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Random products of automorphisms of Heisenberg nilmanifolds and Weil’s representation

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Abstract

For $n \geq 1$, let $H$ be the $(2n+1)$-dimensional real Heisenberg group, and let $\Lambda$ be a lattice in $H$. Let $\Gamma$ be a group of automorphisms of the corresponding nilmanifold $\Lambda \backslash H$ and $U$ the associated unitary representation of $\Gamma$ on $L^2(\Lambda \backslash H)$. Denote by $T$ the maximal torus factor associated to $\Lambda \backslash H$. Using Weil’s representation (also known as the metaplectic representation), we show that a dense set of matrix coefficients of the restriction of $U$ to the orthogonal complement of $L^2(T)$ in $L^2(\Lambda \backslash H)$ belong to $\ell^{4n+2+\varepsilon}(\Gamma)$ for every $\varepsilon > 0$.

We give the following application to random walks on $\Lambda \backslash H$ defined by a probability measure $\mu$ on $\text{Aut}(\Lambda \backslash H)$. Denoting by $\Gamma$ the subgroup of $\text{Aut}(\Lambda \backslash H)$ generated by the support of $\mu$ and by $U^0$ and $V^0$ the restrictions of $U$ respectively to the subspaces of $L^2(\Lambda \backslash H)$ and $L^2(T)$ with zero mean, we prove the following inequality:

$$\|U^0(\mu)\| \leq \max \left\{ \|V^0(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/(2n+2)} \right\},$$

where $\lambda_\Gamma$ is the left regular representation of $\Gamma$ on $\ell^2(\Gamma)$. In particular, the action of $\Gamma$ on $\Lambda \backslash H$ has a spectral gap if and only if the corresponding action of $\Gamma$ on $T$ has a spectral gap.

1 Introduction

Let $(X, m)$ be a probability space and $G$ a locally compact group of measure preserving transformations of $X$. Given a probability measure $\mu$ on $G$,
consider a sequence of independent $\mu$-distributed random variables $X_n^\omega$ with values in $G$ and the corresponding random products $S_n^\omega = X_n^\omega \ldots X_1^\omega$ for $n \in \mathbb{N}$. This defines a random walk on $X$ with initial distribution $m$ and trajectories $S_n^\omega(x)$ for $x \in X$ and $n \in \mathbb{N}$. A question of interest is whether this random walk has a spectral gap. To define this notion, let $U$ be the unitary representation of $G$ on $L^2(X, m)$ defined by $U_g(\xi) = \xi(g^{-1}(x))$ for $g \in G$, $\xi \in L^2(X, m)$, and $x \in X$. Let $L^2_0(X, m)$ be the $U(G)$-invariant subspace of functions $\xi$ in $L^2(X, m)$ with zero mean, that is, with $\int_X \xi(x)dm(x) = 0$. Denote by $U^0$ the restriction of $U$ to $L^2_0(X, m)$. Let $U^0(\mu)$ be the convolution operator defined on $L^2_0(X, m)$ by

$$U^0(\mu)\xi = \int_G U^0_g(\xi)d\mu(g) \quad \text{for all} \quad \xi \in L^2_0(X, m).$$

Observe that $\|U^0(\mu)\| \leq 1$. We say that $\mu$ has a spectral gap in $(X, m)$ if $\|U^0(\mu)\| < 1$. This spectral gap property has several interesting applications; the most immediate one is the exponentially fast convergence of the sequence of functions $x \mapsto E(\xi(S_n^\omega(x)))$ to $\int_X \xi dm$ in the $L^2$-norm for every $\xi \in L^2(X, m)$. Other applications include the existence of a rate of convergence for random ergodic theorems, a central limit theorem, and the uniqueness of invariant means on $L^\infty(X, m)$; see [FuSh99], [Guiv05], [Lubo94], [Sarn90].

The spectral gap property can be formulated in terms of weak containment of group representations (see [BeHV08, G.4.2]). Assume that the subgroup generated by the support of $\mu$ is dense in $G$. Assume moreover that $\mu$ is aperiodic (that is, the support of $\mu$ is not contained in the coset of a proper closed subgroup of $G$). Then $\mu$ has a spectral gap in $(X, m)$ if and only if there is no $G$-almost invariant vectors in $L^2_0(X, m)$. If this is the case, we say for short that the $G$-action on $X$ has a spectral gap.

We emphasize that the existence of a spectral gap property is a phenomenon which can occur only in the context of non-amenable groups: when $G$ is a discrete amenable group and $m$ is non-atomic, then $G$ has never a spectral gap on $X$ (see [JuRo79] or [Schmi80]).

When $X$ is the $n$-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$, equipped with the normalized Lebesgue measure $m$, sufficient conditions were given in [FuSh99] for the existence of a spectral gap for the action of a subgroup of $GL_n(\mathbb{Z})$ by automorphisms on $\mathbb{R}^n/\mathbb{Z}^n$ (see also Example 4 below). In this paper, we will consider the case where $X$ is a Heisenberg nilmanifold and $G$ a group of automorphisms of $X$.  

