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# Keller-Osserman estimates for some quasilinear elliptic systems

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## Abstract

In this article we study quasilinear systems of two types, in a domain  $\Omega$  of  $\mathbb{R}^N$  : with absorption terms, or mixed terms:

$$(A) \begin{cases} \mathcal{A}_p u = v^\delta, \\ \mathcal{A}_q v = u^\mu, \end{cases} \quad (M) \begin{cases} \mathcal{A}_p u = v^\delta, \\ -\mathcal{A}_q v = u^\mu, \end{cases}$$

where  $\delta, \mu > 0$  and  $1 < p, q < N$ , and  $D = \delta\mu - (p-1)(q-1) > 0$ ; the model case is  $\mathcal{A}_p = \Delta_p, \mathcal{A}_q = \Delta_q$ . Despite of the lack of comparison principle, we prove a priori estimates of Keller-Osserman type:

$$u(x) \leq Cd(x, \partial\Omega)^{-\frac{p(q-1)+q\delta}{D}}, \quad v(x) \leq Cd(x, \partial\Omega)^{-\frac{q(p-1)+p\mu}{D}}.$$

Concerning system  $(M)$ , we show that  $v$  always satisfies Harnack inequality. In the case  $\Omega = B(0, 1) \setminus \{0\}$ , we also study the behaviour near 0 of the solutions of more general weighted systems, giving a priori estimates and removability results. Finally we prove the sharpness of the results.

**Keywords.** Quasilinear elliptic systems, a priori estimates, large solutions, asymptotic behaviour, Harnack inequality

**Mathematic Subject Classification (2010)** 35B40, 35B45, 35J47, 35J92, 35M30

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# 1 Introduction

In this article we study the nonnegative solutions of quasilinear systems in a domain  $\Omega$  of  $\mathbb{R}^N$ , either with absorption terms, or mixed terms, that is,

$$(A) \begin{cases} \mathcal{A}_p u = v^\delta, \\ \mathcal{A}_q v = u^\mu, \end{cases} \quad (M) \begin{cases} \mathcal{A}_p u = v^\delta, \\ -\mathcal{A}_q v = u^\mu, \end{cases} \quad (1.1)$$

where  $\delta, \mu > 0$  and  $1 < p, q < N$ . The operators are given in divergence form by

$$\mathcal{A}_p u := \operatorname{div} [A_p(x, u, \nabla u)], \quad \mathcal{A}_q v := \operatorname{div} [A_q(x, v, \nabla v)],$$

where  $A_p$  and  $A_q$  are Carathéodory functions. In our main results, we suppose that  $\mathcal{A}_p$  is S- $p$ -C (strongly- $p$ -coercive), that means

$$A_p(x, u, \eta) \cdot \eta \geq K_{1,p} |\eta|^p \geq K_{2,p} |A_p(x, u, \eta)|^{p'}, \quad \forall (x, u, \eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N.$$

for some  $K_{1,p}, K_{2,p} > 0$ , and similarly for  $\mathcal{A}_q$ . The model type for  $\mathcal{A}_p$  is the  $p$ -Laplace operator

$$u \longmapsto \Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

We prove *a priori estimates of Keller-Osserman type* for such operators, under a natural condition of "superlinearity":

$$D = \delta\mu - (p-1)(q-1) > 0, \quad (1.2)$$

and we deduce Liouville type results of nonexistence of entire solutions. We also study the behaviour near 0 of nonnegative solutions of possibly weighted systems of the form

$$(A_w) \begin{cases} \mathcal{A}_p u = |x|^a v^\delta, \\ \mathcal{A}_q v = |x|^b u^\mu, \end{cases} \quad (M_w) \begin{cases} \mathcal{A}_p u = |x|^a v^\delta, \\ -\mathcal{A}_q v = |x|^b u^\mu, \end{cases}$$

in  $\Omega \setminus \{0\}$ , where  $a, b \in \mathbb{R}$ , with  $a > -p$  and  $b > -q$ . In particular we discuss about the *Harnack inequality* for  $u$  or  $v$ .

Recall some classical results in the scalar case. For the model equation with an absorption term

$$\Delta_p u = u^Q, \quad (1.3)$$

in  $\Omega$ , with  $Q > p-1$ , the first estimate was obtained by Keller [19] and Osserman [24] for  $p = 2$ , and extended to the case  $p \neq 2$  in [29]: any nonnegative solution  $u \in C^2(\Omega)$  satisfies

$$u(x) \leq C d(x, \partial\Omega)^{-p/(Q-p+1)}, \quad (1.4)$$

where  $d(x, \partial\Omega)$  is the distance to the boundary, and  $C = C(N, p, Q)$ . For the equation with a source term

$$-\Delta_p u = u^Q,$$

up to now estimate (1.4), valid for any  $Q > p-1$  in the radial case, has been obtained only for  $Q < Q^*$ , where  $Q^* = \frac{N(p-1)+p}{N-p}$  is the Sobolev exponent, with difficult proofs, see [18], [9]

in the case  $p = 2$  and [27] in the general case  $p > 1$ . For  $p = 2$ , the estimate, with a universal constant, is not true for  $Q = \frac{N+1}{N-3}$ , and the problem is open between  $Q^*$  and  $\frac{N+1}{N-3}$ .

Up to our knowledge all the known estimates for systems are related with systems for which some comparison properties hold, of competitive type, see [16], or of cooperative type, see [11]; or with quasilinear operators in [17], [32]. Problems (A) and (M) have been the object of very few works because such properties do not hold. The main ones concern systems  $(A_w)$  and  $(M_w)$  in the linear case  $p = q = 2$ , see [5] and [6]; the proofs rely on the inequalities satisfied by the mean values  $\bar{u}$  and  $\bar{v}$  on spheres of radius  $r$ , they cannot be extended to the quasilinear case. A radial study of system (A) was introduced in [15], and recently in [7].

The problem with two source terms

$$(S) \begin{cases} -\mathcal{A}_p u = |x|^a v^\delta, \\ -\mathcal{A}_q v = |x|^b u^\mu, \end{cases}$$

was analyzed in [8]. The results are based on integral estimates, still valid under weaker assumptions:  $\mathcal{A}_p$  is called W- $p$ -C (weakly- $p$ -coercive) if

$$\mathcal{A}_p(x, u, \eta) \cdot \eta \geq K_p |\mathcal{A}_p(x, u, \eta)|^{p'}, \quad \forall (x, u, \eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \quad (1.5)$$

for some  $K_p > 0$ ; similarly for  $\mathcal{A}_q$ . When  $\delta, \mu < Q_1$ , where  $Q_1 = \frac{N(p-1)}{N-p}$ , punctual estimates were deduced for S- $p$ -C, S- $q$ -C operators and it was shown that  $u$  and  $v$  satisfy the Harnack inequality.

In **Section 2**, we give our main tools for obtaining a priori estimates. First we show that the technique of integral estimates is fundamental, and can be used also for systems (A) and (M). In Proposition 2.1 we consider both equations with absorption or source terms

$$-\mathcal{A}_p u + f = 0, \quad \text{or} \quad -\mathcal{A}_p u = f, \quad (1.6)$$

in a domain  $\Omega$ , where  $f \in L^1_{loc}(\Omega)$ ,  $f \geq 0$ , and obtain local integral estimates of  $f$  with respect to  $u$  in a ball  $B(x_0, \rho)$ . When  $\mathcal{A}_p$  is S- $p$ -C, they imply minorizations by the Wölf potential of  $f$  in the ball

$$W_{1,p}^f(B(x_0, \rho)) = \int_0^\rho \left( t^p \oint_{B(x_0, t)} f \right)^{\frac{1}{p-1}} \frac{dt}{t}, \quad (1.7)$$

extending the first results of [20], [21]. The second tool is the well known weak Harnack inequalities for solutions of (1.6) in case of S- $p$ -C operators, and a more general version in case of equation with absorption, which appears to be very useful. The third one is a bootstrap argument given in [5] which remains essential.

In **Section 3** we study both systems (A) and (M). When  $\mathcal{A}_p = \Delta_p$  and  $\mathcal{A}_q = \Delta_q$ , they admit particular radial solutions

$$u^*(x) = A^* |x|^{-\gamma}, \quad v^*(r) = B^* |x|^{-\xi},$$

where

$$\gamma = \frac{p(q-1) + q\delta}{D}, \quad \xi = \frac{q(p-1) + p\mu}{D},$$

whenever  $\gamma > \frac{N-p}{p-1}$  and  $\xi > \frac{N-q}{q-1}$  for system (A), and  $\gamma > \frac{N-p}{p-1}$  and  $\xi < \frac{N-q}{q-1}$  for system (M).

Our main result for the system with absorption term (A) extends precisely the Osserman-Keller estimate of the scalar case (1.3):

**Theorem 1.1** *Assume that*

$$\mathcal{A}_p \text{ is } S\text{-}p\text{-}C, \quad \mathcal{A}_q \text{ is } S\text{-}q\text{-}C, \quad (1.8)$$

*and (1.2) holds. Let  $u \in W_{loc}^{1,p}(\Omega) \cap C(\Omega)$ ,  $v \in W_{loc}^{1,q}(\Omega) \cap C(\Omega)$  be nonnegative solutions of*

$$\begin{cases} -\mathcal{A}_p u + v^\delta \leq 0, \\ -\mathcal{A}_q v + u^\mu \leq 0, \end{cases} \quad \text{in } \Omega.$$

*Then for any  $x \in \Omega$*

$$u(x) \leq C d(x, \partial\Omega)^{-\gamma}, \quad v(x) \leq C d(x, \partial\Omega)^{-\xi}, \quad (1.9)$$

*with  $C = C(N, p, q, \delta, \mu, K_{1,p}, K_{2,p}, K_{1,q}, K_{2,q})$ .*

Our second result shows that the mixed system (M) also satisfies the Osserman-Keller estimate, *without any restriction on  $\delta$  and  $\mu$* , and moreover the second function  $v$  *always satisfies Harnack inequality*:

**Theorem 1.2** *Assume (1.2), (1.8). Let  $u \in W_{loc}^{1,p}(\Omega) \cap C(\Omega)$ ,  $v \in W_{loc}^{1,q}(\Omega) \cap C(\Omega)$  be nonnegative solutions of*

$$\begin{cases} -\mathcal{A}_p u + v^\delta \leq 0, \\ -\mathcal{A}_q v \geq u^\mu, \end{cases} \quad \text{in } \Omega.$$

*Then (1.9) still holds for any  $x \in \Omega$ .*

*Moreover, if  $(u, v)$  is any nonnegative solution of system (M), then  $v$  satisfies Harnack inequality in  $\Omega$ , and there exists another  $C > 0$  as above, such that the punctual inequality holds*

$$u^\mu(x) \leq C v^{q-1}(x) d(x, \partial\Omega)^{-q}. \quad (1.10)$$

Notice that the results are new even for  $p = q = 2$ . As a consequence we deduce Liouville properties:

