence, we see that the more distant the Green bundles are, the greater the sum of the positive Lyapunov exponents is. This gives an upper bound to the positive Lyapunov exponents.

PROOF A consequence of the linearized Hamilton equations is that if the graph $\mathcal{G}$ of a symmetric matrix G is invariant by the linearized flow, then any infinitesimal orbit $(\delta q, G\delta q)$ satisfies the following equation: $\delta\dot{q} = (\frac{\partial^2 H}{\partial p^2}G + \frac{\partial^2 H}{\partial q \partial p})\delta q$.
Hence, we have: $\frac{d}{dt}\det(D\pi \circ D\varphi_{t|\mathcal{G}}) = \mathrm{tr}(\frac{\partial^2 H}{\partial p^2}G + \frac{\partial^2 H}{\partial q\partial p})\det(D\pi \circ D\varphi_{t|\mathcal{G}})$; we deduce:

$$\begin{aligned}\tfrac{1}{T}\log\det(D\pi \circ D\varphi_{T|\mathcal{G}} \quad & (q,p)) \\ & = \tfrac{1}{T}\log\det(D\pi(q,p)_{|\mathcal{G}}) + \tfrac{1}{T}\int_0^T \mathrm{tr}(\tfrac{\partial^2 H}{\partial p^2}G + \tfrac{\partial^2 H}{\partial q\partial p})(\varphi_t(q,p))dt.\end{aligned}$$

Via ergodic Birkhoff's theorem, we deduce for (q,p) generic that:

$$\liminf_{T\to+\infty}\frac{1}{T}\log\det(D\pi \circ D\varphi_{T|\mathcal{G}}(q,p)) = \int \mathrm{tr}(\frac{\partial^2 H}{\partial p^2}G + \frac{\partial^2 H}{\partial q\partial p})d\mu.$$

We have noticed that $D\pi_{G_\pm}$ is uniformly bilipschitz above $\mathrm{supp}\mu$, hence we can remove $D\pi$ in the previous formula when $\mathcal{G}$ is one of the two Green bundles.
Moreover, we know that $E^s \subset G_- \subset E^{s\perp} = E^c \oplus E^s$ and that $E^u \subset G_+ \subset E^{u\perp} = E^c \oplus E^u$. Hence, the sum of the Lyapunov exponents of the restricted cocycle $(D\varphi_{t|G_+})$ is exactly $\Lambda_+(\mu)$ and the sum of the Lyapunov exponents of the restricted cocycle $(D\varphi_{t|G_-})$ is $\Lambda_-(\mu) = -\Lambda_+(\mu)$. Then we have:

$$\Lambda_+(\mu) = \int \mathrm{tr}(\frac{\partial^2 H}{\partial p^2}\mathbb{U} + \frac{\partial^2 H}{\partial q\partial p})d\mu \quad \text{and} \quad -\Lambda_+(\mu) = \int \mathrm{tr}(\frac{\partial^2 H}{\partial p^2}\mathbb{S} + \frac{\partial^2 H}{\partial q\partial p})d\mu.$$

We obtain the conclusion by subtracting the two equalities. □

3.3 Twist maps

Theorem. 2 *Let $f : \mathbb{A}_d \to \mathbb{A}_d$ be a twist map and let μ be a locally minimizing ergodic measure with compact support. Then, if $\Lambda(\mu)$ is the sum of the non-negative exponents of μ, if S_-, S_+ designate the symmetric matrices whose graphs are the two Green bundles G_- and G_+ and S_k designates the symmetric matrix whose graph is G_k, we have:*

$$\Lambda(\mu) = \frac{1}{2}\int \log\left(\frac{\det(S_+(x) - S_{-1}(x))}{\det(S_-(x) - S_{-1}(x))}\right)d\mu(x).$$

In this case again, we see that the closer the two Green bundles are to each other, the closer to 0 the Lyapunov exponents are.