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For \( n \geq 1 \), let \( H = H_{2n+1}(\mathbb{R}) \) be the \((2n+1)\)-dimensional real Heisenberg group. This is a two step nilpotent Lie group with one-dimensional centre \( Z \) (see Section 2 below). Let \( \Lambda \) be a lattice in \( H \): \( \Lambda \) is a discrete subgroup of \( H \) such that there exists a (unique) probability measure \( m \) on the Borel sets of the corresponding nilmanifold \( \Lambda \setminus H \) which is invariant under right translation by elements from \( H \). (Observe that \( \Lambda \) is cocompact in \( H \).) Denote by \( \text{Aut}(H) \) the group of continuous automorphisms of \( H \) and by \( \text{Aut}(\Lambda \setminus H) \) the subgroup of all \( g \in \text{Aut}(H) \) such that \( g(\Lambda) = \Lambda \); every automorphism \( g \in \text{Aut}(\Lambda \setminus H) \) induces a homeomorphism of \( \Lambda \setminus H \).

Let \( \Gamma \) be a subgroup of \( \text{Aut}(\Lambda \setminus H) \). The action of \( \Gamma \) on \( \Lambda \setminus H \) preserves the \( H \)-invariant probability measure \( m \) on \( \Lambda \setminus H \). Let \( U \) be the associated unitary representation of \( \Gamma \) on \( L^2(\Lambda \setminus H, m) \). Let \( T = \Lambda Z \setminus H \) be the maximal torus factor of \( \Lambda \setminus H \). Observe that \( T \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n} \) and \( \text{Aut}(T) \cong GL_{2n}(\mathbb{Z}) \). Since \( \text{Aut}(\Lambda \setminus H) \) preserves \( Z \Lambda \), we have a homomorphism \( p : \text{Aut}(\Lambda \setminus H) \to \text{Aut}(T) \) and an induced action of \( \Gamma \) on \( T \). This defines a unitary representation of \( \Gamma \) on \( L^2(T) \), where \( T \) is equipped with normalized Lebesgue measure. We can (and will) identify \( L^2(T) \), as \( \Gamma \)-space, with a closed \( U(\Gamma) \)-invariant subspace of \( L^2(\Lambda \setminus H) \). Denote by \( \mathcal{H} \) the orthogonal complement of \( L^2(T) \) in \( L^2(\Lambda \setminus H) \), so that we have an orthogonal decomposition

\[
L^2(\Lambda \setminus H) = L^2(T) \oplus \mathcal{H}
\]

into \( U(\Gamma) \)-invariant subspaces. Here is our main result.

**Theorem 1** The matrix coefficients of the restriction of \( U \) to \( \mathcal{H} \) are strongly \( L^{4n+2+\varepsilon} \): there are dense subspaces \( D_1 \) and \( D_2 \) of \( \mathcal{H} \) such that, for any \( v \in D_1 \) and \( w \in D_2 \), the matrix coefficient \( \gamma \mapsto \langle U_\gamma v, w \rangle \) belongs to \( \ell^{4n+2+\varepsilon}(\Gamma) \), for every \( \varepsilon > 0 \).

Concerning the proof of the previous theorem, we first show that the representation \( U \) is linked with Weil’s representation, which is also known as Segal-Shale-Weil, metaplectic, or oscillator representation (see [Shal62], [Weil64]). The crucial tool is then a result from [HoMo79] about the decay of the matrix coefficients of Weil’s representation.

Here is an immediate consequence of Theorem 1. Recall that, if \( X \) is a locally compact space, \( C_0(X) \) denotes the space of complex-valued continuous functions on \( X \) which tend to zero at infinity.

**Corollary 2** The restriction of the unitary representation \( U \) to \( \mathcal{H} \) is mixing: the matrix coefficients \( \gamma \mapsto \langle U_\gamma v, w \rangle \) belong to \( C_0(\Gamma) \) for all \( v, w \in \mathcal{H} \).
The previous corollary immediately implies that the ergodicity or mixing of
the \( \Gamma \)-action on \( \Lambda \setminus H \) is equivalent to the ergodicity or mixing of the \( \Gamma \)-action
on \( T \) (see Corollary 6 below).

We apply Theorem 1 to the existence of a spectral gap for the random
walk on \( \Lambda \setminus H \) associated to a probability measure \( \mu \) on \( \text{Aut}(\Lambda \setminus H) \).

**Theorem 3** Let \( \mu \) be a probability measure on \( \text{Aut}(\Lambda \setminus H) \). Denote by \( \Gamma \) be
the subgroup of \( \text{Aut}(\Lambda \setminus H) \) generated by the support of \( \mu \). Let \( U^0 \) and \( V^0 \) be the
associated unitary representations of \( \Gamma \) on \( L^2_0(\Lambda \setminus H) \) and \( L^2_0(T) \) respectively. Then
\[
\|U^0(\mu)\| \leq \max\{\|V^0(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/(2n+2)}\},
\]
where \( \lambda_\Gamma \) is the left regular representation of \( \Gamma \) on \( \ell^2(\Gamma) \). In particular, the
action of \( \Gamma \) on \( \Lambda \setminus H \) has a spectral gap if and only if the corresponding action
of \( \Gamma \) on \( T \) has a spectral gap.