**Corollary 1.3** *Assume (1.2), (1.8). Then there exist no entire nonnegative solutions of systems (A) or (M).*

**Section 4** concerns the behaviour near 0 of systems with possible weights  $(A_w)$  and  $(M_w)$ , where  $\gamma, \xi$  are replaced by

$$\gamma_{a,b} = \frac{(p+a)(q-1) + (q+b)\delta}{D}, \quad \xi_{a,b} = \frac{(q+b)(p-1) + (p+a)\mu}{D},$$

in other terms  $\delta\xi_{a,b} = (p-1)\gamma_{a,b} + p+a$ ,  $\mu\gamma_{a,b} = (q-1)\xi_{a,b} + q+b$ . We set  $B_r = B(0, r)$  and  $B'_r = B_r \setminus \{0\}$  for any  $r > 0$ . Our results extend and simplify the results of [5], [6] in a significant way:

**Theorem 1.4** Assume (1.2), (1.8). Let  $u \in W_{loc}^{1,p}(B'_1) \cap C(B'_1)$ ,  $v \in W_{loc}^{1,q}(B'_1) \cap C(B'_1)$  be nonnegative solutions of

$$\begin{cases} -\mathcal{A}_p u + |x|^a v^\delta \leq 0, \\ -\mathcal{A}_q v + |x|^b u^\mu \leq 0, \end{cases} \quad \text{in } B'_1. \quad (1.11)$$

Then there exists  $C = C(N, p, q, a, b, \delta, \mu, K_{1,p}, K_{2,p}, K_{1,q}, K_{2,q}) > 0$  such that

$$u(x) \leq C |x|^{-\gamma_{a,b}}, \quad v(x) \leq C |x|^{-\xi_{a,b}} \quad \text{in } B'_{\frac{1}{2}}. \quad (1.12)$$

**Theorem 1.5** Assume (1.2), (1.8). Let  $u \in W_{loc}^{1,p}(B'_1) \cap C(B'_1)$ ,  $v \in W_{loc}^{1,q}(B'_1) \cap C(B'_1)$  be nonnegative solutions of

$$\begin{cases} -\mathcal{A}_p u + |x|^a v^\delta \leq 0, \\ -\mathcal{A}_q v \geq |x|^b u^\mu, \end{cases} \quad \text{in } B'_1. \quad (1.13)$$

in  $B'_1$ . Then there exists  $C > 0$  as in theorem 1.4 such that

$$u(x) \leq C |x|^{-\gamma_{a,b}}, \quad v(x) \leq C \min(|x|^{-\xi_{a,b}}, |x|^{-\frac{N-q}{q-1}}), \quad \text{in } B'_{\frac{1}{2}}. \quad (1.14)$$

Moreover if  $(u, v)$  is any nonnegative solution of  $(M_w)$ , then  $v$  satisfies Harnack inequality in  $B'_{\frac{1}{2}}$ , and there exist another  $C > 0$  as above, such that

$$|x|^{b+q} u^\mu(x) \leq C v^{q-1}(x), \quad \text{in } B'_{\frac{1}{2}}. \quad (1.15)$$

Moreover we give removability results for the two systems  $(A_w)$  and  $(M_w)$ , see Theorems 4.1, 4.2, whenever  $\mathcal{A}_p$  and  $\mathcal{A}_q$  satisfy monotonicity and homogeneity properties, extending to the quasilinear case [5, Corollary 1.2] and [6, Theorem 1.1].

In **Section 5** we show that our results on Harnack inequality are optimal, even in the radial case. And we prove the sharpness of the removability conditions.

## 2 Main tools

For any  $x \in \mathbb{R}^N$  and  $r > 0$ , we set  $B(x, r) = \{y \in \mathbb{R}^N / |y - x| < r\}$  and  $B_r = B(0, r)$ . For any function  $w \in L^1(\Omega)$ , and for any weight function  $\varphi \in L^\infty(\Omega)$  such that  $\varphi \geq 0$ ,  $\varphi \neq 0$ , we denote by

$$\oint_\varphi w = \frac{1}{\int_\Omega \varphi} \int_\Omega w \varphi$$

the mean value of  $w$  with respect to  $\varphi$  and by

$$\oint_\Omega w = \frac{1}{|\Omega|} \int_\Omega w = \oint_1 w.$$

For any function  $g \in L^1_{loc}(\Omega)$ , we shall say that a function  $u \in W^{1,p}_{loc}(\Omega)$  satisfies

$$-\mathcal{A}_p u \geq g \quad \text{in } \Omega, \quad (\text{resp. } \leq, \text{ resp. } =)$$

if  $\mathcal{A}_p(x, u, \nabla u) \in L^{p'}_{loc}(\Omega)$  and

$$-\int_\Omega \mathcal{A}_p(x, u, \nabla u) \cdot \nabla \phi \geq \int_\Omega g \phi, \quad (\text{resp. } \leq, \text{ resp. } =) \quad (2.1)$$

for any nonnegative  $\phi \in W^{1,\infty}(\Omega)$  with compact support in  $\Omega$ .

## 2.1 Integral estimates under weak conditions

Next we prove integral inequalities for equations (1.6), either with source or with absorption terms. Such type of estimates for source terms were initiated in [8, Proposition 2.1]. They were used for specific  $f$  for proving Liouville type results in [23] and more recently in [10, Theorem 2.1], and in [14] for the case of absorption terms. They are obtained by multiplication by  $u^\alpha$ , with  $\alpha < 0$  for the source case,  $\alpha > 0$  for the absorption case.

**Proposition 2.1** *Let  $\mathcal{A}_p$  be  $W$ - $p$ -C. Let  $f \in L^1_{loc}(\Omega)$ ,  $f \geq 0$  and let  $u \in W^{1,p}_{loc}(\Omega)$  be any nonnegative solution of inequality*

$$-\mathcal{A}_p u \geq f, \quad \text{in } \Omega, \quad (2.2)$$

or

$$-\mathcal{A}_p u + f \leq 0, \quad \text{in } \Omega. \quad (2.3)$$

Let  $\xi \in \mathcal{D}(\Omega)$ , with values in  $[0, 1]$ , and  $\varphi = \xi^\lambda$ ,  $\lambda > 0$ , and  $S_\xi = \text{supp}|\nabla \xi|$ . Then for any  $\ell > p - 1$ , there exists  $\lambda(p, \ell)$  such that for  $\lambda \geq \lambda(p, \ell)$ , there exists  $C = C(N, p, K_p, \ell, \lambda) > 0$  such that

$$\int_{\Omega} f \varphi \leq C |S_\xi| \max_{\Omega} |\nabla \xi|^p \left( \oint_{S_\xi} u^\ell \right)^{\frac{p-1}{\ell}}. \quad (2.4)$$

**Proof.** (i) First assume that  $\ell > p - 1 + \alpha$ , with  $\alpha \in (1 - p, 0)$  in case of equation (2.2),  $\alpha \in (0, 1)$  (any  $\alpha > 0$  if  $u \in L^\infty_{loc}(\Omega)$ ) in case of equation (2.3). We claim that there exists  $\lambda(p, \alpha, \ell)$  such that for any  $\lambda \geq \lambda(p, \alpha, \ell)$

$$\int_{\Omega} f u^\alpha \varphi \leq C |S_\xi| \max_{\Omega} |\nabla \xi|^p \left( \oint_{S_\xi} u^\ell \varphi \right)^{\frac{p-1+\alpha}{\ell}}, \quad (2.5)$$

for some  $C = C(N, p, K_p, \alpha, \ell, \lambda)$ . For proving (2.5), one can assume that  $u^\ell \in L^1(B(x_0, \rho))$ . Let  $\varphi = \xi^\lambda$ , where  $\lambda > 0$  will be chosen after. Let  $\delta > 0, k \geq 1$ , and  $(\eta_n)$  be a sequence of mollifiers; we set  $u_\delta = u + \delta$ ,  $u_{\delta,k} = \min(u, k) + \delta$  and approximate  $u$  by  $u_{\delta,k,n} = u_{\delta,k} * \eta_n$ , and we take  $\phi = u_{\delta,k,n}^\alpha \varphi$  as a test function. Then in any case, from (1.5) and Hölder inequality,

$$\begin{aligned} & |\alpha| \int_{\Omega} u_{\delta,k,n}^{\alpha-1} \varphi \mathcal{A}_p(x, u, \nabla u) \cdot \nabla u_{\delta,k,n} + \int_{\Omega} f u_{\delta,k,n}^\alpha \varphi \\ & \leq \lambda \int_{S_\xi} u_{\delta,k,n}^\alpha \xi^{\lambda-1} |\mathcal{A}_p(x, u, \nabla u)| |\nabla \xi| \\ & \leq \lambda K_p^{-1/p'} \int_{S_\xi} u_{\delta,k,n}^\alpha \xi^{\lambda-1} (\mathcal{A}_p(x, u, \nabla u) \cdot \nabla u)^{1/p'} |\nabla \xi| \\ & \leq \lambda K_p^{-1/p'} \left( \int_{S_\xi} u_{\delta,k,n}^{\alpha-1} \xi^\lambda \mathcal{A}_p(x, u, \nabla u) \cdot \nabla u \right)^{\frac{1}{p'}} \left( \int_{S_\xi} u_{\delta,k,n}^{\alpha+p-1} \xi^{\lambda-p} |\nabla \xi|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Otherwise  $(\nabla u_{\delta,k,n})$  tends to  $\chi_{\{u \leq k\}} \nabla u$  in  $L_{loc}^p(\Omega)$ , and up to subsequence *a.e.* in  $\Omega$ , and  $A_p(x, u, \nabla u) \in L_{loc}^{p'}(\Omega)$ . By letting  $n \rightarrow \infty$ , we obtain

$$\begin{aligned} & |\alpha| \int_{\{u \leq k\}} u_{\delta,k}^{\alpha-1} \xi^\lambda A_p(x, u, \nabla u) \cdot \nabla u + \int_{\Omega} f u_{\delta,k}^\alpha \xi^\lambda \\ & \leq \lambda K_p^{-1/p'} \left( \int_{S_\xi} u_{\delta,k}^{\alpha-1} \xi^\lambda A_p(x, u, \nabla u) \cdot \nabla u \right)^{\frac{1}{p'}} \left( \int_{S_\xi} u_{\delta,k}^{\alpha+p-1} \xi^{\lambda-p} |\nabla \xi|^p \right)^{\frac{1}{p}} \\ & \leq \frac{|\alpha|}{2} \int_{\{u \leq k\} \cap S_\xi} u_{\delta,k}^{\alpha-1} \xi^\lambda A_p(x, u, \nabla u) \cdot \nabla u + C \int_{S_\xi} u_{\delta,k}^{\alpha+p-1} \xi^{\lambda-p} |\nabla \xi|^p, \end{aligned}$$

