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A. Halanay, A. Ioniţă, C.-A. Safta

HOPF BIFURCATIONS THROUGH DELAY IN PILOT REACTION IN A LONGITUDINAL FLIGHT

by A. Halanay[1]*, A. Ioniţă[2], C-A. Safta[3]

Abstract

The paper is devoted to the study of pilot induced oscillations in the landing transition between the approach task and flare to touch-down. These oscillations are proved to appear in a longitudinal flight model when the delay in pilot's reactions exceeds a certain threshold for which the stability of equilibria is lost and a Hopf bifurcation appears. The formulae needed to compute the Lyapunov coefficient and an approximation of the solution are developed for the delay differential equations that model the pilot-vehicle interaction in landing task. These are applied for a concrete model.

Key words: delay differential equations, Hopf bifurcation, Lyapunov coefficient, pilot induced oscillations

AMS 2000 subject classification: 34K18, 34K20, 34G15

§1. Introduction

The so-called Pilot Induced Oscillations (PIO) phenomena are now recognized to belong to a certain subclass of Aircraft Pilot Coupling (APC) (see [1]). The oscillatory APC event is defined as "inadvertent, unwanted aircraft motion that is associated with anomalous interactions between the aircraft and the pilot" ([2]). In what follows a generic model (see [3]) will be considered that outlines the interaction between effective aircraft dynamics and pilot's characteristics in the longitudinal motion since PIO are dominant in terminal flight conditions. Famous PIO in landing phase were reported for YF22, Saab Grippen, C-17A, Airbus A321, Boeing 777 aircraft, some of which are using advanced highly augmentation flight control systems. Yet, the Hopf bifurcations

[1]corresponding author, Department of Mathematics I, University Politehnica of Bucharest, 313 Splaiul Independentei, RO-060042, Bucharest, ROMANIA, e-mail :halanay@mathem.pub.ro

[2]Institute for Theoretical and Experimental Analysis of Aeronautical Structures, 220 Bd. Iuliu Maniu, RO-061126, Bucharest, ROMANIA, e-mail:achim.ionita@straero.ro

[3]Department of Hydraulics and Hydraulic Machinery, University Politehnica of Bucharest, 313 Splaiul Independenţei, RO-060042, Bucharest, ROMANIA, e-mail :carmen123safta@yahoo.com

of equilibria and the study of the stability of the resulting limit cycle taking as parameter the time delay in pilot's action were not considered until much later (see [4], [5], [6], [7]).

In the study of PIO based on the qualitative theory of delay differential equations, two points of view are met: one is pessimistic with respect to the use of such a model due to its inherent complexity: functional initial conditions, an infinite number of roots for the characteristic equation, etc. A second point of view, more optimistic, hopes to use such models in order to choose closed-loop system's parameters such that the bifurcating value be high with respect to certain requirements (see [7]).

The time-delay model that will be studied below is a conventional one and can be seen as an equivalent system obtained from the high order augmentation dynamics of an airplane.

The paper is organized as follows: in Section 2 the model of the pilot-vehicle interaction in landing tasks is introduced. In Section 3 the system of equations for equilibria is analyzed in connection with Hopf bifurcation theorem in [8]. Section 4 is devoted to the study of the stability of the limit cycle provided by the Hopf bifurcation theorem. The first Lyapunov coefficient, as well as others constants involved in the description of the limit cycle, depend, of course, on the equilibrium point where bifurcation appears. In Section 5 a case study is presented. Some concluding remarks are given in Section 6.

§2. The Pilot-Vehicle Model in the Landing Task

The ADMIRE generic model (see [3]) is a single engine aircraft with a delta-canard configuration. The coefficients in the nonlinear aerodynamic model are calculated on the basis of the Generic Aerodynamic Model (GAM). The aircraft dynamics is completed by a flight control system in order to provide stability and affordable handling qualities within the operational envelope. The longitudinal controller provides the pitch rate control and angle of attack control. It consists of two parts

$$u(t) = u_1(t) + u_2(t-\tau)$$

where u_1 is the contribution of the flight control system with stability augmentation and u_2 is the contribution of pilot's action. So

$$u_1(t) = k_\alpha \alpha(t) + k_q q(t), \quad u_2(t) = k\theta(t-\tau).$$

The model for pilot's action is in the frame of classical crossover model: pure pilot gain k and pilot delay τ.

The mathematical model for a longitudinal landing flight becomes

$$\begin{aligned} \dot\alpha &= m_{11}\,\alpha + m_{12}\,q + c\,\cos\theta + b_1\,k\,\theta(t-\tau) \\ \dot q &= m_{21}\,\alpha + m_{22}\,q + c\,m_0\cos\theta - c\,c_1\sin\theta + b_2\,k\,\theta(t-\tau) \\ \dot\theta &= q. \end{aligned} \tag{2.1}$$

The state vector consists of the incidence angle α, the pitch attitude θ and the pitch rate q. The coefficients are supposed constant but depend on flight conditions and aerodynamic characteristics such as mass or moment of inertia of the generic aircraft.

S 3. Hopf bifurcation of equilibria

Equilibria for (2.1) are obtained from the system

$$\begin{cases} q = 0 \\ m_{11}\alpha + b_1\, k\, \theta + c\, \cos\theta = 0 \\ m_{21}\alpha + b_2\, k\, \theta + c\, m_0 \cos\theta - c\, c_1 \sin\theta = 0. \end{cases} \tag{3.1}$$

Let $(\alpha_0, 0, \theta_0)$ be an equilibrium point for (2.1).
Perform a translation into zero through

$$\alpha_1 = \alpha - \alpha_0,\ q_1 = q,\ \theta_1 = \theta - \theta_0 \tag{3.2}$$

