Adaptive feedback control and synchronization of non-identical chaotic fractional order systems

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Abstract

This paper addresses the reliable synchronization problem between two non-identical chaotic fractional order systems. In this work, we present an adaptive feedback control scheme for the synchronization of two coupled chaotic fractional order systems with different fractional orders. Based on the stability results of linear fractional order systems and Laplace transform theory, using the master-slave synchronization scheme, sufficient conditions for chaos synchronization are derived. The designed controller ensures that fractional order chaotic oscillators that have non-identical fractional orders can be synchronized with suitable feedback controller applied to the response system. Numerical simulations are performed to asses the performance of the proposed adaptive controller in synchronizing chaotic systems.

Keywords: Chaos synchronization; Fractional order system; Caputo fractional derivative; Stability; Feedback control.

1 Introduction

As a kind of characteristic of nonlinear systems, chaos is a bounded unstable dynamic behavior that exhibits sensitive dependence on initial conditions and includes infinite unstable periodic motions. Chaos has been investigated and studied in mathematical and physical communities in the last few decades because of its grate applications in many fields such as secure communication, data encryption, flow dynamics and biomedical engineering [1]. The research efforts have devoted to the chaos control and chaos synchronization problems in many dynamical systems [2-5].

On the other hand, the development of models based on fractional order differential systems has recently gained popularity in the investigation of dynamical systems. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. The main reason for using the integer-order models was the absence of solution methods for fractional differential equations. The advantages or the real objects of the fractional order systems [6] are that we have more degrees of freedom in the model and that a "memory" is included in the model.

Recently, studying fractional order systems has become an active research area. The chaotic dynamics of fractional order systems began to attract much attention in recent years. It has been shown that fractional order systems, as generalizations of many well-known systems, can also behave chaotically, such as the fractional Duffing system [7], the fractional Chua system [6, 8], the fractional Rössler system [9], the fractional Chen system [10, 11, 12], the fractional Lorenz system [13], the fractional Arneodo's system [14, 15], the fractional Lü system [16], the fractional Newton-Leipnik [17] and the fractional Chen-Lee system [18]. In [8, 9, 10, 11] it has been shown that some fractional order systems can produce chaotic attractors with order less than 3.

Meanwhile, chaotic dynamics of fractional order systems began to attract much attention in recent years.

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A challenging problem is the control and synchronization of chaotic systems. Recent studies show that chaotic fractional order systems can also be synchronized [19]-[32]. In many literatures, synchronization among fractional order systems is only investigated through numerical simulations that are based on the stability criteria of linear fractional order systems, such as the work presented in [24]-[27], or based on Laplace transform theory, such as the work presented in [28]-[32].

In the present paper, we study the synchronization of fractional order chaotic systems with different fractional orders via nonlinear control. By taking the fractional version of Chen system (which belongs to the double scroll attractor family) and Rössler system (which belongs to the one scroll attractor family) as examples, we show that fractional order systems with different fractional orders can be synchronized. Based on stability results of fractional order systems, using the drive-response concept, an adaptive feedback control is constructed to achieve synchronization between two 3D fractional order systems that have non-identical orders. The synchronization controllers are investigated theoretically and then, numerical simulations are presented to verify the theoretical analysis.

1.1 Basic concepts

There are several definitions of a fractional derivative of order $\alpha > 0$ [33]-[37]. The two most commonly used are Riemann-Liouville and Caputo definitions. Each definition uses Riemann-Liouville fractional integration and derivatives of whole order. The difference between the two definitions is in the order of evaluation. Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of the function $f(t)$ is defined as,

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0. \tag{1}$$

Some properties of the operator J^α can be found, for example, in [34, 36]. We recall only the following, for $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$, we have,

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t),$$

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}.$$

The Laplace transform of Riemann-Liouville fractional integral satisfies,

$$L\{J^\alpha f(t)\} = s^{-\alpha} L\{f(t)\}. \tag{2}$$

In this study, Caputo definition is used and the fractional derivative of $f(t)$ is defined as, $D^\alpha f(t) = J^{m-\alpha} D^m f(t)$,

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \tag{3}$$

for $m-1 < \alpha \leq m$, $m \in I\!N$, $t > 0$. Caputo's definition has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables with their integer order which is the case in most physical processes. Fortunately, the Laplace transform of Caputo fractional derivative satisfies,

$$L\{D^\alpha f(t)\} = s^\alpha L\{f(t)\} - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-1-k}, \tag{4}$$

where $m-1 < \alpha \leq m$. The Laplace transform of Caputo fractional derivative requires the knowledge of the (bounded) initial values of the function and of its integer derivatives of order $k = 1, 2, \cdots, m-1$.

