Moments and Semi-Moments for fuzzy portfolios selection

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February 18, 2011

Abstract

The aim of this paper is to consider the moments and the semi-moments (i.e semi-kurtosis) for portfolio selection with fuzzy risk factors (i.e. trapezoidal risk factors). In order to measure the leptokurtocity of fuzzy portfolio return, notions of moments (i.e. Kurtosis) kurtosis and semi-moments(i.e. Semi-kurtosis) for fuzzy portfolios are originally introduced in this paper, and their mathematical properties are studied. As an extension of the mean-semivariance-skewness model for fuzzy portfolio, the mean-semivariance-skewness-semikurtosis is presented and its four corresponding variants are also considered. We briefly designed the genetic algorithm integrating fuzzy simulation for our optimization models.

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1 INTRODUCTION

In portfolio analysis, an asset return is usually characterized as a random variable on a probability space (see Bachelier [1] and Markowitz [14]). An important area of finance research is portfolio selection which is to select a combination of assets under the constraints of the investor objectives. Since returns are uncertain in nature, allocation capital in different risky assets to minimize risk and to maximize the return is the main concern of portfolio selection. Modern portfolio selection theory has been introduced by the seminal work of [14] which consider trade-off between return and risk. As in Markowitz[15], variance has been widely accepted as a risk measure by numerous portfolio selection models. However variance as a risk measure has some shortcomings and limitations (see [15]).

One important shortcoming is that analysis based on variance considers high returns as equally undesirable low returns (i.e. it does not take into account the asymmetry of the probability distribution). Then there is a controversy over the issue of whether higher moments should be considered in portfolio selection models. Some authors such as Samuelson [18], Krauss et al. [11], Konno et al.[9]-[10], Briec et al.[4] have argued that it is important to take into account higher moments than the first and second ones (see also For instance Samuelson [18] showed that investors would prefer a portfolio with a larger third order moment if first and second moments are same.

The above literature assumed that the securities returns are random variables with fixed expected returns and variances values. However, since investors receive efficient or inefficient information from the real world, ambiguous factors usually exist in it. Consequently, we need to consider not only random conditions but also ambiguous and subjective conditions for portfolio selection problems. A recent litterature has recognized the fuziness and the uncertainty of portfolios returns. As discussed in [6], investors can make use of fuzzy set to reflect the vagueness and ambiguity of securities (i.e. incompleteness of information due to the lack of data). Therefore, the probability theory becomes difficult to used. For example, some authors such as Tanaka and Guo [19] quantified mean and variance of a portfolio through fuzzy probability and possibility distributions, Carlsson et al.[2]-[3] used their own definitions of mean and variance of fuzzy numbers. In particular, Huang [7] quantified portfolio return and risk by the expected value and variance based on credibility measure. Recently, Huang [7] has proposed the mean-semivariance model for portfolio selection and Li et al.[5], Kar et al.[8] introduced mean-variance-skewness for portfolio selection with fuzzy returns.

Different from Huang [7] and Li et al.[5], after recalling the definition of mean, variance, semi-variance and skewness, this paper consider the k-moments (i.e. Kurtosis for $k = 4$) and semi-moments (i.e. semi-Kurtosis for $k = 4$) for portfolio selection with fuzzy risk factors (i.e. returns). Several empirical studies show that portfolio returns have fat tails. Generally investors would prefer a portfolio return with smaller kurtosis which indicates the leptokurtosis (fat-tails or thin-tails) when the mean value, the variance and the asymmetry are the same. In order to measure the leptokurtocity of fuzzy portfolio return, notions of moments and the semi-moments of fuzzy portfolio are originally introduced in this paper, and their mathematical properties are studied. As an extension of the mean-semivariance-skewness model for fuzzy portfolio, the mean-semivariance-skewness-semikurtosis is presented and the corresponding variants (the mean-variance-skewness-kurtosis, the mean-variance-skewness-semikurtosis and the mean-variance-skewness-semikurtosis models) are also considered. We briefly designed the genetic algorithm integrating fuzzy simulation for our optimization models.

The paper is organized as follows. In Section 2, we review some preliminary knowledge on fuzzy variable and credibility measure. In Section 3, we recall the notions of mean, variance, and skewness of a fuzzy variable. We introduce kurtosis for fuzzy variables, study some of its properties and determine, for an integer $k > 1$, the k-moment of a symmetric fuzzy trapezoidal fuzzy variable. We compute variance, skewness and kurtosis of trapezoidal numbers and triangular numbers. In Section 4, we introduce the notion of semi-moment of order n=2p ($p \in \mathbb{N}^*$) of a fuzzy variable. We justify that the particular cases of the semi-moment are the known notion of semi variance and the new notion of semi-Kurtosis for $p = 1$ and $p = 2$ respectively. We compute the semi-variance and the semi-kurtosis of a trapezoidal fuzzy variable. We establish some links between moment and semi-moment of fuzzy variable. After a brief introduction of fuzzy-simulation-based genetic algorithm, Section 5 suggests some determinist optimization programs with a family of independent triangular fuzzy numbers and, proposes a genetic algorithm to compute Kurtosis and semi-kurtosis of a fuzzy variable. Section 6 contains some concluding remarks and the proofs are in Section 7.

2 Fuzzy variable and credibility

Let ξ be a fuzzy variable with membership function μ. For any $x \in \mathbb{R}$, $\mu(x)$ represents the possibility that ξ takes value x. For any set B, Liu defined the credibility measure as the average of possibility measure and necessity measure as follows:

$$Cr(\{\xi \in B\}) = \frac{1}{2}\left(\sup_{x\in B}\mu(x) \;-\; \sup_{x\in B^c}\mu(x) + 1\right). \quad (1)$$

It is easy to show that credibility measure is self-dual. That is,

$$Cr(\{\xi \in B\}) + Cr(\{\xi \in B^c\}) = 1.$$

Remark 1. Note that for ξ taking values in B, Zadeh has defined the possibility measure of B by

$$Pos(\{\xi \in B\}) = \sup_{x\in B}\mu(x)$$

and the necessity measure of ξ by

$$Nec(\{\xi \in B\}) = 1 - \sup_{x\in B^c}\mu(x).$$

But neither, of these measures are self-dual. That reason also justified the introduction of the credibility measure by Liu [12].

Example 1. *1. Let $\xi = (a, b, c, d)$ be a trapezoidal fuzzy number (with $a \leq b \leq c \leq d$). For any $r \in \mathbb{R}$, $Cr(\xi \leq r)$ is defined as follows:*

$$Cr(\{\xi \leq r\}) = \begin{cases} 0 \textit{ if } \; r < a \\ \frac{1}{2}(\frac{r-a}{b-a}) \; \textit{ if } a \leq r < b \\ \frac{1}{2} \textit{ if } b \leq r < c \\ 1 - \frac{1}{2}(\frac{r-d}{c-d}) \textit{ if } c \leq r < d \\ 1 \textit{ if } d \leq r \end{cases}$$

and

$$Cr(\{\xi \geq r\}) = \begin{cases} 1 \textit{ if } r < a \\ 1 - \frac{1}{2}(\frac{r-a}{b-a}) \textit{ if } a \leq r < b \\ \frac{1}{2} \textit{ if } \; b \leq r < c \\ \frac{1}{2}(\frac{r-d}{c-d}) \textit{ if } c \leq r < d \\ 0 \textit{ if } d \leq r \end{cases}.$$

2. For all $r \in \mathbb{R}$, the credibility of a triangular fuzzy variable $\xi = (a, b, c)$ (with $a \leq b \leq c$) is given by:

$$Cr(\{\xi \leq r\}) = \begin{cases} 0 \textit{ if } r < a \\ \frac{1}{2}(\frac{r-a}{b-a}) \textit{ if } a \leq r < b \\ 1 - \frac{1}{2}(\frac{r-c}{b-c}) \textit{ if } b \leq r < c \\ 1 \textit{ if } \; c \leq r. \end{cases}$$

and

$$Cr(\{\xi \geq r\}) = \begin{cases} 1 \ \mathit{if}\ r < a \\ 1 - \frac{1}{2}(\frac{r-a}{b-a}) \ \mathit{if}\ a \leq r < b \\ \frac{1}{2}(\frac{r-c}{b-c}) \ \mathit{if}\ b \leq r < c \\ 0 \ \mathit{if}\ c \leq r \end{cases}.$$

Let us end this Section by giving some notations useful throughout this paper.

- For a trapezoidal fuzzy variable $\xi = (a, b, c, d)$ such that $a \neq b$ and $c \neq d$, $supp(\xi) = [a, d]$ its support, $cor(\xi) = [b, c]$ its core, l_s the length of $supp(\xi)$ and l_c the length of $cor(\xi)$. We set:

$$\alpha = b - a, \beta = d - b, l_s(\xi) = d - a \text{ and } l_c(\xi) = c - b.$$

- For a triangular fuzzy variable $\xi = (a, b, c)$ such that $b \neq a$ and $c \neq a$, we set:

$$\alpha_1 = \max\{b - a, c - b\} \text{ and } \gamma = \min\{b - a, c - b\}.$$

- $\xi = (a, b, c, d)$ is symmetric (that is $\exists t \in \mathbb{R}, \forall r \in \mathbb{R}, \mu(t - r) = \mu(t + r)$) if $\alpha = \beta$, and $\xi = (a, b, c)$ is symmetric if $\alpha_1 = \gamma$,

3 Moments of the trapezoidal of fuzzy variables

3.1 Expected Value, Variance and Skewness of fuzzy random variables

The definitions of the expected value, variance and skewness of fuzzy variables are obtained from Li et al. [5].