System (2.1) becomes

$$\begin{aligned} \dot{\alpha}_1 &= m_{11}\alpha_1 + m_{12}q_1 + c\, \cos\theta_0 \cos\theta_1 - \\ &- c\sin\theta_0 \sin\theta_1 + b_1\, k\, \theta_1(t-\tau) + c\, \cos\theta_0 \\ \dot{q}_1 &= m_{21}\alpha_1 + m_{22}q_1 + (c\, m_0 \cos\theta_0 - c\, c_1 \sin\theta_0)\cos\theta_1 - \\ &- (c\, m_0 \sin\theta_0 + c\, c_1 \cos\theta_0)\sin\theta_1 + b_2\, k\, \theta_1(t-\tau) - \\ &- (c\, m_0 \cos\theta_0 - c\, c_1 \sin\theta_0) \\ \dot{\theta}_1 &= q \end{aligned} \tag{3.3}$$

The characteristic equation for the trivial equilibrium of (3.3) is given by

$$\begin{aligned} &\Delta(\lambda, \tau) := \\ &= \det \begin{pmatrix} \lambda - m_{11} & -m_{12} & c\sin\theta_0 - b_1 k\, \mathrm{e}^{-\lambda\tau} \\ -m_{21} & \lambda - m_{22} & c\, m_0 \sin\theta_0 + c\, c_1 \cos\theta_0 - b_2 k\, \mathrm{e}^{-\lambda\tau} \\ 0 & -1 & \lambda \end{pmatrix} = 0. \end{aligned} \tag{3.4}$$

It follows that

$$\Delta(\lambda, \tau) = P(\lambda) + Q(\lambda)e^{-\lambda\tau} \tag{3.5}$$

where $P,\ Q \in \mathbb{R}[X]$, $deg\, P = 3$ and $deg\, Q = 1$.

Recall from [8] the following theorem

Theorem 3.1 ([8], Ch11, §11.1) *Suppose that there exists $\tau_c > 0$ such that the characteristic equation (3.4), $\Delta(\lambda, \tau_c) = 0$, has a pair of simple purely imaginary roots $\lambda_0 = i\nu_0,\ \overline{\lambda}_0 = -i\nu_0$ and all other roots have negative real parts. Suppose also that*

$$Re\, \lambda_0'(\tau_c) \neq 0. \tag{3.6}$$

Then a Hopf bifurcation occurs for $\tau = \tau_c$, that is, for τ close to τ_c every system (3.3) (thus system (2.3) too) has a periodic solution.

In order to find ν_0 and τ_c we follow [9] and begin with the equation

$$F(y) := |P(iy)|^2 - |Q(iy)|^2 = 0 \tag{3.7}$$

(P and Q are given in (3.5)). We suppose that conditions of Theorem 1 in [9] are fulfilled so, if $\nu_0 > 0$ is a simple root of (3.7) it is a simple root of $|P(iy)| = |Q(iy)|$.

Introduce the real and imaginary parts of $P(iy)$ and $Q(iy)$ through $P = P_1 + iP_2$, $Q = Q_1 + iQ_2$ so, $P_1(y) = Re\,P(iy)$, $P_2(y) = Im\,P(iy)$, $Q_1(y) = Re\,Q(iy)$, $Q_2(y) = Im\,Q(iy)$. Remark that the solution of (3.1) is present in the coefficients of P.

Being a solution of (3.4) is equivalent for $\lambda = i\nu_0$ to

$$\begin{aligned} \cos\tau\,\nu_0 &= -\frac{P_1(\nu_0)\,Q_1(\nu_0) + P_2(\nu_0)Q_2(\nu_0)}{|Q(i\nu_0)|^2} \\ \sin\tau\,\nu_0 &= -\frac{P_1(\nu_0)\,Q_2(\nu_0) - P_2(\nu_0)\,Q_1(\nu_0)}{|Q(i\nu_0)|^2} \end{aligned} \tag{3.8}$$

(by the first hypothesis in Theorem 1 in [9] and by (3.7), $Q(i\nu_0) \neq 0$).

If τ_c is a solution of (3.8), $\lambda(\tau_c)$ crosses the imaginary axis at $\lambda = i\nu_0$ in the direction given by

$$s = \operatorname{sgn}\left[\operatorname{Re}\frac{d\lambda}{d\tau}\Big|_{\lambda=i\nu_0}\right] = \operatorname{sgn} F'(\nu_0). \tag{3.9}$$

As remarked in [9] the condition that ν_0 is a simple root of $F(y) = 0$ assures that, for $\tau = \tau_c$, $i\nu_0$ is a simple root of (3.4).

A particular case of the formulae given in [9] implies that if (3.7) has only one positive root, this one crosses the imaginary axis, as τ increases to τ_c, from left to right. So, if for $\tau = 0$ the roots of the characteristic equation (3.4) are in the left half plane they will remain there until $\tau = \tau_c$, thus equilibria given by (3.1), $(\alpha_0, 0, \theta_0)$, are stable for $\tau < \tau_c$.

The stability properties of the limit cycle are determined using the first Lyapunov coefficient.

§4. Stability of limit cycles

The basic reference for the formulae to compute the first Lyapunov coefficient is [10]. Similar computations were carried on in [11].

With

$$\mu = \tau - \tau_c \tag{4.1}$$

equation (3.3) can be rewritten as

$$\dot{x} = G_\mu(x_t),\ t \geq 0 \tag{4.2}$$

where $x_t(s) = x(t+s)$ and G_μ is a functional that acts on the space $C := C\left([-\mu-\tau_c, 0], \mathbb{C}^3\right)$of continuous functions defined on $[-\mu-\tau_c, 0]$ with values in $\mathbb{C}^3$. G_μ is defined through

$$G_\mu^{(1)}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} = m_{11}\varphi_1(0) + m_{12}\,\varphi_2(0) + c\,\cos\theta_0\cos\varphi_3(0) - \tag{4.3}$$
$$-c\,\sin\theta_0\sin\varphi_3(0) + b_1\,k\,\varphi_3(-\mu-\tau_c) + a_0$$
$$G_\mu^{(2)}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} = m_{21}\,\varphi_1(0) + m_{22}\,\varphi_2(0) + d_1\cos\varphi_3(0) -$$
$$-d_2\sin\varphi_3(0) + b_2\,k\,\varphi_3(-\mu-\tau_c) + b_0$$
$$G_\mu^{(3)}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} = \varphi_2(0)$$

where

$$d_1 = c\,m_0\cos\theta_0 - c\,c_1\sin\theta_0,\ \ d_2 = c\,m_0\sin\theta_0 + c\,c_1\cos\theta_0. \tag{4.4}$$