1.2 Stability analysis of fractional systems

Stability analysis of fractional order systems, which is of main interest in control theory, has been thoroughly investigated where necessary and sufficient conditions have been derived [38]-[41] (see also references therein). In this section, we recall the main stability results. For this object, we consider the following n dimensional fractional order system,

$$\begin{cases} \dfrac{d^{\alpha_1} x_1}{dt^{\alpha_1}} = f_1(x_1, x_2, \cdots, x_n), \\ \dfrac{d^{\alpha_2} x_2}{dt^{\alpha_2}} = f_2(x_1, x_2, \cdots, x_n), \\ \vdots \\ \dfrac{d^{\alpha_n} x_n}{dt^{\alpha_n}} = f_n(x_1, x_2, \cdots, x_n), \end{cases} \tag{5}$$

where α_i is a rational number between 0 and 1 and $\frac{d^{\alpha_i}}{dt^{\alpha_i}}$ is the Caputo fractional derivative of order α_i, for $i = 1, 2, \cdots, n$. Assume that $\alpha_i = k_i/m_i$, $(k_i, m_i) = 1$,

$k_i, m_i \in I\!N$, for $i = 1, 2, \cdots, n$. Let m be the least common multiple of the denominators m_i's of α_i's.

First, if the system (5) is a linear system, that is $[f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_n(\mathbf{x})]^T = [a_{ij}]_{i,j=1}^n \mathbf{x} = A\mathbf{x}$, where $\mathbf{x} \in I\!R^n$, then we have the following results:

- If $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_n$, then the fractional order system (5) is asymptotically stable iff $|arg(spec(A))| > \alpha\pi/2$. In this case the components of the state decay towards 0 like $t^{-\alpha}$ [38].

- If α_i's are rational numbers between 0 and 1, then the system (5) is asymptotically stable if all roots λ of the equation $det\big(diag(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \cdots, \lambda^{m\alpha_n}) - A\big) = 0$ satisfy $|arg(\lambda)| > \gamma\pi/2$, where $\gamma = 1/m$ [39].

Second, if function f_i has second continuous partial derivatives in a ball centered at an equilibrium point $\mathbf{x}^* = (x_1, x_2, \cdots, x_n)$, that is $f_i(x_1, x_2, \cdots, x_n) = 0$, for $i = 1, 2, \cdots, n$, then we have the following results:

- If $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_n$, then the equilibrium point $\mathbf{x}^*$ of system (5) is asymptotically stable if $|arg(spec(J|_{\mathbf{x}^*}))| > \alpha\pi/2$, where the matrix J is the Jacobian matrix of the system (5) that is defined as $J = [\frac{\partial f_i}{\partial x_j}]_{i,j=1}^n$ [40].

- If α_i's are rational numbers between 0 and 1, then the equilibrium point $\mathbf{x}^*$ of system (5) is asymptotically stable if all roots λ of the equation $det\big(diag(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \cdots, \lambda^{m\alpha_n}) - J|_{\mathbf{x}^*}\big) = 0$ satisfy $|arg(\lambda)| > \gamma\pi/2$, where $\gamma = 1/m$ [41].

Fractional order systems are, at least, as stable as their integer order counterpart, because systems with memory are typically more stable than their memoryless counterpart [40]. The previous stability results play an important role in studying the existence of chaotic attractors and the synchronization of fractional order systems.

2 fractional Chen system

Chen and Ueta [42] introduced, in 1999, the Chen system which is similar but not topologically equivalent to Lorenz system. It is a chaotic system with a double scroll attractor. The fractional version of Chen system reads as,

$$\begin{cases} \dfrac{d^{\alpha_1} x}{dt^{\alpha_1}} = a(y - x), \\ \dfrac{d^{\alpha_2} y}{dt^{\alpha_2}} = (c - a)x - xz + cy, \\ \dfrac{d^{\alpha_3} z}{dt^{\alpha_3}} = xy - bz, \end{cases} \tag{6}$$

where $0 < \alpha_1, \alpha_2, \alpha_3 \leq 1$. Integer order Chen system displays chaotic attractors, for example, when $(a, b, c) = (35, 3, 28)$. Simulations are performed to obtain chaotic behavior of fractional order Chen system for different fractional orders α, when $(a, b, c) = (35, 3, 28)$. For example, chaotic attractors are found in [25] when $\alpha = (0.95, 0.95, 0.95)$. In [11] chaotic behaviors are found when $\alpha = (0.9, 0.9, 0.9)$. Moreover, in [28] and [43], it is found that for the parameters $\alpha = (0.985, 0.99, 0.98)$ and $\alpha = (0.8, 1, 0.9)$, respectively, fractional order Chen system can display chaotic attractors.