PROOF We use coordinates such that G_+ becomes the horizontal bundle, i.e. we use the change of symplectic coordinates whose matrix is $\begin{pmatrix} 1 & S_+ \\ 0 & 1 \end{pmatrix}$. This change of coordinates is not continuous, but it is uniformly bounded because $S_1 \leq S_+ \leq S_1$, and S_{-1} and S_1 vary continuously in the compact set suppμ. The matrix of Df at x is then:

$$M = \begin{pmatrix} B_1(x)(S_+(x) - S_{-1}(x)) & B_1(x) \\ 0 & B_1(x)(S_1(fx) - S_+(fx)) \end{pmatrix}.$$

We know that $E^u \subset G_+ \subset E^u \oplus E^c$, hence along G_+ we see the non-negative Lyapunov exponents. Then we have:

$$\Lambda(\mu) = \int \log \left|\det Df_{|G_+}\right| d\mu = \int \log |\det B_1(x)(S_+(x) - S_{-1}(x))| \, d\mu.$$

In the same way, we have:

$$-\Lambda(\mu) = \int \log \left|\det Df_{|G_-}\right| d\mu = \int \log |\det B_1(x)(S_-(x) - S_{-1}(x))| \, d\mu.$$

By subtracting these two equalities, we obtain the equality of the theorem.

□

4 Lower bounds for the positive Lyapunov exponents

Here we prove results that are specific to the twisting dynamics.

4.1 Tonelli Hamiltonians

Lemma 11. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. Let (x_t) be a locally minimizing orbit and let U and S be two Lagrangian bundles along this orbit that are invariant by the linearized Hamilton flow and transverse to the vertical. Let $\delta x_U \in U$ be an infinitesimal orbit contained in the bundle U and let us denote by δx_S the unique vector of S such that $\delta x_U - \delta x_S \in V$ (hence δx_S is* not *an infinitesimal orbit). Then:*

$$\frac{d}{dt}(\omega(x_t)(\delta x_S(t), \delta x_U(t)) = (\delta x_U(t) - \delta x_S(t)) H_{pp}(x_t)(\delta x_U(t) - \delta x_S(t)) \geq 0.$$

REMARK. Let us notice that $\omega(x_t)(\delta x_S(t), \delta x_U(t))$ is nothing else but the relative height $Q(S, U)$ of U above S at the vector $\delta q_U = D\pi.\delta x_U = D\pi.\delta x_S$.

PROOF As the result that we want to prove is local, we can assume that we are in the domain of a dual chart and express all the things in the corresponding dual linearized coordinates.
We consider an invariant Lagrangian linear bundle G that is transverse to the vertical along the orbit of $x = (q, p)$. We denote the symmetric matrix whose graph is G by G again. An infinitesimal orbit contained in this bundle satisfies: $\delta p = G\delta q$. We deduce from the linearized Hamilton equations (if we are along the orbit $(q(t), p(t)) = x(t)$, $\dot{G}$ designates $\frac{d}{dt}(G(x(t)))$ that:

$$\delta\dot{q} = (H_{qp} + H_{pp}G)\delta q; \quad \delta\dot{p} = (\dot{G} + GH_{qp} + GH_{pp}G)\delta q = -(H_{qq} + H_{pq}G)\delta q.$$

We deduce from these equations the classical Ricatti equation (it is given for example in [7] for Tonelli Hamiltonians, but the reader can find the initial and simpler Ricatti equation given by Green in the case of geodesic flows in [12]):

$$\dot{G} + GH_{pp}G + GH_{qp} + H_{pq}G + H_{pp} = 0.$$

Let us assume now that the graphs of the symmetric matrices $\mathbb{U}$ and $\mathbb{S}$ are invariant by the linearized flow along the same orbit. We denote by $(\delta q_U, \mathbb{U}\delta q_U)$ an infinitesimal orbit that is contained in the graph of $\mathbb{U}$. Then we have:

$$\frac{d}{dt}(\delta q_U(\mathbb{U} - \mathbb{S})\delta q_U) = 2\delta q_U(\mathbb{U} - \mathbb{S})\delta\dot{q}_U + \delta q_U(\dot{\mathbb{U}} - \dot{\mathbb{S}})\delta q_U$$

$$= 2\delta q_U(\mathbb{U}-\mathbb{S})(H_{qp}+H_{pp}\mathbb{U})\delta q_U+\delta q_U(\mathbb{S}H_{pp}\mathbb{S}-\mathbb{U}H_{pp}\mathbb{U}+\mathbb{S}H_{qp}+H_{pq}\mathbb{S}-\mathbb{U}H_{qp}-H_{pq}\mathbb{U})\delta q_U$$

$$= \delta q_U(\mathbb{U}H_{qp} - \mathbb{S}H_{qp} + \mathbb{U}H_{pp}\mathbb{U} - 2\mathbb{S}H_{pp}\mathbb{U} + \mathbb{S}H_{pp}\mathbb{S} + H_{pq}\mathbb{S} - H_{pq}\mathbb{U})\delta q_U$$

$$= \delta q_U(\mathbb{U} - \mathbb{S})H_{pp}(\mathbb{U} - \mathbb{S})\delta q_U \geq 0.$$

To finish the proof, we just need to notice that in coordinates:

$$\omega(\delta x_S, \delta x_U) = \omega(\delta x_U, \delta x_U - \delta x_S) = (\delta q_U, \mathbb{U}\delta q_U)\begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ (\mathbb{U} - \mathbb{S})\delta q_U \end{pmatrix} = \delta q_U(\mathbb{U}-\mathbb{S})\delta q_U$$

□

NOTATIONS. If S is a positive semi-definite matrix that is not the null matrix, then $q_+(S)$ is its smallest positive eigenvalue.

Theorem. 3 *Let μ be an ergodic measure with no conjugate points and with at least one non zero Lyapunov exponent; then its smallest positive Lyapunov exponent $\lambda(\mu)$ satisfies: $\lambda(\mu) \geq \frac{1}{2}\int m(\frac{\partial^2 H}{\partial p^2}).q_+(\mathbb{U}-\mathbb{S})d\mu$.*

Hence, the gap between the two Green bundles gives a lower bound of the smallest positive Lyapunov exponent. It is not surprising that when E^s and E^u collapse, the Lyapunov exponents are 0. What is more surprising and specific to the case of Tonelli Hamiltonians is the fact that the bigger the gap between E^s and E^u is, the greater the Lyapunov exponents are: in general, along a hyperbolic orbit, you may have a big angle between the Oseledet's bundles and some very small Lyapunov exponents.

Proof. Let μ be an ergodic Borel probability measure with no conjugate points; its support K is compact and then, by the first remark of section 3.2, there exists a constant $C > 0$ such that $\mathbb{U}$ and $\mathbb{S}$ are bounded by C above K. We choose a point (q,p) that is generic for μ and $(\delta q, \mathbb{U}\delta q)$ in the Oseledet's bundle corresponding to the smallest positive Lyapunov exponent $\lambda(\mu)$ of μ. Using the linearized Hamilton equations (see lemma 11), we obtain:

$$\frac{d}{dt}((\delta q(\mathbb{U}-\mathbb{S})\delta q) = \delta q(\mathbb{U}-\mathbb{S})\frac{\partial^2 H}{\partial p^2}(q_t,p_t)(\mathbb{U}-\mathbb{S})\delta q.$$

Let us notice that $(\mathbb{U}-\mathbb{S})^{\frac{1}{2}}\delta q$ is contained in the orthogonal space to the kernel of $\mathbb{U}-\mathbb{S}$. Hence:

$$\frac{d}{dt}((\delta q(\mathbb{U}-\mathbb{S})\delta q) \geq m(\frac{\partial^2 H}{\partial p^2})q_+(\mathbb{U}-\mathbb{S})\delta q(\mathbb{U}-\mathbb{S})\delta q.$$