Definition 1. *Let ξ be a fuzzy variable. Then its expected value is defined as*

$$E[\xi] = e = \int_0^{+\infty} Cr\{\xi \geq r\}\, dr \ - \int_{-\infty}^{0} Cr\{\xi \leq r\}\, dr \qquad (2)$$

provided that at least one of the above integrals is finite.

Remark 2. Note that, expected value is one of the most important concepts of fuzzy variable, which gives the center of its distribution.

Example 2. *The expected value of a trapezoidal fuzzy variable denoted $\xi = (a, b, c, d)$ is given by $E[\xi] = \frac{a+b+c+d}{4}$ and the expected value of a triangular fuzzy variable denoted $\xi = (a, b, c)$ is given by $E[\xi] = \frac{a+2b+c}{4}$.*

Definition 2. *Let ξ be a fuzzy variable with finite expected value e. Then its variance is defined as*

$$V[\xi] = E[(\xi - e)^2]. \tag{3}$$

Let us determine the variance of a trapezoidal fuzzy variable and that of a triangular fuzzy variable.

Example 3. *1. Let $\xi = (a, b, c, d)$ be a fuzzy trapezoidal variable with expected value $E[\xi] = \frac{a+b+c+d}{4} = e$. The variance $V[\xi]$ of ξ is given by:*

$$V[\xi] = -[\frac{1}{4}(l_s(\xi) + l_c(\xi))]^3(\frac{|\alpha - \beta|}{3\alpha\beta}) + \max(\frac{(\frac{|\alpha-\beta|}{4} - \frac{1}{2}l_c(\xi))^3}{6\alpha \vee \beta}, 0)+$$

$$\frac{|\alpha - \beta|}{2\alpha\beta}[\frac{1}{2}l_s(\xi) - \frac{(\alpha + \beta)}{4}][\frac{1}{4}(l_s(\xi) + l_c(\xi))]^2 + \frac{(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_s(\xi))^3}{6\alpha \vee \beta} - \frac{(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_c(\xi))^3}{6\alpha \wedge \beta}.$$

2. We can easily check that if ξ is symmetric ($\alpha = \beta$), $V[\xi]$ simply becomes

$$V[\xi] = \frac{3[l_c(\xi) + \beta]^2 + \beta^2}{24}.$$

3. Let $\xi = (a, b, c)$ be a triangular fuzzy variable such that $E[\xi] = \frac{a+2b+c}{4} = e$.
The variance $V[\xi]$ of ξ can be deduced from the variance of a trapezoidal one by this way :

$$V[\xi] = \frac{33\alpha_1^3 + 21\alpha_1^2\gamma + 11\alpha_1\gamma^2 - \gamma^3}{384\alpha_1}.$$

4. More precisely:
- The variances of the following three trapezoidal fuzzy variables are: $V[(-1, 2, 3, 4)] = \frac{41}{24}$, $V[(1, 2, 3, 4)] = \frac{13}{24}$ and $V[(-1, 0, 1, 4)] = \frac{41}{24}$.
- The variances of the following three triangular fuzzy variables are: $V[(-1, 0, 4)] = \frac{2491}{1536}$ and $V[(-1, 1, 2)] = V[(1, 2, 4)] = \frac{123}{256}$.

Let us end this Subsection with some useful preliminaries on the Skewness of a fuzzy variable.

Definition 3. *Let ξ be a fuzzy variable with finite expected value e. Then its skewness is defined as*

$$Sk(\xi) = E[(\xi - e)^3]. \tag{4}$$

Remark 3. If ξ has a symmetric membership, then $Sk[\xi] = 0$, see [5].

In the following example, we determine the skewness of trapezoidal and triangular fuzzy variable respectively.

Example 4. 1. *The skewness of a trapezoidal fuzzy variable $\xi = (a, b, c, d)$ is given by*

$$Sk[\xi] = \frac{1}{8(b-a)}[(\frac{b-e}{4})^4 - (\frac{a-e}{4})^4] + \frac{1}{8(c-d)}[(\frac{c-e}{4})^4 - (\frac{d-e}{4})^4].$$

2. *The skewness of a triangular fuzzy variable $\xi = (a, b, c)$ is given by*

$$Sk[\xi] = \frac{1}{8(b-a)}[(\frac{b-e}{4})^4 - (\frac{a-e}{4})^4] + \frac{1}{8(b-c)}[(\frac{b-e}{4})^4 - (\frac{c-e}{4})^4].$$

that is,

$$Sk[\xi] = \frac{(c-a)^2}{32}(c + a - 2b).$$

In the following Subsection, we determine, for an integer $k > 1$, the k-moment of a symmetric trapezoidal fuzzy variable.

3.2 k-moment of a symmetric trapezoidal fuzzy variable

Proposition 1. *Let $\xi = (a, b, c, d)$ be a symmetric trapezoidal fuzzy variable with expected value $E[\xi] = e$. For an integer $k > 1$, the k-moment $m_k = E[(\xi - e)^k]$ is given by:*

$$m_k = \begin{cases} 0 \text{ if } k \text{ is odd} \\ \frac{\sum_{i=0}^{\frac{k}{2}} C_{k+1}^{2i+1}[(c-b)+\alpha]^{k-2i}}{2^{k+1}(k+1)} \text{ if } k \text{ is even} \end{cases}$$

Corollary 1. *Let $\xi = (a, b, c)$ be a symmetric triangular fuzzy variable with expected value $E[\xi] = e$. For an integer $p \geq 1$, the k-moment $m_k = E[(\xi - e)^k]$ is given by:*

- *If $k = 2p + 1$, then*

$$m_k = m_{2p+1} = 0 \tag{5}$$

- *If $k = 2p$, then $m_k = m_{2p} = \frac{\alpha^k}{4p+2}$.*

3.3 Kurtosis: definitions, first properties and some particular cases

In this section, we introduce the kurtosis of a fuzzy variable. We study its properties and give some examples.

Definition 4. *Let ξ be a fuzzy variable such that $E[\xi] = e < \infty$.*

- *The kurtosis of ξ, denoted $K[\xi]$, is given by:*

$$K[\xi] = E[(\xi - e)^4].$$

- *The normalized kurtosis of ξ, denoted $K^1[\xi]$, is given by:*

$$K_1[\xi] = \frac{E[(\xi - e)^4]}{(\sigma[\xi])^4}.$$

We can rewrite $K[\xi]$ and $K_1[\xi]$ by means of a credibility measure. For such, we have:

Let ξ be a fuzzy variable such that $E[\xi] = e < \infty$.

- The kurtosis $K[\xi]$ is given by:

$$K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\}\, dr. \tag{6}$$

- The normalized kurtosis $K_1[\xi]$ is given by:

$$K_1[\xi] = \frac{\int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\} dr}{[\int_0^{+\infty} Cr\{(\xi - e)^2 \geq r\} dr]^2}. \tag{7}$$

The following result determines the Kurtosis by means of a credibility measure. It also establishes some properties on the linearity of the Kurtosis.

Proposition 2. *Let ξ be a fuzzy variable such that $E[\xi] = e$.*

1. *The kurtosis of ξ is defined by*

$$K[\xi] = \int_0^{+\infty} Cr\{\xi - e \geq \sqrt[4]{r}\} \vee Cr\{\xi - e \leq \sqrt[4]{r}\} dr. \tag{8}$$

2. *The normalized kurtosis of ξ is defined by*

$$K^1[\xi] = \frac{\int_0^{+\infty} Cr\{\xi - e \geq \sqrt[4]{r}\} \vee Cr\{\xi - e \leq \sqrt[4]{r}\} dr}{[\int_0^{+\infty} Cr\{\xi - e \geq \sqrt[2]{r}\} \vee Cr\{\xi - e \leq \sqrt[2]{r}\} dr]^2}. \tag{9}$$

3. $\forall a, b \in \mathbb{R}, K[a\xi + b] = a^4 K[\xi]$.

4. $\forall a, b \in \mathbb{R}, K^1[a\xi + b] = K^1[\xi]$.

When ξ becomes a symmetric fuzzy variable, then the previous formulas become

Corollary 2. *If ξ is a symmetric fuzzy variable, then (8) becomes*

$$K[\xi] = \int_0^{+\infty} Cr\{\xi - e \geq \sqrt[4]{r}\}dr. \tag{10}$$

Then (9) becomes

$$K^1[\xi] = \frac{\int_0^{+\infty} Cr\{\xi - e \geq \sqrt[4]{r}\}dr}{[\int_0^{+\infty} Cr\{\xi - e \geq \sqrt[2]{r}\}dr]^2}. \tag{11}$$

Let us end this Section with the following Proposition which determine the kurtosis of trapezoidal and triangular fuzzy variable.