Fractional Chen system (6), when $(a, b, c) = (35, 3, 28)$, has three equilibrium points,

$$\begin{cases} P_1 : (0, 0, 0), \\ P_2 : (\sqrt{63}, \sqrt{63}, 21), \\ P_3 : (-\sqrt{63}, -\sqrt{63}, 21). \end{cases} \tag{7}$$

The Jacobian matrix of system (6), evaluated at the equilibrium point $\mathbf{x}^* = (x^*, y^*, z^*)$, is

$$J_C = \begin{pmatrix} -35 & 35 & 0 \\ -7 - z^* & 28 & -x^* \\ y^* & x^* & -3 \end{pmatrix} \tag{8}$$

Now, according to stability results and the results presented in [43], a necessary condition for fractional Chen system to exhibit chaotic attractors similar to its integer order counterpart is:

$$\min_i \{|arg(\lambda_i)|\} \leq \gamma\pi/2, \tag{9}$$

where $\gamma = 1/m$ and λ_i's are the roots of the equation $det\big(diag(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \lambda^{m\alpha_3}) - J_C|_{\mathbf{x}^*}\big) = 0$, for every equilibrium point $\mathbf{x}^*$. Otherwise, one of these equilibrium points becomes asymptotically stable and attracts the nearby trajectories.

Now, if $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, then the eigenvalues of the equilibrium points are,

$$\begin{cases} P_1 : \lambda_1 = -30.83587, \ \lambda_2 = 23.83587, \ \lambda_3 = -3, \\ P_2 : \lambda_1 = -18.42796, \ \lambda_{2,3} = 4.21398 \pm 14.88464\, i, \\ P_3 : \lambda_1 = -18.42796, \ \lambda_{2,3} = 4.21398 \pm 14.88464\, i. \end{cases} \tag{10}$$

According to (9), the equilibrium points P_2 and P_3 are asymptotically stable if,

$$\alpha < \frac{2}{\pi} \tan^{-1}\Big(\frac{14.88464}{4.21398}\Big) = 0.82436. \tag{11}$$

Therefore, system (6) has the necessary condition $\alpha > 0.82436$ for exhibiting double scroll chaotic attractor.

For $\alpha = (1, 0.9, 0.9)$ and according to the last two equilibrium points P_2 and P_3, the equation $det\big(diag(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \lambda^{m\alpha_3}) - J_C|_{\mathbf{x}^*}\big) = 0$ becomes,

$$\lambda^{28} - 25\lambda^{19} + 35\lambda^{18} - 21\lambda^{10} + 105\lambda^9 + 4410 = 0, \tag{12}$$

and so we get,

$$\min_i\{|arg(\lambda_i)|\} = 0.12458 < \pi/20. \tag{13}$$

Hence, system (6) satisfies the necessary condition for exhibiting a double scroll attractor when $\alpha = (1, 0.9, 0.9)$. Using similar analysis we can confirm that fractional Chen system (6) satisfies the necessary condition (9) for exhibiting a double scroll attractor when $\alpha = (0.95, 0.95, 0.95)$, $\alpha = (0.9, 0.9, 0.9)$, $\alpha = (0.985, 0.99, 0.98)$ and $\alpha = (0.8, 1, 0.9)$, in case of $(a, b, c) = (35, 3, 28)$.

In order to observe synchronization behavior in two coupled chaotic fractional order Chen systems with different fractional orders, we build the master and the slave fractional order Chen systems as,

$$M : \begin{cases} \dfrac{d^{\alpha_1} x_m}{dt^{\alpha_1}} = a(y_m - x_m), \\ \dfrac{d^{\alpha_2} y_m}{dt^{\alpha_2}} = (c-a)x_m - x_m z_m + c y_m, \\ \dfrac{d^{\alpha_3} z_m}{dt^{\alpha_3}} = x_m y_m - b z_m, \end{cases} \tag{14}$$

$$S : \begin{cases} \dfrac{d^{\beta_1} x_s}{dt^{\beta_1}} = a(y_s - x_s) + u_1(t), \\ \dfrac{d^{\beta_2} y_s}{dt^{\beta_2}} = (c-a)x_s - x_s z_s + c y_s + u_2(t), \\ \dfrac{d^{\beta_3} z_s}{dt^{\beta_3}} = x_s y_s - b z_s + u_3(t), \end{cases} \tag{15}$$

where $\alpha_i \geq \beta_i$, for $i = 1, 2, 3$, subscripts m and s stand for the master system and slave system, respectively, and $\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t))$ is the nonlinear controller to be designed. Our aim is to determine the controller $\mathbf{u}(t)$ for the global synchronization of the non-identical fractional order Chen systems (14) and (15). For this purpose, we define the synchronization error as,

$$\begin{cases} e_1(t) = x_s(t) - x_m(t), \\ e_2(t) = y_s(t) - y_m(t), \\ e_3(t) = z_s(t) - z_m(t). \end{cases} \tag{16}$$