The linearized equation is given by the Fréchet derivative of G_μ in zero. $L_\mu = G'_\mu(0)$ is explicitly given by

$$L_\mu\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} = \tag{4.5}$$

$$= \begin{pmatrix} m_{11}\varphi_1(0) + m_{12}\,\varphi_2(0) - (c\,\sin\theta_0)\,\varphi_3(0) + b_1\,k\,\varphi_3(-\mu-\tau_c)\\ m_{21}\,\varphi_1(0) + m_{22}\,\varphi_2(0) - d_2\,\varphi_3(0) + b_2\,k\,\varphi_3(-\mu-\tau_c)\\ \varphi_2(0)\end{pmatrix}.$$

Introducing

$$F_\mu := G_\mu - L_\mu \tag{4.6}$$

equation (4.2) becomes

$$\dot{x} = L_\mu\,x_t + F_\mu(x_t),\ \ t \geq 0. \tag{4.7}$$

Obviously $F_\mu(0) = 0,\ F'_\mu(0) = 0$.

Define as in [10], [1], $X_0(\theta) = \begin{cases} 0, & -\mu-\tau_c \leq \theta < 0\\ 1, & \theta = 0\end{cases}$, and then define

$$A_\mu\,\varphi = \varphi' + X_0\left(L_\mu\,\varphi - \varphi'(0)\right),\ \ \varphi \in C. \tag{4.8}$$

Equation (4.7) can be written as

$$\dot{x}_t = A_\mu\,x_t + X_0\,F_\mu(x_t). \tag{4.9}$$

The initial data is $x(s) = \varphi(s),\ s \in [-\mu - \tau_c, 0]$. More details on operator A_μ and initial problem (4.9) can be found in [8], [10], [12].

An eigenvector of A_0 with respect to $\lambda = i\nu_0$ is given by $\gamma(\theta) = \mathrm{e}^{i\,\nu_0\,\theta} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$ where

$$\gamma_1 = \frac{c\,\sin\theta_0 + b_1\,k\,\mathrm{e}^{-i\,\nu_0\,\tau_c} - i\,\nu_0\,m_{12}}{m_{11} - i\,\nu_0},\ \gamma_2 = i\,\nu_0,\ \gamma_3 = 1. \tag{4.10}$$

For $\varphi \in C$ and $\psi \in C\left([0, \tau_c + \mu], \mathbb{C}^3\right)$ define, according to [8], [10], [12], the bilinear form

$$\langle \psi, \varphi \rangle = \sum_{j=1}^{3} \overline{\psi_j(0)}\,\varphi_j(0) - \sum_{j=1}^{3} \int_0^{-\mu-\tau_c} \overline{\psi}_j(\xi + \tau_c + \mu)\,\varphi_j(\xi)d\xi. \tag{4.11}$$

With respect to (4.11) the adjoint of A_μ given in (4.8) is

$$A_\mu^* \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = -\begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} +$$
$$+X_0^* \left[\begin{pmatrix} m_{11}\psi_1(0) + m_{21}\,\psi_2(0) \\ m_{12}\,\psi_1(0) + m_{22}\,\psi_2(0) + \psi_3(0) \\ -(c\sin\theta_0)\psi_1(0) + b_1 k\,\psi_1(\mu + \tau_c) - d_1\psi_2(0) - b_2\,k\,\psi_2(\mu + \tau_c) \end{pmatrix} - \right.$$
$$\left. - \begin{pmatrix} \psi_1'(0) \\ \psi_2'(0) \\ \psi_3'(0) \end{pmatrix} \right].$$

It follows that $\gamma^*(\theta) = \mathrm{e}^{i\,\nu_0\,\theta}\, d \begin{pmatrix} \tilde{\gamma_1} \\ \tilde{\gamma_2} \\ \tilde{\gamma_3} \end{pmatrix}$ is an eigenvector of A_0^* for

$$\tilde{\gamma_1} = 1,\ \tilde{\gamma_2} = \frac{i\,\nu_0 - m_{11}}{m_{12}},\ \tilde{\gamma_3} = -m_{12} + \frac{(i\,\nu_0 - m_{22})\,(i\,\nu_0 - m_{11})}{m_{21}} \tag{4.12}$$

and for every $d \in \mathbb{C}$. We choose d such that the normalizing condition $\langle \gamma^*, \gamma \rangle = 1$ be satisfied, thus $\bar{d} \cdot (\sum_{j=1}^{3} \gamma_j\,\bar{\tilde{\gamma}}_j)\left(1 + \tau_c\,e^{-i\,\nu_0\,\tau_c}\right) = 1$ so

$$\bar{d} = \left[(\sum_{j=1}^{3} \gamma_j\,\bar{\tilde{\gamma}}_j)\left(1 + \tau_c\,e^{-i\,\nu_0\,\tau_c}\right) \right]^{-1}. \tag{4.13}$$

Remark that, from $\Delta\,(i\,\nu_0, \tau_c) = 0$ and from (4.11) it follows that

$$\langle \gamma^*, \bar{\gamma} \rangle = 0. \tag{4.14}$$

The vectors γ and γ^* are used to define the restriction of equation (4.9) to the section $\mathcal{C}_0$ of the center manifold that corresponds to $\pm i\,\nu_0$ and to $\mu = 0$ (see [10], [11], [13], [14]). If x_t is a solution of (4.9) define, for $t \geq 0$,

$$z(t) = \left\langle \gamma^*, x_t \right\rangle \tag{4.15}$$

z and $\bar{z}$ will be used as local coordinates on $\mathcal{C}_0$ in the directions γ^* and $\bar{\gamma}^*$ respectively. (4.11) implies that

$$z(t) = \bar{d} \left(\sum_{j=1}^{3} \bar{\bar{\gamma}}_j \, x_j(t) - \sum_{j=1}^{3} \bar{\bar{\gamma}}_j \int_0^{-\tau_c} e^{-i\,\nu_0\,(\xi+\tau_c)} \cdot x_j(t+\xi)\, d\xi \right).$$