**Example 4** Let \( \Gamma \) be a subgroup of \( \text{Aut}(\Lambda \setminus H) \) such that its image \( p(\Gamma) \subset
GL_{2n}(\mathbb{Z}) \) under the homomorphism \( p : \text{Aut}(\Lambda \setminus H) \to \text{Aut}(T) \cong GL_{2n}(\mathbb{Z}) \) acts
irreducibly on \( \mathbb{R}^{2n} \) and does not have an abelian subgroup of finite index (this
is for instance the case if \( p(\Gamma) \) is Zariski dense in \( GL_{2n}(\mathbb{R}) \)). Then, as shown
in [FuSh99, Theorem 6.5], the action of \( \Gamma \) on \( T \) has a spectral gap.

In the case \( n = 1 \), we have the following more precise result.

**Corollary 5** Let \( H_3(\mathbb{R}) \) be the 3–dimensional Heisenberg group and \( \Lambda \) a
lattice in \( H \). Let \( \mu \) be a probability measure on \( \text{Aut}(\Lambda \setminus H) \). Then
\[
\|U^0(\mu)\| \leq \|\lambda_\Gamma(\mu)\|^{1/4},
\]
where \( \Gamma \) is the subgroup generated by the support of \( \mu \). In particular, if \( \mu \) is
aperiodic, \( \|U^0(\mu)\| < 1 \) if and only if \( \Gamma \) is non-amenable.

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2 Proofs

We first recall the definition of the Heisenberg group \( H = H_{2n+1}(\mathbb{R}) \); we then describe the automorphism group of \( H \) as well as its irreducible unitary representations.

Let \( n \geq 1 \) be an integer. Consider the symplectic form \( \beta \) on \( \mathbb{R}^{2n} \) given by
\[
\beta((x, y), (x', y')) = (x, y)^t J (x', y') \quad \text{for all} \quad (x, y), (x', y') \in \mathbb{R}^{2n},
\]
where \( J \) is the \((2n \times 2n)\)-matrix
\[
J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}
\]
and \( I_n \) is the \( n \times n \)-identity matrix. The \((2n + 1)\)-dimensional Heisenberg group (over \( \mathbb{R} \)) is the group \( H = H_{2n+1}(\mathbb{R}) \) with underlying set \( \mathbb{R}^{2n} \times \mathbb{R} \) and product
\[
((x, y), s)((x', y'), t) = \left((x + x', y + y'), s + t + \frac{1}{2}\beta((x, y), (x', y'))\right),
\]
for \( (x, y), (x', y') \in \mathbb{R}^{2n}, s, t \in \mathbb{R} \). This is a two-step nilpotent Lie group. Its centre \( Z \) coincides with its commutator subgroup and is given by
\[
Z = \{((0, 0), s) : s \in \mathbb{R}\}.
\]

The symplectic group \( Sp_{2n}(\mathbb{R}) \), which is the subgroup of \( GL_{2n}(\mathbb{R}) \) of all matrices \( g \) with \( ^t g J g = J \), acts by automorphisms on \( H = H_{2n+1}(\mathbb{R}) \):
\[
g((x, y), t) = (g(x, y), t) \quad \text{for all} \quad g \in Sp_{2n}(\mathbb{R}), (x, y) \in \mathbb{R}^{2n}, \quad t \in \mathbb{R}.
\]
As is well-known (see [Foll89, 1.22]), the automorphism group \( \text{Aut}(H) \) of \( H \) is generated by:

- the inner automorphisms,
- the automorphisms defined by matrices from \( Sp_{2n}(\mathbb{R}) \) as above,
- the dilations \( ((x, y), t) \mapsto ((rx, ry), r^2t) \) for \( r > 0 \), and
- the inversion \( i : ((x, y), t) \mapsto ((y, x), -t) \).
The connected component \( \text{Aut}(H)_0 \) of the identity in \( \text{Aut}(H) \) is a subgroup of index two and can be viewed as the group of \((2n + 1) \times (2n + 1)\)-matrices of the form
\[
\begin{pmatrix}
  rA & 0 \\
  a^t & r^{2n}
\end{pmatrix}
\]
with \( A \in \text{Sp}_{2n}(\mathbb{R}) \), \( r > 0 \), and \( a \) a column vector in \( \mathbb{R}^{2n} \) (the action on \( H \) corresponding to the usual action on \( \mathbb{R}^{2n+1} \)). The subgroup of automorphisms of \( H \) fixing pointwise the centre can be identified with the group of matrices of the form
\[
\begin{pmatrix}
  A & 0 \\
  a^t & 1
\end{pmatrix}
\]
and is hence isomorphic to the semi-direct product \( \text{Sp}_{2n}(\mathbb{R}) \ltimes \mathbb{R}^{2n} \), for the standard action of \( \text{Sp}_{2n}(\mathbb{R}) \) on \( \mathbb{R}^{2n} \).