Proposition 3. *1. Let $\xi = (a, b, c, d)$ a fuzzy trapezoidal variable with expected value $E[\xi] = e$. The kurtosis $K[\xi]$ of ξ is given by:*

$$K[\xi] = -[\frac{1}{4}(l_s(\xi)+l_c(\xi))]^5(\frac{|\alpha-\beta|}{5\alpha\beta})+\max(\frac{(\frac{|\alpha-\beta|}{4} - \frac{1}{2}l_c(\xi))^5}{10\alpha \vee \beta},0)+\frac{(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_s(\xi))^5}{10\alpha \vee \beta}$$

$$\frac{|\alpha-\beta|}{2\alpha\beta}[\frac{1}{2}l_s(\xi) - \frac{(\alpha+\beta)}{4}][\frac{1}{4}(l_s(\xi)+l_c(\xi))]^4 - \frac{(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_c(\xi))^5}{10\alpha \wedge \beta}$$

2. *If $\xi = (a, b, c, d)$ is symmetric, the previous expression of $K[\xi]$ becomes:*

$$K[\xi] = \frac{5[l_c(\xi)+\beta]^4 + 10\beta^2[l_c(\xi)+\beta]^2 + \beta^4}{160}.$$

3. *Let $\xi = (a, b, c)$ be a triangular fuzzy variable such that $E[\xi] = \frac{a+2b+c}{4} = e$.*
The kurtosis $K[\xi]$ of ξ can be deduced from the kurtosis of a trapezoidal one by this way :

$$K[\xi] = \frac{253\alpha_1^5 + 395\alpha_1^4\gamma + 17\alpha_1\gamma^4 + 290\alpha_1^3\gamma^2 + 70\alpha_1^2\gamma^3 - \gamma^5}{10.240\alpha_1}.$$

4. *Let $\xi = (a, b, c, d)$ a fuzzy trapezoidal variable with expected value $E[\xi] = e$.*

- *The normalized kurtosis of ξ is given by:*

$$K_1[\xi] = \frac{K[\xi]}{V^2[\xi]}$$

where $K[\xi]$ and $V[\xi]$ have been already given before.

- *For $\alpha = \beta$ the new expression of $K_1[\xi]$ is:*

$$K_1[\xi] = \frac{5[l_c(\xi)+\beta]^4 + 10\beta^2[l_c(\xi)+\beta]^2 + \beta^4}{160[\frac{3[l_c(\xi)+\beta]^2+\beta^2}{24}]^2}$$

We deduce from the previous formulae that:
The normalized Kurtosis of some examples of trapezoidal fuzzy variables are: $K^1[(-1,2,3,4)] = \frac{27414}{8405}$, $K^1[(1,2,3,4)] = \frac{2178}{845}$, $K^1[(-2,-1,3,4)] = \frac{3798}{1805}$ and $K^1[(1,2,2,4)] = \frac{90928}{25215}$.

We notice that: for $\xi = (a,b,c)$ a triangular fuzzy number, we have:
- if b=a, then $K[\xi] = \frac{253}{10.240}\gamma^4$ with $E[\xi] = \frac{3b+c}{4}$.
- if b=c, then $K[\xi] = \frac{253}{10.240}\alpha^4$ with $E[\xi] = \frac{a+3b}{4}$.

3.4 Moments of portfolio

Example 5. *Let $(\xi_i = (a_i, b_i, c_i, d_i))_{i=1,2,\ldots,n}$ be a family of n independent trapezoidal fuzzy variables and $x = (x_1, \ldots, x_n)$ a family of n positive reals. The portfolio $\xi = \sum_{i=1}^n \xi_i$ defined by*

$$\xi(x) = \sum_{i=1}^n x_i \xi_i = (\sum_{i=1}^n x_i a_i, \sum_{i=1}^n x_i b_i, \sum_{i=1}^n x_i c_i, \sum_{i=1}^n x_i d_i)$$

is a fuzzy variable and its expectation is:

$$e(x) = E[\xi(x)] = \frac{1}{4}\sum_{i=1}^n (a_i + b_i + c_i + d_i)\, x_i. \tag{12}$$

Proposition 4.
- *The variance of ξ is*

$$V[\xi] = -\frac{1}{192\sum_{k=1}^n \sum_{l=1}^n x_k x_l \alpha_k \beta_l}[\sum_{k=1}^n x_k(l_s(\xi_k)+l_c(\xi_k))]^3 |\sum_{k=1}^n x_k(\alpha_k-\beta_k)| +$$

$$(\frac{1}{32\sum_{k=1}^n \sum_{l=1}^n x_k x_l \alpha_k \beta_l}[\sum_{k=1}^n x_k(l_s(\xi_k)+l_c(\xi_k))]^2 |\sum_{k=1}^n x_k(\alpha_k-\beta_k)|)\times$$

$$([\frac{1}{4}\sum_{k=1}^{n} x_k(2l_s(\xi_k)-(\alpha_k+\beta_k))])+\frac{(\frac{|\sum_{k=1}^n x_k(\alpha_k-\beta_k)|}{4}+\frac{1}{2}\sum_{k=1}^n x_k l_s(\xi_k))^3}{3\sum_{k=1}^n x_k(\alpha_k+\beta_k+|\alpha_k-\beta_k|)}-$$

$$\frac{(\frac{|\sum_{k=1}^n x_k(\alpha_k-\beta_k)|}{4}+\frac{1}{2}\sum_{k=1}^n x_k l_c(\xi_k))^3}{3\sum_{k=1}^n x_k(\alpha_k+\beta_k-|\alpha_k-\beta_k|)}+$$

$$\frac{(\frac{|\sum_{k=1}^n x_k(\alpha_k-\beta_k)|}{4}-\frac{1}{2}\sum_{k=1}^n x_k l_c(\xi_k))^3+|\frac{(|\sum_{k=1}^n x_k(\alpha_k-\beta_k)|}{4}-\frac{1}{2}\sum_{k=1}^n x_k l_c(\xi_k))^3|}{6\sum_{k=1}^n x_k(\alpha_k+\beta_k+|\alpha_k-\beta_k|)}.$$

- *The skewness of ξ is*

$$Sk[\xi]=\frac{1}{8\sum_{k=1}^n x_k(b_k-a_k)}[(\frac{\sum_{k=1}^n x_k(b_k-e_k}{4})^4-(\frac{\sum_{k=1}^n x_k(a_k-e_k)}{4})^4]+$$

$$\frac{1}{8\sum_{k=1}^n x_k(c_k-d_k)}[(\frac{\sum_{k=1}^n x_k(c_k-e_k)}{4})^4-(\frac{\sum_{k=1}^n x_k(d_k-e_k)}{4})^4].$$

- *The kurtosis of ξ is*

$$K[\xi]=-\frac{1}{5120\sum_{k=1}^n\sum_{l=1}^n x_k x_l\alpha_k\beta_l}[\sum_{k=1}^n x_k(l_s(\xi_k)+l_c(\xi_k))]^5|\sum_{k=1}^n x_k(\alpha_k-\beta_k)|+$$

$$(\frac{1}{512\sum_{k=1}^n\sum_{l=1}^n x_k x_l\alpha_k\beta_l}[\sum_{k=1}^n x_k(l_s(\xi_k)+l_c(\xi_k))]^4|\sum_{k=1}^n x_k(\alpha_k-\beta_k)|)\times$$

$$([\frac{1}{4}\sum_{k=1}^{n} x_k(2l_s(\xi_k)-(\alpha_k+\beta_k))])+\frac{(\frac{|\sum_{k=1}^n x_k(\alpha_k-\beta_k)|}{4}+\frac{1}{2}\sum_{k=1}^n x_k l_s(\xi_k))^5}{5\sum_{k=1}^n x_k(\alpha_k+\beta_k+|\alpha_k-\beta_k|)}-$$

$$\frac{(\frac{|\sum_{k=1}^n x_k(\alpha_k-\beta_k)|}{4}+\frac{1}{2}\sum_{k=1}^n x_k l_c(\xi_k))^5}{5\sum_{k=1}^n x_k(\alpha_k+\beta_k-|\alpha_k-\beta_k|)}+$$

$$\frac{(\frac{|\sum_{k=1}^n x_k(\alpha_k-\beta_k)|}{4}-\frac{1}{2}\sum_{k=1}^n x_k l_c(\xi_k))^5+|\frac{(|\sum_{k=1}^n x_k(\alpha_k-\beta_k)|}{4}-\frac{1}{2}\sum_{k=1}^n x_k l_c(\xi_k))^5|}{10\sum_{k=1}^n x_k(\alpha_k+\beta_k+|\alpha_k-\beta_k|)}.$$