with  $C = C(\alpha, K_p, p, \lambda)$ ; otherwise, for  $\alpha < 1$  (or  $u \in L_{loc}^\infty(\Omega)$  and taking  $k \geq \sup_{S_\xi} u$ )

$$\begin{aligned} \int_{\Omega} u_{\delta,k}^{\alpha-1} \xi^\lambda A_p(x, u, \nabla u) \cdot \nabla u &= \int_{\{u \leq k\}} u_{\delta,k}^{\alpha-1} \xi^\lambda A_p(x, u, \nabla u) \cdot \nabla u + \int_{\{u > k\}} u_{\delta,k}^{\alpha-1} \xi^\lambda A_p(x, u, \nabla u) \cdot \nabla u \\ &\leq \int_{\{u \leq k\}} u_{\delta,k}^{\alpha-1} \xi^\lambda A_p(x, u, \nabla u) \cdot \nabla u + M k^{\alpha-1} \end{aligned}$$

where  $M = \int_{\Omega} \xi^\lambda A_p(x, u, \nabla u) \cdot \nabla u$  (or  $M = 0$ ) is independent of  $k$  and  $\delta$ . Then, for any  $\theta > 1$ ,

$$\begin{aligned} & \frac{|\alpha|}{2} \int_{\{u \leq k\}} u_{\delta,k}^{\alpha-1} \xi^\lambda A_p(x, u, \nabla u) \cdot \nabla u + \int_{\Omega} f u_{\delta,k}^\alpha \xi^\lambda \leq C \int_{S_\xi} u_{\delta,k}^{\alpha+p-1} \xi^{\lambda-p} |\nabla \xi|^p + M k^{\frac{\alpha-1}{p'}} \\ & \leq C \left( \int_{S_\xi} u_{\delta,k}^{(\alpha+p-1)\theta} \xi^\lambda \right)^{\frac{1}{\theta}} \left( \int_{S_\xi} \xi^{\lambda-p\theta'} |\nabla \xi|^{p\theta'} \right)^{\frac{1}{\theta'}} + M k^{\frac{\alpha-1}{p'}}. \end{aligned}$$

Choosing  $\theta = \ell/(\alpha + p - 1) > 1$ , and  $\lambda \geq \lambda(p, \alpha, \ell) = p\theta'$ , we find

$$\begin{aligned} & \frac{|\alpha|}{2} \int_{\{u \leq k\}} u_{\delta,k}^{\alpha-1} \xi^\lambda A_p(x, u, \nabla u) \cdot \nabla u + \int_{\Omega} f u_{\delta,k}^\alpha \xi^\lambda \\ & \leq C \left( \int_{S_\xi} u_{\delta,k}^\ell \varphi \right)^{\frac{\alpha+p-1}{\ell}} \left( \int_{S_\xi} |\nabla \xi|^{p\theta'} \right)^{\frac{1}{\theta'}} + M k^{\frac{\alpha-1}{p'}} \\ & \leq C |S_\xi|^{\frac{1}{\theta'}} \max_{\Omega} |\nabla \xi|^p \left( \int_{S_\xi} u_{\delta,k}^\ell \varphi \right)^{\frac{\alpha+p-1}{\ell}} + M k^{\frac{\alpha-1}{p'}}, \end{aligned}$$

with a new constant  $C = C(N, p, K, \alpha, \ell)$ . As  $k \rightarrow \infty$ , we deduce

$$\frac{|\alpha|}{2} \int_{\Omega} u_{\delta}^{\alpha-1} \varphi A_p(x, u, \nabla u) \cdot \nabla u + \int_{\Omega} f u_{\delta}^\alpha \varphi \leq C |S_\xi|^{\frac{1}{\theta'}} \max_{\Omega} |\nabla \xi|^p \left( \int_{\Omega} u_{\delta}^\ell \varphi \right)^{\frac{\alpha+p-1}{\ell}}. \quad (2.6)$$

Finally as  $\delta \rightarrow 0$  we get (2.5) with a new constant  $C$ . Moreover we obtain an estimate of the gradient terms:

$$\frac{|\alpha|}{2} \int_{\Omega} u^{\alpha-1} \varphi A_p(x, u, \nabla u) \cdot \nabla u \leq C |S_\xi|^{\frac{1}{\theta'}} \max_{\Omega} |\nabla \xi|^p \left( \int_{\Omega} u^\ell \varphi \right)^{\frac{\alpha+p-1}{\ell}}. \quad (2.7)$$



(ii) Next we only assume that  $\ell > p - 1$ ,  $u^\ell \in L^1(B(x_0, \rho))$ . Let  $\varphi$  as above, and fix some  $\alpha = \alpha(p, \ell)$  such that  $\alpha \in (1 - p, 0)$  and  $(1 - \alpha)(p - 1) < \ell$  for (2.2),  $\alpha \in (0, 1)$  and  $\alpha + p - 1 < \ell$  for (2.3). In any case  $\tau = \ell/(1 - \alpha)(p - 1) > 1$ , and  $1/\theta p' + 1/p\tau = (p - 1)/\ell$ . Let  $\lambda \geq \lambda(p, \alpha(p, \ell), \ell) \geq p\tau'$ . We take  $\varphi$  as a test function and from (2.6) we deduce successively, with new constants  $C$ ,

$$\begin{aligned}
\int_{\Omega} f\varphi &\leq \lambda \int_{\Omega} \xi^{\lambda-1} |A_p(x, u, \nabla u)| |\nabla \xi| \leq C \int_{\Omega} \xi^{\lambda-1} |A_p(x, u, \nabla u)| |\nabla \xi| u_{\delta}^{\frac{\alpha-1}{p'}} u_{\delta}^{\frac{1-\alpha}{p'}} \\
&\leq C \left( \int_{S_{\xi}} u_{\delta}^{\alpha-1} |A_p(x, u, \nabla u)|^{p'} \varphi \right)^{\frac{1}{p'}} \left( \int_{S_{\xi}} u_{\delta}^{(1-\alpha)(p-1)} \xi^{\lambda-p} |\nabla \xi|^p \right)^{\frac{1}{p}} \\
&\leq C \left( \int_{S_{\xi}} u_{\delta}^{\alpha-1} \varphi A_p(x, u, \nabla u) \cdot \nabla u \right)^{\frac{1}{p'}} \left( \int_{S_{\xi}} u_{\delta}^{\ell} \right)^{\frac{1}{p\tau}} \left( \int_{S_{\xi}} \xi^{\lambda-p\tau'} |\nabla \xi|^{p\tau'} \right)^{\frac{1}{p\tau'}} \\
&\leq C |S_{\xi}|^{\frac{1}{\theta'p'} + \frac{1}{p\tau'}} \max_{\Omega} |\nabla \xi|^p \left( \int_{S_{\xi}} u_{\delta}^{\ell} \right)^{\frac{1}{p'\theta} + \frac{1}{p\tau}} \\
&\leq C |S_{\xi}|^{1 - \frac{p-1}{\ell}} \max_{\Omega} |\nabla \xi|^p \left( \int_{S_{\xi}} u_{\delta}^{\ell} \right)^{\frac{p-1}{\ell}} ;
\end{aligned}$$

and (2.9) follows as  $\delta \rightarrow 0$ . ■

**Corollary 2.2** *Under the assumptions of Proposition 2.1, consider any ball  $B(x_0, 2\rho) \subset \Omega$ , and any  $\varepsilon \in (0, \frac{1}{2}]$ . Let  $\varphi = \xi^{\lambda}$  with  $\xi$  such that*

$$\xi = 1 \text{ in } B(x_0, \rho), \quad \xi = 0 \text{ in } \Omega \setminus \bar{B}_{\rho(1+\varepsilon)}, \quad |\nabla \xi| \leq \frac{C_0}{\varepsilon \rho}. \quad (2.8)$$

*Then for any  $\ell > p - 1$ , there exists  $\lambda(p, \ell)$  such that for  $\lambda \geq \lambda(p, \ell)$ , there exists  $C = C(N, p, K, \ell, \lambda) > 0$  such that*

$$\oint_{\varphi} f \leq C(\varepsilon \rho)^{-p} \left( \oint_{\varphi} u^{\ell} \right)^{\frac{p-1}{\ell}}. \quad (2.9)$$

**Remark 2.3** *If  $S_{\xi} = \cup_{i=1}^k S_{\xi}^i$  where the  $S_{\xi}^i$  are 2 by 2 disjoint, then (2.4) can be replaced by*

$$\int_{\Omega} f\varphi \leq C \sum_{i=1}^k |S_{\xi}^i| \max_{S_{\xi}^i} |\nabla \xi|^p \left( \oint_{S_{\xi}^i} u^{\ell} \right)^{\frac{p-1}{\ell}}. \quad (2.10)$$

## 2.2 Punctual estimates under strong conditions

When  $\mathcal{A}_p$  is S-p-C, the estimate (2.7) of the gradient is the beginning of the proof of the well-known weak Harnack inequalities:

**Theorem 2.4** ([25], [28]) *Let  $\mathcal{A}_p$  be  $S$ - $p$ -C, and  $u \in W_{loc}^{1,p}(\Omega)$  be nonnegative, such that  $-\mathcal{A}_p u \leq 0$  in  $\Omega$ ; then for any ball  $B(x_0, 3\rho) \subset \Omega$ , and any  $\ell > p - 1$ ,*

$$\sup_{B(x_0, \rho)} u \leq C \left( \oint_{B(x_0, 2\rho)} u^\ell \right)^{\frac{1}{\ell}}, \quad (2.11)$$

*with  $C = C(N, p, \ell, K_{1,p}, K_{2,p})$ . Let  $w \in W_{loc}^{1,p}(\Omega)$  be nonnegative, such that  $-\mathcal{A}_p w \geq 0$ ; then for any ball  $B(x_0, 3\rho) \subset \Omega$ , for any  $\ell \in (0, N(p-1)/(N-p))$*

$$\left( \oint_{B(x_0, 2\rho)} v^\ell \right)^{\frac{1}{\ell}} \leq C \inf_{B(x_0, \rho)} v. \quad (2.12)$$

Next we give a more precise version of weak Harnack inequality (2.11). Such a kind of inequality was first established in the parabolic case in [12].

