Monotone stability of quadratic semimartingales with applications to general quadratic BSDEs and unbounded existence result

Pauline Barrieu, Nicole El Karoui

Monotone stability of quadratic semimartingales with applications to general quadratic BSDEs and unbounded existence result

Pauline Barrieu
Department of Statistics
London School of Economics
Houghton Street,
WC2A 2AE London
United Kingdom
p.m.barrieu@lse.ac.uk

Nicole El Karoui*
LPMA/ UPMC (Paris 6)
Case courrier 188
4 Place Jussieu,
75252 Paris Cedex 05
France
nicole.elkaroui@cmap.polytechnique.fr

January 27, 2011

Abstract

In this paper, we study the stability and convergence of some general quadratic semimartingales. Motivated by financial applications, we study simultaneously the semimartingale and its opposite. Their characterization and integrability properties are obtained through some useful exponential inequalities on the absolute value of the terminal condition. Then, a general stability result, including the strong convergence of the martingale parts, is derived under some mild integrability condition on the exponential of the terminal value of the semimartingale.

This strong convergence result is then applied to the study of general

*Author partly supported by the "Chaire Financial Risk" of the Risk Foundation, Paris.

quadratic BSDEs, which does not involve the usual exponential transformation but relies on a regularization with both linear-quadratic growth of the quadratic coefficient it-self through inf-convolution. Strong convergence results for BSDEs are then obtained in a general framework using the stability results previously obtained using a forward point of view and considering the quadratic BSDEs as a particular type of quadratic semimartingales.

1 Introduction

The Backward Stochastic Differential Equations (BSDEs) were first introduced by Peng & Pardoux [38] in 1990 in the Lipschitz continuous framework, and soon recognized as powerful tools with many different possible applications. More recently, there has been an accrued interest for quadratic BSDEs, with various fields of application such as dynamic financial risk measures or risk sensitive control problems. In this case, the BSDE is an equation of the following type:

$$-dY_t = g(t, Y_t, Z_t)dt - Z_t dW_t, \quad Y_T = \xi_T,$$

where $W_.$ is a standard Brownian motion, and the coefficient g satisfies the following quadratic $s$tructure condition $\mathcal{Q}(l, a, \delta)$:

$$|\, g(t, y, z)| \leqslant Q(t, y, z) \equiv \frac{1}{\delta}|l_t| + c_t|y| + \frac{\delta}{2}|z|^2 \quad d\mathbb{P} \otimes dt\text{–a.s.},$$

where $\delta > 0$ is some given constant, and $l_t, c_t \geqslant 0$ are predictable processes satisfying some integrability properties.

The question of existence and uniqueness of solutions to these quadratic equations was first examined by Kobylanski [26] in the bounded case (i.e. when the terminal condition ξ is bounded) in a Brownian filtration setting, and was then extended to a continuous filtration setting by Morlais [35]. Recently, Tevzadze [42] has given a direct proof for the existence and uniqueness of a bounded solution in the Lipschitz-quadratic case. More recently, Briand & Hu [8] have extended the existence result to unbounded solution and proved uniqueness for a convex coefficient [9]. Some other authors have obtained further results in some particular situations (see for instance Hu & Schweizer [24]).

In general, the proof relies first on an exponential transformation as to come back to the better known framework of BSDEs with a coefficient with linear growth and then uses a regularization procedure to take the limit. The major difficulty is then about proving the strong convergence of the martingale parts without having to impose too strong assumptions.

Keeping in mind this possible application, we adopt in this paper a complete different approach and consider a forward point of view to treat the question of the convergence of the martingale parts. This direct forward point of view appears to be very efficient. To do so, we introduce general quadratic semimartingales and study their characterization just as their integrability properties using some interesting exponential inequalities. Mainly motivated by financial applications, where a seller price and a buyer price have to be given simultaneously, we apply systematically the same assumptions on the semimartingale and its opposite. These integrability properties prove to be essential in the estimation of their quadratic variations and gives us a method to construct a priori estimates. Precise estimated on the maximum of the exponential of the semimartingale is proved involving the Shannon entropy of their terminal value.
Then we obtain one of obtain some general stability results, in particular regarding the convergence of their martingale parts as presented in Theorem 3.5. The results are very general and simply require some integrability assumptions on the exponential of the terminal value of the semimartingale. The convergence for the martingale parts is obtained under various assumptions on the space of martingales that is considered, from $\mathbb{H}^1$ to BMO. In the BSDE framework, we also obtained the convergence in total variation of the finite variation part.
The BSDEs become a possible application of this stability result. More precisely, coming back to our initial motivation of quadratic BSDEs, we first regularize the quadratic coefficient of the BSDE through inf-convolution as to transform it into a coefficient with linear growth. The convergence results obtained for quadratic semimartingales can then be applied as the quadratic BSDEs can be seen as a particular type of quadratic semimartingales. The power of the forward point of view is striking as existence results are easily obtained in a far more general framework than the existing literature. The standard techniques first introduced by Kobylanski [26] are no longer needed and a much simpler and quicker methodology is obtained.

The paper is organized as follows. In Section 2, we introduce quadratic semimartingales, satisfying a quadratic structure condition similar to (2), derive some key properties, discuss their integrability and obtain their characterization using exponential inequalities. Section 3 is dedicated to the question of stability and convergence of the quadratic semimartingales. Quadratic variation estimates are first obtained, and then a general stability result for the quadratic semimartingales is presented in Theorem 3.5. Finally, Section 4 comes back to the study of general quadratic BSDEs and in particular the existence of a solution in the light of the previous forward results of this paper.

2 Quadratic semimartingales

The study of BSDEs, especially when it comes to obtain existence and uniqueness results, relies upon a precise definition of the space of processes on which solutions are considered, but also on sophisticated a priori estimates coming from the martingale theory (see for instance El Karoui & Huang [17]). These estimates typically arise from a forward point of view, allowing in particular for the use of localization procedures.
In this section, after having meticulously defined quadratic BSDEs, we adopt a forward point of view, introducing quadratic Itô's semimartingales, with a similar structure condition, studying their main properties and deriving some characterization results, which depend on various integrability assumptions. These results will be very useful to derive some stability and convergence results in the next section.

2.1 Definition of quadratic BSDEs

Quadratic BSDEs have recently received a lot of attention, mainly due to the wide range of possible applications, involving optimization problems such as indifference pricing with exponential utility (see for instance Rouge & El Karoui [40] or Mania & Schweizer [32] among many other references) and risk sensitive control (see for instance Barrieu & El Karoui for an application in terms of dynamic entropic risk measures [7] or El Karoui & Hamadène for an application to risk-sensitive zero-sum stochastic functional games [15]).
Let us start by briefly recalling the definition of a quadratic BSDE. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ be a filtered probability space, where the filtration $(\mathcal{F}_t)$ satisfies the usual conditions of completeness and right-continuity. The σ-field on $\Omega \times \mathbb{R}^+$ generated by the adapted and left continuous processes is called

the predictable σ-field and denoted by $\mathcal{P}$.
A quadratic BSDE is an equation of the following type:

$$-dY_t = g(t, Y_t, Z_t)dt - Z_t dW_t, \quad Y_T = \xi_T, \tag{1}$$

where $T > 0$ is a future time, $W_.$ is a standard d-dimensional $(\mathbb{P}, (\mathcal{F}_t))$-Brownian motion, and $Z_t dW_t$ simply denotes the scalar product. The random variable $\xi_T \in \mathcal{F}_T$ is the terminal condition, and the coefficient g satisfies the following quadratic $s$tructure condition $\mathcal{Q}(l, a, \delta)$:

$$|\, g(., t, y, z)| \leqslant Q(t, y, z) \equiv |l_t| + c_t|y| + \frac{\delta}{2}|z|^2 \quad d\mathbb{P} \otimes dt\text{−a.s.}, \tag{2}$$

where $\delta > 0$ is some given constant, and $l_t, c_t \geqslant 0$ are predictable processes satisfying some integrability properties that would be specified later when required.

By solution to the BSDE(g, ξ_T) defined in Equation (1), we mean a pair of predictable processes taking values in $\mathbb{R} \times \mathbb{R}^d$, $(Y, Z) = \{(Y_t, Z_t); t \in [0, T]\}$, such that the paths of Y are continuous, $\int_0^T |Z_t|^2 dt < \infty$, $\int_0^T |g(t, Y_t, Z_t)|dt < \infty$ hold $\mathbb{P}$-a.s., and

$$Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \tag{3}$$

This minimal definition of a quadratic BSDE will be completed later on by some further integrability assumptions, allowing us to obtain some stability results and some conditions for the existence of a solution.
Adopting a forward point of view, a solution of a quadratic BSDE is a quadratic Itô's semimartingale $Y_.$, with a decomposition satisfying the same quadratic structure condition (2). Such a condition needs to be further specified when considering a more general framework of quadratic semimartingales, as we will see in the next subsection.

2.2 Definition and first properties of quadratic semimartingales

All the semimartingales we consider in this paper are defined on a *continuous* filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$, i.e. a filtered probability space such that any locally bounded martingale is a continuous martingale.

Definition 2.1 (Quadratic semimartingale). *Let $Y_.$ be a continuous semimartingale, with the decomposition $Y_. = Y_0 - V_. + M_.$, where $V_.$ is a predictable process with finite total variation $|V|$ and $M_.$ is a (continuous) local martingale with quadratic variation $\langle M \rangle$.*
$Y_.$ is a quadratic semimartingale *if there exist two adapted continuous increasing processes $\Lambda.$ and $C.$ and a positive constant δ, such that the structure condition $\mathcal{Q}(\Lambda, C, \delta)$ holds true:*

$$d|V_t| \ll \frac{1}{\delta} d\Lambda_t + |Y_t|\, dC_t + \frac{\delta}{2} d\langle M \rangle_t, \quad d\mathbb{P}-\text{a.s.}. \tag{4}$$

The symbol $\ll$ stands for the absolute continuity of the increasing processes. The simplifying notation

$$D_.^{\Lambda, C}(Y, \delta) = \frac{1}{\delta}\Lambda_. + |Y_.| * C_. \tag{5}$$

will also be used in the sequel.

For the sake of clarity of the presentation, we make a distinction between the $\mathcal{Q}_\delta$-semi-martingales for which the processes $C_.$ and $\Lambda_.$ are identically equal to 0, and the general $\mathcal{Q}(\Lambda, C, \delta)$-semimartingales.
Moreover, we will see in Proposition 2.4 that the normalization $\delta = 1$ does not lead to any loss of generality in the results but allows for much simpler expressions. Therefore, the situation where $\delta = 1$ will be our typical framework unless otherwise specified and we will then use the shorter notation $\mathcal{Q}$-semimartingale or more generally $\mathcal{Q}(\Lambda, C)$-semimartingales.
By analogy with the BSDE framework, it is natural to specify a sub-class of finite variation processes $V_.$ by relaxing the reference to the Brownian motion in the following manner. As in El Karoui & Huang [17], let us consider a general BSDE framework with a given d-dimensional continuous orthogonal martingale $N_.$, for which the diagonal predictable quadratic variation matrix is dominated by some continuous predictable increasing process $K_.$, with $d\langle N_t^i \rangle = \gamma_t^i dK_t$. We also assume for the sake of simplicity that $d\Lambda_t = l_t dK_t$ and $dC_t = c_t dK_t$. Note we can still assume that the density processes c_t and l_t with respect to the process $K_.$, of $C_.$ and $\Lambda_.$ respectively, are *bounded* by some universal constant, since we can still add $\Lambda_. + C_.$ to $K_.$ if necessary.

Definition 2.2 (BSDE-like quadratic semimartingale). *A quadratic semimartingale $(Y,\ M, V)$ is said to have a quadratic coefficient g, if $dY_t =$*

$-dV_t + dM_t$, *with*

$$\left\{ \begin{array}{l} dV_t = g(t, Y_t, Z_t)\, dK_t,\;\; dM_t = Z_t dN_t + dM^{\perp},\;\; \forall i\;\; d\langle N^i, M^{\perp}\rangle_t = 0 \\ |g(t,y,z)| \leqslant \frac{1}{\delta} l_t + |y| c_t + \frac{\delta}{2} |\sqrt{\gamma}_t z|^2, \quad |\sqrt{\gamma}_t z|^2 = \sum_{i=1}^{d} \gamma_t^i |z^i|^2 \end{array} \right. \tag{6}$$

The local martingale $Z.N$ is the orthogonal projection of the local martingale $M_.$ onto the space of stochastic integral generated by the local martingale $N_.$. Note that $d|V|_t \ll (\frac{1}{\delta} l_t + |Y_t| c_t + \frac{\delta}{2} |\sqrt{\gamma}_t Z_t|^2) dK_t \ll d\Lambda_t + |Y_t| dC_t + \delta d\langle M\rangle_t$, and $Y_.$ is a quadratic semimartingale.