Now, we design the controller $\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t))$ as follows,

$$\begin{cases} u_1 = a(J^{\alpha_1-\beta_1} - 1)[y_s - x_s] + k_1 J^{\alpha_1-\beta_1} \\ \qquad [x_s - x_m], \\ u_2 = x_s z_s - J^{\alpha_2-\beta_2}[x_m z_m] + (J^{\alpha_2-\beta_2} - 1) \\ \qquad [(c-a)x_s + c y_s] + k_2 J^{\alpha_2-\beta_2}[y_s - y_m], \\ u_3 = J^{\alpha_3-\beta_3}[x_m y_m] - x_s y_s - b(J^{\alpha_3-\beta_3} - 1) \\ \qquad [z_s] + k_3 J^{\alpha_3-\beta_3}[z_s - z_m], \end{cases} \tag{17}$$

where J^γ is the Riemann-Liouville fractional integral operator of order γ, defined in (1), $\mathbf{k} = (k_1, k_2, k_3)$ is the coupling matrix. The constants k_1, k_2 and k_3 will be determined such that $\|\mathbf{e}(t)\| \longrightarrow 0$ as $t \longrightarrow +\infty$. Applying Laplace transform to systems (14), (15) and (17), letting $X(s) = L((x(t))$, $Y(s) = L(y(t))$, $Z(s) = L(z(t))$, $E_1(s) = L((e(t))$, $E_2(s) = L((e_2(t))$, $E_3(s) = L((e_3(t))$, and using formulas (2) and (4), we obtain,

$$\begin{cases} s^{\alpha_1} X_m(s) = s^{\alpha_1 - 1} x_m(0) + a(Y_m - X_m), \\ s^{\alpha_2} Y_m(s) = s^{\alpha_2 - 1} y_m(0) + (c-a) X_m \\ \qquad\qquad -L\{x_m z_m\} + c Y_m, \\ s^{\alpha_3} Z_m(s) = s^{\alpha_3 - 1} z_m(0) + L\{x_m y_m\} - b Z_m, \end{cases} \tag{18}$$

$$\begin{cases} s^{\beta_1} X_s(s) = s^{\beta_1 - 1} x_s(0) + a(Y_s - X_s) + U_1, \\ s^{\beta_2} Y_s(s) = s^{\beta_2 - 1} y_s(0) + (c-a) X_s \\ \qquad\qquad -L\{x_s z_s\} + cY_s + U_2, \\ s^{\beta_3} Z_s(s) = s^{\beta_3 - 1} z_s(0) + L\{x_s y_s\} - bZ_s + U_3, \end{cases} \tag{19}$$

$$\begin{cases} U_1 = a(s^{\beta_1-\alpha_1} - 1)(Y_s - X_s) + k_1 s^{\beta_1-\alpha_1} \\ \quad (X_s - X_m), \\ U_2 = L\{x_s z_s\} - s^{\beta_2-\alpha_2} L\{x_m z_m\} + (s^{\beta_2-\alpha_2} - 1) \\ \quad ((c-a)X_s + cY_s) + k_2 s^{\beta_2-\alpha_2}(Y_s - Y_m), \\ U_3 = s^{\beta_3-\alpha_3} L\{x_m y_m\} - L\{x_s y_s\} - b(s^{\beta_3-\alpha_3} - 1) \\ \quad (Z_s) + k_3 s^{\beta_3-\alpha_3}(Z_s - Z_m). \end{cases} \tag{20}$$

Multiplying the first equation in (19) by $s^{\alpha_1-\beta_1}$, the second equation by $s^{\alpha_2-\beta_2}$ and the third equation by $s^{\alpha_3-\beta_3}$ then, from the definition of the error functions (16), we get,

$$\begin{cases} s^{\alpha_1} E_1(s) = s^{\alpha_1-1} e_1(0) + a(E_2 - E_1) + k_1 E_1, \\ s^{\alpha_2} E_2(s) = s^{\alpha_2-1} e_2(0) + (c-a)E_1 + cE_2 + k_2 E_2, \\ s^{\alpha_3} E_3(s) = s^{\alpha_3-1} e_3(0) - bE_3 + k_3 E_3. \end{cases} \tag{21}$$

Applying the inverse Laplace transform, using formula (4), to system (21), we obtain the following linear fractional order system,

$$\begin{cases} \dfrac{d^{\alpha_1} e_1}{dt^{\alpha_1}} = a(e_2 - e_1) + k_1 e_1, \\ \dfrac{d^{\alpha_2} e_2}{dt^{\alpha_2}} = (c-a)e_1 + ce_2 + k_2 e_2, \\ \dfrac{d^{\alpha_3} e_3}{dt^{\alpha_3}} = -be_3 + k_3 e_3. \end{cases} \tag{22}$$

In case of $\alpha_i = \beta_i$, for $i = 1, 2, 3$, then the controller $\mathbf{u}(t)$, defined in (17), reduces to

$$\begin{cases} u_1 = k_1[x_s - x_m], \\ u_2 = x_s z_s - x_m z_m + k_2[y_s - y_m], \\ u_3 = x_m y_m - x_s y_s + k_3[z_s - z_m]. \end{cases} \tag{23}$$