**Lemma 2.5** *Let  $\mathcal{A}_p$  be  $S$ - $p$ -C, and  $u \in W_{loc}^{1,p}(\Omega)$  be nonnegative, such that  $-\mathcal{A}_p u \leq 0$ . Then for any  $s > 0$ , there exists a constant  $C = C(N, p, s, K_{1,p}, K_{2,p})$ , such that for any ball  $B(x_0, 2\rho) \subset \Omega$  and any  $\varepsilon \in (0, \frac{1}{2}]$ ,*

$$\sup_{B(x_0, \rho)} u \leq C \varepsilon^{-\frac{Np^2}{s^2}} \left( \oint_{B(x_0, \rho(1+\varepsilon))} u^s \right)^{\frac{1}{s}}. \quad (2.13)$$

**Proof.** From a slight adaptation of the usual case where  $\varepsilon = \frac{1}{2}$ , for any  $\ell > p - 1$ , there exists  $C = C(N, \ell) > 0$  such that for any  $\varepsilon \in (0, \frac{1}{2}]$ ,

$$\sup_{B(x_0, \rho)} u \leq C \varepsilon^{-N} \left( \oint_{B(x_0, \rho(1+\varepsilon))} u^\ell \right)^{\frac{1}{\ell}}. \quad (2.14)$$

Thus we can assume  $s \leq p - 1$ . We fix for example  $\ell = p$ , and define a sequence  $(\rho_n)$  by  $\rho_0 = \rho$ , and  $\rho_n = \rho(1 + \frac{\varepsilon}{2} + \dots + (\frac{\varepsilon}{2})^n)$  for any  $n \geq 1$ , and we set  $M_n = \sup_{B(x_0, \rho_n)} u^p$ . From (2.14) we obtain, with new constants  $C = C(N, p)$ ,

$$M_n \leq C \left( \frac{\rho_{n+1}}{\rho_n} - 1 \right)^{-Np} \oint_{B(x_0, \rho_{n+1})} u^p \leq C \left( \frac{\varepsilon}{2} \right)^{-(n+1)Np} \oint_{B(x_0, \rho_{n+1})} u^p.$$

From the Young inequality, for any  $\delta \in (0, 1)$ , and any  $r < 1$ , we obtain

$$\begin{aligned} M_n &\leq C \left( \frac{\varepsilon}{2} \right)^{-(n+1)Np} M_{n+1}^{1-r} \oint_{B(x_0, \rho_{n+1})} u^{pr} \\ &\leq \delta M_{n+1} + r \delta^{1-1/r} \left( C \left( \frac{\varepsilon}{2} \right)^{-(n+1)Np} \right)^{\frac{1}{r}} \left( \oint_{B(x_0, \rho_{n+1})} u^p \right)^{\frac{1}{r}}. \end{aligned}$$

Defining  $\kappa = r \delta^{1-1/r} C^{\frac{1}{r}}$  and  $b = \left( \frac{\varepsilon}{2} \right)^{-Np/r}$ , we find

$$M_n \leq \delta M_{n+1} + b^{n+1} \kappa \left( \oint_{B(x_0, \rho_{n+1})} u^{pr} \right)^{\frac{1}{r}}.$$

Taking  $\delta = \frac{1}{2b}$  and iterating, we obtain

$$\begin{aligned} M_0 &= \sup_{B(x_0, \rho)} u^p \leq \delta^{n+1} M_{n+1} + b\kappa \sum_{i=0}^n (\delta b)^i \left( \oint_{B(x_0, \rho_{n+1})} u^{pr} \right)^{\frac{1}{r}} \\ &\leq \delta^{n+1} M_{n+1} + 2b\kappa \left( \oint_{B(x_0, \rho_{n+1})} u^{pr} \right)^{\frac{1}{r}}. \end{aligned}$$

Since  $B(x_0, \rho_{n+1}) \subset B(x_0, \rho(1+\varepsilon))$ , going to the limit as  $n \rightarrow \infty$ , and returning to  $u$ , we deduce

$$\sup_{B(x_0, \rho)} u \leq (2b\kappa)^{1/p} \left( \oint_{B(x_0, \rho(1+\varepsilon))} u^{pr} \right)^{\frac{1}{rp}},$$

and the conclusion follows by taking  $r = s/p$ . ■

It is interesting to make the link between Proposition 2.1, with the powerful estimates issued from the potential theory, involving *Wölf potentials*, proved in [20], [21] and [22]. Here we show that the lower estimates hold for any S- $p$ -C operator.

**Corollary 2.6** *Suppose that  $\mathcal{A}_p$  is S- $p$ -C. If  $u$  is a nonnegative solution of (2.2), then for any ball  $B(x_0, 2\rho) \subset \Omega$ ,*

$$CW_{1,p}^f(B(x_0, \rho)) + \inf_{B(x_0, 2\rho)} u \leq \liminf_{x \rightarrow x_0} u(x), \quad (2.15)$$

where  $W_{1,p}^f$  is the Wölf potential of  $f$  defined at (1.7), and  $C = C(N, p, K_{1,p}, K_{2,p})$ . If  $u$  satisfies (2.3), then

$$CW_{1,p}^f(B(x_0, \rho)) + \limsup_{x \rightarrow x_0} u(x) \leq \sup_{B(x_0, 2\rho)} u. \quad (2.16)$$

**Proof.** (i) The function  $w = u - m_{2\rho}$ , where  $m_\rho = \inf_{B(x_0, \rho)} u$ , is nonnegative in  $B(x_0, 2\rho)$ , and satisfies the inequality  $-\mathcal{B}_p w \geq f$ , where

$$w \mapsto \mathcal{B}_p w = \operatorname{div} \mathbf{A}_p(x, w + m_{2\rho}, \nabla w)$$

is also a S- $p$ -C operator. Then from Proposition 2.1 with  $\xi$  as in (2.8), fixing  $\ell \in \left(0, \frac{N(p-1)}{N-p}\right)$  and  $\varepsilon = \frac{1}{2}$ , and applying Harnack inequality (2.12), there exists  $C = C(N, p, K_{1,p}, K_{2,p})$  such that

$$2C \left( \rho^{1-N} \int_{B(x_0, \rho)} f \right)^{\frac{1}{p-1}} \leq \rho^{-1} \left( \oint_{B(x_0, 2\rho)} (u - m_{2\rho})^\ell \right)^{\frac{1}{\ell}} \leq \rho^{-1} (m_\rho - m_{2\rho}).$$

Setting  $\rho_j = 2^{1-j}\rho$ , as in [20],

$$CW_{1,p}^f(B(x_0, \rho)) \leq \sum_{j=1}^{\infty} (m_{\rho_j} - m_{\rho_{j-1}}) = \lim m_{\rho_j} - \inf_{B(x_0, 2\rho)} u = \liminf_{x \rightarrow x_0} u - \inf_{B(x_0, 2\rho)} u.$$

(ii) The function  $y = M_{2\rho} - u$  where  $M_{2\rho} = \sup_{B(x_0, 2\rho)} u$  satisfies the inequality  $-\mathcal{C}_p w \geq f$  in  $B(x_0, 2\rho)$ , where

$$w \mapsto \mathcal{C}_p w := \operatorname{div} [\mathbf{A}_p(x, M_{2\rho} - w, \nabla w)]$$

is still S- $p$ -C. Then

$$W_{1,p}^f(B(x_0, \rho)) \leq C \left( \sup_{B(x_0, 2\rho)} u - \limsup_{x \rightarrow x_0} u \right),$$

and (2.16) follows. ■

**Remark 2.7** *The minorizations by Wölf potentials (2.15) and (2.16) have been proved in [20] and [22] for S- $p$ -C operators of type  $\mathcal{A}_p u := \operatorname{div} [\mathcal{A}_p(x, \nabla u)]$  independent of  $u$ , satisfying moreover monotonicity and homogeneity properties, in particular  $\mathcal{A}_p(-u) = -\mathcal{A}_p u$ . The solutions are defined in the sense of potential theory, and may not belong to  $W_{loc}^{1,p}(\Omega)$ ,  $f$  can be a Radon measure; majorizations by Wölf potentials are also given, with weighted operators, see [21] and [22]. In the same way Proposition 2.1 can also be extended to weighted operators, see [8, Remark 2.4] and [14], or to the case of a Radon measure when  $\mathcal{A}_p$  is S- $p$ -C by using the notion of local renormalized solution introduced in [3].*

### 2.3 A bootstrap result

Finally we give a variant of a result of [5, Lemma 2.2]:

**Lemma 2.8** *Let  $d, h \in \mathbb{R}$  with  $d \in (0, 1)$  and  $y, \Phi$  be two positive functions on some interval  $(0, R]$ , and  $y$  is nondecreasing. Assume that there exist some  $K, M > 0$  and  $\varepsilon_0 \in (0, \frac{1}{2}]$  such that, for any  $\varepsilon \in (0, \varepsilon_0]$ ,*

$$y(\rho) \leq K \varepsilon^{-h} \Phi(\rho) y^d[\rho(1 + \varepsilon)] \quad \text{and} \quad \max_{\tau \in [\rho, 3\frac{\rho}{2}]} \Phi(\tau) \leq M \Phi(\rho), \quad \forall \rho \in \left(0, \frac{R}{2}\right].$$

*Then there exists  $C = C(K, M, d, h, \varepsilon_0) > 0$  such that*

$$y(\rho) \leq C \Phi(\rho)^{\frac{1}{1-d}}, \quad \forall \rho \in \left(0, \frac{R}{2e}\right]. \quad (2.17)$$

**Proof.** Let  $\varepsilon_m = \varepsilon_0/2^m$  ( $m \in \mathbb{N}$ ), and  $P_m = (1 + \varepsilon_1) \dots (1 + \varepsilon_m)$ . Then  $(P_m)$  has a finite limit  $P > 0$ , and more precisely  $P \leq e^{2\varepsilon_0} \leq e$ . For any  $\rho \in (0, \frac{R}{2e}]$  and any  $m \geq 1$ ,

$$y(\rho P_{m-1}) \leq K \varepsilon_m^{-h} \Phi(\rho P_{m-1}) y^d(\rho P_m).$$

By induction, for any  $m \geq 1$ ,

$$y(\rho) \leq K^{1+d+\dots+d^{m-1}} \varepsilon_1^{-h} \varepsilon_2^{-hd} \dots \varepsilon_m^{-hd^{m-1}} \Phi(\rho) \Phi^d(\rho P_1) \dots \Phi^{d^{m-1}}(\rho P_{m-1}) y^{d^m}(\rho P_m).$$

Hence from the assumption on  $\Phi$ ,

$$y(\rho) \leq (K \varepsilon_0^{-h})^{1+d+\dots+d^{m-1}} 2^{k(1+2d+\dots+md^{m-1})} M^{d+2d^2+\dots+(m-1)d^{m-1}} \Phi(\rho)^{1+d+\dots+d^{m-1}} y^{d^m}(\rho P_m);$$

and  $y^{d^m}(\rho P_m) \leq y^{d^m}(e\rho) \leq y^{d^m}(\frac{R}{2})$ , and  $\lim y^{d^m}(\frac{R}{2}) = 1$ , because  $d < 1$ . Hence (2.17) follows with  $C = (K \varepsilon_0^{-h})^{1/(1-d)} 2^{h/(1-d)^2} M^{d/(1-d)^2}$ . ■