Basic properties of quadratic semimartingales Let us start by presenting some key properties of $\mathcal{Q}(\Lambda, C, \delta)$-semimartingales, focusing first on the $\mathcal{Q}$-semimartingales.
The simplest $\mathcal{Q}$-semimartingales are those for which the structure condition $\mathcal{Q}$ is saturated, i.e. $V_. = \frac{\delta}{2}\langle M\rangle_.$ or $\underline{V}_. = -\frac{\delta}{2}\langle M\rangle_.$. Because of their importance, we give them a specific denomination and refer to them as q_δ-(resp. $\underline{q}_\delta$-) semimartingales. In particular, when $\delta = 1$ (resp. $\delta = -1$), q (resp. $\underline{q}$) semimartingales are also denoted by:

$$\left\{ \begin{array}{ll} r_.(r_0, M) & \equiv r_0 + M_. - \frac{1}{2}\langle M\rangle_. \equiv r_0 + r_.(M), \\ \underline{r}_.(r_0, M) & \equiv \underline{r}_0 + M_. + \frac{1}{2}\langle M\rangle_. = \underline{r}_0 - r_.(-M). \end{array} \right. \tag{7}$$

Taking the exponential of $r_.(M)$ immediately leads to the exponential martingale $\mathcal{E}(M)_.$, so that

$$\left\{ \begin{array}{ll} \mathcal{E}(M)_. & = \exp(M_. - \frac{1}{2}\langle M\rangle_.) \\ e^{r_.(M)} & = \mathcal{E}(M)_. \\ e^{\underline{r}_.(M)} & = (\mathcal{E}(-M)_.)^{-1} \end{array} \right. \tag{8}$$

It will also be useful to introduce some asymmetry in the previous definition of $\mathcal{Q}$-semimartingales, with the notion of $\mathcal{Q}$-submartingales, especially useful when characterizing the former.

Definition 2.3. *A $\mathcal{Q}$-submartingale is a continuous semimartingale $X_. = X_0 - V_. + M_.$ such that $A. \equiv -V_. + \frac{1}{2}\langle M\rangle_.$ is a predictable increasing process. Equivalently, $X_. = X_0 + r_.(M) + A_.$ and $e^{X_.} = e^{X_0 + A_.}\mathcal{E}_.(M)$ is a continuous submartingale.*

Note that, if $Y_\cdot$ is a $\mathcal{Q}$-semimartingale, then both $Y_\cdot$ and $-Y_\cdot$ are $\mathcal{Q}$-submartingales.

The set of quadratic semimartingales is stable by elementary transformations as presented in the next proposition.

Proposition 2.4. *Let $Y_\cdot$ be a $\mathcal{Q}(\Lambda, C, \delta)$-semimartingale. Then:*
(i) THE ROLE OF δ: *For any $\lambda \neq 0$, the process $\lambda Y_\cdot$ is a $\mathcal{Q}(\Lambda, C, \frac{\delta}{|\lambda|})$-semimartingale, and a $\mathcal{Q}(\lambda\Lambda, C, \delta)$-semimartingale when $\lambda > 1$. In particular, $\delta Y_\cdot$ and $-\delta Y_\cdot$ are $\mathcal{Q}(\Lambda, C)$-semimartingales.*
(ii) STRUCTURE SIMPLIFICATION: *Let $X_\cdot^{\Lambda,C}(Y)$ and $\bar{X}_\cdot^{\Lambda,C}(|Y|)$ be the two transformations:*

$$X_\cdot^{\Lambda,C}(Y) = Y_\cdot + \Lambda_\cdot + |Y| * C_\cdot, \quad \bar{X}_\cdot^{\Lambda,C}(|Y|) = e^{C_\cdot}|Y_\cdot| + e^C * \Lambda_\cdot \tag{9}$$

Then both processes $X_\cdot^{\Lambda,C}(\delta Y)$ and $\bar{X}_\cdot^{\Lambda,C}(|\delta Y_\cdot|)$ are $\mathcal{Q}$-submartingales.

(iii) ANOTHER EXPONENTIAL TRANSFORMATION: *Let $U^{\Lambda,C}(e^Y)$ be the transformation*

$$U_t^{\Lambda,C}(e^Y) = e^{Y_t} + \int_0^t e^{Y_s} d\Lambda_s + \int_0^t e^{Y_s}|Y_s| dC_s. \tag{10}$$

Then $U^{\Lambda,C}(e^Y)$ is a positive submartingale such that:
$dU_t^{\Lambda,C}(e^Y) = e^{-(\Lambda_t + |Y| * C_t)} d\, e^{X_t^{\Lambda,C}(Y)}$.

The first property (i) brings a way to relate the study and characteristics of $\mathcal{Q}(\Lambda, C, \delta)$-semimartingales to those of $\mathcal{Q}(\Lambda, C)$-semimartingales and therefore justifies the normalization and the choice of restricting our study to $\mathcal{Q}(\Lambda, C)$-semimartingale without any loss of generality. In the general structure Condition (4), the presence of the term $|Y_\cdot| * C_\cdot$ makes the characterization of quadratic semimartingales more difficult to obtain. The transformations proposed in (ii) and (iii) can partially reduce the problem to $\mathcal{Q}$-submartingales. In particular, the last exponential-type transformation $U^{\Lambda,C}$ will be essential to obtain useful inequalities for the derivation of some stability and convergence results.

Proof. (i) The semimartingale $Y_\cdot^\lambda = \lambda Y_\cdot$ is associated with the martingale $M_\cdot^\lambda = \lambda M_\cdot$ and the finite variation process $V_\cdot^\lambda = \lambda V_\cdot$. Then the structure condition becomes $d|V^\lambda|_t \ll \frac{|\lambda|}{\delta} d\Lambda_t + |Y_t^\lambda|\, dC_t + |\lambda|\frac{\delta}{2} d\langle M\rangle_\cdot$ Since $|\lambda|\, d\langle M\rangle_\cdot = \frac{1}{|\lambda|} d\langle M^\lambda\rangle_\cdot$, then $\lambda Y_\cdot \in \mathcal{Q}(\Lambda, C, \frac{\delta}{|\lambda|})$ and to $\mathcal{Q}(|\lambda|\Lambda, C, \delta)$ if $|\lambda| \geqslant 1$. In particular, $\delta Y_\cdot$ and $-\delta Y_\cdot$ are $\mathcal{Q}(\Lambda, C)$ semimartingales.

(ii) a) Since δY is a $\mathcal{Q}(\Lambda, C)$ semimartingale, it is sufficient to study the case $\delta = 1$. The semimartingale $X^{\Lambda,C}_.(Y) = Y_. + \Lambda_. + |Y| * C_. = Y_. + D^{\Lambda,C}_.(Y)$ is associated with the martingale M and the finite variation process $V^X = V - D^{\Lambda,C}_.(Y) = (\alpha^V - 1) * (D^{\Lambda,C}_.(Y) + \frac{1}{2}\alpha * \langle M \rangle_.$ where $\alpha^V_. \in [-1, 1]$ is a predictable process such that $V = \alpha^V * (D^{\Lambda,C}_.(Y) + \frac{1}{2}\langle M \rangle_.)$. Since the process $-V^X + \frac{1}{2}\langle M \rangle_. = (1-\alpha) * D^{\Lambda,C}_.(Y) + \frac{1}{2}\langle M \rangle_.)$ is a non decreasing process, the semimartingale $X^{\Lambda,C}_.(Y)$ is a $\mathcal{Q}$-submartingale.

(ii) b) We first study the dynamics of $|Y_.|$, using Itô-Tanaka formula involving the sign function defined by $\text{sign}(x) = x/|x|$ (with the convention $\text{sign}(0) = 1$), and the local time $L_.(Y)$ of $Y_.$ at 0: $|Y_.| = |Y_0| + \text{sign}(Y) * Y_. + L_.(Y)$. This decomposition leads to the following representation of $\bar{X}^{\Lambda,C}_.(Y) = e^{C_.}|Y_.| + e^{C_.} * \Lambda_.$, where we simply write $D^{\Lambda,C}_.$ for $D^{\Lambda,C}_.(Y)$:

$$
\begin{aligned}
& d\bar{X}^{\Lambda,C}_.(Y) = e^{C_.}[|Y_.|dC_. + d\Lambda_. + \text{sign}(Y)_. dM_. - \text{sign}(Y)_. dV_. + dL_.(Y)] \\
& = e^{C_.}[dD^{\Lambda,C}_. + \tfrac{1}{2}d\langle M \rangle_. - \text{sign}(Y)_. dV_.] + e^{C_.}(\text{sign}(Y)_. dM_. - \tfrac{1}{2}d\langle M \rangle_.) + dL_.(Y) \\
& = e^{C_.}(dA^s_. + dL_.(Y) + \tfrac{1}{2}(e^{C_.} - 1)d\langle M \rangle_t) + dr_.(e^{C_.}\text{sign}(Y_.) * M_.),
\end{aligned}
$$

where $A^s_. = D^{\Lambda,C}_. + \frac{1}{2}\langle M \rangle_. - \text{sign}(Y_.) * V_.$ is an increasing process.

We conclude by observing that $e^{C_.} * (A^s_. + L_. + \frac{1}{2}(e^{C_.} - 1) * \langle M \rangle_.)$ is a increasing process since $e^{C_.} - 1 \geqslant 0$.

(iii) Let $Y_.$ be a $\mathcal{Q}(\Lambda, C)$-semimartingale. Then $U^{\Lambda,C}_.(e^Y) = e^{Y_.} + e^{Y_.} * D^{\Lambda,C}_.$. Since $X^{\Lambda,C}_.(Y) = Y_. + D^{\Lambda,C}_.$, we have $e^{Y_.} = e^{-D^{\Lambda,C}_.} e^{X^{\Lambda,C}_.(Y)}$ and from the classical Itô's formula,

$$
de^{Y_t} = e^{-D^{\Lambda,C}_t} de^{X^{\Lambda,C}_t(Y)} - e^{Y_t} dD^{\Lambda,C}_t \quad \text{and} \quad dU^{\Lambda,C}_t(e^Y) = e^{-D^{\Lambda,C}_t} de^{X^{\Lambda,C}_t(Y)}.
$$

Given that $\exp(X^{\Lambda,C}_.(Y))$ is a submartingale, $U^{\Lambda,C}_.(e^Y)$ is also a submartingale. $\square$

2.3 Exponential transformations and algebraic characterization of quadratic semimartingales

We are now interested in a direct characterization of quadratic semimartingales. The various exponential transformations introduced in Proposition 2.3 lead to different characterizations based on submartingale properties, the key point being to apply these properties to both processes $Y_.$ and $-Y_.$. We start by adopting an algebraic point of view, based on the additive or multiplicative Doob-Meyer decomposition of submartingales. We also strive to relax the path regularity assumptions for the considered processes in order

to obtain some further characterizations, which are stable for the almost sure convergence. Not surprisingly, the most intuitive characterization is obtained for $\mathcal{Q}$-semimartingales.
But before looking at the characterization question more in details, let us first recall the general definition of submartingales, for which, as in Protter [39], Dellacherie & Meyer [12] or Lenglart, Lépingle & Pratelli [27] for more detailed properties, the paths of the process are assumed to be only with left and right limits (làdlàg in the French denomination). We recall that, by assumption, all considered martingales are continuous processes.

Definition 2.5. (i) *A submartingale $S_.$ is a làdlàg optional process $S_. = S_0 + N_. + K_.$ where $N_.$ is a local martingale and $K_.$ a predictable làdlàg increasing process*[1]*. The pair $(N_., K_.)$ is called the additive decomposition of S.*
(ii) *When $S_.$ is a positive submartingale, $(M_., A_.)$ is said to be the multiplicative decomposition of $S_.$ if $S_. = S_0\, \mathcal{E}_.(M) \exp(A_.)$, where $M_.$ is a local martingale and $A_.$ a predictable làdlàg increasing process.*

$\mathcal{Q}(\Lambda, C)$-semimartingale characterization via exponential submartingales From Definition 2.3 above, the exponential of a $\mathcal{Q}$-submartingale $X_.$ is a continuous positive submartingale, characterized by its multiplicative decomposition $\exp X_. = \exp(X_0 + A_.)\mathcal{E}_.(M)$. From Proposition 2.4, we know that for any $\mathcal{Q}(\Lambda, C)$-semimartingale $Y_.$, the process$X^{\Lambda,C}_.(Y)$ is also a $\mathcal{Q}$-submartingale. The same result is still valid for exponential transform of the process $Y_.$. The converse property holds also true, when applying these properties at both processes Y and $-Y$.