The Jacobian matrix for the error system (22) is

$$J = \begin{pmatrix} -a + k_1 & a & 0 \\ c - a & c + k_2 & 0 \\ 0 & 0 & -b + k_3 \end{pmatrix}, \tag{24}$$

and so, the characteristic equation $det\big(diag(\lambda^{r_1}, \lambda^{r_2}, \lambda^{r_3}) - J\big) = 0$ can be written as,

$$((\lambda^{r_1} + a - k_1)(\lambda^{r_2} - c - k_2) + a(a-c))(\lambda^{r_3} + b - k_3) = 0, \tag{25}$$

where $r_i = m\alpha_i$, for $i = 1, 2, 3$. According to the stability results, the drive system (14) and the response system (15) are synchronized ($\|\mathbf{e}(t)\| \longrightarrow 0$ as $t \longrightarrow +\infty$) if all roots λ of Eq. (25) satisfy $|arg(\lambda)| > \gamma\pi/2$, where $\gamma = 1/m$.
Now, in case of $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ and $(a, b, c) = (35, 3, 28)$, then Eq. (25) reduces to,

$$((\lambda + 35 - k_1)(\lambda - 28 - k_2) + 245)(\lambda + 3 - k_3) = 0, \tag{26}$$

and so, systems (14) and (15) are synchronized if k_1, k_2 and k_3 satisfy the laws,

$$\begin{cases} k_1 + k_2 - 7 \mp w < 0, & if \;\; A = w \in I\!R \\ \left|\frac{w}{k_1 + k_2 - 7}\right| < \tan(\alpha\pi/2), & if \;\; A = iw \in iI\!R \end{cases} \tag{27}$$

$$k_3 < 3, \tag{28}$$

where $A = \big((7 - k_1 - k_2)^2 - 4(245 - (35 - k_1)(28 + k_2))\big)^{1/2}$.

Example 1 *Taking* $(a, b, c) = (35, 3, 28)$, $\alpha = (1, 1, 1)$ *and* $\beta = (0.95, 0.95, 0.95)$. *If we select* $k_1 = 20$, $k_2 = -15$ *and* $k_3 = 2$, *then they satisfy the laws (27) and (28).*

Therefore, under the controller,

$$\begin{cases} u_1 = 35(J^{0.05} - 1)[y_s - x_s] + k_1 J^{0.05} \\ \quad [x_s - x_m], \\ u_2 = x_s z_s - J^{0.05}[x_m z_m] + (J^{0.05} - 1) \\ \quad [-7x_s + 28y_s] + k_2 J^{0.05}[y_s - y_m], \\ u_3 = J^{0.05}[x_m y_m] - x_s y_s - 3(J^{0.05} - 1) \\ \quad [z_s] + k_3 J^{0.05}[z_s - z_m], \end{cases} \tag{29}$$

the drive system (14) and the response system (15) are synchronized. The error functions evolution, in this case, is shown in Fig. 1. From Fig. 1, it is obvious that the components of the error system (22) decay towards zero as $t \longrightarrow +\infty$. So, we can numerically conclude that the designed controller can effectively control the chaotic fractional order Chen systems (14) and (15) with non-identical orders to achieve synchronization.

Example 2 *If we take* $(a, b, c) = (35, 3, 28)$, $\alpha = (1, 0.9, 0.9)$, $\beta = (0.9, 0.9, 0.9)$, $k_1 = 35$, $k_2 = -28$ *and* $k_3 = 2$, *then Eq. (25) reduces to,*

$$(\lambda^{19} + 245)(\lambda^9 + 1) = 0. \tag{30}$$

Simply, we can show that all roots of Eq. (30) lie in the region $|arg(\lambda)| > \pi/20$. Therefore, under the controller

$$\begin{cases} u_1 = 35(J^{0.1} - 1)[y_s - x_s] + k_1 J^{0.1} \\ \qquad [x_s - x_m], \\ u_2 = x_s z_s - x_m z_m + k_2(y_s - y_m), \\ u_3 = x_m y_m - x_s y_s + k_3(z_s - z_m), \end{cases} \tag{31}$$

the drive system (14) and the response system (15) are synchronized. The error functions evolution, in this case, is shown in Fig. 2. It is clear, from Fig. 2, that the components of the error system (22) decay towards zero as $t \longrightarrow +\infty$.

3 fractional Rössler system

Now, we consider Rössler system [44], which is a nonlinear system that can exhibit one scroll chaotic attractor. Its fractional version reads as,

$$\begin{cases} \dfrac{d^{\alpha_1} x}{dt^{\alpha_1}} = -(y + z), \\ \dfrac{d^{\alpha_2} y}{dt^{\alpha_2}} = x + ay, \\ \dfrac{d^{\alpha_3} z}{dt^{\alpha_3}} = z(x - c) + b, \end{cases} \tag{32}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is subject to $0 < \alpha_1, \alpha_2, \alpha_3 \leq 1$. Integer Rössler system, when $\alpha = (1, 1, 1)$, is chaotic, for example, when $(a, b, c) = (0.2, 0.2, 5)$. Simulations are performed to obtain chaotic behavior of fractional order Rössler system and the results demonstrate that chaos indeed exists with order less than 3. For example, chaotic attractors are found in [9] when $\alpha = (0.9, 0.9, 0.9)$ and $(a, b, c) = (0.4, 0.2, 10)$. Also, in [23], it is found that for the parameters $\alpha = (1, 1, 0.8)$ and $(a, b, c) = (0.2, 0.2, 5)$ fractional order Rössler system can display chaotic behaviors.