Moreover $\delta q \notin \ker(\mathbb{U}-\mathbb{S})$ because $(\delta q, \mathbb{U}\delta q)$ corresponds to a positive Lyapunov exponent and then $(\delta q, \mathbb{U}\delta q) \notin G_- \cap G_+$. Then :

$$\begin{array}{ll}\frac{2}{T}\log(\|\delta q(T)\|)+ & \frac{\log 2C}{T} \geq \frac{1}{T}\log(\delta q(T)(\mathbb{U}-\mathbb{S})(q_T,p_T)\delta q(T)) \geq \\ & \frac{1}{T}\log(\delta q(0)(\mathbb{U}-\mathbb{S})(q,p)\delta q(0)) + \frac{1}{T}\int_0^T m(\frac{\partial^2 H}{\partial p^2}(q_t,p_t))q_+((\mathbb{U}-\mathbb{S})(q_t,p_t))dt.\end{array}$$

Using Birkhoff's ergodic theorem, we obtain:

$$\lambda(\mu) \geq \frac{1}{2}\int m(\frac{\partial^2 H}{\partial p^2})q_+(\mathbb{U}-\mathbb{S})d\mu.$$

□

4.2 Twist maps: the weakly hyperbolic case

Theorem. 4 *Let $f : \mathbb{A}_d \to \mathbb{A}_d$ be a symplectic twist map and let μ be a locally minimizing ergodic measure with no zero Lyapunov exponents. We denote the smallest Lyapunov exponent of μ by $\lambda(\mu)$ and an upper bound for $\|s_1 - s_{-1}\|$ above* $\mathrm{supp}\mu$ *by C. Then we have:*

$$\lambda(\mu) \geq \frac{1}{2} \int \log \left(1 + \frac{1}{C} m(\mathbb{U}(x) - \mathbb{S}(x)) \right) d\mu(x).$$

Proof We assume then that $(x_n) = (q_n, p_n)$ is a generic orbit for μ. Hence there exists $v_0 \in G_+(x_0) \backslash \{0\}$ such that:

$$\lim_{n \to \infty} \frac{1}{n} \log \left(\|Df^n(x_0)v_0\| \right) = \lambda(\mu).$$

The Lagrangian bundles G_- and G_+ being transverse to the vertical at every point of $\mathrm{supp}\mu$, there exist two symmetric matrices $\mathbb{S}$ and $\mathbb{U}$ such that G_- (resp. G_+) is the graph of $\mathbb{S}$ (resp. $\mathbb{U}$) in the usual coordinates of $\mathbb{R}^d \times \mathbb{R}^d = T_x\mathbb{A}_d$. As G_- and G_+ are transverse μ-almost everywhere, we know that there exists $\varepsilon > 0$ such that $A_\varepsilon = \{x \in \mathrm{supp}\mu; \mathbb{U} - \mathbb{S} \geq \varepsilon \mathbf{1}\}$ has positive μ-measure. We may then assume that $x_0 \in A_\varepsilon$ and that $\{n; \mathbb{U}(x_n) - \mathbb{S}(x_n) > \varepsilon \mathbf{1}\}$ is infinite. Let us notice that in this case, G_- and G_+ are transverse along the whole orbit of x_0 (but $\mathbb{U} - \mathbb{S}$ can be very small at some points of this orbit).
Hence, for every $n \in \mathbb{N}$, there exists a unique positive definite matrix $S_0(x_n)$ such that: $S_0(x_n)^2 = \mathbb{U}(x_n) - \mathbb{S}(x_n)$. Let us recall that a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of dimension $2d$ is symplectic if and only if its entries satisfy the following equalities:

$$^t ac = {}^t ca; \quad {}^t bd = {}^t db; \quad {}^t da - {}^t bc = \mathbf{1}.$$