Theorem 2.6. (i) *Let $X_.$ be a làdlàg optional process. Then, $X_.$ is a $\mathcal{Q}$-semimartingale if and only if both processes $\exp(X_.)$ and $\exp(-X_.)$ are submartingales. In all cases, $X_.$ is a continuous process.*
(ii) *Let $Y_.$ be a làdlàg optional process. $Y_.$ is a $\mathcal{Q}(\Lambda, C)$-semimartingale if and only if both processes $\exp(X^{\Lambda,C}_.(Y))$ and $\exp(X^{\Lambda,C}_.(-Y))$ are submartingales, or equivalently if and only if both processes $U^{\Lambda,C}_.(e^Y)$ and $U^{\Lambda,C}_.(e^{-Y})$ are submartingales. In all cases, $Y_.$ is a continuous process.*

Proof. Let us first note that all these conditions are necessary from Proposition 2.4 since Y and $-Y$ are $\mathcal{Q}(\Lambda, C)$-semimartingales. We have to prove their sufficiency.

[1] In the classical setting, $K_.$ is assumed to be right continuous.

(i) Assume now that $\exp(X_.)$ and $\exp(-X_.)$ are two làdlàg submartingales, with respective multiplicative decomposition $(\overline{M}_., \overline{A}_.)$, and $(\underline{M}_., \underline{A}_.)$. Taking the logarithm leads to two different decomposition of X, which are compatible only if X is continuous. Indeed, from the multiplicative submartingale decomposition, we have:

$$X_. = X_0 + \overline{M}_. - \frac{1}{2}\langle \overline{M}\rangle_. + \overline{A}_. \quad \text{and} \quad -X_. = -X_0 + \underline{M}_. - \frac{1}{2}\langle \underline{M}\rangle_. + \underline{A}_.$$

Since the martingales and their quadratic variations are continuous processes, the jumps of X are the same as the positive jumps of the increasing process $\overline{A}_.$. The same remark holds true for the jumps of the process $-X$. Then, the jumps of X being simultaneous positive and negative, the process $X_.$ is continuous.

Moreover, from the uniqueness of the predictable decomposition of $X_.$ we know that $\underline{M}_. = -\overline{M}_.$. Hence, $\langle \underline{M}\rangle = \langle \overline{M}\rangle$, and $\overline{A}_. + \underline{A}_. = \langle M\rangle_.$. From Radon-Nikodym's Theorem, there exists a predictable process $\alpha_.$, with $0 \leqslant \alpha_t \leqslant 2$, such that $d\overline{A}_t = \frac{1}{2}\alpha_t d\langle M\rangle_t$. Substituting $\overline{A}$ into the decomposition of X, we get $dX_t = -\frac{1}{2}(1-\alpha_t)d\langle M\rangle_t + dM_t$ with $|1-\alpha_t| \leqslant 1$. Therefore, $X_.$ is a $\mathcal{Q}$-semimartingale.

(ii) Assume now that both processes $e^{\overline{X}_.}$ and $e^{\underline{X}_.}$ are submartingales, where $\overline{X}_. = X_.^{\Lambda,C}(Y)$ and $\underline{X}_. = X_.^{\Lambda,C}(-Y)$. The processes $\overline{X}_.$ and $\underline{X}_.$ satisfies the following relations:

$$\frac{1}{2}(\overline{X}_. - \underline{X}_.) = Y_., \text{ and } \quad \frac{1}{2}(\overline{X}_. + \underline{X}_.) = D_.^{\Lambda,C} = \Lambda_. + \frac{1}{2}|\overline{X} - \underline{X}| * C_.$$

Using the same notation and arguments as above, the processes $\overline{X}_.$ and $\underline{X}_.$, whose exponentials are submartingales, can only have positive jumps. This contradicts the fact that their sum is a continuous increasing process. Hence, both processes are continuous. For the same reasons, the sum $\underline{M}_. + \overline{M}_.$ is identically equal to 0, and the sum of increasing processes $\frac{1}{2}(\underline{A}_. + \overline{A}_.) = D_.^{\Lambda,C} + \frac{1}{2}\langle \overline{M}\rangle_. \equiv G_.^{\Lambda,C}$. As in (i), there exists a predictable process $\alpha_.$, with $0 \leqslant \alpha_t \leqslant 1$, such that $\frac{1}{2}\overline{A}_. = \alpha_. * G_.^{\Lambda,C}$. Substituting $\overline{A}_.$ in the decomposition of $Y_. = \frac{1}{2}(\overline{X}_. - \underline{X}_.)$, we get $dY_t = -(1-2\alpha_t)dG_t^{\Lambda,C} + d\overline{M}_t$. Therefore, $Y_.$ is a $\mathcal{Q}(\Lambda, C)$-semimartingale.

(iii) Assume now that both processes $U_.(e^Y)$ and $U_.(e^{-Y})$ are làdlàg submartingales. Let $U_.(e^Y) = U_0 + \overline{N}_. + \overline{K}_.$ and $U_.(e^{-Y}) = U_0 + \underline{N}_. + \underline{K}_.$ be their respective additive decompositions. The jumps of $U_.(e^Y)$ and $U_.(e^{-Y})$

are the same as the jumps of $e^{Y_\cdot}$ and $e^{-Y_\cdot}$, and also as those of the increasing processes $\underline{K}_\cdot$ and $\overline{K}_\cdot$. Since the jumps of $e^{Y_\cdot}$ and $e^{-Y_\cdot}$ are positive, the jumps of $Y_\cdot$ and $-Y_\cdot$ are also positive. Hence, the process $Y_\cdot$ is continuous. As in Proposition 2.4, the processes $e^{X^{\Lambda,C}_\cdot(Y)} = e^{(\Lambda+|Y|*C)} * U_\cdot(e^{Y})$, and $e^{X^{\Lambda,C}_\cdot(-Y)} = e^{(\Lambda+|Y|*C)} * U_\cdot(e^{-Y})$ are two submartingales and we can apply the previous results to conclude that $Y_\cdot$ is a $\mathcal{Q}(\Lambda, C)$-semimartingale. $\square$

2.4 Characterization via exponential inequalities

We have just obtained a simple characterization of $\mathcal{Q}(\Lambda, C)$-semimartingales using an exponential transformation, leading naturally to submartingales defined by their multiplicative or additive decomposition. Whenever submartingales have good integrability properties, the existence of an additive decomposition is equivalent to the submartingale inequalities. It is the famous Doob-Meyer decomposition. The main objective of this subsection is to precise such integrability properties and the consequent inequalities.

Uniform integrability, class $(\mathcal{D})$ and their exponential equivalents
(i) In the classical martingale theory, uniformly integrable (u.i.) martingales play a key role as martingale equalities are then valid between two stopping times. Recall that the conditional expectation of some positive integrable random variable is still a uniformly integrable martingale. The class of such martingales is denoted by $\mathcal{U}$.
(ii) In the exponential framework, any exponential martingale $\mathcal{E}(M)_\cdot$ of a continuous martingale $M_\cdot$ is a positive local martingale, with expectation $\leqslant 1$, hence a supermartingale. The process $\mathcal{E}_\cdot(M)$ is a u.i. martingale on $[0, T]$ if and only $\mathcal{E}_t(M) = \mathbb{E}\big[\mathcal{E}_T(M)|\mathcal{F}_t\big]$ $\mathbb{P}$ *a.s.*. It is therefore natural to introduce *the class* $\mathcal{U}_{\exp}$ of continuous martingales M such that $\mathcal{E}_\cdot(M)$ *is a uniformly integrable martingale.*
(iii) The optional processes X for which the absolute value is dominated by a uniformly integrable martingale are said to be in the class [2] $(\mathcal{D})$. They are also characterized by the fact that the associated family of random variables $\{X_\sigma; \sigma \leqslant T, \sigma \text{ stopping times }\}$ is uniformly integrable.
When adopting the exponential point of view, we can extend this notion into:

$X_\cdot$ is said to be in the class $(\mathcal{D}_{\exp})$ if $e^{X_\cdot}$ belongs to the class $(\mathcal{D})$.

[2] P.A.Meyer used the term "class $(\mathcal{D})$", in the honor of Doob.

Observe that $|X_{.}|$ belongs to the class $(\mathcal{D}_{\exp})$ if and only if $X_{.}$ and $-X_{.}$ belongs to the class $(\mathcal{D}_{\exp})$, which is also equivalent to $\cosh(X_{.}) = \cosh(|X_{.}|)$ is in the class $(\mathcal{D})$.

(iv) The class of $(\mathcal{D})$-submartingales S has a particular importance in our study, since such processes are characterized through "submartingale inequalities", i.e.

$$\text{for any stopping times } \sigma \leqslant \tau \leqslant T, \quad S_\sigma \leqslant \mathbb{E}[S_\tau|\mathcal{F}_\sigma], \ a.s..$$

If the submartingale is positive (or bounded by below) and if $S_T \in \mathbb{L}^1$, the inequality $S_t \leqslant \mathbb{E}[S_T|\mathcal{F}_t]$ a.s. implies that $S_{.}$ is a $(\mathcal{D})$-submartingale, hence the inequality holds true for any pair of stopping times $\sigma \leqslant \tau$.

(v) The Doob-Meyer decomposition (see for e.g. Protter [39]) of a $(\mathcal{D})$-submartingale involves a u.i martingale $N_{.}$, and a predictable increasing process $K_{.}$. If the submartingale is positive, a multiplicative decomposition also still exists, with a u.i. exponential martingale. Taking the logarithm $X_{.} = \ln S_{.}$ yields to the so-called *entropic inequality* :

$$\forall\, \sigma \leqslant \tau \leqslant T, \quad X_\sigma \leqslant \rho_\sigma(X_\tau) \ a.s. \text{ where } \rho_\sigma(X_\tau) = \ln \mathbb{E}\big[\exp(X_\tau)|\mathcal{F}_\sigma\big]. \quad (11)$$

The operator $\rho_{.}$ is known as the *entropic process* and has been intensively studied in the framework of risk measures (see for instance Barrieu & El Karoui [6] or [7]).

Entropic inequalities and $\mathcal{Q}(\Lambda, C)$-semimartingales. An example of $\mathcal{Q}$ submartingale in the class $(\mathcal{D}_{\exp})$ is the simple process $r_{.}(M)$ defined in Equation (7) with $M_{.} \in \mathcal{U}_{\exp}$. In this case, $\exp r_{.}(M) = \mathcal{E}_{.}(M)$ is a positive u.i. martingale, equal to the conditional expectation of its terminal value $\exp(r_T(M))$. Since $\xi_T \equiv r_T(M) \in \mathbb{L}^1_{\exp}$, we can recover $r_t(M)$ from its terminal condition from the following identity[3] based on the entropic process $\rho_{.}(\xi_T)$:

$$r_t(M) = \ln \mathbb{E}\big[\exp(\xi_T)|\mathcal{F}_t\big] = \rho_t(\xi_T)$$

The conditional properties of the u.i martingale $\mathbb{E}\big[\exp(\xi_T)|\mathcal{F}_t\big] = \mathbb{E}[\exp(\xi_T)]\mathcal{E}_t(M)$ are translated into the time consistency property of the entropic process over any pair of stopping times (σ, τ) such that $\sigma \leqslant \tau$, $\rho_\sigma(\xi_T) = \rho_\sigma(\rho_\tau(\xi_T))$.

Finally, let us observe that $\rho_{.}(\xi_T)$ is the smallest q-semimartingale $X_{.} = X_0 + r_{.}(N)$ with the terminal value $X_T = \xi_T$. This is a simple consequence of

[3] Note that the identity $\rho_t(\xi_T) = r_t(\rho_0(\xi_T), M)$ has suggested the notation $r_t(M)$ for the logarithm of some exponential martingale.

the fact that $\exp(X_.)$ is a positive local martingale and hence a supermartingale.