Define, for $t \geq 0$ and $s \in [-\tau_c, 0]$,

$$w(t,s) = x_t(s) - 2Re\,[z(t)\,\gamma(s)] = \begin{pmatrix} w^{(1)}(t,s) \\ w^{(2)}(t,s) \\ w^{(3)}(t,s) \end{pmatrix}. \tag{4.16}$$

On $\mathcal{C}_0$ we have $w(t,s) = W\,[z(t), \bar{z}(t), s]$ where

$$W(z,\bar{z},s) = \begin{pmatrix} w_{20}^{(1)}(s) \\ w_{20}^{(2)}(s) \\ w_{20}^{(3)}(s) \end{pmatrix} \frac{z^2}{2} + \begin{pmatrix} w_{11}^{(1)}(s) \\ w_{11}^{(2)}(s) \\ w_{11}^{(3)}(s) \end{pmatrix} z\,\bar{z} + \begin{pmatrix} w_{02}^{(1)}(s) \\ w_{02}^{(2)}(s) \\ w_{02}^{(3)}(s) \end{pmatrix} \frac{\bar{z}^2}{2} + \dots \tag{4.17}$$

For real x_t, w is real so $w_{02} = \bar{w}_{20}$. From (4.14) and (4.15) it follows that $\left\langle \gamma^*, w \right\rangle = \left\langle \gamma^*, x_t \right\rangle - \left(\left\langle \gamma^*, \left\langle \gamma^*, x_t \right\rangle \gamma \right\rangle + \left\langle \gamma^*, \overline{\left\langle \gamma^*, x_t \right\rangle} \bar{\gamma} \right\rangle \right) = 0$.

On the section $\mathcal{C}_0$,

$$\begin{aligned} \dot{z}(t) &= \left\langle \gamma^*, A_0\,x_t + X_0\,[F_0(x_t)] \right\rangle = i\,\nu_0\,z(t) + \left\langle \gamma^*, F_0[x(t)] \right\rangle \overset{(4.16)}{=} \\ &= i\,\nu_0\,z(t) + \left\langle \gamma^*, F_0[w(t,0) + 2Re z(t)\,\gamma(0)] \right\rangle = \\ &= i\,\nu_0\,z(t) + \overline{\gamma^*(0)}^T \cdot f_0\,(z(t), \bar{z}(t)) := \\ &:= i\,\nu_0\,z(t) + g\,[z(t), \bar{z}(t)] \end{aligned} \tag{4.18}$$

where dot means the scalar product in $\mathbb{C}^3$, T means transposed and $f_0(z,\bar{z})$ is given by the Taylor expansion of (3.3) around zero.

Namely

$$\begin{aligned} f_1(\alpha_1, q_1, \theta_1) &= m_{11}\,\alpha_1 + m_{12}\,q_1 + (c\,\cos\theta_0) \cdot \left(1 - \frac{\theta_1^2}{2!} + \frac{\theta_1^4}{4!} - \dots\right) - \\ &\quad - (c\,\sin\theta_0) \cdot \left(\theta_1 - \frac{\theta_1^3}{3!} + \dots\right) + a_0 + k\,b_1\,\theta_1\,(t-\tau_c) \\ f_2(\alpha_1, q_1, \theta_1) &= m_{21}\,\alpha_1 + m_{22}\,q_1 + d_1 \cdot \left(1 - \frac{\theta_1^2}{2!} + \frac{\theta_1^4}{4!} - \dots\right) - \\ &\quad - d_2 \cdot \left(\theta_1 - \frac{\theta_1^3}{3!} + \dots\right) + b_0 + k\,b_2\,\theta_1\,(t-\tau_c) \end{aligned}$$

We infer from (4.6) that

(4.19)
$$
\begin{aligned}
& F_0^{(1)}\left[\begin{pmatrix} w^{(1)}(t,0)+(z+\bar z)\,\gamma_1\\ w^{(2)}(t,0)+(z+\bar z)\,\gamma_2\\ w^{(3)}(t,0)+(z+\bar z)\,\gamma_3\end{pmatrix}\right] = c\cos\theta_0\cos\left[w^{(3)}(t,0)+(z+\bar z)\,\gamma_3\right]+\\
&\quad +c\sin\theta_0\left\{w^{(3)}(t,0)+(z+\bar z)\,\gamma_3-\sin\left[w^{(3)}(t,0)+(z+\bar z)\,\gamma_3\right]\right\}+a_0=\\
&= -\tfrac{c\cos\theta_0}{2}\left[w^{(3)}(t,0)+z+\bar z\right]^2+\tfrac{c\sin\theta_0}{6}\left[w^{(3)}(t,0)+z+\bar z\right]^3+\\
&\quad +O\left(\left[w^{(3)}(0)+z+\bar z\right]^4\right)\\
& F_0^{(2)}\left[\begin{pmatrix} w^{(1)}(t,0)+(z+\bar z)\,\gamma_1\\ w^{(2)}(t,0)+(z+\bar z)\,\gamma_2\\ w^{(3)}(t,0)+(z+\bar z)\,\gamma_3\end{pmatrix}\right] = d_1\cos\left[w^{(3)}(t,0)+(z+\bar z)\,\gamma_3\right]-\\
&\quad -d_2\left\{w^{(3)}(t,0)+(z+\bar z)\,\gamma_3-\sin\left[w^{(3)}(t,0)+(z+\bar z)\,\gamma_3\right]\right\}+b_0=\\
&= -\tfrac{d_1}{2}\left[w^{(3)}(t,0)+z+\bar z\right]^2-\tfrac{d_2}{6}\left[w^{(3)}(t,0)+z+\bar z\right]^3+O\left(\left[w^{(3)}(0)+z+\bar z\right]^4\right)\\
& F_0^{(3)}\left[\begin{pmatrix} w^{(1)}(t,0)+(z+\bar z)\,\gamma_1\\ w^{(2)}(t,0)+(z+\bar z)\,\gamma_2\\ w^{(3)}(t,0)+(z+\bar z)\,\gamma_3\end{pmatrix}\right] = 0
\end{aligned}
$$

(recall from (4.10) that $\gamma_3 = 1$).