We define along the orbit of x_0 the following change of basis: $P = \begin{pmatrix} S_0^{-1} & S_0^{-1} \\ \mathbb{S}S_0^{-1} & \mathbb{U}S_0^{-1} \end{pmatrix}$.
Then it defines a symplectic change of coordinates, whose inverse is:

$$Q = P^{-1} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} {}^t P \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} S_0^{-1}\mathbb{U} & -S_0^{-1} \\ -S_0^{-1}\mathbb{S} & S_0^{-1} \end{pmatrix}.$$

We use this symplectic change of coordinates along the whole orbit of x_0. More precisely, if we denote the matrix of Df^k in the usual canonical base $e = (e_i)$ by M_k, then the matrix of Df^k in the base $Pe = (Pe_i)$ is denoted by $\tilde{M}_k$; we have then: $\tilde{M}_k(x_n) = P^{-1}(x_{n+k}) M_k(x_n) P(x_n)$. Let use notice that the image of the horizontal (resp. vertical) Lagrangian plane by P is G_- (resp. G_+). As the bundles G_- and G_+

are invariant by f, we deduce that $\tilde{M}_k = \begin{pmatrix} \tilde{a}_k & 0 \\ 0 & \tilde{d}_k \end{pmatrix}$; we have ${}^t\tilde{a}_k\tilde{d}_k = \mathbf{1}$ because this matrix is symplectic.

Moreover, we know that: $M_k(x_n) = \begin{pmatrix} -b_k(x_n)s_{-k}(x_n) & b_k(x_n) \\ c_k(x_n) & s_k(x_{k+n})b_k(x_n) \end{pmatrix}$ where $G_k(x_n) = Df^k.V(x_{n-k})$ is the graph of $s_k(x_n)$.

Writing that $\tilde{M}_k(x_n) = \begin{pmatrix} \tilde{a}_k(x_n) & 0 \\ 0 & \tilde{d}_k(x_n) \end{pmatrix} = P^{-1}(x_{n+k})M_k(x_n)P(x_n)$, we obtain firstly:

$$S_0(x_{n+k})^{-1\,t}b_k(x_n)S_0(x_n)^{-1} = S_0(x_{n+k})^{-1}(\mathbb{S}(x_{n+k})-s_k(x_{n+k}))b_k(x_n)(s_{-k}(x_n)-\mathbb{S}(x_n))S_0(x_n)^{-1};$$

$$-S_0(x_{n+k})^{-1\,t}b_k(x_n)S_0(x_n)^{-1} = S_0(x_{n+k})^{-1}(\mathbb{U}(x_{n+k})-s_k(x_{n+k}))b_k(x_n)(\mathbb{U}(x_n)-s_{-k}(x_n))S_0(x_n)^{-1}.$$

We deduce that: $\tilde{a}_k(x_n) = S_0(x_{n+k})b_k(x_n)(\mathbb{S}(x_n) - s_{-k}(x_n))S_0(x_n)^{-1}$ and:
$\tilde{d}_k(x_n) = S_0(x_{n+k})b_k(x_n)(\mathbb{U}(x_n) - s_{-k}(x_n))S_0(x_n)^{-1}$.

Because of the changes of basis that we used, $(\tilde{a}_k(x_n))_k$ represents the linearized dynamics $(Df^k_{|G_-(x_n)})_k$ restricted to G_- and $(\tilde{d}_k(x_n))_k$ the linearized dynamics restricted to G_+. Hence we need to study $(\tilde{d}_k(x_n))$ to obtain some information about the positive Lyapunov exponents of μ. Let us compute:
${}^t\tilde{d}_k(x_n) = \tilde{a}_k(x_n)^{-1} = S_0(x_n)(\mathbb{S}(x_n) - s_{-k}(x_n))^{-1}b_k(x_n)^{-1}S_0(x_{n+k})^{-1}$; we deduce:

$$\begin{aligned} {}^t\tilde{d}_k(x_n)\tilde{d}_k(x_n) &= S_0(x_n)(\mathbb{S}(x_n) - s_{-k}(x_n))^{-1}(\mathbb{U}(x_n) - s_{-k}(x_n))S_0(x_n)^{-1} \\ &= S_0(x_n)(\mathbb{S}(x_n) - s_{-k}(x_n))^{-1}(\mathbb{U}(x_n) - \mathbb{S}(x_n) + \mathbb{S}(x_n) - s_{-k}(x_n))S_0(x_n)^{-1} \\ &= \mathbf{1} + S_0(x_n)(\mathbb{S}(x_n) - s_{-k}(x_n))^{-1}S_0(x_n) \\ &= \mathbf{1} + (\mathbb{U}(x_n) - \mathbb{S}(x_n))^{\frac{1}{2}}(\mathbb{S}(x_n) - s_{-k}(x_n))^{-1}(\mathbb{U}(x_n) - \mathbb{S}(x_n))^{\frac{1}{2}}. \end{aligned}$$

Let us denote the conorm of a (for the usual Euclidean norm of $\mathbb{R}^d$) by: $m(a) = \|a^{-1}\|^{-1}$. Then we have:

$$m(\tilde{d}_k(x_n))^2 = m({}^t\tilde{d}_k(x_n)\tilde{d}_k(x_n));$$

and then: $m(\tilde{d}_k(x_n))^2 \geq 1 + \frac{1}{C}m((\mathbb{U}-\mathbb{S})(x_n))$ where C designates $\sup \|s_1 - s_{-1}\|$ above the (compact) support of μ; indeed, we know that: $s_1 - s_{-1} \geq \mathbb{S} - s_{-k} > 0$.

The entry $\tilde{d}_k$ being multiplicative, we deduce that:

$$m(\tilde{d}_k(x_0))^2 \geq \prod_{n=0}^{k-1}(1 + \frac{1}{C}m(\mathbb{U}(x_n) - \mathbb{S}(x_n)))$$

and:

$$\frac{1}{k}\log m(\tilde{d}_k(x_0)) \geq \frac{1}{2k}\sum_{n=0}^{k-1}\log(1 + \frac{1}{C}m(\mathbb{U}(x_n) - \mathbb{S}(x_n))).$$

When k tends to $+\infty$, we deduce from Birkhoff's ergodic theorem that:

$$(*) \quad \liminf_{k\to\infty} \frac{1}{k} \log m(\tilde{d}_k(x_0)) \geq \frac{1}{2} \int \log \left(1 + \frac{1}{C} m(\mathbb{U}(x) - \mathbb{S}(x))\right) d\mu(x).$$

Let us recall that $(\tilde{d}_k(x_0))$ represents the dynamics along G_+, but the change of basis that we have done is not necessarily bounded. To obtain a true information about the Lyapunov positive exponents, we need to have a result for the matrix D_k of $(Df^k_{|G_+(x_0)})$ in the base of G_+ whose matrix in the usual coordinates is: $\begin{pmatrix} \mathbf{1} \\ \mathbb{U} \end{pmatrix}$. Since $(\tilde{d}_k)$ is the matrix of Df^k in the base whose matrix is $\begin{pmatrix} S_0^{-1} \\ \mathbb{U}S_0^{-1} \end{pmatrix}$, we deduce that: $D_k(x_0) = S_0(x_k)\tilde{d}_k(x_0)S_0(x_0)^{-1}$ and:
$m(D_k(x_0)) \geq m(S_0(x_k))m(\tilde{d}_k(x_0))m(S_0(x_0)^{-1}) = (m(\mathbb{U}(x_k) - \mathbb{S}(x_k)))^{\frac{1}{2}} m(\tilde{d}_k(x_0))m(S(x_0)^{-1})$.
We have $(*)$ and we know that: $\liminf_{k\to\infty} m(\mathbb{U}(x_k) - \mathbb{S}(x_k)) \geq \varepsilon$. We deduce:

$$\lambda(\mu) \geq \liminf_{k\to\infty} \frac{1}{k} \log m(D_k(x_0)) \geq \frac{1}{2} \int \log \left(1 + \frac{1}{C} m(\mathbb{U}(x) - \mathbb{S}(x))\right) d\mu(x).$$

□

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