Corollary 3. *Let $(\xi_i=(a_i,b_i,c_i))_{i=1,2,\ldots,n}$ be a family of n independent triangular fuzzy variables, $x=(x_1,\ldots,x_n)$ a family of n positive reals, and $\xi=\sum x_i xi_i$ be the portfolio.*
Then

1. *The mean of the portfolio return is :*

$$E[\xi(x)]=\sum_{i=1}^n x_i(a_i+2b_i+c_i).$$

2. *The variance of the portfolio return is:*

$$V[\xi(x)] = \frac{33\alpha_1^3 + 21\alpha_1^2\gamma + 11\alpha_1\gamma^2 - \gamma^3}{384\alpha_1}.$$

3. *The Skewness of the portfolio return is:*

$$SK[\xi(x)] = (\sum_{i=1}^{n} x_i(c_i - a_i))^2 . \sum_{i=1}^{n} x_i(c_i - 2b_i + a_i)$$

4. *The Kurtosis of the portfolio return is:*

$$K[\xi] = \frac{253\alpha_1^5 + 395\alpha_1^4\gamma + 17\alpha_1\gamma^4 + 290\alpha_1^3\gamma^2 + 70\alpha_1^2\gamma^3 - \gamma^5}{10.240\alpha_1}.$$

4 Semi-Moment of fuzzy variable

Let ξ be a fuzzy variable with finite expected value e. We define the variable $(\xi - e)^-$ as follows:

$$(\xi - e)^- = \begin{cases} \xi - e \text{ if } \xi \leq e \\ 0 \text{ if } \xi > e \end{cases} . \tag{13}$$

4.1 Definitions

Definition 5. 1. *The semi-Moment of order $n = 2p$ with $p \in \mathbb{N}^*$ is*

$$M_{2p}^S[\xi] = M_n^S[\xi] = E[[(\xi - e)^-]^{2p}] = \int_0^{+\infty} Cr\{[(\xi - e)^-]^{2p} \geq r\} dr. \tag{14}$$

2. *The normalized semi-moment of ξ is defined by:*

$$M_{2p}^S[\xi] = \frac{M_{2p}^S[\xi]}{(M_2^S[\xi])^p} = \frac{M_{2p}^S[\xi]}{(V^S[\xi])^p}.$$

In the case where $p = 1$, we obtain the well-known semivariance of ξ described as follows.

Definition 6. *Let ξ be a fuzzy variable with expected value e.*

1. *The semivariance of ξ is defined as*

$$V^S[\xi] = E[[(\xi - e)^-]^2] = \int_0^{+\infty} Cr\{[(\xi - e)^-]^2 \geq r\} dr. \tag{15}$$

Remark 4. The variance of ξ is used to measure the spread of its distribution about $e = E[\xi]$. Note that, variance concerns not only the part "ξ is less than e", but also the part "ξ is greater than e". If we are only interested with the first part, then we should use the concept of semi-variance.

For the example of semivariance of triangular and trapezoidal fuzzy variables, we have the following example:

Example 6. *1. The semivariance of a trapezoidal fuzzy number $\xi = (a, b, c, d)$ (where $a, b, c, d \in \mathbb{R}$ such that $a \neq b$ and $c \neq d$) with expected value $e = \frac{a+b+c+d}{4}$ is given by:*

$$V^S[\xi] = \frac{1}{6(b-a)}[(\frac{e-a}{4})^3 + \min(0, (\frac{b-e}{4})^3)] + \frac{1}{6(d-c)} \max(0, (\frac{e-c}{4})^3)$$

2. The semivariance of a triangular fuzzy number $\xi = (a, b, c)$ with expected value $e = \frac{a+2b+c}{4}$ is deduced from the semivariance of a trapezoidal one by this way:

$$V^S[\xi] = \frac{1}{6(b-a)}[(\frac{e-a}{4})^3 + \frac{1}{(b-c)}(\frac{b-e}{4})^3 \min(0, (b-e))].$$

4.2 Semi-kurtosis: Definitions and examples

In this Subsection, we focus on the semi-Kurtosis (i.e. p=2 in (14))

Definition 7. *Let ξ be a fuzzy variable with finite expected value e. Then the semikurtosis of ξ is defined*

$$K^S[\xi] = E[[(\xi - e)^-]^4] = \int_0^{+\infty} Cr\{[(\xi - e)^-]^4 \geq r\} dr. \tag{16}$$

Let us give the semikurtosis of a trapezoidal fuzzy number and a triangular fuzzy number.

Example 7. *1. The semikurtosis of a trapezoidal fuzzy variable $\xi = (a, b, c, d)$ with expected value $e = \frac{a+b+c+d}{4}$ is given by:*

$$K^S[\xi] = \frac{1}{10(b-a)}[(\frac{e-a}{4})^5 + \min(0, (\frac{b-e}{4})^5)] + \frac{1}{10(d-c)} \max(0, (\frac{e-c}{4})^5).$$

2. The semikurtosis of a triangular fuzzy number $\xi = (a, b, c)$ with expected value $e = \frac{a+2b+c}{4}$ is deduced from the semikurtosis of a trapezoidal one by this way:

$$K^S[\xi] = \frac{1}{10(b-a)}[(\frac{e-a}{4})^5 + \frac{1}{(b-c)}(\frac{b-e}{4})^5 \min(0, (b-e))]$$

Definition 8. *Let ξ a fuzzy variable with expected value e. The normalized semi-kurtosis of ξ is defined by:*

$$K_1^S[\xi] = \frac{K^S[\xi]}{(V^S[\xi])^2}.$$

Example 8. 1. *The normalized semikurtosis of a trapezoïdal fuzzy variable $\xi = (a, b, c, d)$ with expected value e is defined as follows:*

$$K_1^S[\xi] = \frac{\frac{1}{10(b-a)}[(\frac{e-a}{4})^5 + \min(0, (\frac{b-e}{4})^5)] + \frac{1}{10(d-c)}\max(0, (\frac{e-c}{4})^5)}{[\frac{1}{6(b-a)}[(\frac{e-a}{4})^3 + \min(0, (\frac{b-e}{4})^3)] + \frac{1}{6(d-c)}\max(0, (\frac{e-c}{4})^3)]]^2}$$

2. *The normalized semikurtosis of a triangular fuzzy variable $\xi = (a, b, c)$ with expected value e is defined as follows:*

$$K_1^S[\xi] = \frac{\frac{1}{10(b-a)}[(\frac{e-a}{4})^5 + \frac{1}{(b-c)}(\frac{b-e}{4})^5 \min(0, (b-e))]}{[\frac{1}{6(b-a)}[(\frac{e-a}{4})^3 + \frac{1}{(b-c)}(\frac{b-e}{4})^3 \min(0, (b-e))]]^2}$$

Proposition 5. *Let $(\xi_k)_{k=1,\ldots,n}$ be a family of independent trapezoidal fuzzy variables with finite expected values $(e_k)_{k=1,\ldots,n}$, $(x_k)_{k=1,\ldots,n}$be a family of n positive reals and $\xi = \sum_{k=1}^n x_k\xi_k$ be a portfolio. Then*

- *The semivariance of ξ is*

$$V^S[\xi] = \frac{1}{6\sum_{k=1}^n x_k(b_k - a_k)}[(\frac{\sum_{k=1}^n x_k(e_k - a_k)}{4})^3 + \min(0, (\frac{\sum_{k=1}^n x_k(b_k - e_k)}{4})^3)] + \frac{1}{6\sum_{k=1}^n x_k(d_k - c_k)}\max(0, (\frac{\sum_{k=1}^n x_k(e_k - c_k)}{4})^3)$$

- *The semikurtosis of ξ is*

$$K^S[\xi] = \frac{1}{10\sum_{k=1}^n x_k(b_k - a_k)}[(\frac{\sum_{k=1}^n x_k(e_k - a_k)}{4})^5 + \min(0, (\frac{\sum_{k=1}^n x_k(b_k - e_k)}{4})^5)] + \frac{1}{10\sum_{k=1}^n x_k(d_k - c_k)}\max(0, (\frac{\sum_{k=1}^n x_k(e_k - c_k)}{4})^5).$$

We end this section by establishing a link between Moment and Semi-Moment.

4.3 Links between Moments and Semi-Moments

Proposition 6. *Let ξ be a fuzzy variable with finite expected value e, $M^S_{2p}[\xi]$ and $M_{2p}[\xi]$ the semi-kurtosis and kurtosis of ξ respectively. Then*

$$0 \leq M^S_{2p}[\xi] \leq M_{2p}[\xi]. \tag{17}$$

Proposition 7. *Let ξ be a fuzzy variable with finite expected value e. Then*

$$M_{2p}[\xi] = 0 \textit{ if and only if } Cr\{\xi = e\} = 1. \tag{18}$$

Proposition 8. *Let ξ be a fuzzy variable with finite expected value e. Then*

$$M^S_{2p}[\xi] = 0 \textit{ if and only if } Cr\{\xi = e\} = 1, i.e., M_{2p}[\xi] = 0. \tag{19}$$

Proposition 9. *Let ξ be a symmetric fuzzy variable with finite expected value e. Then*

$$M^S_{2p}[\xi] = M_{2p}[\xi]. \tag{20}$$

Remark 5. *The previous results generalize those established by Huang [7] when we consider moment and semi-moment as variance and semi-variance.*

Furthermore, we can deduce the links between Kurtosis and semi-Kurtosis of a fuzzy variable.