For example, fractional Rössler system, when $(a, b, c) = (0.2, 0.2, 5)$, has two equilibrium points,

$$\begin{cases} P_1 : (0.00801, -0.04006, 0.04006), \\ P_2 : (4.99199, -24.95994, 24.95994). \end{cases} \tag{33}$$

The Jacobian matrix of system (32), evaluated at the equilibrium point $\mathbf{x}^* = (x^*, y^*, z^*)$, is

$$J_R = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ z^* & 0 & x^* - 5 \end{pmatrix} \tag{34}$$

Now, according to stability results and the results presented in [43], a necessary condition for fractional Rössler system to exhibit chaotic attractor similar to its integer order counterpart is:

$$\min_i \{|arg(\lambda_i)|\} \leq \gamma\pi/2, \tag{35}$$

where $\gamma = 1/m$ and λ_i's are the roots of the equation $det\big(diag(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \lambda^{m\alpha_3}) - J_R|_{\mathbf{x}^*}\big) = 0$, for every equilibrium point $\mathbf{x}^*$. Otherwise, one of these equilibrium points becomes asymptotically stable and attracts the nearby trajectories.

For $\alpha = (1, 1, 0.8)$ and according to the first equilibrium point P_1, the equation $det\big(diag(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \lambda^{m\alpha_3}) - J_R|_{\mathbf{x}^*}\big) = 0$ becomes,

$$\lambda^{14} + 4.99199\lambda^{10} - 0.2\lambda^9 - 0.95833\lambda^5 + \lambda^4 + 4.98397 = 0, \tag{36}$$

and so we get,

$$\min_i \{|arg(\lambda_i)|\} = 0.29486 < \pi/10. \tag{37}$$

Hence, system (32) satisfies the necessary condition for exhibiting a one scroll attractor when $\alpha = (1, 1, 0.8)$ and $(a, b, c) = (0.2, 0.2, 5)$.

For $\alpha = (0.9, 0.9, 0.9)$ and $(a, b, c) = (0.4, 0.2, 10)$, then the equilibrium point $(0.00801, -0.02002, 0.02002)$ of system (32) has the following eigenvalues

$$\lambda_1 = -9.99001, \ \lambda_{2,3} = 0.199008 \mp 0.979690\, i,$$

and so, $|arg(\lambda_{2,3})| = 1.370390 < (0.9)\pi/2$. Therefore, system (32) satisfies the necessary condition for exhibiting one scroll attractor when $\alpha = (0.9, 0.9, 0.9)$

and $(a,b,c) = (0.4, 0.2, 10)$.

In order to observe synchronization behavior in two coupled chaotic fractional order Rössler systems with different fractional orders, we build the master and the slave fractional order Rössler systems as,

$$M: \begin{cases} \dfrac{d^{\alpha_1} x_m}{dt^{\alpha_1}} = -(y_m + z_m), \\ \dfrac{d^{\alpha_2} y_m}{dt^{\alpha_2}} = x_m + a y_m, \\ \dfrac{d^{\alpha_3} z_m}{dt^{\alpha_3}} = z_m(x_m - c) + b, \end{cases} \tag{38}$$

$$S: \begin{cases} \dfrac{d^{\beta_1} x_s}{dt^{\beta_1}} = -(y_s + z_s) + u_1(t), \\ \dfrac{d^{\beta_2} y_s}{dt^{\beta_2}} = x_s + a y_s + u_2(t), \\ \dfrac{d^{\beta_3} z_s}{dt^{\beta_3}} = z_s(x_s - c) + u_3(t) + b, \end{cases} \tag{39}$$

where $\alpha_i \geq \beta_i$, for $i = 1, 2, 3$, subscripts m and s stand for the master system and slave system, respectively, and $\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t))$ is the nonlinear controller to be designed. Our aim is to determine the controller $\mathbf{u}(t)$ for the global synchronization of non-identical fractional order Rössler systems (38) and (39). For this purpose, we design the controller $\mathbf{u}(t)$ as follows,

$$\begin{cases} u_1 = (J^{\alpha_1-\beta_1} - 1)[-x_s - z_s] + k_1 J^{\alpha_1-\beta_1} \\ \qquad [x_s - x_m], \\ u_2 = (J^{\alpha_2-\beta_2} - 1)[x_s + a y_s] \\ \qquad + k_2 J^{\alpha_2-\beta_2}[y_s - y_m], \\ u_3 = J^{\alpha_3-\beta_3}[x_m z_m] - x_s z_s - c(J^{\alpha_3-\beta_3} - 1)[z_s] \\ \qquad -b + \frac{b\, t^{\alpha_3-\beta_3}}{\Gamma(1+\alpha_3-\beta_3)} + k_3 J^{\alpha_3-\beta_3}[z_s - z_m], \end{cases} \tag{40}$$

where J^γ is the Riemann-Liouville fractional integral operator of order γ, defined in (1), $\mathbf{k} = (k_1, k_2, k_3)$ is the coupling matrix. Following the same analysis presented in the previous section, if we apply Laplace transform to systems (38), (39) and (40), using formulas (2) and (4), we obtain,

$$\begin{cases} s^{\alpha_1} E_1(s) = s^{\alpha_1-1} e_1(0) - (E_1 + E_3) + k_1 E_1, \\ s^{\alpha_2} E_2(s) = s^{\alpha_2-1} e_2(0) + E_1 + a E_2 + k_2 E_2, \\ s^{\alpha_3} E_3(s) = s^{\alpha_3-1} e_3(0) - c E_3 + k_3 E_3. \end{cases} \tag{41}$$