The unitary dual \( \hat{H} \) of \( H \) (that is, the set of classes of irreducible unitary representations of \( H \) under unitary equivalence) consists of the equivalence classes of the following representations (see [Foll89, 1.50]):

- the unitary characters of the abelianized group \( H/Z \);
- for every \( t \in \mathbb{R} \setminus \{0\} \), the infinite dimensional representation \( \pi_t \) defined on \( L^2(\mathbb{R}^n) \) by the formula
  \[
  \pi_t((a, b), s)\xi(x) = \exp(2\pi its) \exp \left( 2\pi \langle a, x - \frac{b}{2} \rangle \right) \xi(x - b)
  \]
  for \((a, b), s) \in H, \xi \in L^2(\mathbb{R}^n)\), and \( x \in \mathbb{R}^n \).

For \( t \neq 0 \), the representation \( \pi_t \) is, up to unitary equivalence, the unique irreducible unitary representation of \( \hat{H} \) whose restriction to the centre \( Z \) is a multiple of the unitary character \( s \mapsto \exp(2\pi its) \).

The group \( \text{Aut}(H) \) acts on \( \hat{H} \) by
\[
\pi^g(h) = \pi(g^{-1}(h)) \quad \text{for all } \pi \in \hat{H}, g \in \text{Aut}(H), h \in H.
\]

Let \( g \in \text{Sp}_{2n}(\mathbb{R}) \). For \( t \in \mathbb{R} \setminus \{0\} \), the representation \( \pi^g_t \) is unitary equivalent to \( \pi_t \), since both representations have the same restriction to \( Z \). Therefore, there exists a unitary operator \( \sigma(g) \) on \( L^2(\mathbb{R}^n) \) such that
\[
\sigma(g)\pi_t(g^{-1}(h))\sigma(g)^{-1} = \pi_t(h) \quad \text{for all } h \in H.
\]
By Schur’s lemma, \( \sigma(g) \) is unique up to a scalar multiple of the identity operator. Hence, for \( g_1, g_2 \in Sp_{2n}(\mathbb{R}) \), there exists a complex number \( c(g_1, g_2) \) of modulus one such that \( \sigma(g_1)\sigma(g_2) = c(g_1, g_2)\sigma(g_1g_2) \). This means that \( g \mapsto \sigma(g) \) is a projective unitary representation of \( Sp_{2n}(\mathbb{R}) \). We extend \( \sigma \) to a projective unitary representation \( \omega_t \), called Weil’s representation, of \( Sp_{2n}(\mathbb{R}) \ltimes \mathbb{R}^{2n} \) by setting 

\[
\omega_t(g, a) = \sigma(g)\pi_t(a) \quad \text{for all} \quad (g, a) \in Sp_{2n}(\mathbb{R}) \ltimes \mathbb{R}^{2n}.
\]

Although we will not need this fact, it is worth mentioning that \( \omega_t \) lifts to an ordinary representation of a two-fold cover of \( Sp_{2n}(\mathbb{R}) \ltimes \mathbb{R}^{2n} \) (see [Foll89, Chapter 4]).

How, let \( \Lambda \) be a lattice in \( H \). (As an example, \( \Lambda \) can be the standard lattice \( \{(x, y), s/2) : x, y \in \mathbb{Z}^n, s \in \mathbb{Z}\} \); a full classification of the lattices in \( H \) is given in [Ausl77, I. 2].)

The Lebesgue measure on \( \mathbb{R}^{2n} \times \mathbb{R} \) is a Haar measure on \( H \) and induces an invariant measure \( m \) on the nilmanifold \( \Lambda \backslash H \). (For the classification of \( \Gamma \)-invariant measures on \( \Lambda \backslash H \) for “large” groups \( \Gamma \subset \text{Aut}(\Lambda \backslash H) \), see [Heu09].)

**Proof of Theorem 1.**

Let \( \Gamma \) be a subgroup of \( \text{Aut}(\Lambda \backslash H) \). Then \( \Gamma \) is a discrete subgroup of \( \text{Aut}(H) \), for the topology of uniform convergence on compact subsets of \( H \). Moreover, the subgroup of \( \Gamma \) consisting of the automorphisms fixing pointwise the centre of \( H \) has finite index in \( \Gamma \). Indeed, the mapping

\[
\text{Aut}(H)_0 \to \mathbb{R}^*, \quad \left( \begin{array}{cc} rA & 0 \\ a^t & r^{2n} \end{array} \right) \mapsto r
\]

is a homomorphism and the image of \( \Gamma \cap \text{Aut}(H)_0 \) is a discrete subgroup of \( \mathbb{R}^* \).