Introduction of (4.19) into (4.18) gives

$$
\begin{aligned}
\dot z(t) = {} & i\,\nu_0\,z(t)+\\
& +\bar d\,\bar{\bar\gamma}_1\left\{-\frac{c\cos\theta_0}{2}\left[w^{(3)}(t,0)+z+\bar z\right]^2+\right.\\
& \left.+\frac{c\sin\theta_0}{6}\left[w^{(3)}(t,0)+z+\bar z\right]^3+\dots\right\}+\\
& +\bar d\,\bar{\bar\gamma}_2\left\{-\frac{d_1}{2}\left[w^{(3)}(t,0)+z+\bar z\right]^2-\frac{d_2}{6}\left[w^{(3)}(t,0)+z+\bar z\right]^3+\dots\right\}
\end{aligned}
$$

Since $w^{(3)}(t,0) = w^{(3)}_{20}(0)\frac{z^2}{2} + w^{(3)}_{11}(0)z\,\bar z + w^{(3)}_{02}(0)\frac{\bar z^2}{2} + \dots$ it follows that

$$
\begin{aligned}
g(z,\bar z) \;=\; & \bar d\,\frac{\bar{\bar\gamma}_1}{2}\left\{(-c\cos\theta_0)\left[w^{(3)}_{20}(0)\frac{z^2}{2}+w^{(3)}_{11}(0)z\,\bar z+w^{(3)}_{02}(0)\frac{\bar z^2}{2}+z+\bar z\right]^2+\right.\\
& +\;\left.\frac{c\sin\theta_0}{3}\left[w^{(3)}_{20}(0)\frac{z^2}{2}+w^{(3)}_{11}(0)z\,\bar z+w^{(3)}_{02}(0)\frac{\bar z^2}{2}+z+\bar z\right]^3+\dots\right\}-\\
& -\;\bar d\,\frac{\bar{\bar\gamma}_2}{2}\left\{d_1\left[w^{(3)}_{20}(0)\frac{z^2}{2}+w^{(3)}_{11}(0)z\,\bar z+w^{(3)}_{02}(0)\frac{\bar z^2}{2}+z+\bar z\right]^2+\right.\\
& +\;\left.\frac{d_2}{3}\left[w^{(3)}_{20}(0)\frac{z^2}{2}+w^{(3)}_{11}(0)z\,\bar z+w^{(3)}_{02}(0)\frac{\bar z^2}{2}+z+\bar z\right]^3+\dots\right\}:=\\
& :=\frac{1}{2}\,g_{20}\,z^2+g_{11}\,z\,\bar z+\frac{1}{2}g_{02}\,\bar z^2+\frac{1}{2}g_{21}\,z^2\,\bar z+\dots
\end{aligned}
\tag{4.20}
$$

(4.18) is called the normal form obtained by restricting the flow to the center manifold. The Liapunov coefficients are computed using the coefficients of the Taylor expansion of g around $(0,0)$. Define

$$L_1(0) = \frac{i}{2\nu_0}\left(g_{20}\,g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{1}{2}g_{21} \tag{4.21}$$

The first Lyapunov coefficient is

$$l_1(0) = Re L_1(0) \tag{4.22}$$

So, for $l_1(0)$, we need $g_{20},\ g_{11}$ and g_{21}.
It follows straightly from (4.20) that

$$\begin{array}{l} g_{20} = \frac{\bar{d}\,\bar{\bar{\gamma}}_1}{2}(-c\cos\theta_0) - \frac{\bar{\bar{\gamma}}_2\,\bar{d}\,d_1}{2} \\ g_{02} = \overline{g_{20}} \\ g_{11} = \bar{d}\,\bar{\bar{\gamma}}_1(-c\cos\theta_0) - \bar{\bar{\gamma}}_2\,\bar{d}\,d_1 \end{array} \tag{4.23}$$

and

$$g_{21} = \bar{d}\,\bar{\bar{\gamma}}_1(-c\cos\theta_0)\left[w_{20}^{(3)}(0) + 2w_{11}^{(3)}(0)\right] - \bar{d}\,\bar{\bar{\gamma}}_2 d_1\left[w_{20}^{(3)}(0) + 2w_{11}^{(3)}(0)\right] \tag{4.24}$$

To compute $w_{20}^{(3)}(0)$ and $w_{11}^{(3)}(0)$ remark that it follows from (4.9), (4.16) and (4.18) that

$$\begin{array}{l} \frac{d}{dt}w(t,\cdot) = \frac{d}{dt}x_t(\cdot) - \frac{d}{dt}\left[z(t)\gamma(\cdot) + \bar{z}(t)\bar{\gamma}(\cdot)\right] = \\ \quad = A_0\,w(t,\cdot) + X_0F_0\left[w(t,\cdot) + 2Re\left(z(t)\gamma(\cdot)\right)\right] - \\ \quad\ -2Re\left[g(z(t),\bar{z}(t))\gamma(\cdot)\right] = \\ \quad = A_0\,w(t,\cdot) + H\left[z(t),\bar{z}(t),\cdot\right] \end{array} \tag{4.25}$$

where, for $s \in [-\tau_c, 0]$ we define

$$H(z,\bar{z},s) = -2Re\left[g(z,\bar{z})\gamma(s)\right] + X_0(s)F_0\left[W(z,\bar{z},s) + 2Re\left[z\gamma(s)\right]\right] \tag{4.26}$$