Applying the inverse Laplace transform, using formula (4), to system (41), we obtain the following linear fractional order system,

$$\begin{cases} \dfrac{d^{\alpha_1} e_1}{dt^{\alpha_1}} = -(e_1 + e_3) + k_1 e_1, \\ \dfrac{d^{\alpha_2} e_2}{dt^{\alpha_2}} = e_1 + a e_2 + k_2 e_2, \\ \dfrac{d^{\alpha_3} e_3}{dt^{\alpha_3}} = -c e_3 + k_3 e_3. \end{cases} \tag{42}$$

In case of $\alpha_i = \beta_i$, for $i = 1, 2, 3$, then the controller $\mathbf{u}(t)$, defined in (40), reduces to

$$\begin{cases} u_1 = k_1[x_s - x_m], \\ u_2 = k_2[y_s - y_m], \\ u_3 = x_m z_m - x_s z_s + k_3[z_s - z_m]. \end{cases} \tag{43}$$

The Jacobian matrix for the error system (42) is

$$J = \begin{pmatrix} -1 + k_1 & 0 & -1 \\ 1 & a + k_2 & 0 \\ 0 & 0 & -c + k_3 \end{pmatrix}, \tag{44}$$

and so, the characteristic equation $det\big(diag(\lambda^{r_1}, \lambda^{r_2}, \lambda^{r_3}) - J\big) = 0$ can be written as,

$$(\lambda^{r_1} + 1 - k_1)(\lambda^{r_2} - a - k_2)(\lambda^{r_3} + c - k_3) = 0, \tag{45}$$

where $r_i = m\alpha_i$, for $i = 1, 2, 3$. According to the stability results, the drive system (38) and the response system (39) are synchronized ($\|\mathbf{e}(t)\| \longrightarrow 0$ as $t \longrightarrow +\infty$) if all roots λ of Eq. (45) satisfy $|arg(\lambda)| > \gamma\pi/2$, where $\gamma = 1/m$.
Now, in case of $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, then Eq. (45) reduces to,

$$(\lambda + 1 - k_1)(\lambda - a - k_2)(\lambda + c - k_3) = 0, \tag{46}$$

and so, systems (38) and (39) are synchronized if k_1, k_2 and k_3 satisfy the law,

$$\begin{cases} k_1 < 1, \\ k_2 < -a, \\ k_3 < c. \end{cases} \tag{47}$$

Example 3 *Taking* $(a, b, c) = (0.4, 0.2, 10)$, $\alpha = (1, 1, 1)$ *and* $\beta = (0.9, 0.9, 0.9)$. *If we select* $k_1 = 0.5$, $k_2 = -1$ *and* $k_3 = 8$, *then they satisfy the law (47).*

Therefore, under the controller,

$$\begin{cases} u_1 = (J^{0.1} - 1)[-x_s - z_s] + k_1 J^{0.1}[x_s - x_m], \\ u_2 = (J^{0.1} - 1)[x_s + a y_s] + k_2 J^{0.1}[y_s - y_m], \\ u_3 = J^{0.1}[x_m z_m] - x_s z_s - c(J^{0.1} - 1)[z_s] \\ \qquad -0.2 + \frac{0.2\, t^{0.1}}{\Gamma(1.1)} + k_3 J^{0.1}[z_s - z_m], \end{cases} \tag{48}$$

the drive system (38) and the response system (39) are synchronized. The error functions evolution, in this case, is shown in Fig. 3. From Fig. 3, it is obvious that the components of the error system (42) decay towards zero as $t \longrightarrow +\infty$. So, we can numerically conclude that the designed controller can effectively control the chaotic fractional order Rössler systems (38) and (39) with non-identical orders to achieve synchronization.

Example 4 *If we take* $(a, b, c) = (0.2, 0.2, 5)$, $\alpha = (1, 1, 0.8)$, $\beta = (1, 1, 0.8)$, $k_1 = 0$, $k_2 = -1.2$ *and* $k_3 = 3$, *then Eq. (45) reduces to,*

$$(\lambda^5 + 1)(\lambda^5 + 1)(\lambda^4 + 2) = 0. \tag{49}$$

Simply, we can show that all roots of Eq. (49) lie in the region $|arg(\lambda)| > \pi/10$. Therefore, under the controller

$$\begin{cases} u_1 = k_1[x_s - x_m], \\ u_2 = k_2[y_s - y_m], \\ u_3 = x_m z_m - x_s z_s + k_3[z_s - z_m], \end{cases} \tag{50}$$

the drive system (38) and the response system (39) are synchronized. The error functions evolution, in this case, is shown in Fig. 4. It is clear, from Fig. 4, that the components of the error system (42) decay towards zero as $t \longrightarrow +\infty$.