### 3 Keller-Osserman estimates

#### 3.1 The scalar case

First consider the solutions of inequality

$$-\mathcal{A}_p u + cu^Q \leq 0, \quad \text{in } \Omega, \quad (3.1)$$

with  $Q > p - 1$  and  $c > 0$ . From the integral estimates of Proposition 2.1 we get easily Keller-Osserman estimates in the scalar case of the equation with absorption, without any hypothesis of monotonicity on the operator:

**Proposition 3.1** *Let  $Q > p - 1$ ,  $c > 0$ . If  $\mathcal{A}_p$  is S-p-C, and  $u \in W_{loc}^{1,p}(\Omega) \cap C(\Omega)$  is a nonnegative solution of (3.1), there exists with  $C = C(N, p, K_{1,p}, K_{2,p}, Q)$  such that, for any  $x \in \Omega$ ,*

$$u(x) \leq Cc^{-1/(Q+1-p)}d(x, \partial\Omega)^{-p/(Q+1-p)}. \quad (3.2)$$

**Proof.** Let  $B(x_0, \rho_0) \subset \Omega$ , and  $u \in W^{1,p}(B(x_0, \rho_0))$ . From Corollary 2.2 with  $\rho \leq \frac{\rho_0}{2}$ ,  $\varepsilon = \frac{1}{2}$ , and  $\ell = Q$  and a function  $\varphi$  satisfying (2.8), we obtain for  $\lambda = \lambda(p, Q)$

$$\oint_{\varphi} u^Q \leq c^{-1}C\rho^{-p} \left( \oint_{\varphi} u^Q \right)^{\frac{p-1}{Q}}, \quad (3.3)$$

where  $C = C(N, p, K_{1,p}, K_{2,p}, Q)$ . Then with another  $C > 0$  as above,

$$\left( \oint_{B(x_0, \rho)} u^Q \right)^{\frac{1}{Q}} \leq Cc^{-\frac{1}{Q+1-p}}\rho^{-\frac{p}{Q+1-p}}.$$

Since  $\mathcal{A}_p$  is S-p-C, from the weak Harnack inequality (2.11), with another constant  $C$  as above,

$$u(x_0) \leq C \left( \oint_{B(x_0, \rho)} u^Q \right)^{\frac{1}{Q}} \leq c^{-\frac{1}{Q+1-p}}\rho^{-\frac{p}{Q+1-p}},$$

and (3.2) follows by taking  $\rho_0 = d(x_0, \partial\Omega)$ . ■

#### 3.2 The system (A)

**Proof of Theorem 1.1.** Consider a ball  $B(x_0, \rho_0) \subset \Omega$ ,  $\varepsilon \in (0, \frac{1}{2}]$ , and a function  $\varphi$  satisfying (2.8) with  $\lambda$  large enough.

(i) Case  $\mu > p - 1$ ,  $\delta > q - 1$ . Here  $C$  denotes different constants which only depend on  $N, p, q, \delta, \mu$ , and  $K_{1,p}, K_{2,p}, K_{1,q}, K_{2,q}$ . We take  $\varepsilon = \frac{1}{2}$  and apply Proposition 2.1 with  $\rho \leq \frac{\rho_0}{2}$  to the solution  $u$  with  $f = v^\delta$ , and with  $\ell = \mu > p - 1$ ; since  $\mathcal{A}_p$  is W-p-C, from (2.9), we obtain

$$\oint_{\varphi} v^\delta \leq C\rho^{-p} \left( \oint_{\varphi} u^\mu \right)^{\frac{p-1}{\mu}}, \quad (3.4)$$

and similarly we apply it to the solution  $v$  with now  $f = u^\mu$  and  $\ell = \delta > q - 1$  : since  $\mathcal{A}_q$  is W- $q$ -C, we obtain

$$\oint_{\varphi} u^\mu \leq C \rho^{-q} \left( \oint_{\varphi} v^\delta \right)^{\frac{q-1}{\delta}}. \quad (3.5)$$

We can assume that  $\oint_{\varphi} u^\mu > 0$ . Indeed if  $\oint_{\varphi} u^\mu = 0$ , then  $u = 0$  in  $B(x_0, \rho_0)$ . Then  $\nabla u = 0$ , thus  $v^\delta = 0$  and then the estimates are trivially verified. Replacing (3.5) in (3.4) we deduce

$$\oint_{\varphi} v^\delta \leq C \rho^{-p-q} \frac{\rho^{-1}}{\mu} \left( \oint_{\varphi} v^\delta \right)^{\frac{(q-1)(p-1)}{\mu\delta}},$$

and similarly for  $u$ , hence

$$\left( \oint_{\varphi} v^\delta \right)^{\frac{1}{\delta}} \leq C \rho^{-\xi}, \quad \left( \oint_{\varphi} u^\mu \right)^{\frac{1}{\mu}} \leq C \rho^{-\gamma}. \quad (3.6)$$

Moreover, since  $\mathcal{A}_q$  is S- $q$ -C, then from the usual weak Harnack inequality, since  $v \in L_{loc}^\infty(\Omega)$ , and  $\varphi(x) = 1$  in  $B(x_0, \rho)$ , with values in  $[0, 1]$ ,

$$\sup_{B(x_0, \frac{\rho}{2})} v \leq C \left( \oint_{B(x_0, \rho)} v^\delta \right)^{\frac{1}{\delta}} \leq \left( \oint_{\varphi} v^\delta \right)^{\frac{1}{\delta}} \leq C \rho^{-\xi}.$$

Similarly for  $u$ , since  $\mathcal{A}_p$  is S- $p$ -C.

(ii) Case  $\mu > p - 1$ , and  $\delta \leq q - 1$ . Here we still apply Proposition 2.1 with  $\rho \leq \frac{\rho_0}{2}$ ,  $\varepsilon \in (0, 1/4]$ , and a function  $\varphi$  satisfying (2.8). Since  $\mu > p - 1$ , we still obtain (3.4); and for any  $k > q - 1$ , and  $\lambda$  large enough,

$$\oint_{\varphi} u^\mu \leq C(\varepsilon \rho)^{-q} \left( \oint_{\varphi} v^k \right)^{(q-1)/k}, \quad (3.7)$$

and from Lemma 2.5,

$$\left( \oint_{\varphi} v^k \right)^{1/k} \leq \sup_{B(x_0, \rho(1+\varepsilon))} v \leq C \varepsilon^{-\frac{Nq^2}{\delta^2}} \left( \oint_{B(x_0, \rho(1+2\varepsilon))} v^\delta \right)^{\frac{1}{\delta}}.$$

Then with new constants  $C$ , setting  $m = q + \delta^{-2} N q^2 (q - 1)$ , and  $h = (p - 1) \mu^{-1} m$ ,

$$\oint_{\varphi} u^\mu \leq C \varepsilon^{-m} \rho^{-q} \left( \oint_{B(x_0, \rho(1+2\varepsilon))} v^\delta \right)^{\frac{(q-1)}{\delta}}, \quad (3.8)$$

hence from (3.4) and (3.8),

$$\oint_{B(x_0, \rho)} v^\delta \leq C \oint_{\varphi} v^\delta \leq C \rho^{-p} \left( \oint_{\varphi} u^\mu \right)^{\frac{p-1}{\mu}} \leq C \varepsilon^{-h} \rho^{-\frac{p\mu + q(p-1)}{\mu}} \left( \oint_{B(x_0, \rho(1+2\varepsilon))} v^\delta \right)^{\frac{(p-1)(q-1)}{\delta\mu}},$$

for any  $\rho \leq \frac{\rho_0}{2}$ . Next we apply the bootstrap Lemma 2.8 with  $R = \rho_0$ ,  $y(\rho) = \oint_{B(x_0, \rho)} v^\delta$ ,  $\Phi(r) = r^{-\frac{p\mu+q(p-1)}{\mu}}$  and  $2\varepsilon$ . We deduce that

$$\left( \oint_{B(x_0, \rho)} v^\delta \right)^{1/\delta} \leq C\rho^{-\xi},$$

for any  $\rho < \frac{\rho_0}{2}e$ , and thus also

$$\sup_{B(x_0, \frac{\rho}{2})} v \leq C \left( \oint_{B(x_0, \rho)} v^\delta \right)^{\frac{1}{\delta}} \leq C\rho^{-\xi}, \quad \sup_{B(x_0, \frac{\rho}{2})} u \leq C \left( \oint_{B(x_0, \rho)} u^\mu \right)^{1/\mu} \leq C\rho^{-\gamma}.$$

In particular

$$u(x_0) \leq C\rho_0^{-\gamma}, \quad v(x_0) \leq C\rho_0^{-\xi}, \quad (3.9)$$

for any ball  $B(x_0, \rho_0) \subset \Omega$ , and the estimates (1.9) follow by taking  $\rho_0 = d(x_0, \partial\Omega)$ . ■

### 3.3 The system (M)

**Proof of Theorem 1.2.** We consider a ball  $B(x_0, \rho_0)$  such that  $B(x_0, 2\rho_0) \subset \Omega$ . From Proposition 2.1, we have the same estimates: for any  $\ell > p-1, k > q-1, \rho \leq \rho_0$ ,

$$\oint_{\varphi} u^\mu \leq C\rho^{-q} \left( \oint_{\varphi} v^k \right)^{\frac{q-1}{k}}, \quad \oint_{\varphi} v^\delta \leq C\rho^{-p} \left( \oint_{\varphi} u^\ell \right)^{\frac{p-1}{\ell}}.$$