It is clear that, if Theorem 1 is true for a subgroup of finite index in \( \Gamma \), then it is true for \( \Gamma \). So, we can (and will) assume that \( \Gamma \) is a subgroup of \( Sp_{2n}(\mathbb{R}) \ltimes \mathbb{R}^{2n} \).

Since every \( \gamma \in \Gamma \) preserves the measure \( m \) on \( \Lambda \backslash H \), we have an associated unitary representation \( U : \gamma \mapsto U_\gamma \) of \( \Gamma \) on \( L^2(\Lambda \backslash H, m) \).

Let \( \rho_{\Lambda \backslash H} \) be the unitary representation of \( H \) on \( L^2(\Lambda \backslash H, m) \) given by right translation:

\[
\rho_{\Lambda \backslash H}(h)\xi(x) = \xi(xh) \quad \text{for all} \quad h \in H, \xi \in L^2(\Lambda \backslash H, m), \ x \in \Lambda \backslash H.
\]
The representations $U$ and $\rho_{\Lambda \setminus H}$ are linked in the following way. For every $\gamma \in \Gamma$, we have:

(1) \quad U_{\gamma} \rho_{\Lambda \setminus H}(h) U_{\gamma^{-1}} = \rho_{\Lambda \setminus H}(\gamma(h)) \quad \text{for all} \quad h \in H.

We have a decomposition of $L^2(\Lambda \setminus H, m)$ into $\rho_{\Lambda \setminus H}$-invariant subspaces

$$L^2(\Lambda \setminus H, m) = \bigoplus_{m \in \mathbb{Z}} H_m,$$

where

$$H_m = \{ \xi \in L^2(\Lambda \setminus H) : \rho_{\Lambda \setminus H}(0, 0, s) \xi = e^{2\pi i m s} \xi \quad \text{for all} \quad t \in \mathbb{R} \}.$$

The space $H_0$ coincides with the space $L^2(T)$, where $T \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is the maximal torus factor associated to $\Lambda \setminus H$. Moreover, for every $m \in \mathbb{Z} \setminus \{0\}$, the subspace $H_m$ is an isotypical component for $\rho_{\Lambda \setminus H}$ and is equivalent to a finite multiple of the irreducible representation $\pi_m$ from above. (For a computation of the multiplicities, see [Tol78], [Moo65].)

Let $m \in \mathbb{Z} \setminus \{0\}$. Since $\Gamma$ fixes pointwise $Z$, we see from (1) that

$$U_{\gamma}(H_m) = H_m \quad \text{for all} \quad \gamma \in \Gamma.$$

Denote by $U^{(m)}$ the restriction of $U$ to $H_m$.

Since $H_m$ is equivalent to a finite multiple of the irreducible representation $\pi_m$, we can assume that $H_m$ is the tensor product

$$H_m = K_m \otimes L_m$$

of the Hilbert space $K_m$ of $\pi_m$ with a finite dimensional Hilbert space $L_m$, in such a way that

(2) \quad \rho_{\Lambda \setminus H}(h)|_{H_m} = \pi_m(h) \otimes I_{L_m} \quad \text{for all} \quad h \in H.

Let $\gamma \in \Gamma$. By (1) and (2) above, we have

(3) \quad U_{\gamma}^{(m)} (\pi_m(h) \otimes I_{L_m}) U_{\gamma^{-1}}^{(m)} = \pi_m(\gamma(h)) \otimes I_{L_m} \quad \text{for all} \quad x \in H.

Let $\omega_m$ be Weil’s representation from above. Recall that $\omega_m$ is a projective representation of $Sp_{2n}(\mathbb{R}) \ltimes \mathbb{R}^{2n}$ defined on $K_m = L^2(\mathbb{R}^n)$ and that $\omega_m$ extends $\pi_m$. We have

(4) \quad \omega_m(\gamma) \pi_m(h) \omega_m(\gamma)^{-1} = \pi_m(\gamma(h)) \quad \text{for all} \quad h \in H.
It follows from (3) and (4) that, on $H_m$, the operator $(\omega_m(\gamma)^{-1} \otimes I_{L_m}) U_{\gamma}^{(m)}$ commutes with $(\pi_m(h) \otimes I_{L_m})$ for all $h \in H$. Hence, since $\pi_m$ is irreducible, there exists a bounded operator $V_{\gamma}^{(m)}$ on $L_m$ such that 

\[(\omega_m(\gamma)^{-1} \otimes I_{L_m}) U_{\gamma}^{(m)} = I_{K_m} \otimes V_{\gamma}^{(m)},\]

that is,

\[(5) \quad U_{\gamma}^{(m)} = \omega_m(\gamma) \otimes V_{\gamma}^{(m)}.\]

Since $U^{(m)}$ is a unitary representation, it is clear that $\gamma \mapsto V_{\gamma}^{(m)}$ is a projective unitary representation of $\Gamma$.