4 Conclusion

In the present paper, we study the control and synchronization problems of non-identical chaotic fractional order systems. We present theoretical results for drive-response synchronization between fractional order systems with different fractional orders, based on stability results of fractional order systems and Laplace transform theory, such as the fractional extension of Chen and Rössler systems. The designed adaptive nonlinear controller that applied to the response system affects the system dynamics to realize synchronization. The controller is designed such that the components of the error system decay towards zero as the time variable, t, tends to infinity. The numerical simulations show the effectiveness and the feasibility of the proposed scheme.

Finally, the recent appearance of fractional order systems as models in many fields makes it necessary to investigate the issues of chaos, control and synchronization of such systems and we hope that this work is a step in this direction.

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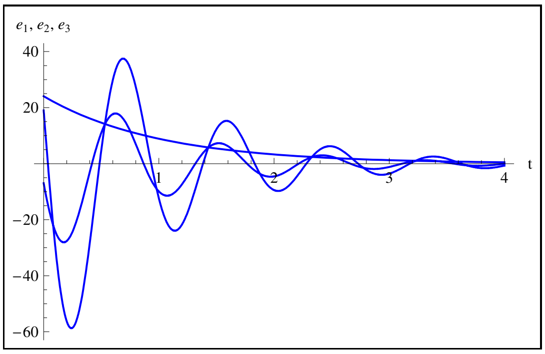

Figure 1: Synchronization of fractional Chen systems (14) and (15), when $(a, b, c) = (35, 3, 28)$, $\alpha = (1, 1, 1)$, $\beta = (0.95, 0.95, 0.95)$ and $(k_1, k_2, k_3) = (20, -15, 2)$.

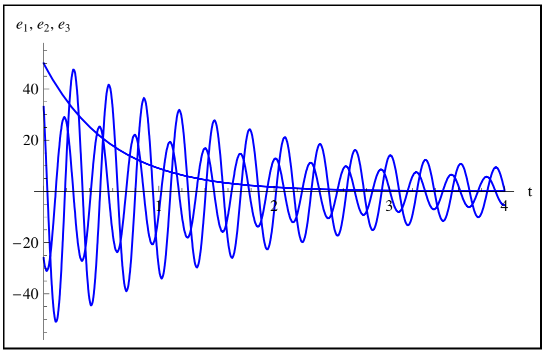

Figure 2: Synchronization of fractional Chen systems (14) and (15), when $(a, b, c) = (35, 3, 28)$, $\alpha = (1, 0.9, 0.9)$, $\beta = (0.9, 0.9, 0.9)$ and $(k_1, k_2, k_3) = (35, -28, 2)$.

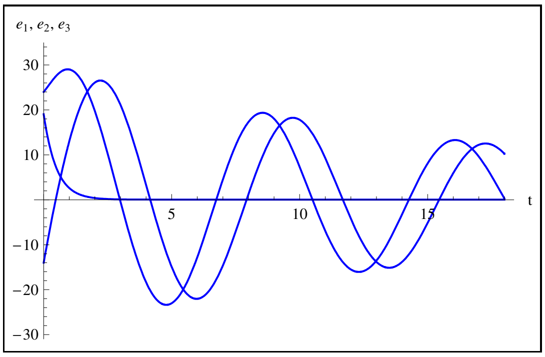

Figure 3: Synchronization of fractional Rössler systems (38) and (39), when $(a, b, c) = (0.4, 0.2, 10)$, $\alpha = (1, 1, 1)$, $\beta = (0.9, 0.9, 0.9)$ and $(k_1, k_2, k_3) = (0.5, -1, 8)$.

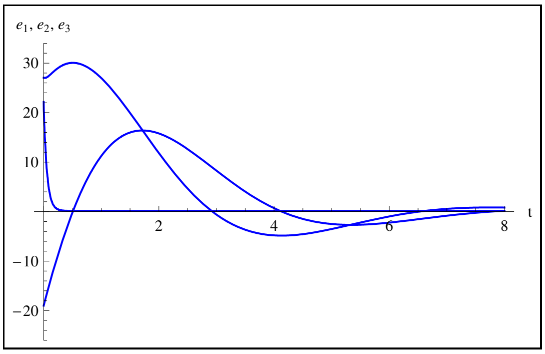

Figure 4: Synchronization of fractional Rössler systems (38) and (39), when $(a, b, c) = (0.2, 0.2, 5)$, $\alpha = (1, 1, 0.8)$, $\beta = (1, 1, 0.8)$ and $(k_1, k_2, k_3) = (0, -1.2, 3)$.