We are now able to give another formulation for the characterization of $\mathcal{Q}$-semimartingales in the class $(\mathcal{D}_{\exp})$ in terms of inequalities involving the entropic process, or submartingale inequalities. This formulation is better suited than that of Theorem 2.6 when taking limits as we will see in a later section:

Theorem 2.7. (i) *Let $X_.$ be a làdlàg optional process such that $|X_T| \in \mathbb{L}^1_{\exp}$. Then $X_.$ is a $\mathcal{Q}$-semimartingale such that $|X_.| \in (\mathcal{D}_{\exp})$ if and only if $X_.$ satisfies the following entropic inequalities, for any pair of stopping times $0 \leqslant \sigma \leqslant \tau \leqslant T$,*

$$\underline{\rho}_\sigma\big(X_\tau\big) := -\rho_\sigma\big(-X_\tau\big) \leqslant X_\sigma \leqslant \rho_\sigma\big(X_\tau\big) \quad \mathbb{P}-a.s. \tag{12}$$

(ii) *Let $Y_.$ be a làdlàg optional process such that $X^{\Lambda,C}_T(|Y|) \in \mathbb{L}^1_{\exp}$. Then $Y_.$ is a $\mathcal{Q}(\Lambda, C)$-semimartingale such that $X^{\Lambda,C}_.(|Y|)$ is in $(\mathcal{D}_{\exp})$ if and only if for any pair of stopping times $0 \leqslant \sigma \leqslant \tau \leqslant T$,*

$$-\rho_\sigma\big(-Y_\tau+\Lambda_{\sigma,\tau}+|Y|*C_{\sigma,\tau}\big) \leqslant Y_\sigma \leqslant \rho_\sigma\big(Y_\tau+\Lambda_{\sigma,\tau}+|Y|*C_{\sigma,\tau}\big) \quad \mathbb{P}-a.s. \tag{13}$$

(iii) *Let $Y_.$ be a làdlàg optional process such that $U^{\Lambda,C}_T(e^{|Y|}) \in \mathbb{L}^1$. Then $Y_.$ is a $\mathcal{Q}(\Lambda, C)$-semimartingale such that $U^{\Lambda,C}_.(e^{|Y|}) \in (\mathcal{D})$ if and only if for any pair of stopping times $0 \leqslant \sigma \leqslant \tau \leqslant T$*

$$\begin{cases} \exp(Y_\sigma) & \leqslant \mathbb{E}[\exp(Y_\tau) + \int_\sigma^\tau e^{Y_s} d\Lambda_s + \int_\sigma^\tau e^{Y_s}|Y_s| dC_s|\mathcal{F}_\sigma]\,, \\ \exp(-Y_\sigma) & \leqslant \mathbb{E}[\exp(-Y_\tau) + \int_\sigma^\tau e^{-Y_s} d\Lambda_s + \int_\sigma^\tau e^{-Y_s}|Y_s| dC_s.|\mathcal{F}_\sigma] \end{cases} \tag{14}$$

Proof. We would like to prove that $X_.$ is a $\mathcal{Q}$-semimartingale such that $|X_.| \in (\mathcal{D}_{\exp})$, i.e. both processes $X_.$ and $-X_.$ are $(\mathcal{D}_{\exp})$-submartingales if and only if $X_.$ satisfies the entropic inequalities:
$0 \leqslant \sigma \leqslant \tau \leqslant T \quad \Rightarrow -\rho_\sigma\big(-X_\tau\big) \leqslant X_\sigma \leqslant \rho_\sigma\big(X_\tau\big) \quad \mathbb{P}-a.s.$ or equivalently if and only if $e^{X_.}$ and $e^{-X_.}$ are $(\mathcal{D})$-submartingales.
(i) Let us first focus on the process $X_.$ as the proof for the process $-X_.$ will be similar. On the one hand, let us assume that the entropic inequality holds true for the process $X_.$. Since $X_T \in \mathbb{L}^1_{\exp}$, $\exp(\rho_.(X_T))$ is a u.i. martingale. Therefore, as a straightforward consequence of the entropic inequality, the process $\exp(X_.)$ is dominated by $\exp(\rho_.(X_T))$, and $X_.$ itself is also in the class $(\mathcal{D}_{\exp})$. Moreover, taking the exponential on both sides of the entropic

inequality leads to the "submartingale inequality" for the process $\exp(X_.)$. Hence the result.
(ii) On the other hand, let us assume that the process $X_.$ is a $(\mathcal{D}_{\exp})$-submartingale. Then $\exp(X_.)$ is a positive submartingale in the class $(\mathcal{D})$. From the Doob-Meyer multiplicative decomposition, we have $\exp(X_.) = \mathcal{E}(M_.)\exp(A_.)$ where $M_.$ is a $\mathcal{U}_{\exp}$-martingale, and $A_.$ is a predictable increasing process. Therefore,

$$\forall\, 0 \leqslant \sigma \leqslant \tau \leqslant T, \qquad \exp(X_\tau) = \exp(X_\sigma)\mathcal{E}_\sigma^\tau(M)\exp(A_\sigma^\tau) \geqslant \exp(X_\sigma)\mathcal{E}_\sigma^\tau(M),$$

where K_σ^τ demotes the increments of the process $K_.$ between σ and τ. Taking the conditional expectation leads immediately to the entropic inequality. Hence the result.
The same arguments can be applied to the process $-X_.$ and so at this stage, we have proved the equivalence between the fact that both processes $X_.$ and $-X_.$ are $(\mathcal{D}_{\exp})$-submartingales and the fact that they satisfy the entropic inequalities. Theorem 2.6 allows us to conclude.
(ii) and (iii) Very similar proof. □

Sufficient condition for $U_.^{\Lambda,C}(e^{|Y|})$ to be in the class $(\mathcal{D})$ As to make Theorem 2.7 tractable, it is essential to have sufficient conditions only for the data $(Y_T, \Lambda_., C_.)$ implying that the processes $\exp(X_.^{\Lambda,C}(|Y|))$ and / or $U_.^{\Lambda,C}(e^{|Y|})$ are in the class $(\mathcal{D})$. A possible way is to give a central place to the submartingale $\exp(\bar{X}_t^{\Lambda,C}(Y)) = \exp(e^{C_t}|Y_t| + \int_0^t e^{C_s} d\Lambda_s)$ as a generalization of $\exp(|Y_t|)$ by assuming that $\exp(\bar{X}_.^{\Lambda,C}(|Y|))$ is a $(\mathcal{D})$-submartingale. This assumption implies in particular that $\bar{X}_T^{\Lambda,C}(|Y|) \in \mathbb{L}^1_{\exp}$, and that $|Y_0| \leqslant \rho_0(\bar{X}_T^{\Lambda,C}(|Y|))$. The same inequality is true if we start at time t by considering the t-conditional expectation so that $|Y_t| \leqslant \rho_t(e^{C_{t,T}}|Y_T| + \int_t^T e^{C_{t,s}} d\Lambda_s)$.
In other words, the assumption we will make in the following can be formulated as follows, using the non-adapted decreasing process $\phi_{.,T} = e^{C_{.,T}}|Y_T| + \int_.^T e^{C_{.,s}} d\Lambda_s$ and related to Briand & Hu [8] (Lemma 1):

Hypothesis 2.8. *The random variable $\bar{X}_T^{\Lambda,C}(|Y_T|) = e^{C_T}|Y_T| + \int_0^T e^{C_s} d\Lambda_s$ belongs to $\mathbb{L}^1_{\exp}$ and*

$$|Y_t| \leqslant \rho_t(e^{C_{t,T}}|Y_T| + \int_t^T e^{C_{t,s}} d\Lambda_s) = \rho_t(\phi_{t,T}(|Y_T|)) \tag{15}$$

A straightforward consequence of this assumption is the converse property:

Lemma 2.9. *Hypothesis 2.8 is a necessary and sufficient condition for the process $\bar{X}^{\Lambda,C}_{.}(|Y|)$) to be in the class ($\mathcal{D}_{\exp}$).*

Proof. Recall that the entropic process $\rho_{\delta,t}(\xi_T) = \frac{1}{\delta}\rho_t(\delta\xi_T)$ is monotonous with respect to the parameter δ (Hölder inequality for the exponential). Then, since $e^{C_t} \geqslant 1$, the following inequality holds true: $\rho_t(e^{C_{t,T}}|Y_T| + \int_t^T e^{C_{t,s}} d\Lambda_s) \leqslant e^{-C_t}\rho_t(e^{C_{0,T}}|Y_T| + \int_t^T e^{C_{0,s}} d\Lambda_s)$.
So under Hypothesis 2.8, $\bar{X}^{\Lambda,C}_{.}(|Y_{.}|) = e^{C_{.}}|Y_{.}| + e^{C} * \Lambda_{.}$ is dominated by the entropic process $\rho_{.}(\bar{X}^{\Lambda,C}_T(|Y_T|))$ in ($\mathcal{D}_{\exp}$). Hence the result. □

The properties of the dominating process $\rho_{.}(e^{C_{.,T}}|Y_T| + \int_{.}^T e^{C_{t,s}} d\Lambda_s) = \rho_{.}(\phi_{.,T}(|Y_T|))$, or equivalently of $\Phi_{.}(|Y_T|) \equiv \mathbb{E}[\exp(\phi_{.,T}(|Y_T|)|\mathcal{F}_{.}]$, are therefore essential to obtain results for the process $Y_{.}$. To study the process $\Phi_{.}(|Y_T|)$, we adopt the point of view proposed by Briand & Hu [8], and often omit the reference to Y_T in $\phi_{.,T}(|Y_T|)$ or $\Phi_{.}(|Y_T|)$ for the sake of clarity.

Proposition 2.10. *Assume $\mathbb{E}[\exp(\bar{X}^{\Lambda,C}_T(|Y_T|))] < \infty$, and let $\Phi_t(|Y_T|) \equiv \mathbb{E}[e^{\phi_{t,T}(|Y_T|)}|\mathcal{F}_t]$.*
(i) *The process $\Phi_{.}$ is a supermartingale dominated by the martingale $\mathbb{E}[e^{\phi_{0,T}}|\mathcal{F}_t] = N^0_t$, with the additive decomposition $\Phi_{.} = \Phi_0 + N^{\Phi}_{.} - A^{\Phi}_{.}$. The predictable increasing process is $A^{\Phi}_{.} = \int_0^{.} \Phi_s d\Lambda_s + \int_0^{.} \mathbb{E}[e^{\phi_{s,T}}|\phi_{s,T}||\mathcal{F}_s]dC_s$, when the process $N^{\Phi}_{.}$ is a uniformly integrable martingale.*
(ii) *The process $U^{\Lambda,C}_{.}(\Phi) = \Phi_{.} + \int_0^{.} \Phi_s d\Lambda_s + \int_0^{.} \Phi_s \ln(\Phi_s) dC_s$ is a positive supermartingale, generated by the increasing process $A^U_{.} = \int_0^{.}(\mathbb{E}[e^{\phi_{s,T}}|\phi_{s,T}||\mathcal{F}_s] - \Phi_s \ln(\Phi_s))dC_s$ and the u.i. martingale $N^{\Phi}_{.}$.*
The quantity $H^{\text{ent}}_s(e^{\phi_{s,T}}) \equiv \mathbb{E}[e^{\phi_{s,T}}|\phi_{s,T}||\mathcal{F}_s] - \Phi_s \ln(\Phi_s)$ is well-known in statistics as the conditional Shannon entropy of the random variable $e^{\phi_{s,T}}$.
(iii) *Assume Hypothesis 2.8. The submartingale $U^{\Lambda,C}_{.}(e^Y)$ is a ($\mathcal{D}$)-submartingale dominated by the ($\mathcal{D}$)-supermartingale $U^{\Lambda,C}_{.}(\Phi)$.*

Proof. As suggested by Briand & Hu [8] (Lemma 1), we adopt a backward non-adapted point of view based on the non-adapted process $\phi_{t,T} \equiv e^{C_{t,T}}|Y_T| + \int_t^T e^{C_{t,s}} d\Lambda_s$, decreasing solution of the ordinary differential equation $d\phi_t = -(d\Lambda_t + |\phi_t| dC_t)$ with terminal condition $\phi_T = |Y_T|$. The non adapted process $U^{\Lambda,C}_t(e^{\phi_{t,T}})$ is constant (deterministic version of the martingale property) and equal to $e^{\phi_{0,T}}$, with $\phi_{0,T} = \bar{X}^{\Lambda,C}_T(|Y_T|)$.
(i) We are in fact interested in the adapted version of the decreasing process $e^{\phi_{.,T}}$, i.e. in the conditional expectation $\mathbb{E}[e^{\phi_{t,T}}|\mathcal{F}_t] \equiv \Phi_t$, (which is well-defined since $\mathbb{E}[e^{\phi_{0,T}}] = \mathbb{E}[\exp(\bar{X}^{\Lambda,C}_0(|Y_T|))] < \infty$) and its $U^{\Lambda,C}_t$ transform.