Thus, for $s \in [-\tau_c, 0)$
$H(z,\bar{z},s) = -2Re\left[g(z,\bar{z})\gamma(s)\right] = H_{20}(s)\frac{z^2}{2} + H_{11}(s)z\bar{z} + H_{02}(s)\frac{\bar{z}^2}{2} + \ldots$
with

$$\begin{array}{l} H_{20}(s) = -g_{20}\,\gamma(s) - \bar{g}_{02}\,\bar{\gamma}(s) = -2Re\left[g_{20}\,\gamma(s)\right] \\ H_{11}(s) = -2Re\left[g_{11}\,\gamma(s)\right] \\ H_{02}(s) = \overline{H_{20}}(s) \end{array} . \tag{4.27}$$

By (4.25) and (4.17)

$$\begin{array}{l} A_0\,w(t,s) + H\left[z(t),\bar{z}(t),s\right] = \frac{d}{dt}w(t,s) = \\ = \frac{d}{dt}\left[w_{20}(s)\frac{z^2(t)}{2} + w_{11}(s)z(t)\bar{z}(t) + w_{02}(s)\frac{\bar{z}^2(t)}{2} + \ldots\right] = \\ = w_{20}(s)z(t)\dot{z}(t) + w_{11}(s)\left[\dot{z}(t)\bar{z}(t) + z(t)\dot{\bar{z}}(t)\right] + w_{02}(s)\bar{z}(t)\dot{\bar{z}}(t) + \ldots = \\ = w_{20}(s)z\left[i\,\nu_0\,z + g(z,\bar{z})\right] + w_{11}(s)\left[\bar{z}\left(i\,\nu_0\,z + g(z,\bar{z})\right) + \right. \\ \quad \left. + \ z\left(-i\,\nu_0\,\bar{z} + \overline{g(z,\bar{z})}\right)\right] + w_{02}(s)\bar{z}\left[-i\,\nu_0\,\bar{z} + \overline{g(z,\bar{z})}\right] \end{array} \tag{4.28}$$

(A_0 is acting on variable s).

Identifying the terms corresponding to $z^2,\ z\,\bar{z},\ \bar{z}^2$ in (4.28) and recalling (4.17) we get

$$\begin{array}{l} (A_0 - 2i\,\nu_0)\,w_{20}(s) = -H_{20}(s) \\ A_0\,w_{11}(s) = -H_{11}(s) \\ (A_0 + 2i\,\nu_0)\,w_{02}(s) = -H_{02}(s) \end{array} \quad . \tag{4.29}$$

So, $\dot{w}_{20} = 2i\,\nu_0\,w_{20} + g_{20}\,e^{i\,\nu_0\,s}\begin{pmatrix}\gamma_1\\ i\,\nu_0\\ 1\end{pmatrix} + \bar{g}_{02}\,e^{-i\,\nu_0\,s}\begin{pmatrix}\bar{\gamma}_1\\ -i\,\nu_0\\ 1\end{pmatrix}$

therefore

$$\dot{w}^{(3)}_{20}(s) = 2i\,\nu_0\,w^{(3)}_{20}(s) + 2Re\left(g_{20}\,e^{i\,\nu_0\,s}\right). \tag{4.30}$$

Making $s = 0$ in (4.25) and identifying again the coefficient of z^2 one gets

$$\begin{cases} i\,\nu_0\,w^{(1)}_{20}(0) = m_{11}w^{(1)}_{20}(0) + m_{12}w^{(2)}_{20}(0) - (c_1\sin\theta_0)w^{(3)}_{20}(0) + \\ \qquad\qquad + kb_1 w^{(3)}_{20}(-\tau_c) + H^{(1)}_{20}(0) \\ \\ i\,\nu_0\,w^{(2)}_{20}(0) = m_{21}w^{(1)}_{20}(0) + m_{22}w^{(2)}_{20}(0) - d_1\,w^{(3)}_{20}(0) + \\ \qquad\qquad + kb_2 w^{(3)}_{20}(-\tau_c) + H^{(2)}_{20}(0) \\ \\ i\,\nu_0\,w^{(3)}_{20}(0) = w^{(2)}_{20}(0) + H^{(3)}_{20} \end{cases} \quad . \tag{4.31}$$

It follows from (4.30) that

$$\begin{aligned} w^{(3)}_{20}(s) &= w^{(3)}_{20}(0)e^{2i\,\nu_0\,s} + 2\int_0^s e^{2i\,\nu_0\,(s-\theta)} Re\left(g_{20}\,e^{i\,\nu_0\,\theta}\right)d\theta = \\ &= w^{(3)}_{20}(0)e^{2\,i\,\nu_0\,s} + e^{2\,i\,\nu_0\,s}\left(g_{20}\int_0^s e^{-i\,\nu_0\,\theta}d\theta + \bar{g}_{20}\int_0^s e^{-3i\,\nu_0\,\theta}d\theta\right) = \\ &= \left(w^{(3)}_{20}(0) + \frac{g_{20}}{i\nu_0} + \frac{\bar{g}_{20}}{3i\nu_0}\right)e^{2\,i\,\nu_0\,s} - \frac{g_{20}}{i\nu_0}e^{i\,\nu_0\,s} - \frac{\bar{g}_{20}}{3i\nu_0}e^{-i\,\nu_0\,s} \end{aligned}$$

Then

$$w^{(3)}_{20}(-\tau_c) = \left(w^{(3)}_{20}(0) + \frac{g_{20}}{i\nu_0} + \frac{\bar{g}_{20}}{3i\nu_0}\right)e^{-2\,i\,\nu_0\,\tau_c} - \frac{g_{20}}{i\nu_0}\,e^{-i\,\nu_0\,\tau_c} - \frac{\bar{g}_{20}}{3i\nu_0}\,e^{i\,\nu_0\,\tau_c} \tag{4.32}$$

and this value is to be introduced in (4.31).