Let $\xi, \eta \in S(\mathbb{R}^n)$ be Schwartz functions on $\mathbb{R}^n$. By [HoMo79, Proposition 6.4], for every $\varepsilon > 0$, the matrix coefficient

\[C_{\xi, \eta}^{\omega_m} : g \mapsto \langle \omega_m(g)\xi, \eta \rangle\]

of the metaplectic representation $\omega_m$ belongs to $L^{4n+2+\varepsilon}(Sp_{2n}(\mathbb{R}) \ltimes \mathbb{R}^{2n})$. Set $G = Sp_{2n}(\mathbb{R}) \ltimes \mathbb{R}^{2n}$; observe that $\Gamma$ is a discrete and hence closed subgroup of $G$. Choosing a Borel subset $X \subset G$ which is a fundamental domain for the quotient space $\Gamma \backslash G$, we can write (compare with the proof of Proposition 6.4 in [Howe82])

\[
\int_G |C_{\xi, \eta}^{\omega_m}(g)|^{4n+2+\varepsilon} dg = \int_G |\langle \omega_m(g)\xi, \eta \rangle|^{4n+2+\varepsilon} dg
\]

\[= \int_X \left( \sum_{\gamma \in \Gamma} |\langle \omega_m(\gamma g)\xi, \eta \rangle|^{4n+2+\varepsilon} \right) dg < \infty.
\]

Therefore, by Fubini’s theorem, for almost every $g \in X$, we have

\[
\sum_{\gamma \in \Gamma} |\langle \omega_m(\gamma)\omega_m(g)\xi, \eta \rangle|^{4n+2+\varepsilon} < \infty,
\]

that is, $C_{\omega_m|_{\omega_m(\gamma)\xi, \eta}}^{\omega_m} \in \ell^{4n+2+\varepsilon}(\Gamma)$.

Since $S(\mathbb{R}^n)$ contains a countable set which is dense in $L^2(\mathbb{R}^n)$, it follows that there exist dense subspaces $D_1^{(m)}$ and $D_2^{(m)}$ of $L^2(\mathbb{R}^n)$ such that

\[C_{\xi, \eta}^{\omega_m|_{\Gamma}} \in \ell^{4n+2+\varepsilon}(\Gamma)\]
for all $\xi \in D_1^{(m)}$ and $\eta \in D_2^{(m)}$.

Since $U^{(m)} = \omega_m(\gamma) \otimes V^{(m)}_\gamma$ and since matrix coefficients of projective unitary representations are bounded, the matrix coefficients $C_{\xi\otimes\xi',\eta\otimes\eta'}^{U^{(m)}}$ of $U^{(m)}$ belong to $\ell^{4n+2+\varepsilon}(\Gamma)$ for $\xi \in D_1^{(m)}$, $\eta \in D_2^{(m)}$ and $\xi' \in \mathcal{L}_m$, $\eta' \in \mathcal{L}_m$.

Let now $D_1$, $D_2$ be the linear subspaces of $\mathcal{H}$ generated respectively by

$$
\{ \xi \otimes \xi' : \xi \in D_1^{(m)}, \xi' \in \mathcal{L}_m, m \in \mathbb{Z} \setminus \{0\} \}
$$

and

$$
\{ \eta \otimes \eta' : \eta \in D_2^{(m)}, \eta' \in \mathcal{L}_m, m \in \mathbb{Z} \setminus \{0\} \}.
$$

Then $D_1$ and $D_2$ are dense in $\mathcal{H}$ and the matrix coefficients $C_{v,w}^{U}$ belong to $\ell^{4n+2+\varepsilon}(\Gamma)$ for $v \in D_1$ and $w \in D_2$. ■

**Proof of Theorem 3**

Let $\mu$ be a probability measure on $\text{Aut}(\Lambda\backslash H)$. Denote by $\Gamma$ the subgroup of $\text{Aut}(\Lambda\backslash H)$ generated by the support of $\mu$.

Let $\lambda_\Gamma$ be the left regular representation of $\Gamma$ on $\ell^2(\Gamma)$. Let $U^0$ and $V^0$ be the corresponding unitary representations of $\Gamma$ on $L_0^0(\Lambda\backslash H)$ and $L_0^0(T)$, respectively. We claim that

$$
\|U^0(\mu)\| \leq \max\{\|V^0(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/k}\}
$$

for $k = 2n + 2$.