The dynamics of the supermartingale $\Phi_t = \mathbb{E}[e^{\phi_{t,T}}|\mathcal{F}_t]$ is obtained by taking conditional expectation in the relation $e^{\phi_{t,T}} + \int_0^t e^{\phi_{s,T}} d\Lambda_s + \int_0^t e^{\phi_{s,T}}|\phi_{s,T}|dC_s = e^{\phi_{0,T}}$.
First observe that the assumption $e^{\phi_{0,T}} \in \mathbb{L}^1$ implies that the random variable $A^\phi_T = \int_0^T e^{\phi_{s,T}} d\Lambda_s + \int_0^T e^{\phi_{s,T}}|\phi_{s,T}|dC_s \in \mathbb{L}^1$. Since $A^\Phi_T = \int_0^T \Phi_s d\Lambda_s + \int_0^T \mathbb{E}[e^{\phi_{s,T}}|\phi_{s,T}||\mathcal{F}_s]dC_s$ has the same expectation than A^ϕ_T, A^Φ_T belongs also to $\mathbb{L}^1$. So the process $N^1_t = \mathbb{E}[A^\phi_T - A^\Phi_T|\mathcal{F}_t] = \mathbb{E}[A^\phi_t - A^\Phi_t|\mathcal{F}_t]$ is a uniformly integrable martingale. Then, taking the conditional expectation in $U^{C,\Lambda}_.(e^\phi)$ implies that $\Phi_t + A^\Phi_t + N^1_t = N^0_t$, and $N^\Phi_t = N^0_t - N^1_t$.
(ii) To show that $U^{\Lambda,C}_t(\Phi)$ is also a supermartingale, we observe, as in Briand & Hu [8], that since $x \ln x$ is convex and increasing for $|x| \geqslant 1$, $\mathbb{E}[e^{\phi_{s,T}}|\phi_{s,T}||\mathcal{F}_s] \geqslant \Phi_s \ln(\Phi_s)$. The difference between these two terms is the well-known conditional Shannon entropy $H^{\text{ent}}_s(e^\phi)$. Then, some simple calculation shows that $U^{\Lambda,C}_t(\Phi) + \int_0^t H^{\text{ent}}_s(e^\phi)dC_s = \Phi_t + \int_0^t \Phi_s d\Lambda_s + \int_0^t \mathbb{E}[e^{\phi_{s,T}}|\phi_{s,T}||\mathcal{F}_s]dC_s = N^\Phi_t$ is a positive uniformly integrable martingale with expectation $U_0(\Phi)$, that provides a precise description on the supermartingale $U^{\Lambda,C}_.(\Phi)$.
(iii) This last statement is a straightforward consequence of the inequality $e^{|Y_.|} \leqslant \Phi_.$. □

Remark 1. *The key condition to obtain these properties is that the process* $U^{\Lambda,C}_.(\Phi(|Y_T|))$ *is a ($\mathcal{D}$)-supermartingale. Note that this will be also true if we replace* $|Y_T|$ *by any* $\mathcal{F}_T$*-random variable* $|\eta_T| \geqslant |Y_T|$, *such that* $e^{C_T}|\eta_T| + \int_0^T e^{C_s} d\Lambda_s \in \mathbb{L}^1_{\exp}$. *Therefore, in the next section, we will consider a slightly modified version of Hypothesis 2.8 where* $|Y_T|$ *is replaced by such a random variable* η_T.

Remark 2. *As observed by Briand & Hu [8], the same kind of estimates as those introduced in the linear growth case by Lepeltier & San Martin in [29], may be obtained in the superlinear quadratic case. The linear growth condition in* Y, $|Y|.C_.$, *is then replaced by* $h(|Y|).C_.$, *where* h *is an increasing convex* C^1 *function, with* $h(0) > 0$, *satisfying the integrability condition* $\int_0^T du \frac{|u|}{h(u)} = +\infty$. *The function* $\phi(t)$ *is replaced by the solution of the ODE* $\phi'(t) = -h(\phi_t)$ *with a terminal condition* $\phi_T = z \leqslant 0$.

$L \log L$-integrability and maximal inequality The so-called $L \log L$-Doob inequality (see for e.g. Protter [39], or Dellacherie & Meyer [12]) gives a necessary and sufficient condition on the terminal value of a u.i.

exponential martingale for it to be in $\mathbb{H}^1$. This condition naturally appears when considering this martingale as the likelihood of a probability measure $\mathbb{Q}$, absolutely continuous with respect to $\mathbb{P}$, as it measures the Shannon entropy $H(\mathbb{Q}|\mathbb{P}) = H^{\text{ent}}(d\mathbb{Q}/d\mathbb{P}) = \mathbb{E}\big(d\mathbb{Q}/d\mathbb{P} \ln(d\mathbb{Q}/d\mathbb{P})\big)$ of $\mathbb{Q}$ with respect to $\mathbb{P}$. More precise sharp estimates have been recently proposed by Harremoës [22] in the discrete time context. We give this result with proof to be self-contained.

Proposition 2.11. *Let $L_.$ be a positive submartingale and* $\max L_. \in [1,\infty[$ *its running supremum. For any $m > 0$, let $u_m(x)$ be the convex function defined on $\mathbb{R}^+$ by $u_m(x) = x - m - m\ln(x)$, $u_1(x) := u(x)$.*
(i) a) *(Doob) Assume that $L_.$ is a u.i. integrable martingale with $L_0 = 1$, then $\max L_T$ is an integrable variable if and only if $H^{\text{ent}}(L_T) = \mathbb{E}\big(L_T \ln(L_T)\big) < \infty$.*
b) *Using the representation of the martingale $L_t = \mathbb{E}(L_T|\mathcal{F}_t)$ as $L_t = \exp(M_t - \frac{1}{2}\langle M\rangle_t)$,*

$$H^{\text{ent}}(L_T) = \mathbb{E}\big(L_T \ln(L_T)\big) = \mathbb{E}\big(L_T \frac{1}{2}\langle M\rangle_T\big). \tag{16}$$

(ii) a) *(Harremoës) Then, the following sharp inequality holds true*

$$u\big(\mathbb{E}(\max L_T)\big) \leqslant \mathbb{E}\big(L_T \ln(L_T)\big) = H^{\text{ent}}(L_T), \tag{17}$$

b) *Assume L to be a positive $(\mathcal{D})$-submartingale. The previous inequality becomes:*

$$u_m\big(\mathbb{E}(\max L_T)\big) - u_m(L_0) \leqslant H^{\text{ent}}(L_T) = \mathbb{E}\big(L_T \ln(L_T)\big) - \mathbb{E}(L_T)\ln\big(\mathbb{E}(L_T)\big),$$

with $m = \mathbb{E}(L_T)$.

Proof. The proof use ideas from Dellacherie [11] and Harremoës [22].
(i) a) Since L is a continuous process, $\max L_s$ only increases on the set $\{L_. = \max L_.\}$ and $\max L_t = 1 + \int_0^t d\max L_s = 1 + \int_0^t \frac{L_s}{\max L_s} d\max L_s$. Taking the expectation and using the fact that L is the conditional expectation of its terminal value leads to the equality $\mathbb{E}\big(\max L_T\big) - 1 = \mathbb{E}\big(L_T \ln(\max L_T)\big) = H^{\text{ent}}(L_T)$.
(i) b1) Moreover, since $\ln(\max L_T) \geqslant \ln(L_T)^+$, and $L_t \ln(L_t)^- \leqslant 1/e$, then $L_T \ln(L_T) \in \mathbb{L}^1$ when $\max L_T \in \mathbb{L}^1$. This establishes the necessary condition.
(i) b2) Under this assumption, let T_K be an increasing sequence of stopping times, such that the martingale $M_.$, such that $L_t = \exp(M_t - \frac{1}{2}\langle M\rangle_t)$, is bounded by K. The sequence T_K is increasing and goes to infinity with

K. Thanks to the Girsanov theorem, $N^{\mathbb{Q}}_{.} = M_{.} - \langle M\rangle_{.}$ is a martingale with respect to $\mathbb{Q} = L_T.\mathbb{P}$, and $\mathbb{E}\big(L_T\frac{1}{2}\langle M\rangle_T\big) = \lim_K \mathbb{E}\big(L_T\frac{1}{2}\langle M\rangle_{T\wedge T_K})\big) = \lim_K \mathbb{E}\big(L_{T\wedge T_K}\frac{1}{2}\langle M\rangle_{T\wedge T_K}\big)$. Using the fact that $\mathbb{E}\big(L_{T\wedge T_K} N^{\mathbb{Q}}_{T\wedge T_K}\big) = 0$,

$$\begin{aligned}\mathbb{E}\big(L_{T\wedge T_K}\frac{1}{2}\langle M\rangle_{T\wedge T_K}\big) &= \mathbb{E}\big(L_{T\wedge T_K}(M_{T\wedge T_K} - \langle M\rangle_{T\wedge T_K} + \frac{1}{2}\langle M\rangle_{T\wedge T_K})\big)\\ &= \mathbb{E}\big(L_{T\wedge T_K}\ln(L_{T\wedge T_K})\big) \leqslant \mathbb{E}\big(\max L_{T\wedge T_K}\big) - 1 \leqslant \mathbb{E}\big(\max L_T\big) - 1\end{aligned}$$

Then $N^{\mathbb{Q}}_{.}$ is a square integrable $\mathbb{Q}$-martingale and $\mathbb{E}\big(L_T\ln(L_T)\big) = \mathbb{E}\big(L_T\frac{1}{2}\langle M\rangle_T\big)$. The sufficient condition is a by product of the inequality (17) proved below.
(ii) a) To prove Inequality (17), we start by comparing $\mathbb{E}\big(L_T\ln(\max L_T)\big)$ and $\mathbb{E}\big(L_T\ln(L_T)\big)$ using the concavity of the function ln and the random variable $\max L_T/L_T$. Given that $x^* = \mathbb{E}(\max L_T) = \mathbb{E}_{\mathbb{Q}}(\max L_T/L_T)$ if $\mathbb{Q} = L_T.\mathbb{P}$,

$$\mathbb{E}\big(L_T\,\ln(\max L_T/L_T)\big) = \mathbb{E}_{\mathbb{Q}}\big(\ln(\max L_T/L_T)\big) \leqslant \ln\big(\mathbb{E}_{\mathbb{Q}}(\max L_T/L_T)\big) = \ln x^*.$$

Inequality (17) is then obtained by reorganizing the terms in the inequality $x^* - 1 = \mathbb{E}(\max L_T) - 1 \leqslant \mathbb{E}\big(L_T\ln(L_T)\big) + \ln x^*$.
(ii) b) To prove that this inequality is sharp, Harremoës considers the martingale generated as follows: let U a random variable uniformly distributed on $[0,1]$ equipped with the filtration $\mathcal{F}_t = \sigma(U\wedge t)$ and completed with the negligible sets. For a given increasing function density f such that $f(0) > 0, f(1) = +\infty$, and $\mathbb{E}\big(f(U)\big) = 1$, the martingale $X_.$, defined as $X_t = \mathbb{E}\big(f(U)|\mathcal{F}_t\big) = f(U)\mathbf{1}_{U<t} + \frac{\int_t^1 f(y)dy}{1-t}\mathbf{1}_{U\geqslant t}$, is maximal at time $U \leqslant 1$, with supremum $\max X_1 = \frac{\int_U^1 f(y)dy}{1-U}$. Simple calculations show that $\mathbb{E}(\max X_1) = \int_0^1 f(y)|\ln(1-y)|dy$, $\mathbb{E}\big(X_1\ln(X_1)\big) = \int_0^1 f(y)\ln f(y)dy$. Let $\beta \in [0,1[$ and $f(x) = (1-\beta)(1-x)^{-\beta}$. Then $\mathbb{E}(\max X_1) = \frac{1}{1-\beta}$ and $\mathbb{E}\big(X_1\ln(X_1)\big) = \ln(1-\beta) + \frac{\beta}{1-\beta} = u(\mathbb{E}(\max X_1))$.
(ii) c) The extension to L being a positive submartingale does not present any specific difficulties other than purely computational, since $\mathbb{E}\big(\max L_T\big) - L_0 \leqslant \mathbb{E}\big(L_T\ln(\max L_T/L_0)\big)$. Taking now $\mathbb{Q} = (L_T/m).\mathbb{P}$, $x^*/m = \mathbb{E}(\max L_T)/m = \mathbb{E}_{\mathbb{Q}}(\max L_T/L_T)$, the convexity inequality becomes:
$\mathbb{E}_{\mathbb{Q}}(\max L_T/L_T) \leqslant \ln\big(\mathbb{E}_{\mathbb{Q}}(\max L_T/L_T)\big) = \ln(x^*/m)$. Some elementary algebra gives the final result. □

The following proposition, coming as a straightforward consequence of Proposition 2.11, gives a potential application of the $L\log L$ inequality to our framework and enables us to have some conditions for the integrability of the running maximum process of different processes of interest.