Identification in (4.28) of coefficients of $z\bar{z}$ when $s=0$ leads to

$$(4.33)\qquad \begin{cases} m_{11}w_{11}^{(1)}(0)+m_{12}w_{11}^{(2)}(0)-(c\sin\theta_0)w_{11}^{(3)}(0)+ \\ +kb_1w_{11}^{(3)}(-\tau_c)+H_{11}^{(1)}(0)=0 \\ m_{21}w_{11}^{(1)}(0)+m_{22}w_{11}^{(2)}(0)-d_1\,w_{11}^{(3)}(0)+ \\ +kb_2w_{11}^{(3)}(-\tau_c)+H_{11}^{(2)}(0)=0 \\ w_{11}^{(2)}(0)+H_{11}^{(3)}(0)=0 \end{cases}.$$

We infer from the second equation in (4.29) and from (4.27) that

$$\dot{w}_{11}(s)=g_{11}\,\gamma(s)+\bar{g}_{11}\,\bar{\gamma}(s)=g_{11}\,e^{i\,\nu_0\,s}\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}+\bar{g}_{11}\,e^{-i\,\nu_0\,s}\begin{pmatrix}\bar{\gamma}_1\\ \bar{\gamma}_2\\ \bar{\gamma}_3\end{pmatrix}$$

so

$$w_{11}(s)=g_{11}\frac{e^{i\,\nu_0\,s}-1}{i\,\nu_0}\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}-\bar{g}_{11}\frac{e^{-i\,\nu_0\,s}-1}{i\,\nu_0}\begin{pmatrix}\bar{\gamma}_1\\ \bar{\gamma}_2\\ \bar{\gamma}_3\end{pmatrix}+w_{11}(0).$$

Then, using (4.10), it follows that

$$(4.34)\qquad w_{11}^{(3)}(-\tau_c)=g_{11}\frac{e^{-i\,\nu_0\,\tau_c}-1}{i\,\nu_0}-\bar{g}_{11}\frac{e^{i\,\nu_0\,\tau_c}-1}{i\,\nu_0}+w_{11}^{(3)}(0)$$

and this value will be introduced in (4.33).

In order to solve system (4.31) to find $w_{20}^{(3)}(0)$ and system (4.33) to find $w_{11}^{(3)}(0)$ we need $H_{20}(0)$ and $H_{11}(0)$.

From definition (4.26), using (4.19), one gets

$$\begin{aligned} H(z,\bar{z},0) &= -2Re\left[g(z,\bar{z})\,\gamma(0)\right]+F_0\left[W(z,\bar{z},0)+2Re\left[z\,\gamma(0)\right]\right]= \\ &= -2Re\left[\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}\frac{1}{2}g_{20}\,z^2+g_{11}\,z\bar{z}+\frac{1}{2}g_{02}\,\bar{z}^2+\frac{1}{2}g_{21}\,z^2\bar{z}+\ldots\right]+ \\ &+\begin{pmatrix} -\frac{c\,\cos\theta_0}{2}\left[w^{(3)}(0)+z+\bar{z}\right]^2+\frac{c\,\sin\theta_0}{6}\left[w^{(3)}(0)+z+\bar{z}\right]^3+\ldots \\ -\frac{d_1}{2}\left[w^{(3)}(0)+z+\bar{z}\right]^2-\frac{d_2}{6}\left[w^{(3)}(0)+z+\bar{z}\right]^3+\ldots \\ 0\end{pmatrix}\end{aligned}$$

With $w^{(3)}(0)=w_{20}^{(3)}(0)\frac{z^2}{2}+w_{11}^{(3)}(0)z\bar{z}+w_{02}^{(3)}(0)\frac{\bar{z}^2}{2}$, identification of coefficients for z^2 and $z\bar{z}$ gives

$$(4.35)\qquad H_{20}(0)=-Re\left[g_{20}\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}\right]-\begin{pmatrix}\frac{c\,\cos\theta_0}{2}\\ \frac{d_1}{2}\\ 0\end{pmatrix}$$

and

$$H_{11}(0) = -Re\left[g_{11}\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}\right] - \begin{pmatrix}c\cos\theta_0\\ d_1\\ 0\end{pmatrix} \tag{4.36}$$

Systems (4.31) and (4.33) can be solved now and (4.24) gives g_{21} and calculation of the Lyapunov coefficient $l_1(0)$ is completed.

Define

$$\mu_2 = -\frac{l_1(0)}{Re\,[\lambda'(\tau_c)]},\ T_2 = -\frac{Im\,[L_1(0)] + \mu_2\, Im\,[\lambda'(\tau_c)]}{\nu_0} \tag{4.37}$$

we have

Theorem 4.1 ([10]) *If the Lyapunov coefficient $l_1(0)$ defined in* (4.22) *is negative, periodic solutions (limit cycles) exist, for equation (3.3), if $\tau > \tau_c$, τ close to τ_c, and are orbitally stable. They exist for $\tau < \tau_c$ and are unstable if $l_1(0) > 0$. Their period increases if $T_2 > 0$ and decreases for $T_2 < 0$ (T_2 given in* (4.37)*).*

By [10] the periodic solutions are approximated by

$$\varphi(t,\mu) = 2\left(\frac{\mu}{\mu_2}\right)^{\frac{1}{2}} Re\left[e^{i\,\nu_0\,t}\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}\right] + \begin{pmatrix}\alpha_0\\ 0\\ \theta_0\end{pmatrix} \tag{4.38}$$

(μ is defined in (4.1)).

§5. Case study (M = 0,25; H = 500 m)

The formulae developed in paragraphs 3 and 4 are applied for a specific model based on ADMIRE (AeroData Model In a Research Environment, see [3]). Specifically

$$m_{11} = -1.3892;\quad m_{12} = 1.6688;\quad c = 0.1161;\quad b_1 = -0.5209;$$

$$m_{21} = 4.2859;\quad m_{22} = -14.4124;\quad c_1 = 0.7153;\quad b_2 = -6.3859;$$

$$m_0 = -5.2641$$

In Figure 1, the dependence of τ_c on k is depicted.

The higher k is the smaller is the critical time for pilot reaction. The limit cycles are depicted in Figure 2.