From Lemma 2.5 (even if  $\mu < p-1$ ), we have

$$\sup_{B(x_0, \frac{\rho}{2})} u^\mu \leq C \oint_{B(x_0, \rho)} u^\mu.$$

Taking  $k < \frac{N(q-1)}{N-q}$ , and using the weak Harnack inequality for  $v$ , we obtain

$$\begin{aligned} \sup_{B(x_0, \frac{\rho}{2})} u^\mu &\leq C \oint_{B(x_0, \rho)} u^\mu \leq C \oint_{\varphi} u^\mu \leq C\rho^{-q} \left( \oint_{\varphi} v^k \right)^{\frac{q-1}{k}} \\ &\leq C\rho^{-q} \left( \oint_{B(x_0, 2\rho)} v^k \right)^{\frac{q-1}{k}} \leq C\rho^{-q} \inf_{B(x_0, \rho)} v^{(q-1)}; \end{aligned}$$

hence (1.10) holds in  $B(x_0, \frac{\rho}{2})$ . Moreover if  $v(x_0) = 0$ , then  $u = 0$  in  $B(x_0, \frac{\rho}{2})$ , then also  $v = 0$  in  $B(x_0, \frac{\rho}{2})$ . Since  $\Omega$  is connected, it implies that  $v \equiv 0$ , and then  $u \equiv 0$ . If  $v \not\equiv 0$ , then  $v$  stays positive in  $\Omega$ , and we can write

$$-\mathcal{A}_q v = dv^{q-1}, \quad \text{in } \Omega, \quad (3.10)$$

with  $d(x) = u^\mu/v^{(q-1)} \leq C\rho^{-q}$  in  $B(x_0, \frac{\rho}{2})$ ; in particular

$$d(x_0) = \frac{u^\mu(x_0)}{v^{q-1}(x_0)} \leq C\rho^{-q}, \quad (3.11)$$

thus (1.10) holds and  $v$  satisfies Harnack inequality in  $\Omega$  : there exists a constant  $C > 0$  such that

$$\sup_{B(x_0, \rho)} v \leq C \inf_{B(x_0, \rho)} v.$$

Therefore

$$\begin{aligned} v^\delta(x_0) &\leq \sup_{B(x_0, \rho)} v^\delta \leq C \inf_{B(x_0, \rho)} v^\delta \leq C \oint_{\varphi} v^\delta \leq C \rho^{-p} \left( \oint_{\varphi} u^\ell \right)^{\frac{p-1}{\ell}} \\ &\leq C \rho^{-p} \sup_{B(x_0, 2\rho)} u^{p-1} \leq C \rho^{-p} \rho^{-q \frac{p-1}{\mu}} \inf_{B(x_0, 4\rho)} v^{\frac{(q-1)(p-1)}{\mu}} \\ &\leq C \rho^{-(p+q \frac{p-1}{\mu})} v^{\frac{(q-1)(p-1)}{\mu}}(x_0); \end{aligned} \quad (3.12)$$

and then (3.9) follows again from (3.12) and (3.11). ■

**Remark 3.2** Once we have proved (3.11) we can obtain the estimate on  $u$  in another way: we have the relation in the ball

$$\mathcal{A}_p u = v^\delta \geq c u^{\frac{\delta \mu}{q-1}}, \quad \text{in } B(x_0, \rho_0)$$

with  $c = C_1 \rho_0^{\frac{q\delta}{q-1}}$ ; then from Osseman-Keller estimates of Proposition 3.1 with  $Q = \frac{\delta \mu}{q-1} > p-1$ , we deduce that

$$u(x) \leq C_2 c^{-1/Q} \rho_0^{-\frac{p}{Q+1-p}} = C_3 \rho_0^{-\gamma}, \quad \text{in } B(x_0, \frac{\rho_0}{2}).$$

The Liouville results are a direct consequence of the estimates:

**Proof of Corollary 1.3.** Let  $x \in \mathbb{R}^N$  be arbitrary. Applying the estimates in a ball  $B(x, R)$ , we deduce that  $u(x) \leq C R^{-\gamma}$ ,  $v(x) \leq C R^{-\xi}$ . Then we get  $u(x) = v(x) = 0$  by making  $R$  tend to  $\infty$ . ■

**Remark 3.3** In the scalar case of inequality (3.1) it was proved in [14] that the Liouville result is also valid for a  $W$ - $p$ - $C$  operator. In the case of systems (A) or (M), the question is open. Indeed the method is based on the multiplication of the inequality by  $u^\alpha$  with  $\alpha$  large enough, and cannot be extended to the system.

## 4 Behaviour near an isolated point

### 4.1 The system $(A_w)$

**Proof of Theorem 1.4.** It is a variant of Theorem 1.1: we consider  $\Omega = B'_1$  and  $x_0 \in B'_{\frac{1}{2}}$ , and take  $\rho_0 = \frac{|x_0|}{4}$ . Here we apply Proposition 2.1 in the ball  $B(x_0, \rho)$  with  $\rho \leq \frac{\rho_0}{2}$  and  $\varepsilon \in (0, \frac{1}{4}]$ . The estimates (3.4) and (3.7) are replaced by

$$\oint_{\varphi} |x|^a v^\delta \leq C(\varepsilon \rho)^{-p} \left( \oint_{\varphi} u^\ell \right)^{\frac{p-1}{\ell}}, \quad \oint_{\varphi} |x|^b u^\mu \leq C(\varepsilon \rho)^{-q} \left( \oint_{\varphi} v^k \right)^{\frac{q-1}{k}}, \quad (4.1)$$



for any  $\ell > p - 1, k > q - 1$ ; and  $2\rho_0 \leq |x| \leq 6\rho_0$  in  $B(x_0, 2\rho_0)$ , then in any of the cases  $a \leq 0$  or  $a > 0$ , with a new constant  $C$ ,

$$\oint_{\varphi} v^{\delta} \leq C \varepsilon^{-p} \rho^{-(p+a)} \left( \oint_{\varphi} u^{\ell} \right)^{\frac{p-1}{\ell}}, \quad \oint_{\varphi} u^{\mu} \leq C \varepsilon^{-q} \rho^{-(q+b)} \left( \oint_{\varphi} v^k \right)^{\frac{q-1}{k}}. \quad (4.2)$$

Then all the proof is the same up to the change from  $p, q$  into  $p + a$  and  $q + b$ . We deduce the same estimates with  $\gamma, \xi$  replaced by  $\gamma_{a,b}, \xi_{a,b}$ :

$$u(x_0) \leq C |x_0|^{-\gamma_{a,b}}, \quad v(x_0) \leq C |x_0|^{-\xi_{a,b}}, \quad (4.3)$$

where  $C$  depends on  $N, p, q, a, b, \delta, \mu$ , and  $K_{1,p}, K_{2,p}, K_{1,q}, K_{2,q}$ . ■

## 4.2 The system $(M_w)$

**Proof of theorem 1.5.** In the same way we obtain estimate (4.3), then we only need to prove the estimate with respect to  $|x|^{-\frac{N-q}{q-1}}$ . We can apply to the function  $v$  the results of [2], recalled in [8, Propositions 2.2 and 2.3]:  $|x|^b u^{\mu} \in L^1\left(B_{\frac{1}{2}}\right)$ , and for any  $k \in \left(0, \frac{N(q-1)}{N-q}\right)$ , and  $\rho > 0$  small enough,

$$\left( \oint_{B(0,\rho)} v^k \right)^{\frac{1}{k}} \leq C \rho^{-\frac{N-q}{q-1}}. \quad (4.4)$$

Moreover, arguing as in the proof of (1.10), we obtain the punctual inequality

$$u^{\mu}(x_0) \leq C |x_0|^{-(q+b)} v^{q-1}(x_0), \quad \text{in } B'_{\frac{1}{2}}, \quad (4.5)$$

which implies that

$$d(x_0) = |x_0|^b \frac{u^{\mu}(x_0)}{v^{q-1}(x_0)} \leq C |x_0|^{-q}.$$

Then  $v$  satisfies the Harnack inequality in  $B'_{\frac{1}{2}}$ , hence from (4.4)

$$v(x_0) \leq \left( \oint_{B(x_0, \frac{|x_0|}{2})} v^k \right)^{\frac{1}{k}} \leq C |x_0|^{-\frac{N-q}{q-1}},$$

and (1.14) follows. ■

## 4.3 Removability results

Here we suppose that

$$(C_p) \begin{cases} \mathcal{A}_p u := \operatorname{div} [A_p(x, \nabla u)], & \mathcal{A}_p \text{ is S-p-C,} \\ (A_p(x, \xi) - A_p(x, \zeta)) \cdot (\xi - \zeta) > 0, & \text{for } \xi \neq \zeta, \\ A_p(x, \lambda \xi) = |\lambda|^{p-2} \lambda A_p(x, \xi), & \text{for } \lambda \neq 0, \end{cases}$$

and similarly for  $\mathcal{A}_q$ . We give sufficient conditions ensuring that at least one of the functions  $u, v$  or both are bounded. We have the two following results, relative to systems  $(A_w)$  and  $(M_w)$ :

**Theorem 4.1** Assume (1.2),  $(C_p), (C_q)$ . Let  $u \in W_{loc}^{1,p}(B'_1)$ ,  $v \in W_{loc}^{1,q}(B'_1)$  be nonnegative solutions of

$$\begin{cases} -\mathcal{A}_p u + |x|^a v^\delta \leq 0, \\ -\mathcal{A}_q v + |x|^b u^\mu \leq 0, \end{cases} \quad \text{in } B'_1.$$

(i) If  $\gamma_{a,b} \leq \frac{N-p}{p-1}$ , then  $u$  is bounded near 0; if  $\xi_{a,b} \leq \frac{N-q}{q-1}$ , then  $v$  is bounded.

(ii) If moreover  $(u, v)$  is a solution of  $(A_w)$  and  $u$  is bounded near 0 and  $\delta > \frac{(p+a)(q-1)}{N-q}$  (or  $\delta = \frac{(p+a)(q-1)}{N-q}$  if  $\mathcal{A}_p = \Delta_p$ ) then  $v$  is also bounded. In the same way if  $v$  is bounded and  $\mu > \frac{(q+b)(p-1)}{N-p}$  (or  $\mu = \frac{(q+b)(p-1)}{N-p}$  if  $\mathcal{A}_q = \Delta_q$ ) then  $u$  is also bounded.

**Theorem 4.2** Assume (1.2),  $(C_p), (C_q)$ . Let  $u \in W_{loc}^{1,p}(B'_1) \cap C(B'_1)$ ,  $v \in W_{loc}^{1,q}(B'_1) \cap C(B'_1)$  be nonnegative solutions of

$$\begin{cases} -\mathcal{A}_p u + |x|^a v^\delta \leq 0, \\ -\mathcal{A}_q v \geq |x|^b u^\mu, \end{cases} \quad \text{in } B'_1.$$

If  $\gamma_{a,b} \leq \frac{N-p}{p-1}$ , or if  $\gamma_{a,b} > \frac{N-p}{p-1}$  and  $\mu > \frac{(N+b)(p-1)}{N-p}$ , then  $u$  is bounded.

The proofs require some lemmas, adapted to subsolutions of equation  $\mathcal{A}_p u = 0$ .

**Lemma 4.3** Assume  $(C_p)$ . Let  $u \in W_{loc}^{1,p}(B'_1) \cap C(B'_1)$  be nonnegative, such that  $-\mathcal{A}_p u \leq 0$  in  $B'_1$ . Then either there exists  $C > 0$  and  $r \in (0, \frac{1}{2})$  such that

$$\sup_{|x|=\rho} u \geq C \rho^{\frac{p-N}{p-1}}, \quad \text{for any } \rho \in (0, r), \quad (4.6)$$

or  $u$  is bounded near 0.