Proposition 2.12. *Let us assume that* $\mathbb{E}\big(\exp(\bar{X}_T^{\Lambda,C}(|Y_T|))\bar{X}_T^{\Lambda,C}(|Y_T|)\big) < \infty$.
(i) *Then, the running maximum of the entropic process* $\rho_{.}(\bar{X}_T^{\Lambda,C}(|Y_T|)$ *belongs to* $\mathbb{L}^1_{\exp}$ *and the exponential of its norm is dominated by the Shannon entropy of* $\exp(\bar{X}_T^{\Lambda,C})$.
(ii) *The running supremum of the process* $e^{C_t}|Y_t| + \int_0^t e^{C_s} d\Lambda_s$ *belongs to* $\mathbb{L}^1_{\exp}$ *and the exponential of its norm is also dominated by the Shannon entropy of* $\exp \bar{X}_T^{\Lambda,C}$.

3 Quadratic variation estimates and stability results

We are now interested in studying some stability results. To be able to use the previous estimates, we need the family of $\mathcal{Q}(\Lambda, C)$-semimartingales we consider to be uniformly dominated. Therefore, it seems natural to introduce the following class $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$, and to work within this class of quadratic semimartingales:

Definition 3.1. *Let* $|\eta_T|$ *be a* $\mathcal{F}_T$*-random variable, such that* $\bar{X}_T^{C,\Lambda}(|\eta_T|) = e^{C_T}|\eta_T| + \int_0^T e^{C_s} d\Lambda_s$ *belongs to* $\mathbb{L}^1_{\exp}$. *The class* $\mathcal{S}_Q(||, \Lambda, C)$ *is the set of* $\mathcal{Q}(\Lambda, C)$*-semimartingales* $Y_.$ *such that* $|Y_t| \leqslant \rho_t(e^{C_{t,T}}|\eta_T| + \int_t^T e^{C_{t,s}} d\Lambda_s)$ *a.s..*

3.1 Quadratic variation estimates

We now study the quadratic variation of $\mathcal{Q}(\Lambda, C)$-semimartingale $Y_.$ when $Y_.$ belongs to $\mathcal{S}_{\mathcal{Q}}(|\eta_T|, \Lambda, C)$. Following Kobylanski [26], the best way to study the quadratic variation is to use the function $v(x) = e^x - 1 - x$ instead of the simple exponential function. This function is indeed positive, increasing and convex for $x \geqslant 0$, and verifies $v''(x) - v'(x) = 1$. In the following, we use the short notation $\bar{X}_T^{C,\Lambda}(|\eta_T|) = \bar{X}_T^{C,\Lambda}$.

Theorem 3.2 (Quadratic variation estimates)**.** *Let* $Y_. \in \mathcal{S}_Q(|\eta_T|, \Lambda, C)$.
(i) *Then, the quadratic variation* $\langle M \rangle_.$ *of the* $\mathcal{Q}(\Lambda, C)$*-semimartingale* $Y_. = Y_0 + M_. - V_.$ *satisfies for any stopping times* $S \leqslant T$

$$\frac{1}{2}\mathbb{E}[\langle M \rangle_{S,T}|\mathcal{F}_S] \leqslant \Phi_S(|Y_T|)\mathbf{1}_{\{S<T\}} \leqslant \mathbb{E}[\exp(\bar{X}_T^{C,\Lambda})\,\mathbf{1}_{\{S<T\}}|\mathcal{F}_S] \tag{18}$$

In particular, the martingale $M_.$ is in $\mathbb{H}^2$, with a uniform control of the quadratic norm

$$\mathbb{E}[\frac{1}{2}\langle M\rangle_T] \leqslant \mathbb{E}[\exp(\bar{X}_T^{C,\Lambda})] \tag{19}$$

(ii) *Let $p^\eta = \sup\{p;\ \mathbb{E}\big[\exp(p\bar{X}_T^{C,\Lambda})\big] < +\infty\}$. Then $p^\eta \geqslant 1$ and $\forall p \in [1, p^\eta[$, the martingale M belongs to $\mathbb{H}^{2p}$, and*

$$\mathbb{E}[\langle M\rangle_T^p] \leqslant (2p)^p\, \mathbb{E}\big[\exp(p\bar{X}_T^{C,\Lambda})\big]. \tag{20}$$

(iii) *If for any $t \leqslant T$, $\Phi_t(|\eta_T|) = \mathbb{E}[\exp(e^{C_{t,T}}|\eta_T| + \int_t^T e^{C_{t,u}} d\Lambda_u)|\mathcal{F}_t]$ is bounded, then the martingale M is in BMO.*

Proof. (i) – By analogy with the previous notation, when using the function $v(x) = e^x - 1 - x$, we put $V_t^{\Lambda,C}(e^{|Y|}) = v(|Y_t|) + \int_0^t v'(|Y_s|)(d\Lambda_s + |Y_s|dC_s) = \int_0^t v'(|Y_s|)dD_s^{\Lambda,C}$. The semimartingale $|Y_.|$ is associated with the martingale $M_.^s = \mathrm{sign}(Y_.) * M$, the finite variation process $V_.^s = \mathrm{sign}(Y_.) * V$ and the local time at $\{0\}$, that disappears in the Ito's formula since $v'(0) = 0$. Using similar calculation to those of the previous section, and the identity $v''(x) - 1 = v'(x)$, we have, that $V_t^{\Lambda,C}(e^{|Y|}) - \frac{1}{2}\langle M\rangle_t = v(|Y_0|) + \int_0^t v'(|Y_s|)dM_s^s + \int_0^t v'(|Y_s|)(dD_s^{\Lambda,C} - dV_s^s + \frac{1}{2}d\langle M\rangle_s)$ is a submartingale,

for any $S \leqslant T$, $\mathbb{E}[\frac{1}{2}\langle M\rangle_{S,T}|\mathcal{F}_S] \leqslant \mathbb{E}[v(|Y_T|) - v(|Y_S|) + \int_S^T v'(|Y_s|)dD_s^{\Lambda,C}|\mathcal{F}_S]$.

– Since, by definition, $\forall x \geqslant 0, v(x) \leqslant e^x$ and $v'(x) \leqslant e^x$,

$\int_S^T v'(|Y_s|)dD_s^{\Lambda,C} \leqslant \int_t^T \Phi_s(d\Lambda_s + \ln|\Phi_s|dC_s)$ for any $S \leqslant T$.

Thanks to the supermartingale property of $U_t^{\Lambda,C}(\Phi)$ and the inequality ($\exp(\eta_T) = \Phi_T \geqslant \exp(|Y_T|)$, $\mathbb{E}[\int_S^T \Phi_s(d\Lambda_s + \ln|\Phi_s|dC_s)|\mathcal{F}_S] \leqslant \mathbb{E}[\Phi_S - \Phi_T|\mathcal{F}_S]$ and

$$\begin{aligned}
\mathbb{E}[\frac{1}{2}\langle M\rangle_{S,T}|\mathcal{F}_S] &\leqslant \mathbb{E}[v(|Y_T|) - v(|Y_S|) - (\Phi_T - \Phi_S)|\mathcal{F}_S] \\
&\leqslant \mathbb{E}[v(|Y_T|) - v(|Y_S|) - (\exp|\eta_T| - \exp(e^{C_{S,T}}|\eta_T| + \int_S^T e^{C_{S,u}} d\Lambda_u))|\mathcal{F}_S] \\
&\leqslant \mathbb{E}[(v(|Y_T|) - \exp|\eta_T| + \exp(e^{C_{S,T}}|\eta_T| + \int_S^T e^{C_{S,u}} d\Lambda_u))\,\mathbf{1}_{\{S<T\}}|\mathcal{F}_S] \\
&\leqslant \Phi_S \mathbf{1}_{\{S<T\}} \leqslant \mathbb{E}[\exp\bar{X}_T^{\Lambda,C}\,\mathbf{1}_{\{S<T\}}|\mathcal{F}_S].
\end{aligned}$$

(ii) As observed in Lenglart, Lépingle & Pratelli [27], the final result is a simple consequence of the so-called Garsia-Neveu Lemma (Lemma 3.3) (see for instance Neveu [37]) recalled below.

(iii) This is a straightforward consequence of the inequality $\mathbb{E}[\frac{1}{2}\langle M\rangle_{S,T}|\mathcal{F}_S] \leqslant \Phi_S(|\eta_T|)$. □

Lemma 3.3 (Garsia-Neveu Lemma)**.** *Let $A_.$ a predictable increasing process and U a random variable, positive and integrable. If for any stopping times $S \leqslant T$,*

$$\mathbb{E}[A_T - A_S \mathbf{1}_{\{0<S\}} | \mathcal{F}_S] \leqslant \mathbb{E}[U \mathbf{1}_{\{S<T\}} | \mathcal{F}_S],$$
$$\text{then} \quad \forall r \geqslant 1, \quad \mathbb{E}[A_T^r] \leqslant r^r \mathbb{E}[U^r].$$

More generally, for any convex function F such that $p = \sup_{x>0}(x(\ln F)'(x)) < +\infty$,

$$\mathbb{E}[F(A_T)] \leqslant \mathbb{E}[F(pU)].$$

Here we apply this lemma to the power function $F(x) = x^p$ and the random variable $U = \exp\big(\bar{X}_T^{\Lambda,C}(|\eta_T|)\big)$ for any $p \geqslant 1$ such that $U \in \mathbb{L}^p$. As a corollary of this result, uniform estimates may be obtained for the total variation of the process $V_.$:

Corollary 3.4. *Let $Y_. \in \mathcal{S}_Q(|\eta_T|, \Lambda, C)$. The total variation of the process $V_.$ such that $Y_. = Y_0 + M_. - V_.$ satisfies for $1 \leqslant p < p^\eta$*

$$\mathbb{E}[|V|_T^p] \leqslant (2p)^p \, \mathbb{E}\big[\exp(p\bar{X}_T^{C,\Lambda})\big], \tag{21}$$

When $\Phi_t(|\eta_T|) = \mathbb{E}[\exp(e^{C_{t,T}}|\eta_T| + \int_t^T e^{C_{t,u}} d\Lambda_u) | \mathcal{F}_t]$ is bounded by K_C, then $\mathbb{E}[|V|_{S,T} | \mathcal{F}_S] \leqslant 2K_C$.

Proof. Since $V_.$ satisfies the structure condition $\mathcal{Q}(\Lambda, C)$, $\mathbb{E}[|V|_{S,T}|\mathcal{F}_S] \leqslant E[\Lambda_{S,T} + \int_S^T |Y_s| dC_s + \frac{1}{2}\langle M \rangle_{S,T} | \mathcal{F}_S] \leqslant 2\mathbb{E}[\exp(\bar{X}_T^{\Lambda,C}) \mathbf{1}_{\{S<T\}} | \mathcal{F}_S]$. Indeed, $\mathbb{E}[\Lambda_{S,T} + \int_S^T |Y_s| dC_s | \mathcal{F}_S] \leqslant \mathbb{E}[\int_S^T e^{|Y_s|}(d\Lambda_s + |Y_s| dC_s) | \mathcal{F}_S] \leqslant \mathbb{E}[(\Phi_S - \Phi_T) | \mathcal{F}_S] \leqslant \mathbb{E}[\exp(\bar{X}_T^{\Lambda,C}) \mathbf{1}_{\{S<T\}} | \mathcal{F}_S]$.
We conclude with Lemma 3.3. □

3.2 Stability results for $\mathcal{Q}(\Lambda, C)$-semimartingales

We can start by noticing that the class $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ is stable by a.s. convergence, since the submartingale property of both uniformly dominated processes $U_.(e^Y)$ and $U_.(e^{-Y})$ is stable by a.s. convergence. Moreover, Theorem 2.6 implies that the limit process is also in $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$.
However, previous estimates of both the quadratic variation and finite variation processes suggest that a better stability result may hold true, in particular regarding the strong convergence of the martingale parts. The space of

martingales where this convergence takes place depends obviously on the exponential integrability properties of the random variable $X_T^{\Lambda,C}(|\eta_T|)$. When the $\mathcal{Q}(\Lambda, C)$-semimartingales are bounded, this type of results has already been obtained for the $\mathbb{H}^2$-convergence by Kobylanski [26]. In addition, Briand & Hu [8] have shown the convergence for stopped martingales in the unbounded case. Our stability result is novel and direct, and gives better convergence results.