For $k = 4$, equilibria in (2.1) are $\alpha_0 = 0,0954029$; $\theta_0 = -0.00789153$. The solution of (3.8) is $\tau_c = 0.552144$ and $\nu_0 = 2.04975$. It follows that $Re\,\lambda_0'(\tau_c) = 1.8379$ so Theorem 3.1 applies.

Computations on the lines in §4 yield $l_1(0) = -1.63799$ so the limit cycle provided by Theorem 3.1 is a stable one (see Figures 3a and 3b). Also $\mu_2 = 0.891232$ and $T_2 = 2.07495$.

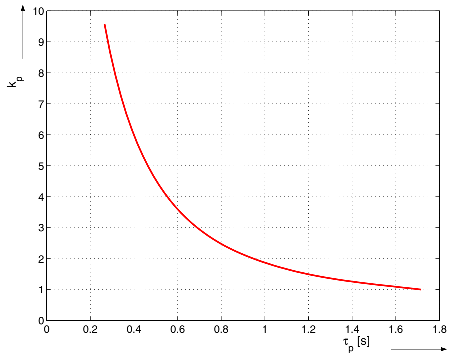
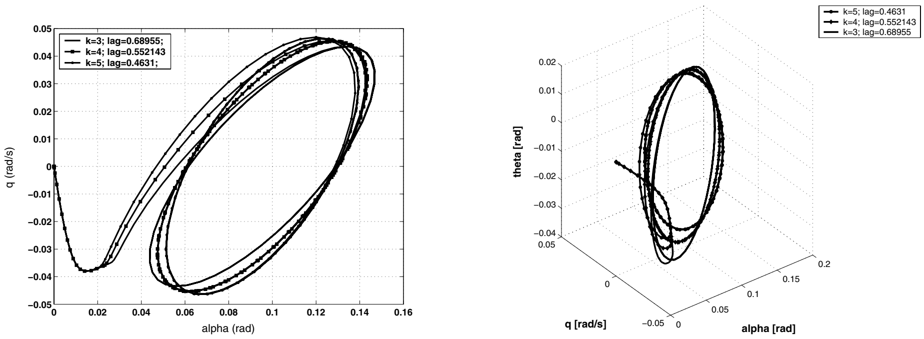
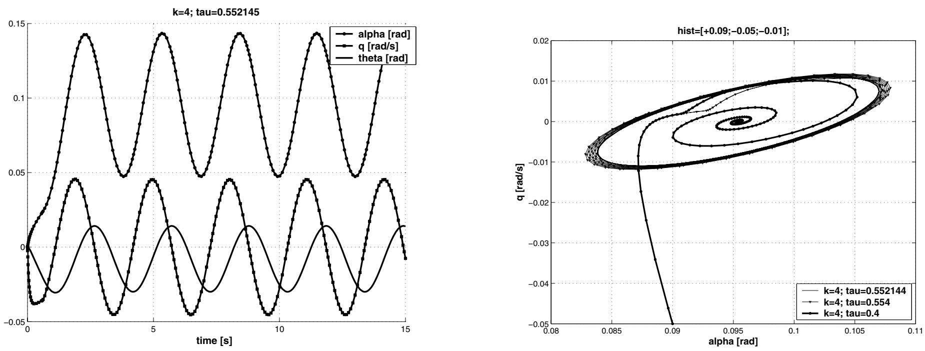


Figure 1: Dependence of τ_c on k



a) Limit cycles in 2 dimensions b) Limit cycles in 3 dimensions

Figure 2: Limit cycles



a) b)

Figure 3: Stable limit cycle

§6. Concluding remarks

The qualitative analysis of the mathematical model for a longitudinal motion with flight control system revealed that high pilot gains reduce pilot's allowed time delay, that is, in highly demanding tasks the pilot's reaction must be quick enough. It follows from the analysis of the stability of equilibria that, after a certain threshold τ_c, the asymptotic stability is lost and a limit cycle appears through a Hopf bifurcation. Its characteristics depend on pilot's gain k. Numerical calculations of the Lyapunov coefficient show that in some situations the limit cycle is stable (for example when $k \in \{3, 4, 5\}$) while in other cases is unstable ($k \in \{1, 2, 6, 7\}$).

Numerical simulations performed for $k = 4$ and τ close to τ_c, $\tau > \tau_c$ reveal a more complex situation: for constant functions as initial data not very close to the limit cycle (see (4.38)) the solution approaches another limit cycle, encircling the one given by Hopf bifurcation theorem (see Fig. 4).

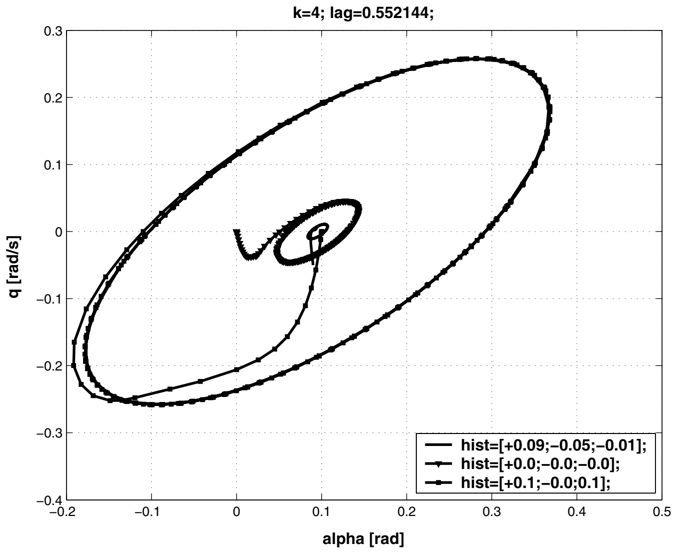


Figure 4: Complex situation: two limit cycles

This is much alike the toric bifurcation (see [10]) and deserves further study both theoretical and numerical.

Acknowledgement

This research was partially supported by Grant CNCSIS ID-919/2007 and by the National Authority for Scientific Research of Romania under contract C143 and was performed in the framework of GARTEUR FM/AG-15 "Pilot-in-the-Loop Oscillations - analysis and test techniques for their prevention" phase II 2004-2007.

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