Corollary 4. *Let ξ be a fuzzy variable with finite expected value e, $K^S[\xi]$ and $K[\xi]$ the semi-kurtosis and kurtosis of ξ respectively. Then*

1. $$0 \leq K^S[\xi] \leq K[\xi]. \tag{21}$$

2. $$K[\xi] = 0 \textit{ if and only if } Cr\{\xi = e\} = 1. \tag{22}$$

3. $$K^S[\xi] = 0 \textit{ if and only if } Cr\{\xi = e\} = 1, i.e., K[\xi] = 0. \tag{23}$$

4. $$K^S[\xi] = K[\xi]. \tag{24}$$

5 An application in finance

5.1 Review, model, and a determinist program with a family of triangular fuzzy numbers

Let ξ_i be a fuzzy variable representing the return of the ith security, and let x_i be the proportion of the total capital invested in security i. In general, ξ_i is given as $\frac{(p'_i+d_i-p_i)}{p_i}$, where p_i is the closing price of the ith security at present, p'_i is the estimated closing price in the next year, and d_i is the estimated dividends during the coming year.
It is clear that p'_i and d_i are unknown at present. If they are estimated as fuzzy variables, then ξ_i is also a fuzzy variable. Thereby, the portfolios $\xi_1, ..., \xi_n$ and the total return $\xi = \xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n$ are also fuzzy variables.

When minimal expected return, minimal skewness and maximal risk are given, the investors prefer an asymmetric portfolio with small kurtosis . Therefore, we propose the following mean-semivariance-skewness-semikurtosis model:

$$\left\{\begin{array}{l} \text{ minimize } K^S[x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n] \\ \text{subject to} \\ E[x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n] \geq s_1 \\ V^S[x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n] \leq s_2 \\ S[x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n] \geq s_3 \\ x_1 + x_2 + ... + x_n = 1 \\ x_i \geq 0, i = 1, 2, ..., n. \end{array}\right. . \tag{25}$$

The first constraint of this model ensures the expected return is no less than some target value s_1, the second one assures that risk does not exceed some given level s_2 the investor can bear, the third one assures that the skewness is no less than some target value s_3. The last two constraints imply that all the capital will be invested to n securities and short-selling is not allowed.
The other variants of this model can be deduced from this one by changing the objective function either by mean or semi-variance or skewness.

Theorem 1. *Let $(\xi_i = (a_i, b_i, c_i))_{i=1,2,...,n}$ be a family of n independent triangular fuzzy variables.*
Then the model (25) becomes the following determinist programm:

$$\begin{cases} min \ \frac{1}{10\sum_{i=1}^n x_i(b_i-a_i)}[(\frac{\sum_{i=1}^n x_i(e_i-a_i)}{4})^5 + \frac{1}{\sum_{i=1}^n x_i(b_i-d_i)}(\frac{\sum_{i=1}^n x_i(b_i-e_i)}{4})^5 \min(0, \sum_{i=1}^n x_i(b_i-e_i))] \\ subject\ to \\ \sum_{i=1}^n x_i(a_i+2b_i+c_i) \geq 4s_1 \\ \frac{1}{5\sum_{i=1}^n x_i(b_i-a_i)}[(\frac{\sum_{i=1}^n x_i(e_i-a_i)}{4})^3 + \frac{1}{\sum_{i=1}^n x_i(b_i-d_i)}(\frac{\sum_{i=1}^n x_i(b_i-e_i)}{4})^3 \min(0, \sum_{i=1}^n x_i(b_i-e_i))] \leq s_2 \\ (\sum_{i=1}^n x_i(c_i-a_i))^2 \sum_{i=1}^n x_i(c_i-2b_i+a_i) \geq 32s_3 \\ x_1+x_2+...+x_n=1 \\ x_i \geq 0, i=1,2,...,n \end{cases}$$

Remark 6. *It is important to notice that, similarly to above, one can write four variants of the previous model and deterministic program. These variants are described as follows.*

1. *This model minimizes risk (semi-variance) when expected return and skewness are both no less than some given target values s_1 and s_3 respectively and the kurtosis is no more than the given target value s_4. If the second and the third constraints (skewness and Kurtosis) do not exist, then the above model degenerates to mean-variance model proposed earlier by Huang [7].*

2. *This model maximizes the expected return. Similarly, if the first and the third constraints (semi-variance and kurtosis) do not exist, then the above model degenerates to mean-variance model proposed earlier by Huang [7].*

3. *The third variant of the first model is a model which maximizes the skewness. If we cancel the third constraint (kurtosis), then the above model degenerates to mean-variance-skewness model proposed by Li [5].*

4. *The fourth and last variant of the first model is the multi-objective nonlinear programming. The aim of this model is to minimize the risk (semi-variance) and the kurtosis, to maximize the expected value and the skewness when the different target values are unknown.*

5.2 Random fuzzy simulation and Genetic algorithm

Genetic algorithm (GA) has been successfully used to solve many industrial optimization problems, and has been well discussed in Goldberg[**?**] and recently in [12]. In [12], the author designed the hybrid intelligent algorithm integrating random fuzzy fuzzy simulation and GA is designed to solve the proposed models. Roughly speaking, in the proposed algorithm, random simulation and fuzzy simulation are employed to computed the credibility of a fuzzy investment return $X = \xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n$, the Kurtosis and semi-Kurtosis of X.

In the following, we provide a method to compute the kurtosis and semi-kurtosis of general fuzzy variables describing security returns. Fuzzy simulation was first introduced by Liu and Iwamura [13], and then was successfully applied to solving fuzzy optimisation problems by Liu [12]. For the computation of other parameters as mean, variance, semivariance and skewness, we can refer to [5] and [7].
Let ξ_j be a fuzzy variable with membership function μ_j, and decision variable x_j, for all $1 \leq j \leq n$. It is obvious to see that the computation of the kurtosis and semikurtosis of the variable $\xi_1 x_1+\xi_2 x_2+...+\xi_n x_n$ depends on the computation of $Cr\{\xi_1 x_1+\xi_2 x_2+...+\xi_n x_n \geq r\}$ where r is a nonnegative real number.

- Computation of the credibility measure
  Randomly generate real numbers θ_{ji} such that $\mu_j(\theta_{ji}) \geq \varepsilon, j = 1, 2, ..., n, i = 1, 2, ..., N$ respectively, where ε is a sufficiently small number, and N is a sufficiently large integer. Then, the value of $Cr\{\xi_1 x_1+\xi_2 x_2+...+\xi_n x_n \geq r\}$ can be estimated by the formula

  $$\frac{1}{2}(\max_{1\leq i\leq N}\{\min_{1\leq j\leq n}\mu_j(\theta_{ji})/\sum_{j=1}^{n}\theta_{ji}x_j \geq r\}+1-\max_{1\leq i\leq N}\{\min_{1\leq j\leq n}\mu_j(\theta_{ji})/\sum_{j=1}^{n}\theta_{ji}x_j \geq r\}).$$

  In the same way, we can deduce the computation of the expression $Cr\{\xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n \leq r\}$.

- Computation of the kurtosis
  Note that we write $\tau = E[\xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n]$ which may be calculated by fuzzy simulation [12].
  The following algorithm is used to compute $K[\xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n]$.
  step 1. Set $e = 0$
  step 2. Randomly generate θ_{jk} such that $\mu_j(\theta_{jk}) \geq \varepsilon, j = 1, 2, ..., n, k = 1, 2, ..., N$ where ε is a sufficiently small number.
  step 3. Set two numbers

  $$a = \min_{1\leq k\leq N}(\theta_{1k}x_1 + \theta_{2k}x_2 + ... + \theta_{nk}x_n - \tau)^4,$$

  $$b = \max_{1\leq k\leq N}(\theta_{1k}x_1 + \theta_{2k}x_2 + ... + \theta_{nk}x_n - \tau)^4.$$

  step 4. Randomly generate a number $r \in [a, b]$.

step 5. Set $e \longleftarrow e + Cr\{\xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n \geq r\}$.
step 6. Repeat the fourth to fifth steps for N times.
step 7. Return $a \vee 0 + b \wedge 0 + \frac{e(b-a)}{N}$ as the target value.