**Proof.** From our assumptions on  $\mathcal{A}_p$ , there exists at least a solution  $E$  of the Dirichlet problem

$$-\mathcal{A}_p E = \delta_0, \quad \text{in } B_1,$$

where  $\delta_0$  is the Dirac mass at 0, in the renormalized sense, see [13, Theorem 3.1]. In particular it satisfies the equation in  $\mathcal{D}'(B_1)$ , and it is a smooth solution of equation  $\mathcal{A}_p E = 0$  in  $B'_1$ . From [25], [26], there exists  $C_1, C_2 > 0$  such that  $C_1 |x|^{-\frac{N-p}{p-1}} \leq E(x) \leq C_2 |x|^{-\frac{N-p}{p-1}}$  near 0. Assume that (4.6) does not hold. Then there exists  $r_n < \min(1/n, r_{n-1})$  such that

$$\sup_{|x|=r_n} u \leq \frac{1}{n} r_n^{\frac{p-N}{p-1}} \leq \frac{1}{nC_1} E(r_n).$$

Next we use the comparison theorem in the annulus  $\mathcal{C}_n = \{x \in \mathbb{R}^N : r_n \leq |x| \leq \frac{1}{2}\}$  for functions in  $W_{loc}^{1,p}(\mathcal{C}) \cap C(\overline{\mathcal{C}_n})$ , and we find that

$$u(x) \leq \frac{1}{nC_1} E(x) + \max_{|x|=\frac{1}{2}} u, \quad \text{in } \mathcal{C}_n.$$

Going to the limit as  $n \rightarrow \infty$ , we deduce that  $u$  is bounded. ■

Our next lemma complements the results of [8, Proposition 2.2]:

**Lemma 4.4** Assume that  $\mathcal{A}_p$  is  $W$ - $p$ -C. Let  $f \in L^1_{loc}(B'_1)$ ,  $f \geq 0$ . Let  $u \in W^{1,p}_{loc}(B'_1)$  be nonnegative, such that

$$-\mathcal{A}_p u + f \leq 0 \quad \text{in } B'_1.$$

If  $|x|^{\frac{N-p}{p-1}} u$  is bounded near 0, then  $f \in L^1_{loc}(B_1)$ .

**Proof.** Let  $0 < \rho < \frac{1}{2}$ . Here we apply Proposition 2.1 with  $\varphi = \xi^\lambda$  given by

$$\xi = 1 \text{ for } \rho < |x| < \frac{1}{2}, \quad \xi = 0 \text{ for } |x| \leq \frac{\rho}{2} \text{ or } |x| \geq \frac{3}{4}, \quad |\nabla \xi| \leq \frac{C_0}{\rho}.$$

From Remark 2.3, we find with for example  $\ell = p$ ,

$$\int_{\rho \leq |x| \leq \frac{1}{2}} f \leq C \rho^{N-p} \left( \oint_{\frac{\rho}{2} \leq |x| \leq \rho} u^\ell \right)^{\frac{p-1}{\ell}} + C \left( \oint_{\frac{1}{2} \leq |x| \leq \frac{3}{4}} u^\ell \right)^{\frac{p-1}{\ell}}. \quad (4.7)$$

Hence from our assumption on  $u$ , the integral is bounded, then  $f \in L^1(B_{\frac{1}{2}})$ . ■

**Proof of Theorem 4.1.** (i) Suppose that  $\gamma_{a,b} \leq \frac{N-p}{p-1}$ . Then  $u(x_0) \leq C |x_0|^{-\frac{N-p}{p-1}}$ . Let us show that  $u$  is bounded. If  $\gamma_{a,b} < \frac{N-p}{p-1}$  it is a direct consequence of Lemma 4.3. Then we can assume  $\gamma_{a,b} = \frac{N-p}{p-1}$ . If  $u$  is not bounded, then (4.6) holds for some  $C > 0$ . Let us set  $f = |x|^a v^\delta$ . From (4.2) with  $\varepsilon = \frac{1}{4}$  then for any  $r_0 \leq \frac{1}{2}$  and any  $x_0$  such that  $|x_0| = r_0$ , and Lemma 2.5, taking  $\rho = \frac{r_0}{4}$ ,

$$\begin{aligned} u^\mu(x_0) &\leq C \oint_{B(x_0, \rho)} u^\mu \leq C r_0^{-(q+b)-N \frac{q-1}{\delta}} \left( \int_{B(x_0, 2\rho)} v^\delta \right)^{\frac{q-1}{\delta}} \\ &\leq C r_0^{-(q+b)-(N+a) \frac{q-1}{\delta}} \left( \int_{\frac{r_0}{2} \leq |x| \leq \frac{3r_0}{2}} f \right)^{\frac{q-1}{\delta}}, \end{aligned}$$

then

$$\begin{aligned} C r_0^{-\mu \gamma_{a,b}} &= C r_0^{-(q-1)\xi_{a,b}-q-b} \leq \sup_{|x|=r_0} u^\mu \leq C r_0^{-(q+b)-(N+a) \frac{q-1}{\delta}} \left( \int_{\frac{r_0}{2} \leq |x| \leq \frac{3r_0}{2}} f \right)^{\frac{q-1}{\delta}}, \\ C r_0^{-(q-1)\xi_{a,b} \frac{\delta}{q-1} + (N+a)} &= C r_0^0 = C \leq \int_{\frac{r_0}{2} \leq |x| \leq \frac{3r_0}{2}} f; \end{aligned}$$

then for any  $n \in \mathbb{N}$ ,

$$C \leq \int_{\frac{r_0}{2 \cdot 3^n} \leq |x| \leq \frac{r_0}{2 \cdot 3^{n-1}}} f.$$

By summation it contradicts Lemma 4.4. Similarly for  $v$ .

(ii) Suppose that  $(u, v)$  is a solution of  $(A_w)$  and  $u$  is bounded and  $\delta \geq \frac{(p+a)(q-1)}{N-q}$ . Here  $v$  satisfies equation  $\mathcal{A}_q v = g$  with  $g = |x|^b u^\mu \leq C |x|^b$ , thus  $g \in L^{N/q+\varepsilon}(\Omega)$  for some  $\varepsilon > 0$ , then from [25], [26], if  $v$  is not bounded near 0, then there exist  $C_1, C_2 > 0$  such that

$$C_1 |x|^{-\frac{N-q}{q-1}} \leq v \leq C_2 |x|^{-\frac{N-q}{q-1}}$$

near 0. If  $\delta > \frac{(p+a)(q-1)}{N-q}$  then

$$\mathcal{A}_p u = |x|^a v^\delta \geq C_1 |x|^{a-\delta \frac{N-q}{q-1}} = C_1 |x|^{-p-\varepsilon},$$

for some  $\varepsilon > 0$ , then from (4.1),

$$\rho^{-p-\varepsilon} \leq C \oint_{\varphi} |x|^{-p-\varepsilon} \leq C \rho^{-p} \left( \oint_{\varphi} u^\ell \right)^{\frac{p-1}{\ell}} \leq C \rho^{-p},$$

which is a contradiction. If  $\delta = \frac{(p+a)(q-1)}{N-q}$ , then

$$C_2 |x|^{-p} \geq \mathcal{A}_p u = |x|^a v^\delta \geq C_1 |x|^{-p}.$$

Otherwise  $u$  is bounded by some  $M$  in a ball  $B'_r$ . Then the function  $w = M - u$  is nonnegative and bounded and satisfies

$$-\mathcal{A}_p w \geq C_1 |x|^{-p} \quad \text{in } B'_r.$$

But for  $\mathcal{A}_p = \Delta_p$ , there is no bounded solution of this inequality, from [8, Proposition 2.7], we reach a contradiction. ■

**Remark 4.5** *The results obviously apply to the scalar case, finding again and improving a result of [31].*

**Proof of Theorem 4.2.** (i) Assume  $\gamma_{a,b} \leq \frac{N-p}{p-1}$ . The proof of part (i) of Theorem 4.1 is still valid and shows that  $u$  is bounded.

(ii) Assume  $\gamma_{a,b} > \frac{N-p}{p-1}$  and  $\mu > \frac{(N+b)(p-1)}{N-p}$ . Then  $\xi_{a,b} > \frac{N-q}{q-1}$ , thus the estimate (1.14) for  $v$  gives  $v(x_0) \leq C |x_0|^{-\frac{N-q}{q-1}}$ , then

$$u^\mu(x_0) \leq C |x_0|^{-(q+b)} v^{(q-1)}(x_0) \leq C |x_0|^{-(N+b)}.$$

Then  $\rho^{\frac{N-p}{p-1}} \sup_{|x|=\rho} u$  tends to 0, hence  $u$  is bounded from Lemma 4.3.

**Remark 4.6** *Let us give an alternative proof of (i): the punctual inequality (4.5) implies that near 0,*

$$\mathcal{A}_p u \geq |x|^a v^\delta \geq C |x|^{a+\delta(q+b)/(q-1)} u^{\mu\delta/(q-1)};$$

*then we are reduced to a simple scalar inequality:*

$$-\mathcal{A}_p u + |x|^m u^Q \leq 0, \tag{4.8}$$

*with  $Q = \frac{\mu\delta}{q-1} > p-1$  and  $m = a + \frac{\delta(q+b)}{q-1} > -p$ . And  $\gamma_{a,b} = \frac{m+p}{Q+1-p} \leq \frac{N-p}{p-1}$ ; applying Theorem 4.1 to the scalar inequality (4.8), we find again that  $u$  is bounded.*

■

## 5 Sharpness of the results

In this last section we show the optimality of our results by constructing some radial solutions of systems  $(A_w)$  or  $(M_w)$  in case  $\mathcal{A}_p = \Delta_p, \mathcal{A}_q = \Delta_q$ . They are based on the transformation introduced in [4], valid for systems with any sign:

$$\begin{cases} -\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \varepsilon_1 |x|^a v^\delta, \\ -\Delta_q v = -\operatorname{div}(|\nabla v|^{q-2} \nabla v) = \varepsilon_2 |x|^b u^\mu, \end{cases}$$

with  $\varepsilon_1 = -1 = \varepsilon_2$  for the system with absorption, and  $\varepsilon_1 = -1, \varepsilon_2 = 1$  for the mixed system: setting

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = -\varepsilon_1 r^{1+a} u^s v^\delta \frac{u'}{|u'|^p}, \quad W(t) = -\varepsilon_2 r^{1+b} u^\mu v^m \frac{v'}{|v'|^q},$$

where  $t = \ln r$ , and we obtain the system

$$(\Sigma) \begin{cases} X_t = X \left[ X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = Y \left[ Y - \frac{N-q}{q-1} + \frac{W}{q-1} \right], \\ Z_t = Z [N + a - \delta Y - Z], \\ W_t = W [N + b - \mu X - W]. \end{cases}$$

And  $u, v$  are recovered from  $X, Y, Z, W$  by the relations

$$u = r^{-\gamma_{a,b}} (|X|^{p-1} Z)^{(q-1)/D} (|Y|^{q-1} W)^{\delta/D}, \quad v = r^{-\xi_{a,b}} (|X|^{p-1} Z)^{\mu/D} (|Y|^{q-1} W)^{(p-1)/D}. \quad (5.1)$$

### 5.1 About Harnack inequality

Here we show that Harnack inequality can be *false* in case of system  $(A_w)$  and also for the function  $u$  of system  $(M_w)$ , even in the radial case; indeed we construct nonnegative radial solutions of system  $(A_w)$  in a ball such that  $u(0) = 0 < v(0)$ , or by symmetry  $u(0) > 0 = v(0)$  and solutions of system  $(M_w)$  such that  $u(0) = 0 < v(0)$ . Such solutions were constructed in [15] by using Schauder theorem, and in [7] in the case of system  $(A_w)$  for  $p = q = 2$  by using system  $(\Sigma)$ . Here we show that the construction of [7] extends to the general case. We consider the radial regular solutions, which are  $C^2$  if  $a, b \geq 0$ , and  $C^1$  if  $a, b > -1$ .