Denoting by $U^\mathcal{H}$ the restriction of $U$ to $\mathcal{H}$, it suffices to show that

$$
\|U^\mathcal{H}(\mu)\| \leq \|\lambda_\Gamma(\mu)\|^{1/k}.
$$

By Theorem 1, the matrix coefficients $C_{v,w}^{U^\mathcal{H}}$ of $U^\mathcal{H}$ are in $\ell^{2k}(\Gamma)$ for $v$ and $w$ in dense subspaces $D_1$ and $D_2$ of $\mathcal{H}$. It follows that the $k$-fold tensor power $(U^\mathcal{H})^\otimes k$ of $U^\mathcal{H}$ is unitarily equivalent to a subrepresentation of an infinite multiple $\infty \lambda_\Gamma$ of $\lambda_\Gamma$ (see [HoTa92, Chapter V, 1.2.4]). Hence,

$$
\| (U^\mathcal{H})^\otimes k (\mu) \| \leq \| \infty \lambda_\Gamma(\mu) \| = \| \lambda_\Gamma(\mu) \|
$$

and Inequality (7) will be proved if we show that

$$
\|U^\mathcal{H}(\mu)\| \leq \| (U^\mathcal{H})^\otimes k (\mu) \|^{1/k}.
$$
To show Inequality (8), we use the following argument from [Nevo98]. Denote by \( \tilde{\mu} \) the probability measure on \( \Gamma \) defined by \( \tilde{\mu}(\gamma) = \mu(\gamma^{-1}) \). For every vector \( v \in H \), using Jensen’s inequality, we have

\[
\|U(\mu)v\|^{2k} = \left| \langle U(\tilde{\mu} \ast \mu)v, v \rangle \right|^{k} = \sum_{\gamma \in \Gamma} \left| \langle U(\gamma)v, v \rangle \right|^{k}(\tilde{\mu} \ast \mu)(\gamma) = \left| \langle U^{\otimes k}(\tilde{\mu} \ast \mu)v^{\otimes k}, v^{\otimes k} \rangle \right| = \|U^{\otimes k}(\mu)v^{\otimes k}\|^{2}.
\]

Hence, \( \|U^{H}(\mu)\| \leq \| (U^{H})^{\otimes k}(\mu) \|^{1/k} \), as claimed.

Assume that \( \mu \) is aperiodic and that the \( \Gamma \) action on \( T \) has a spectral gap. Then, as mentioned in the Introduction, \( \Gamma \) is not amenable. Hence, \( \|\lambda_{\Gamma}(\mu)\| < 1 \) (see [BeHV08, G.4.2]) and therefore \( \|U^{0}(\mu)\| < 1 \), that is, the \( \Gamma \) action on \( \Lambda \setminus H \) has a spectral gap. ■

**Proof of Corollary 5**

Let \( H = H_{3}(\mathbb{R}) \) be the 3-dimensional Heisenberg group and \( \Lambda \) a lattice in \( H \). The unitary representation \( V \) of \( \text{Aut}(\Lambda \setminus H) \) on \( L^{2}(T) \) factors through \( p : \text{Aut}(\Lambda \setminus H) \to \text{Aut}(T) \cong GL_{2}(\mathbb{Z}) \) to the standard representation of \( GL_{2}(\mathbb{Z}) \) on \( L^{2}(T) = L^{2}(\mathbb{R}^{2}/\mathbb{Z}^{2}) \). By Fourier duality, this last representation is unitarily equivalent to the representation of \( GL_{2}(\mathbb{Z}) \) on \( \ell^{2}(\mathbb{Z}^{2}) \) obtained from the dual action of \( GL_{2}(\mathbb{Z}) \) on \( \mathbb{Z}^{2} \). We have an orthogonal decomposition into \( GL_{2}(\mathbb{Z}) \)-invariant subspaces

\[
\ell^{2}(\mathbb{Z}^{2}) = C\delta_{0} \bigoplus \bigoplus_{t \in T} \ell^{2}(GL_{2}(\mathbb{Z})/\Gamma_{t}),
\]

where \( T \) is a set of representatives for the \( GL_{2}(\mathbb{Z}) \)-orbits in \( \mathbb{Z}^{2} \setminus \{0\} \) and \( \Gamma_{t} \) is the stabilizer of \( t \) in \( GL_{2}(\mathbb{Z}) \). Since every \( \Gamma_{t} \) is solvable (and hence amenable), it follows that \( V^{0} \) is weakly contained in \( \lambda_{GL_{2}(\mathbb{Z})} \circ p \) (see [BeHV08, Appendix F]). Hence, for every probability measure \( \mu \) on \( \text{Aut}(\Lambda \setminus H) \), we have

\[
\|V^{0}(\mu)\| \leq \| (\lambda_{GL_{2}(\mathbb{Z})} \circ p)(\mu) \|.
\]
Since
\[ \ker p \subset \left\{ \left( \begin{array}{cc} I_2 & 0 \\ a^2 & 1 \end{array} \right) : a \in \mathbb{R}^2 \right\} \cong \mathbb{R}^2, \]
\[ \ker p \] is amenable and it follows that
\[ \| (\lambda_{GL_2(\mathbb{Z})} \circ p)(\mu) \| \leq \| \lambda_{\text{Aut}(\Lambda \setminus H)}(\mu) \|. \]

Therefore, we have
\[ \| V^0(\mu) \| \leq \| \lambda_{\text{Aut}(\Lambda \setminus H)}(\mu) \|. \]