- <u>Computation of the semikurtosis</u>
  Note that we write $\tau = E[\xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n]$ which may be calculated by fuzzy simulation [12].
  the following algorithm is used to compute $K^S[\xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n]$.
  step 1. Set $e = 0$
  step 2. Randomly generate θ_{jk} such that $\mu_j(\theta_{jk}) \geq \varepsilon, j = 1, 2, ..., n, k = 1, 2, ..., N$, where ε is a sufficiently small number.
  Step 3. If $\theta_{1k}x_1 + \theta_{2k}x_2 + ... + \theta_{nk}x_n - \tau \leq 0$, go to step 4 else go to step 2.
  Step 4. Set two numbers

$$a = \min_{1 \leq k \leq N} (\theta_{1k}x_1 + \theta_{2k}x_2 + ... + \theta_{nk}x_n - \tau)^4,$$

$$b = \max_{1 \leq k \leq N} (\theta_{1k}x_1 + \theta_{2k}x_2 + ... + \theta_{nk}x_n - \tau)^4.$$

  step 5. Randomly generate a number $r \in [a, b]$.
  step 6. Set $e \longleftarrow e + cr\{\xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n \geq r\}$.
  step 7. Repeat the fifth to sixth steps for N times.
  step 8. Return $a \vee 0 + b \wedge 0 + \frac{e(b-a)}{N}$ as the target value.

In the following, we recall general principle of GA.

Liu[12] has successfully applied GA to solve many optimization problems with fuzzy parameters. In in our work, following [5], a solution $\mathbf{x} = (x_1, \ldots, x_n)$ is encoded by a chromosome $\mathbf{c} = (c_1, \ldots, c_n)$ where the genes c_i for $i = 1, \ldots, n$ are non-negative numbers. We can then find the decoding processes by the relation $x_i = c_i / \sum_{j=1}^{n} c_j$. The preceded relation ensures that $\sum_{j=1}^{n} x_j = 1$ always holds.

In GA, we employ the rank-based-evaluation(RBE) function to measure the likelihood of reproduction for each chromosome. The RBE function is defined by

$$Eval(\mathbf{c}_i) = \nu\ (1-\nu)^{i-1} \qquad ; \quad i = 1, \ldots, pop-size, \tag{26}$$

where $\nu \in (0, 1)$. The procedure of the genetic algorithm is summarized as follows:

- *Step 1.* Initialize $pop - size$ feasible chromosomes, in which fuzzy simulation is use d to check the feasibility of the chromosomes (i.e. $x_i \geq 0$, $x_i = c_i / \sum_{i=1}^n c_i$ and therefore $\sum_{j=1}^n x_j = 1$).

- *Step 2.* Employ random fuzzy simulation to compute the objectives of all chromosomes, and then gives an order of the chromosomes based of the objectives values.

- *Step 3.* Find the evaluation function of each chromosome according to the RBE function. Then calculate the fitness of each chromosome by the evaluation function.

- *Step 4.* Select the chromosomes according to spinning roulettes wheel.

- *Step 5.* Update the chromosomes by crossover operation and mutation operation where random fuzzy simulation is utilized to check the feasibility of each child.

- *Step 6.* Repeat Steps 2-5 for a given number of generations.

- *Step 7.* Report the best chromosome, and then decoded into the optimal soplution.

6 Concluding remarks

Different from Huang [7] and Li et al.[5], after recalling the definition of mean, variance, semi-variance and skewness, this paper consider the k-moments (i.e. Kurtosis for $k = 4$) and semi-moments (i.e. semi-Kurtosis for $k = 4$) for portfolio selection with fuzzy risk factors (i.e. returns). In order to measure the leptokurtocity of fuzzy portfolio return, notions of moments and the semi-moments of fuzzy portfolio are originally introduced in this paper, and their mathematical properties are studied. As an extension of the mean-semivariance-skewness model for fuzzy portfolio, the mean-semivariance-skewness-semikurtosis is presented and the corresponding variants (the mean-variance-skewness-kurtosis, the mean-variance-skewness-semikurtosis and the mean-variance-skewness-semikurtosis models) are also considered. We briefly designed the genetic algorithm integrating fuzzy simulation for our optimization models. The next step of our research will be an application to real financial portfolios data.

7 Proof of the results

Throughout this Section, ξ is a fuzzy variable with $E[\xi] = e$.

Proof of Proposition 1: For a symmetric trapezoidal fuzzy variable $\xi = (a, b, c, d)$, we can easily show the following result:
$Cr\{(\xi - e)^k \geq r\} = Cr\{\xi - e \geq \sqrt[k]{r}\} \vee Cr\{\xi - e \geq \sqrt[k]{r}\}$.
$$Cr\{(\xi - e)^k \geq r\} = \begin{cases} \frac{1}{2}, \text{ if } 0 \leq r \leq (\frac{c-b}{2})^k \\ -\frac{\sqrt[k]{r}}{2\beta} + \frac{c-b}{4\beta} + \frac{1}{2}, \text{ if } (\frac{c-b}{2})^k \leq r \leq (\frac{c-b}{2} + \beta)^k \\ 0, \text{ if } r \geq (\frac{c-b}{2} + \beta)^k \end{cases}$$
where $\alpha = d - c = b - a$.
So, we can conclude that:
$m_k[\xi] = \int_0^{(\frac{c-b}{2}+\beta)^k} Cr\{(\xi-e)^k \geq r\} = \frac{\sum_{i=0}^{k}\sum_{j=0}^{k-i} C_{k-i}^{j}(2\beta)^j(c-b)^{k-j}}{2^{k+1}(k+1)} = \frac{\sum_{j=0}^{k} C_{k+1}^{j+1}(2\beta)^j(c-b)^{k-j}}{2^{k+1}(k+1)} = \frac{\sum_{i=0}^{\frac{k}{2}} C_{k+1}^{2i+1}[(c-b)+\alpha]^{k-2i}}{2^{k+1}(k+1)}$. The proof is complete. $\square$

Proof of Corollary 1: We show that, for a symmetric fuzzy variable $\xi, m_k[\xi]$ is nil when k is an odd number.
By definition, we have:
$m_k[\xi] = E[(\xi - E[\xi])^k] = \int_0^{+\infty} Cr\{(\xi - E[\xi])^k \geq r\}dr - \int_{-\infty}^{0} Cr\{(\xi - E[\xi])^k \leq r\}dr, \forall k \in \mathbb{N}^*$.
In [5], X. Li has already proved that for a symmetric fuzzy variable ξ, $E[\xi] = e$ and $Cr\{\xi - e \geq r\} = Cr\{\xi - e \leq -r\}$, where e is a real number such that $\mu(e - r) = \mu(e + r), \forall r \in \mathbb{R}$ and μ is the membership function of ξ.
Furthermore, we have:
$m_k[\xi] = \int_0^{+\infty} Cr\{(\xi-e)^k \geq r\}dr - \int_{-\infty}^{0} Cr\{(\xi-e)^k \leq r\}dr = \int_0^{+\infty} kr^{k-1}Cr\{\xi - e \geq r\}dr - \int_{-\infty}^{0} kr^{k-1}Cr\{\xi - e \leq r\}dr = \int_0^{+\infty} kr^{k-1}Cr\{\xi - e \leq -r\}dr - \int_0^{+\infty} kr^{k-1}Cr\{\xi - e \leq r\}dr = 0$.
Now, we assume that k is an even integer.
For a symmetric triangular fuzzy variable $\xi = (a, b, c)$, we can easily show the following result:
Since $Cr\{(\xi - e)^k \geq r\} = Cr\{\xi - e \geq \sqrt[k]{r}\} \vee Cr\{\xi - e \leq \sqrt[k]{r}\}$, we have:
$$Cr\{(\xi - e)^k \geq r\} = \begin{cases} \frac{\alpha - \sqrt[k]{r}}{2\alpha}, \text{ if } 0 \leq r \leq \alpha^k \\ 0, \text{ if } r \geq \alpha^k \end{cases}$$
where $\alpha = c - b = b - a$.
So, we can conclude that: $m_k[\xi] = \int_0^{\alpha^k} \frac{\alpha - \sqrt[k]{r}}{2\alpha}dr = \frac{1}{2k+2}\alpha^k$. $\square$

Proof of Proposition 2: 1) It is easy to show that: $Cr\{(\xi - e)^4 \geq r\} =$

$Cr\{\xi - e \geq \sqrt[4]{r}\} \vee Cr\{\xi - e \leq \sqrt[4]{r}\}$. Hence we have the following egality:

$$K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\} dr = \int_0^{+\infty} Cr\{\xi - e \geq \sqrt[4]{r}\} \vee Cr\{\xi - e \leq \sqrt[4]{r}\} dr.$$

2) We deduce the second result from the definition of $K^1[\xi]$ and by using the fact that:

$$V[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^2 \geq r\} dr = \int_0^{+\infty} Cr\{\xi - e \geq \sqrt[2]{r}\} \vee Cr\{\xi - e \leq \sqrt[2]{r}\} dr.$$

3) i) Let $a, b \in \mathbb{R}$. We have $K[a\xi + b] = E[(a\xi + b - E[a\xi + b])^4]$. Since $E[a\xi + b] = aE[\xi] + b$, we deduce that $K[a\xi + b] = E[(a\xi + b - aE[\xi] - b)^4] = E[(a\xi - aE[\xi])^4] = a^4 E[(\xi - E[\xi])^4] = a^4 K[\xi]$.
ii) Since $V[a\xi + b] = a^2 V[\xi]$, we deduce $K^1[a\xi + b] = K^1[\xi]$. $\square$