**Proposition 5.1**  $\mathcal{A}_p = \Delta_p, \mathcal{A}_q = \Delta_q$ . *For any  $v_0 > 0$ , there exists a regular radial solution of  $(A_w)$  and  $(M_w)$  such that  $u(0) = 0 < v(0) = v_0$ .*

**Proof.** The regular solutions  $(u, v)$  with nonnegative initial data  $(u_0, v_0) \neq (0, 0)$  are increasing for system  $(A_w)$ , hence  $X, Y < 0 < Z, W$  and  $u$  is increasing and  $v$  is decreasing for system  $(M_w)$ , hence  $X < 0 < Y$  and  $Z, W > 0$ . As shown in [4], the solutions  $(u, v)$  with  $u(0) = u_0 > 0, v(0) = v_0 > 0$  correspond to the trajectories of system  $(\Sigma)$  converging to the fixed point  $N_0 = (0, 0, N + a, N + b)$  as  $t \rightarrow -\infty$ , and local existence and uniqueness holds as in [4, Proposition 4.4]. As in [7] the solutions such that  $u_0 = 0 < v_0$  correspond to a trajectory converging to the

point  $S_0 = (\bar{X}, 0, \bar{Z}, \bar{W}) = \left(-\frac{p+a}{p-1}, 0, N+a, N+b+\mu\frac{p+a}{p-1}\right)$ . The linearization at  $S_0$  gives the eigenvalues

$$\lambda_1 = \bar{X} < 0, \quad \lambda_2 = \frac{1}{q-1}(q+b+\mu\frac{p+a}{p-1}) > 0, \quad \lambda_3 = -\bar{Z} < 0, \quad \lambda_4 = -\bar{W} < 0.$$

Then the unstable manifold  $\mathcal{V}_u$  has dimension 1 and  $\mathcal{V}_u \cap \{Y = 0\} = \emptyset$ , thus there exists a unique trajectory such that  $Y < 0$  (resp.  $Y > 0$ ) and  $Z, W > 0$ . There holds  $\lim_{t \rightarrow -\infty} e^{-\lambda_2 t} Y = c > 0$ ,  $\lim X = \bar{X}$ ,  $\lim Z = \bar{Z}$ ,  $\lim W = \bar{W}$ , then from (5.1)  $v$  has a positive limit  $v_0$ , and  $u$  tends to 0. By scaling we obtain the existence and uniqueness of solutions for any  $v_0 > 0$ . ■

## 5.2 About removability

Here also we show that the results of Theorems 4.1 and 4.2 are optimal, by constructing singular solutions when the assumptions are not satisfied. We begin by system  $(A_w)$ , extending [7, Proposition 3.2]. Obviously it admits a particular singular solution when  $\gamma_{a,b} > \frac{N-p}{p-1}$  and  $\xi_{a,b} > \frac{N-q}{q-1}$ . Moreover we find other types of singular solutions:

**Proposition 5.2** *Consider system  $(A_w)$  with  $\mathcal{A}_p = \Delta_p, \mathcal{A}_q = \Delta_q$ .*

- (i) *If  $\mu < \frac{(q+b)(p-1)}{N-p}$ , there exist solutions such that  $\lim_{\rho \rightarrow 0} \rho^{\frac{N-p}{p-1}} u = \alpha > 0$ ,  $\lim_{\rho \rightarrow 0} v = \beta > 0$ .*
- (ii) *If  $\delta < \frac{(N+a)(q-1)}{N-q}$  and  $\mu < \frac{(N+b)(p-1)}{N-p}$ , there exist solutions such that  $\lim_{\rho \rightarrow 0} \rho^{\frac{N-p}{p-1}} u = \alpha > 0$ ,  $\lim_{\rho \rightarrow 0} \rho^{\frac{N-q}{q-1}} v = \beta > 0$ .*
- (iii) *If  $\gamma_{a,b} > \frac{N-p}{p-1}$ , and either  $\mu > \frac{(N+b)(p-1)}{N-p}$  or  $\mu < \frac{(q+b)(p-1)}{N-p}$ , there exist solutions such that  $\lim_{\rho \rightarrow 0} \rho^{\frac{N-p}{p-1}} u = \alpha > 0$ ,  $\lim_{\rho \rightarrow 0} \rho^{\frac{1}{q-1}(\frac{N-p}{p-1}\mu - (q+b))} v = \beta(\alpha) > 0$ .*

*The results extend by symmetry, after exchanging  $u, v, a, \gamma_{a,b}$  and  $v, u, b, \xi_{a,b}$ .*

**Proof.** As in [5], [7] we prove the existence of trajectories of system  $(\Sigma)$  and return to  $u, v$  by using (5.1).

- (i) Such solutions correspond to trajectories converging to the fixed point  $G_0 = (\frac{N-p}{p-1}, 0, 0, N+b - \frac{N-p}{p-1}\mu)$  of  $(\Sigma)$ . The linearization at  $G_0$  gives the eigenvalues

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_2 = \frac{1}{q-1}(q+b - \frac{N-p}{p-1}\mu), \quad \lambda_3 = N+a > 0, \quad \lambda_4 = \frac{N-p}{p-1}\mu - N-b.$$

If  $\mu < \frac{(q+b)(p-1)}{N-p}$ , then  $\lambda_2, \lambda_4 < 0$ . Then  $\mathcal{V}_u$  has dimension 3, and  $\mathcal{V}_u \cap \{Y = 0\}$  and  $\mathcal{V}_u \cap \{Z = 0\}$  have dimension 2. This implies that  $\mathcal{V}_u$  must contain trajectories such that  $Y, Z < 0 < X, W$ .

- (ii) Such solutions correspond to the fixed point  $A_0 = \left(\frac{N-p}{p-1}, \frac{N-q}{q-1}, 0, 0\right)$ . All the eigenvalues are positive:

$$\lambda_1 = \frac{N-p}{p-1}, \quad \lambda_2 = \frac{N-q}{q-1}, \quad \lambda_3 = N+a - \delta \frac{N-q}{q-1}, \quad \lambda_4 = N+b - \mu \frac{N-p}{p-1}.$$

The unstable manifold  $\mathcal{V}_u$  has dimension 4, then there exists an infinity of trajectories converging to  $A_0$  with  $X, Y, Z, W < 0$ .

(iii) Such solutions correspond to the fixed point  $P_0 = \left(\frac{N-p}{p-1}, Y_*, 0, W_*\right) = \left(\frac{N-p}{p-1}, Y_*, 0, W_*\right)$ , with  $Y_* = \frac{1}{q-1}\left(\frac{N-p}{p-1}\mu - (q+b)\right)$ ,  $W_* = N + b - \frac{N-p}{p-1}\mu$ . the eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_2 = Y_*, \quad \lambda_3 = \frac{D}{q-1}\left(\gamma - \frac{N-p}{p-1}\right) > 0, \quad \lambda_4 = -W_*.$$

If  $\mu > \frac{(N+b)(p-1)}{N-p}$ , then  $\lambda_2, \lambda_4 > 0$  and thus  $\mathcal{V}_u$  has dimension 4, then there exist trajectories, with  $X, Y, Z, W < 0$ , converging to  $P_0$ . If  $\mu < \frac{(q+b)(p-1)}{N-p}$ , then  $\lambda_2, \lambda_4 < 0$ ,  $\mathcal{V}_u$  has dimension 2, and  $\mathcal{V}_u \cap \{Z = 0\}$  has dimension 1, thus there also exist trajectories with  $X, Z, W < 0 < Y$  converging to  $P_0$ . ■

In the same way, system  $(M_w)$  has a particular singular solution when  $\gamma_{a,b} > \frac{N-p}{p-1}$  and  $\xi_{a,b} < \frac{N-q}{q-1}$ , and we find other singular solutions:

**Proposition 5.3** *Consider system  $(M_w)$  with  $\mathcal{A}_p = \Delta_p, \mathcal{A}_q = \Delta_q$ .*

(i) *If  $\gamma_{a,b} > \frac{N-p}{p-1}$ , and  $\xi_{a,b} > \frac{N-q}{q-1}$ , there exist solutions such that  $\lim_{\rho \rightarrow 0} \rho^{\frac{N-q}{q-1}} v = \beta > 0$  and  $\lim_{\rho \rightarrow 0} \rho^{\frac{1}{p-1}\left(\frac{N-q}{q-1}\delta - (q+a)\right)} u = \beta(\alpha) > 0$ .*

(ii) *If  $\delta < \frac{(N+a)(q-1)}{N-q}$  and  $\mu < \frac{(N+b)(p-1)}{N-p}$ , there exist solutions such that  $\lim_{\rho \rightarrow 0} \rho^{\frac{N-p}{p-1}} u = \alpha > 0$ ,  $\lim_{\rho \rightarrow 0} \rho^{\frac{N-q}{q-1}} v = \beta > 0$ .*

**Proof.** (i) These solutions correspond to the fixed point  $Q_0$  deduced from  $P_0$  by symmetry, and our assumptions imply  $\delta > \frac{(N+a)(q-1)}{N-q}$ , hence there exist trajectories, such that  $X, Y, Z < 0 < W$  converging to  $Q_0$ .

(ii) The conclusion follows as in Proposition 5.2, (ii). ■

We refer to [5] and [6] for a description of all the (various) possible behaviours of the solutions in the case  $p = q = 2$ .

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