Denote by \( \Gamma \) the subgroup generated by the support of \( \mu \). Since
\[ \| \lambda_\Gamma(\mu) \| = \| \lambda_{\text{Aut}(\Lambda \setminus H)}(\mu) \|, \]
it follows from Theorem 3 that
\[ \| U^0(\mu) \| \leq \max\{ \| V^0(\mu) \|, \| \lambda_\Gamma(\mu) \|^{1/4} \} = \| \lambda_\Gamma(\mu) \|^{1/4}. \]

Assume that \( \mu \) is aperiodic. If \( \Gamma \) is non-amenable, then \( \| \lambda_\Gamma(\mu) \| < 1 \) and hence \( \| U^0(\mu) \| < 1 \). If \( \Gamma \) is amenable, then \( \Gamma \) has no spectral gap in \( \Lambda \setminus H \). \( \blacksquare \)

### 3 Some further applications

Let \( G \) be a locally compact group acting by measure preserving transformations on a probability space \((X, m)\). Let \( U \) denote the associated unitary representation of \( G \) on \( L^2(X, m) \). The action of \( G \) on \( X \) is weakly mixing if \( L^2_0(X, m) \) contains no non-zero finite dimensional \( U(G) \)-invariant subspace (equivalently: if the diagonal action of \( G \) on \( X \times X \) is ergodic; see [BeMa00, Chapter I, 2.17]). The action is strongly mixing if, for all \( \xi, \eta \in L^2_0(X, m) \), the matrix coefficient \( g \mapsto \langle U_g \xi, \eta \rangle \) belongs to \( c_0(G) \).

With the notation of Theorem 1, all matrix coefficients \( C^{U}_{v, w} \) are in \( c_0(\Gamma) \) for \( v \in D_1 \) and \( w \in D_2 \). By density of \( D_1 \) and \( D_2 \) in \( \mathcal{H} \), the same is true for all \( v, w \in \mathcal{H} \). It follows that \( \mathcal{H} \) contains no non-zero finite dimensional \( U(G) \)-invariant subspace if \( \Gamma \) is infinite (see [BeMa00, Chapter I, 2.15.iii]). Therefore, we immediately obtain the following corollary.

**Corollary 6** Let \( \Gamma \) be a group of automorphisms of the compact Heisenberg nilmanifold \( \Lambda \setminus H \) and \( T \) the maximal \( T \) torus factor associated to \( \Lambda \setminus H \). The following properties are equivalent.
(i) The action of $\Gamma$ on $\Lambda \setminus H$ is ergodic (weakly mixing or strongly mixing, respectively).

(ii) The action of $\Gamma$ on $T$ is ergodic (weakly mixing or strongly mixing, respectively).

Remark 7 In the case where $\Gamma$ is generated by a single automorphism (or even an affine transformation) of an arbitrary compact nilmanifold, the previous corollary was obtained by W. Parry (see [Parr69], [Parr70]). The result concerning ergodicity has been generalized by J.-P. Conze ([Conz09]) to arbitrary groups of affine transformations of a general compact nilmanifold $\Lambda \setminus H$. Moreover, [Conz09] gives an example of an ergodic group $\Gamma$ of automorphisms of the standard 7-dimensional Heisenberg nilmanifold $\Lambda \setminus H_7(\mathbb{R})$ such that no element $\gamma \in \Gamma$ acts ergodically on $\Lambda \setminus H_7(\mathbb{R})$.

Let $G$ be a locally compact group acting by measure preserving transformations on a probability space $(X, m)$ and $U$ the associated unitary representation of $G$ on $L^2(X, m)$. Assume that $\| (U^0 \otimes U^0)(\mu) \| < 1$. This condition, which is formally stronger than the spectral gap condition $\| U^0(\mu) \| < 1$, plays an important role in [FuSh99]. Indeed, it is shown in Theorem 1.4 there that, with the notation as in the Introduction, for every $\xi, \eta \in L^2(X, m)$, the correlation coefficient $\langle U(S^\omega_n)\xi, \eta \rangle$ converge almost surely to $\int_X \xi dm \int_X \eta dm$, with exponentially fast speed.

However, we can see that both conditions are equivalent in our situation.

Corollary 8 With the notation as in Theorem 3, the following properties are equivalent.

(i) $\| (U^0 \otimes U^0)(\mu) \| < 1$;

(ii) $\| U^0(\mu) \| < 1$;

(iii) $\| V^0(\mu) \| < 1$.

Indeed, a proof similar to the one of Theorem 3 shows that the following inequality holds:

$$\|(U^0 \otimes U^0)(\mu)\| \leq \max\{\|(V^0 \otimes V^0)(\mu)\|, \|\lambda(\mu)\|^{1/k}\}.$$

On the other hand, as was shown in [FuSh99, Theorem 6.4], the condition $\| V^0(\mu) \| < 1$ is equivalent to the condition $\|(V^0 \otimes V^0)(\mu)\| < 1$. This shows the equivalence of Conditions (i), (ii), and (iii).
References


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