Proof of Corollary 2: When ξ is a symmetric fuzzy variable, we have: $Cr\{(\xi - e)^4 \geq r\} dr = Cr\{\xi - e \geq \sqrt[4]{r}\}$ and $Cr\{(\xi - e)^2 \geq r\} dr = Cr\{\xi - e \geq \sqrt[2]{r}\}$ and the proof is complete. $\square$

Proof of Proposition 3: 1) Let $\xi = (a, b, c, d)$ be a trapezoidal fuzzy variable such that $E[\xi] = e, \alpha = b - a, \beta = d - c$.
By using the fact that $Cr\{(\xi - e)^4 \geq r\} = Cr\{\xi - e \geq \sqrt[4]{r}\} \vee Cr\{\xi - e \leq \sqrt[4]{r}\}$, we can easily obtain the following results:
i)When $\alpha > \beta$, then $e < c$. We can so distinguish the two following cases as follows:
1^{st}case: $e < b$

$$Cr\{(\xi - e)^4 \geq r\} = \begin{cases} 1 - \frac{\sqrt[4]{r} + e - a}{2\alpha}, \text{ if } 0 \leq r \leq (b - e)^4 \\ \frac{1}{2}, \text{ if } (b - e)^4 \leq r \leq (c - e)^4 \\ -\frac{\sqrt[4]{r} + e - d}{2\beta}, \text{ if } (c - e)^4 \leq r \leq (e - \frac{a+b}{2})^4 \\ \frac{-\sqrt[4]{r} + e - a}{2\alpha}, \text{ if } (e - \frac{a+b}{2})^4 \leq r \leq (e - a)^4 \\ 0, \text{ if } r \geq (e - a)^4. \end{cases}$$

and finally we get:
$K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\} dr = (\frac{(e-a)+(e-b)}{2})^5 . (\frac{\beta - \alpha}{5\alpha\beta}) + (\frac{(e-a)+(e-b)}{2})^4 . (\frac{\alpha(d-e)+\beta(e-a)}{2\alpha\beta}) + \frac{(e-a)^5}{10\alpha} + \frac{(b-e)^5}{10\alpha} - \frac{(c-e)^5}{10\beta}$.
2^{nd}case: $e > b$

$$Cr\{(\xi - e)^4 \geq r\} = \begin{cases} \frac{1}{2}, \text{ if } 0 \leq r \leq (c - e)^4 \\ -\frac{\sqrt[4]{r} + e - d}{2\beta}, \text{ if } (c - e)^4 \leq r \leq (e - \frac{a+b}{2})^4 \\ \frac{-\sqrt[4]{r} + e - a}{2\alpha}, \text{ if } (e - \frac{a+b}{2})^4 \leq r \leq (e - a)^4 \\ 0, \text{ if } r \geq (e - a)^4. \end{cases}$$

and finally we get:
$K[\xi]=\int_0^{+\infty}Cr\{(\xi-e)^4\geq r\}dr=(\frac{(e-a)+(e-b)}{2})^5.(\frac{\beta-\alpha}{5\alpha\beta})+(\frac{(e-a)+(e-b)}{2})^4.(\frac{\alpha(d-e)+\beta(e-a)}{2\alpha\beta})+$
$\frac{(e-a)^5}{10\alpha}-\frac{(c-e)^5}{10\beta}$.
ii) When $\alpha<\beta$, we use a similar way to calculate $K[\xi]$.
iii)When $\alpha=\beta$, we have:

$$Cr\{(\xi-e)^4\geq r\}=\begin{cases}\frac{1}{2},\ \text{if } 0\leq r\leq(\frac{c-b}{2})^4\\ -\frac{\sqrt[4]{r}}{2\beta}+\frac{c-b}{4\beta}+\frac{1}{2},\ \text{if } (\frac{c-b}{2})^4\leq r\leq(\frac{c-b}{2}+\beta)^4\\ 0,\ \text{if } r\geq(\frac{c-b}{2}+\beta)^4\end{cases}$$

$\alpha=d-c=b-a$
and this result implies that:

$$K[\xi]=\int_0^{+\infty}Cr\{(\xi-e)^4\geq r\}dr=\frac{5[(c-b)+\beta]^4+10\beta^2[(c-b)+\beta]^2+\beta^4}{160}.$$

2) Let $\xi=(a,b,c)$ be a triangular fuzzy variable such that $E[\xi]=e,\alpha=b-a,\beta=c-b$. By using the fact that $Cr\{(\xi-e)^4\geq r\}=Cr\{\xi-e\geq\sqrt[4]{r}\}\vee Cr\{\xi-e\leq\sqrt[4]{r}\}$, we can easily obtain the following results:
i)When $\alpha>\beta$, then $e<b$ and

$$Cr\{(\xi-e)^4\geq r\}=\begin{cases}1-\frac{\sqrt[4]{r}+e-a}{2\alpha},\ \text{if } 0\leq r\leq(b-e)^4\\ -\frac{\sqrt[4]{r}+e-c}{2\beta},\ \text{if } (b-e)^4\leq r\leq(\frac{\alpha+\beta}{4})^4\\ \frac{-\sqrt[4]{r}+e-a}{2\alpha},\ \text{if } (\frac{\alpha+\beta}{4})^4\leq r\leq(e-a)^4\\ 0,\ \text{if } r\geq(e-a)^4\end{cases}$$

and finally we get:

$$K[\xi]=\int_0^{+\infty}Cr\{(\xi-e)^4\geq r\}dr=\frac{253\alpha^5+395\alpha^4\beta+17\alpha\beta^4+290\alpha^3\beta^2+70\alpha^2\beta^3-\beta^5}{10.240\alpha}.$$

ii) When $\alpha<\beta$, we use a similar way to calculate $K[\xi]$.
iii)When $\alpha=\beta$, we have:

$$Cr\{(\xi-e)^4\geq r\}=\begin{cases}\frac{\alpha-\sqrt[k]{4}}{2\alpha},\ \text{if } 0\leq r\leq\alpha^k\\ 0,\ \text{if } r\geq\alpha^4.\end{cases}$$

where $\alpha=c-b=b-a$ and this result implies that:

$$K[\xi]=\int_0^{+\infty}Cr\{(\xi-e)^4\geq r\}dr=\frac{\alpha^4}{10}.$$

$\square$

Proof of Corollary 3: We deduce these results from Proposition 3.

Proof of Proposition 6: Let $\theta \in \Theta$ and $r \in \mathbb{R}$. With (13), we have: $[(\xi-e)^{-}]^{2p} = \begin{cases} (\xi - e)^{2p} \text{ si } \xi \leq e \\ 0 \text{ si } \xi > e \end{cases}$. Thus we distinguish two cases as follows:
i) If $\xi(\theta) \leq e$, then $[(\xi(\theta) - e)^{-}]^{2p} = (\xi(\theta) - e)^{2p}$. And $[(\xi(\theta) - e)^{-}]^{2p} \geq r \Leftrightarrow (\xi(\theta) - e)^{2p} \geq r$.
ii) If $\xi(\theta) > e$, then $[(\xi(\theta) - e)^{-}]^{2p} = 0$ and $(\xi(\theta) - e)^{2p} \geq [(\xi(\theta) - e)^{-}]^{2p}$. Thus the inequality $[(\xi(\theta) - e)^{-}]^{2p} \geq r$ implies $(\xi(\theta) - e)^{2p} \geq r$. We deduce that $\forall \theta, r$, $\{\theta/[(\xi(\theta) - e)^{-}]^{2p} \geq r\} \subseteq \{\theta/(\xi(\theta) - e)^{2p} \geq r\}$. Since Cr is monotone, we have: $\forall r, Cr\{[(\xi - e)^{-}]^{2p} \geq r\} \leq Cr\{(\xi - e)^{2p} \geq r\}$. Hence $K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^{2p} \geq r\}dr \geq \int_0^{+\infty} Cr\{[(\xi - e)^{-}]^{2p} \geq r\}dr = K^S[\xi]$. $\square$
For $p = 2$, we show (21).

Proof of Proposition 7: Assume that ξ is symmetric anf let $p \in \mathbb{N}^*$.
$(\Leftarrow)$: Assume that $Cr\{\xi = e\} = 1$. Thus we have: $Cr\{\xi - e = 0\} = 1$ iff $Cr\{(\xi - e)^{2p} = 0\} = 1$. With the self-duality of Cr, we have $Cr\{(\xi - e)^{2p} \neq 0\} = 0$.
Let $r > 0$. We have: $Cr\{(\xi-e)^{2p} \geq r\} \leq Cr\{(\xi-e)^{2p} > 0\} \leq Cr\{(\xi-e)^{2p} \neq 0\} = 0$. That means $\forall r > 0, Cr\{(\xi - e)^{2p} \geq r\} = 0$. And we deduce $K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^{2p} \geq r\}dr = 0$.
$(\Rightarrow:)$ Assume that $K[\xi] = 0$. Since Cr takes values in $[0; 1]$, this equality means $Cr\{(\xi - e)^{2p} \geq r\} = 0, \forall r > 0$. Since Cr is self-dual, we have $Cr\{(\xi - e)^{2p} = 0\} = 1$ and we deduce that $Cr\{\xi - e = 0\} = 1$, that is, $Cr\{\xi = e\} = 1$. $\square$

Assume that ξ is symmetric and replace $p = 2$ in the precede proof to obtain (22).

Proof of Proposition 8: Let $p \in \mathbb{N}^*$. Assume that $M_{2p}[\xi] = 0$. With Proposition 6, we have $M_{2p}^S[\xi] = 0$.
Assume that $M_{2p}^S[\xi] = 0$. that is, $E[[(\xi - e)^{-}]^{2p}] = 0$. Since $E[[(\xi - e)^{-}]^{2p}] = \int_0^{+\infty} Cr\{[(\xi - e)^{-}]^{2p} \geq r\}dr$, and the credibility measure Cr takes its value in $[0; 1]$, then $Cr\{[(\xi - e)^{-}]^{2p} \geq r\} = 0, \forall r > 0$. By the self-duality of Cr, we have $Cr\{[(\xi - e)^{-}]^{2p} = 0\} = 1$ and, deduce that

$$Cr\{(\xi - e)^{-} = 0\} = 1. \tag{27}$$

Since $\xi - e = (\xi - e)^{-} + (\xi - e)^{+}$, then 27 implies $\xi - e = (\xi - e)^{+}$. And $E[(\xi - e)] = E[(\xi - e)^{+}] = \int_0^{+\infty} Cr\{(\xi - e)^{+} \geq r\}dr = 0$. This equality

implies that $Cr\{(\xi-e)^+ \geq r\} = 0, \forall r > 0$. Since Cr is self dual, we obtain $Cr\{(\xi-e)^+ = 0\} = 1$.
With $Cr\{(\xi-e)^- = 0\} = 1$ and $Cr\{(\xi-e)^+ = 0\} = 1$, we deduce $Cr\{(\xi-e) = 0\} = 1$, that is, $Cr\{\xi = e\} = 1$. With Proposition 7, we have $M_{2p}[\xi] = 0$. $\square$
When $p = 2$ and ξ is symmetric we obtain (23).

Proof of Proposition 9: $p \in \mathbb{N}^*$. Assume that ξ is symmetric and let us show (24).
Since $M_{2p}[\xi] = \int_0^{+\infty} Cr\{(\xi-e)^{2p} \geq r\}dr$ and $M^S_{2p}[\xi] = \int_0^{+\infty} Cr\{[(\xi-e)^-]^{2p} \geq r\}dr$, it suffices to show that: $Cr\{(\xi-e)^{2p} \geq r\} = Cr\{[(\xi-e)^-]^{2p} \geq r\}$. For that we distinguish two cases:
- If $r < 0$, then we have $Cr\{(\xi-e)^{2p} \geq r\} = Cr\{[(\xi-e)^-]^{2p} \geq r\} = Cr\{\Theta\} = 1$.
- If $r \geq 0$, then (with $r = r'^{2p}$) and assume that $r' > 0$. We have $(\xi-e)^{2p} \geq r \Leftrightarrow (\xi-e) \in]-\infty; -r'] \cup [r'; +\infty[$, and $[(\xi-e)^-]^{2p} \geq r \Leftrightarrow (\xi-e)^- \in]-\infty; -r'] \cup [r'; +\infty[$. Therefore, we obtain $Cr\{(\xi-e)^{2p} \geq r\} = 1 - Cr\{-r' < \xi - e < r'\}, Cr\{[(\xi-e)^-]^{2p} \geq r\} = 1 - Cr\{-r' < (\xi-e)^- < r'\}$.
It rests to show that $Cr\{-r' < \xi - e < r'\} = Cr\{-r' < (\xi-e)^- < r'\}$.
Let μ be the membership function of $\xi - e$ and μ' be the membership function of $(\xi-e)^-$. Let us recall that $\mu' = \begin{cases} \mu \text{ if } \xi < e \\ 0 \text{ otherwise} \end{cases}$.
We have:
$Cr\{-r' < \xi-e < r'\} = \frac{1}{2}[1+\sup_{x\in]-r';r'[} \mu(x) - \max(\sup_{x\in]-\infty;-r'[} \mu(x), \sup_{x\in]r';+\infty;[} \mu(x))] = \frac{1}{2}[1 + \sup_{x\in]-r';0[} - \sup_{x\in]-\infty;-r'[} \mu(x)]$.
We also have $Cr\{-r' < (\xi-e)^- < r'\} = Cr\{-r' < (\xi-e)^- \leq 0\}$ since $(\xi-e)^- \leq 0$. Therefore

$$\begin{aligned} Cr\{-r' < (\xi-e)^- < r'\} &= \frac{1}{2}\left[1 + \sup_{x\in]-r';0[} \mu'(x) - \max(\sup_{x\in]-\infty;-r'[} \mu'(x), \sup_{x\in]0;+\infty;[} \mu'(x))\right] \\ &= \frac{1}{2}\left[1 + \sup_{x\in]-r';0[} \mu'(x) - \sup_{x\in]-\infty;-r'[} \mu'(x)\right]. \end{aligned} \quad (28)$$

Since $\mu'(x) = 0$, $\forall x \in]0; +\infty[$, hence

$$Cr\{-r' < \xi - e < r'\} = Cr\{-r' < (\xi-e)^- < r'\}.$$

Assume that ξ is symmetric. In the the precede proof, when $p = 2$ we obtain (24). $\square$

References

[1] L. Bachelier (1900): Théorie de la spéculation: Thesis: *Annales Scientifiques de l'Ecole Normale Supérieure* , **(III 17)**, 21–86.

[2] C. Carlsson, R. Fuller (2001): On possibilistic mean value and variance of fuzzy numbers, *Fuzzy Sets and Systems* ,**(122)**, 239–247.

[3] C. Carlsson, R. Fuller and P. Majlender (2002): A approach to selecting portfolios with highest score, *Fuzzy Sets and Systems* ,**(131)**, 13–1.

[4] W. Briec, K. Kerstens and O. Junkung (2007): Mean-variance-skewness portfolio performance gauging: A general shortage function and dual approach, *Management Science* ,**(53)**, 135–149.

[5] X. Li, Z. Qin, S. Kar (2010): Mean-variance-skewness model for portfolio selection with fuzzy returns, *European Journal of Operational Research*, vol. 202; **(1)**, 239–247.

[6] T. Hasuike, H. Katagiri, H. Ishii (2009): Portfolio selection problems with random fuzzy variable returns Fuzzy Sets and Systems, Vol. 160, **(18)**, Pages 2579–2596.

[7] X. Huang (2008): Mean-semivariance models for fuzzy portfolio selection, *Journal of Computational and Applied Mathematics*, Vol. 217, **(1)**, Pages 1–8.

[8] S. Kar, R. Bhattacharyya and D. D. Majumbar (2011): Fuzzy-mean-skewness portfolio selection models by interval analysis , *Computers and Mathematics with applications*, **(61)**, Pages 126–137.

[9] H. Konno, K. Suzuki (1995), A mean-variance-skewness optimization model, *Journal of the Operations Research Society of Japan*, **(38)**, Pages 137–187.

[10] H. Konno, H. Shirakawa, H. Yamazaki (1993): A mean-absolute deviation-skewness portfolio optimization model, *Annals of Operations Research*, **(45)**, Pages 205–220.

[11] A. Kraus, R. Litzenberger (1976): Skewness preference and the valuation of risky assets, *Journal of Finance*, **(21)**, Pages 1085-1094.

[12] B.Liu (2002), *Theory and Practice of Uncertain Programming*, Physica-Verlag, Heidelberg.

[13] B.Liu, K.Iwamura (1998): Chance constrained programming with fuzzy parameters, *Fuzzy Sets and Systems*, 94, Pages 227–237.

[14] H. Markowitz (1952): Portfolio selection, *Journal of Finance* (7), 1952, Pages 77–91.

[15] H. Markowitz (1959): Portfolio selection: Efficient diversification of investments, Wiley, New York.

[16] N.M. Pindoriya, S.N. Singh, S.K. Singh (2010): Multi-objective mean–variance–skewness model for generation portfolio allocation in electricity markets ,*Electric Power Systems Research*, vol. 80, **(10)**, Pages 1314–1321.

[17] A. Saeidifar, E. Pasha (2009): The possibilistic moments of fuzzy numbers and their applications, *Journal of Computational and Applied Mathematics*, Vol. 223, **Issue 2**, Pages 1028–1042.

[18] P. Samuelson (1970): The fundamental approximation theorem of portfolio analysis in terms of means, variances and higher moments, ,*Reviews of Economics Studies*, **(37)**, Pages 537–542.

[19] H. Tanaka, P. Guo (1999): Portfolio selection based based on upper and lower exponential possibility distributions, ,*European Journal of operational research*, **(114)**, Pages 115–126.

[20] Z. Wang , F. Tian (2010): A Note of the expected value and variance of Fuzzy variables, *International Journal of Nonlinear Science*, Vol.9, No.4, pages 486–492.