Theorem 3.5. *Assume the sequence* $(Y^n_.)$ *of* $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$*-semimartingales is a Cauchy sequence for the a.s. uniform convergence, i.e.* $\sup_{t\leqslant T} |Y_t^n - Y_t^{n+p}|$ *tends to 0 almost surely when* $n \to \infty$*. Different types of convergence hold true for the processes* $(M^n_., V^n_.)$ *of the decomposition* $Y^n_. = Y_0^n + M^n_. - V^n_.$
(i) *Martingales convergence:*
a) *The sequence of martingales* $(M^n_.)$ *converges to a martingale* $M_.$ *in* $\mathbb{H}^1$.
b) *If, for some* $p > 1$ $\bar{X}_T^{\Lambda,C}(|\eta_T|) \in \mathbb{L}^p_{\exp}$*, the sequence of martingales* $(M^n_.)$ *converges to a martingale* $M_.$ *in* $\mathbb{H}^{2p}$.
c) *If* $\Phi_S(|\eta_T|)$ *is bounded, the sequence of martingales* $(M^n_.)$ *converges to a martingale* $M_.$ *in the BMO-space.*
(ii) *The sequence of finite variation processes* $(V^n_.)$ *converges uniformly to a process* $V_.$ *satisfying the structure condition* $\mathcal{Q}(\Lambda, C)$ *at least in* $\mathbb{L}^1$.
(iii) *The limit of the sequence of* $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$*-semimartingales* $(Y^n_.)$ *is a quadratic* $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$*-semimartingale* $Y_. = Y_0 + M_. - V_.$.

Proof. We proceed[4] in several steps to prove this convergence result. We first introduce some notations and make some elementary calculations.
Let $Y_t^{i,j} = Y_t^i - Y_t^j$, $\quad M_t^{i,j} = M_t^i - M_t^j$ and $Y_{t,s}^{i,j} = Y_s^i - Y_t^i - Y_s^j - Y_t^j$, and the short notation: $\sup_{t\leqslant u\leqslant s} |\, Y_u^{i,j} - Y_t^{i,j}| = \max |\, Y_{t,s}^{i,j}|$.
Then for any stopping times $\sigma \leqslant \tau \leqslant T$,

$$\langle M_{\sigma,\tau}^{i,j}\rangle = |Y_{\sigma,\tau}^{i,j}|^2 - 2\int_\sigma^\tau Y_{\sigma,s}^{i,j} dX_s^{i,j} \leqslant |Y_{\sigma,\tau}^{i,j}|^2 - 2\int_\sigma^\tau Y_{\sigma,s}^{i,j} dM_s^{i,j} + 2\int_\sigma^\tau |Y_{\sigma,s}^{i,j}| d(|V^j|_s + |V^i|_s)$$

Using either the fact that $Y^{i,j}$ is bounded, or a localization procedure, the stochastic integral $\int_\sigma^{\tau_n} Y_{\sigma,s}^{i,j}\, dM_s^{i,j}$ has null conditional expectation for a well-chosen stopping time τ_n. Then, thanks to the monotonicity of $\langle M \rangle$ and

[4]An earlier proof of this result in the BMO case is due to Nicolas Cazanave, a former PhD student at Ecole Polytechnique

Corollary 3.4, with $B^{i,j} = 2(|V^i| + |V^j|)$,

$$\mathbb{E}\big[\langle M^{i,j}_{\sigma,T}\rangle \mid \mathcal{F}_\sigma\big] \leqslant \mathbb{E}\big[\max|Y^{i,j}_{\sigma,T}|^2\, \mathbf{1}_{\{\sigma<T\}}|\, \mathcal{F}_\sigma\big] + \mathbb{E}\big[\int_\sigma^T \max|Y^{i,j}_{\sigma,s}|\, dB^{i,j}_s \mid \mathcal{F}_\sigma\big]$$
$$\leqslant \mathbb{E}\big[(\max|Y^{i,j}_{0,T}|^2 + \max|Y^{i,j}_{0,T}|B^{i,j}_T)\, \mathbf{1}_{\{\sigma<T\}}|\, \mathcal{F}_\sigma\big].$$

We now start with the simplest proof corresponding to the $\mathbb{H}^p_{\exp}$-case.
(i) b) Thanks to the Garsia-Neveu Lemma (Lemma 3.3), for $r \geqslant 1$,

$$\mathbb{E}\big[\langle M^{i,j}_T\rangle^r\,\big] \leqslant r^r\mathbb{E}\big[(\max|Y^{i,j}_{0,T}|^2 + \max|Y^{i,j}_{0,T}|\; B^{i,j}_T)^r\big]$$
$$\leqslant (2r)^r\{\mathbb{E}\big[(\max|Y^{i,j}_{0,T}|)^{2r}\big] + \mathbb{E}\big[(\max|Y^{i,j}_{0,T}|B^{i,j}_T)^r\big]\}.$$

Then, since $B^{i,j}_T$ belongs to $\mathbb{L}^p$, by Hölder inequalities, for any $1 \leqslant r < p$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\mathbb{E}\big[(\max|Y^{i,j}_{0,T}|B^{i,j}_T)^r\big] \leqslant (\mathbb{E}\big[(\max|Y^{i,j}_{0,T}|)^q\big])^{\frac{r}{q}}(\mathbb{E}\big[(B^{i,j}_T)^p\big])^{\frac{r}{p}}$$
$$\mathbb{E}\big[\langle M^{i,j}_T\rangle^r\,\big] \leqslant \frac{1}{2}(2r)^r\{\mathbb{E}\big[\max|Y^{i,j}_{0,T}|^{2r}\big] + (\mathbb{E}\big[(\max|Y^{i,j}_{0,T}|)^q\big])^{\frac{r}{q}}(\mathbb{E}\big[(B^{i,j}_T)^p\big])^{\frac{r}{p}}\}.$$

From the monotonicity of both sides of this inequality with respect to r, we can take $r = p$. We have used that $\max|Y^{i,j}_{0,T}|$ has finite moments of all orders since $e^{|Y^i_\cdot|}$ and $e^{|Y^j_\cdot|}$ are dominated uniformly by a martingale in $\mathbb{H}^p$. Hence, we have the desired convergence.

(i) c) In the bounded case, thanks to Corollary 3.4, the conditional variation of the process $|V^n|$ is uniformly bounded. To obtain the BMO convergence, we have to modify the previous proof, by using an integration by parts formula involving only the conditional variation of $B^{i,j}$,

$$\begin{aligned}\mathbb{E}[\int_t^T \max|\,Y^{i,j}_{t,s}|\,dB^{i,j}_s \mid \mathcal{F}_t] &= \mathbb{E}[\int_t^T d_u \max|\,Y^{i,j}_{t,u}|\left(\mathbb{E}[\int_u^T dB^{i,j}_s \mid \mathcal{F}_u]\right) \mid \mathcal{F}_t]\\ &\leqslant 2\,C_V\mathbb{E}[\max|\,Y^{i,j}_{t,T}|\mid \mathcal{F}_t].\end{aligned}$$

In terms of quadratic variation, we have: $\mathbb{E}[\langle M^{i,j}_{t,T}\rangle|\mathcal{F}_t] \leqslant 2\,C_V\mathbb{E}[\max|\,Y^{i,j}_{t,T}||\mathcal{F}_t] + \mathbb{E}\big[|Y^{i,j}_{t,T}|^2 \mid \mathcal{F}_t]$. Then, the BMO-convergence holds true.

(i) a) The proof of the general case requires a different argument, based on a result of Barlow & Protter [5] on the convergence of semimartingales. In the framework of quadratic semimartingales, the key points are the uniform estimates of both the quadratic variations and the total variations given in Theorem 3.2, Equation (19) and Corollary 3.4. The proof given in [5] of the $\mathbb{H}^1$-convergence of the martingales is based on the square root of the inequality given at the beginning of the proof,

$$\langle M_t^{i,j}\rangle \leqslant |Y_t^{i,j}|^2 - |Y_0^{i,j}|^2 - 2\int_0^t Y_s^{i,j}\, dM_s^{i,j} + 2\int_0^t |Y_s^{i,j}|\, dB_s^{i,j}.$$

The first step is to estimate the square root of $\max|(Y^{i,j}.M^{i,j})_{0,T}|$ using the Burkolder-Davis-Gundy inequalities for continuous martingales for $p = \frac{1}{2}$: $\mathbb{E}[\max|(Y^{i,j}.M^{i,j})_{0,T}|^{\frac{1}{2}}] \leqslant \bar{C}\mathbb{E}[\langle |Y^{i,j}.M_T^{i,j}|\rangle^{1/4}]$ where $\bar{C}$ is a universal constant. Then,

$$\mathbb{E}[\langle M_T^{i,j}\rangle^{\frac{1}{2}}] \leqslant \mathbb{E}[\max|Y_{0,T}^{i,j}|] + \sqrt{2}\bar{C}\,\mathbb{E}[\max|Y_{0,T}^{i,j}|]^{1/4}\mathbb{E}[\langle M_T^{i,j}\rangle^{1/2}]^{1/4} + \sqrt{2}\mathbb{E}[\max|Y_{0,T}^{i,j}|]^{\frac{1}{2}}\mathbb{E}[B_T^{i,j}|]^{\frac{1}{2}}.$$

Since $\mathbb{E}[\langle M_T^{i,j}\rangle^{1/2}]$ and $\mathbb{E}[B_T^{i,j}]$ are uniformly bounded, and $\mathbb{E}[\max|Y_{0,T}^{i,j}|]$ goes to 0, then $\mathbb{E}[\langle M_T^{i,j}\rangle^{\frac{1}{2}}]$ also goes to 0 and the $\mathbb{H}^1$-convergence of the martingale part is established.

(ii) The next point is to study the convergence of the sequence $(V_.^n)$ to a process $V_.$ satisfying the same structure condition $\mathcal{Q}(\Lambda, C)$. Since, the sequence $(Y_.^n,\, M_.^n, \langle M^n\rangle_.^{\frac{1}{2}})$ converge uniformly at least in $\mathbb{L}^1$ to $(Y_.,\, M_., \langle M\rangle_.^{\frac{1}{2}})$, the sequence $(V_.^n)$ also converges uniformly at least in $\mathbb{L}^1$. Therefore, we can extract a subsequence, still denoted $(Y_.^n, M_.^n, V_.^n,\, \langle M^n\rangle_.^{\frac{1}{2}})$, such that the sequence converges uniformly almost surely.
Thanks to the *structure condition*, the uniformly dominated increasing processes $\overline{A}_.^n = -V_.^n + \frac{1}{2}\langle M^n\rangle_. + \Lambda_. + |Y_.^n| * C_.$ and $\underline{A}_.^n = V_.^n + \frac{1}{2}\langle M^n\rangle_. + \Lambda_. + |Y_.^n| * C_.$ also converge uniformly almost surely to the increasing processes $\overline{A}_.$ and $\underline{A}_.$. The process $V_. = \frac{1}{2}(\overline{A}_. - \underline{A}_.)$, which is the difference of two increasing processes summing up to $\frac{1}{2}\langle M\rangle_. + \Lambda_. + |Y|_. * C_.$, also satisfies the structure condition.
Note that we could use the characterization of $\mathcal{Q}(\Lambda, C)$-semimartingales given in Theorem 2.6 to prove this last property. □

Stability results for BSDE-like quadratic semimartingales The uniform convergence of the quadratic semimartingales needed for these convergence results may seem very strong. We know however from Theorem 2.6 that all the processes obtained by a.s. convergence are continuous. Thanks to Dini's Theorem, the monotone convergence implies uniform convergence for continuous functions on compact spaces. Moreover, when considering sequences of BSDE-like quadratic semimartingales defined in Definition 2.2, under mild assumptions on the sequence of coefficients, the sequence of finite variation processes is converging in finite variation in the appropriate space, and the limit is still a BSDE-like quadratic semimartingale. Therefore, by a localization procedure, we can prove the following very strong result:

Theorem 3.6. *Let assume the sequence $(Y^n_.)$ to be a monotone sequence of $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$- quadratic semimartingales converging almost surely to a process $Y_.$.*
(i) *Then, the limit process $Y_.$ is a continuous $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$- quadratic semimartingale, the convergence is locally uniform and all properties given in Theorem 3.5 hold (locally) true.*
(ii) *Suppose in addition that the processes $(Y^n_.)$ are BSDE-like quadratic semimartingales, associated with a sequence of monotone coefficients g_n such that $Y^n_. = Y^n_0 + Z^n.N_. + M^{n,\perp} - g_n(., Y^n_., Z^n_.).K_.$ and for which the following assumptions are made:*

- *the monotone sequence g_n have uniform quadratic growth:*

$$|g_n(t, y, z)| \leqslant \frac{1}{\delta} l_t + |y| c_t + \frac{\delta}{2} |\sqrt{\gamma}_t z|^2$$

- *the sequence $g_n(t, y_n, z_n)$ converges to $g(t, y, z)$, as soon that $(y_n, z_n) \to (y, z)$.*

Then, the limit process $Y_.$ is a $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$-semimartingale with coefficient $g(t, y, z) = \lim g_n(t, y, z)$

Proof. Note the characterization of $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$-semimartingales given in Theorem 2.6 passes to the limit, since all processes $U^{\Lambda,C}_.(e^{|Y^n|})$ are dominated by the $(\mathcal{D})$-process $U^{\Lambda,C}_.(\Phi)$. The limit process Y is a continuous $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$-semimartingale, with decomposition $Y_. = Y_0 + M_. - V_.$.
(i) The localization procedure is based on the family (T_K) of stopping times as to bind the u.i martingale $N^0_t = \mathbb{E}[\exp(\phi_0(|\eta_T|)|\mathcal{F}_t]$ by K. By the characterization of u.i. continuous martingale (see for instance Azema, Gundy & Yor [3]), the sequence T_K goes to ∞ and for $K \geqslant K_\epsilon$ large enough, $\mathbb{P}(T_K < T) \leqslant \frac{\epsilon}{K}$. Therefore, the sequence $(Y^n_{.\wedge T_K})$ lives on a compact set where the monotone convergence to a *continuous process* is uniform. The sequence of martingales $(M^n_{.\wedge T_K})_n$ strongly converges in the appropriate space to the martingale $M_{.\wedge T_K}$. The same property holds true for the sequence $V^n_{.\wedge T_K}$. Thanks to the previous estimates, for all these processes $Y^n_., M^n_., V^n_.$

the convergence is uniform on $[0, T \wedge T_K]$ in probability.
(ii) Let $Z_t^{n,K} \equiv Z_t^n \mathbf{1}_{\{t \leqslant T_K\}}$ in such way that $(Z^n.N)_{.\wedge T_K} = Z^{n,K}.N_.$. Since the sequence $(M^n_{.\wedge T_K})_n$ strongly converges, the sequences of orthogonal martingales $(M^{n,\perp}_{.\wedge T_K})_n$ and $(Z^{n,K}.N_.)_n$ also strongly converge in the appropriate space, and at least in $\mathbb{H}^1$.
Therefore, we can extract a subsequence still denoted $Z^{n,K}_.$ converging a.s.. By assumption, for $t \leqslant T_K$ the sequence $g^n(t, Y_t^n, Z_t^{n,K})$ goes to $g(t, Y_t, Z_t) dK_. \otimes d\mathbb{P}\, a.s.$. It now remains to show that $\mathbb{E}[\int_0^{T_K} |g_n(s, Y_s^n, Z_s^n - g(s, Y_s, Z_s)| dK_s]$ goes to 0.
But $\mathbb{E}[\int_0^{T_K} |g_n(s, Y_s^n, Z_s^n - g(s, Y_s, Z_s)| \mathbf{1}_{\{|Z_s^n| \leqslant C\}} dK_s]$ goes to 0, by dominated convergence, since $\Phi_.$ and $Y^n_.$ are bounded on $[0, T_K]$. Moreover, since the sequence in n of the quadratic variations at time T_K, $\langle Z^{n,K}.N\rangle_{T_K}$ is bounded in $\mathbb{L}^1$, for $s \leqslant T_K$, $|g_n(s, Y_s^n, Z_s^n - g(s, Y_s, Z_s)| \leqslant \Psi_s + \frac{1}{2}|Z_s^n|^2$, with $\Psi_s \mathbf{1}_{\{t \leqslant T_K\}} \in \mathbb{L}^1(d\mathbb{P} \otimes dK_s)$ and $\mathbb{P}(|Z_s^n| \geqslant C) \leqslant \frac{1}{C^2}\mathbb{E}(|Z_s^n|^2)$. Hence, $\mathbb{E}[\int_0^{T_K} |g_n(s, Y_s^n, Z_s^n - g(s, Y_s, Z_s)| \mathbf{1}_{\{|Z_s^n| > C\}} dK_s]$ goes to 0 when C goes to ∞, uniformly in n. As a consequence, the process $V_.$ in the decomposition of the quadratic semimartingale $Y_.$ is given by $dV_t = g(t, Y_t, Z_t) dK_t$ on $[0, T_K]$ for any K. $\square$

Some comments on the BMO point of view Most of the papers in the literature focusing on the study of quadratic BSDEs consider the situation where the martingale M is BMO, as this gives a well-known framework for the existence of a solution for the BSDE in the space of bounded processes (see for instance the recent papers by Ankrichner, Imkeller & Reis [2], Ankirchner, Imkeller & Popier [1] or Morlais [35] and [36]). In our approach, we do not need this BMO framework and have a stability result prevailing in a wider context. This will allow us to obtain some results about the existence of a solution for a quadratic BSDE outside of the standard framework, moving away from the bounded case to the case where the terminal condition has exponential moment.

4 Existence result for quadratic BSDEs

The question of existence of bounded solutions for the classical quadratic BSDEs in Brownian framework has been solved by Kobylanski in [26], using an exponential transformation as to come back to the standard framework of a coefficient with linear growth. Briand & Hu [8] have been the first to

extend the previous results to unbounded solution. A detailed review of the literature including the comparison theorem and different applications may be found in El Karoui, Hamadène & Matoussi [16].
In all these papers, the main difficulty is to prove the strong convergence of the martingale part. The stability result we have obtained in the previous section opens a new possible direction to tackle this question. The idea is to approximate monotonically the coefficient itself by coefficients with a linear and quadratic growth, for which there are some results on the existence of solution but also for which it is possible to take the limit thanks to the stability Theorem 3.6. Having bounded solutions is naturally replaced by belonging to the class $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ as in the previous section, which reduces to an exponential moment condition for $|\eta_T|$, when Λ and $C \equiv 0$.

We adopt the same notation as in Definition 2.2, that we recall here for the sake of clarity of the exposure: the various involved processes are some continuous predictable increasing process $K_.$, two predictable bounded processes $l_.$ and $c_.$ such that $d\Lambda_t = l_t dK_t$ and $dC_t = c_t dK_t$ and a d-dimensional continuous orthogonal martingale $N_.$, for which the diagonal predictable quadratic variation matrix is dominated by $K_.$, with $d\langle N_t^i\rangle = \gamma_t^i dK_t$ and $d\langle N_t^i, N_t^j\rangle = 0$, if $i \neq j$. Finally $g(t, y, z)$ is a predictable process depending on $(y, z) \in \mathbb{R}\times\mathbb{R}^d$ in a continuous way. In this general framework, when g has a linear growth (i.e. $|g(t, y, z)| \leqslant \frac{1}{\delta} l_t + |y| c_t + \frac{\delta}{2} |\sqrt{\gamma}_t z|$), existence results for Lipschitz coefficients may be found in El Karoui & Huang [17] and easily extended following the arguments of Lepeltier & San Martin [29] to the case of a continuous coefficient with linear growth.

4.1 A canonical example: q_δ-BSDE and entropic process

As to illustrate how general quadratic BSDEs and quadratic semimartingales are interrelated, we start by focusing on simplest quadratic BSDEs and insisting on the various possible points of view in this simple framework. More precisely, there are two different situations which coincide in a Brownian framework.
(i) In the first case, the problem is to find a quadratic q_δ-semimartingale $Y_t = Y_0 + M_t - \frac{\delta}{2}\langle M\rangle_t$ with terminal condition $Y_T = \xi_T$. We refer to the first solution as a GBSDE(q_δ, ξ_T)-solution, where G stands for "generalized".
(ii) In the second case, corresponding to the BSDE general framework, the

problem is to find $(Y_., M = Z.N_. + M^{\perp}_.)$ as $dY_t = -\frac{\delta}{2}|Z_t|^2 dK_t - Z_t.dN_t - dM^{\perp}_t$ with terminal condition $Y_T = \xi_T$.
Since δ is simply a scaling factor, we still make the normalizations $\delta = 1$ or $\delta = -1$, and denote $q \equiv q_1$, $\underline{q} \equiv q_{-1}$. Observe that the function q is a convex function, while the function $\underline{q}$ is concave.

Summary of previous results The entropic process $\rho_t(\xi_T)$ defined earlier in Equation (11) as $\ln \mathbb{E}\big[\exp(\xi_T)|\mathcal{F}_t\big] \equiv \rho_t(\xi_T)$ appears naturally when studying such $(q, \underline{q})$-GBSDEs. Indeed, as presented in the following proposition, if the terminal condition $\xi_T \in \mathbb{L}^1_{\exp}$, then $\rho_.(\xi_T)$ is a $(\mathcal{D}_{\exp})$-solution of q-GBSDE. The stronger assumption on the terminal condition $|\xi_T| \in \mathbb{L}^1_{\exp}$ is used for the estimates of the quadratic variation or for some stability result.

Proposition 4.1. (i) *Assume that $\xi_T \in \mathbb{L}^1_{\exp}$. Then the entropic process $\{\rho_t(\xi_T); t \in [0,T]\}$ is the unique $(\mathcal{D}_{\exp})$-solution of the quadratic* GBSDE(q, ξ_T), *i.e. there exists a martingale $M^\rho_. \in \mathcal{U}_{\exp}$ such that*

$$d\rho_t(\xi_T) = -\tfrac{1}{2} d\langle M^\rho\rangle_t + dM^\rho_t, \quad \rho_T(\xi_T) = \xi_T.$$

Moreover, $\{\rho_t(\xi_T); t \in [0,T]\}$ is the minimal solution in the class of solutions: $\rho_t(\xi_T) \leqslant Y_t$.
(ii) *Assume that $-\xi_T \in \mathbb{L}^1_{\exp}$. The negative entropic process $\{\underline{\rho}_t(\xi_T); t \in [0,T]\}$ is* **a** *solution of the quadratic* GBSDE$(\underline{q}, \xi_T)$, *i.e. there exists a martingale $\underline{M}^\rho$ such that*

$$d\underline{\rho}_t(\xi_T) = \tfrac{1}{2} d\langle \underline{M}^\rho\rangle_t + d\underline{M}^\rho_t \quad \underline{\rho}_T(\xi_T) = \xi_T.$$

but in general $\underline{\rho}_.(\xi_T)$ is not a $(\mathcal{D}_{\exp})$-solution.
(iii) *When $|\xi_T| \in \mathbb{L}^1_{\exp}$, then* a) *$\underline{\rho}_t(\xi_T)$ is the maximal solution of the* GBSDE$(\underline{q}, \xi_T)$.
b) *The martingales M^ρ and $\underline{M}^\rho$ are in $\mathbb{H}^2$ and if ξ_T is bounded, they are BMO-martingales.*
c) *If in addition $|\xi_T| + \ln(|\xi_T|) \in \mathbb{L}^1_{\exp}$, the r.v.* $\max|\rho_{0,T}(\xi_T)|$ *and* $\max|\underline{\rho}_{0,T}(\xi_T)|$ *belong to $\mathbb{L}^1_{\exp}$.*

Proof. (i) From Subsection 2.4 and as $\rho_.(\xi_T) = \rho_0(\xi_T) + r_.(M)$, $\rho_.(\xi_T)$ is the unique $(\mathcal{D}_{\exp})$-solution for the GBSDE(q, ξ_T), and the smallest in the class of the q-semimartingale with the same terminal value.
(ii) Since $-\xi_T \in \mathbb{L}^1_{\exp}$, the process $\rho_.(-\xi_T)$ is well-defined in $(\mathcal{D}_{\exp})$ and $-\rho_.(-\xi_T)$ is solution of the $\underline{q}$-GBSDE, but it is not in general in the class $(\mathcal{D}_{\exp})$.
(iii) Assume both variables ξ_T and $-\xi_T$ are in $\mathbb{L}^1_{\exp}$. Using the convexity of ρ,

its follows that $0 = \rho_\cdot(0) \leqslant \frac{1}{2}(\rho_\cdot(\xi_T) - \underline{\rho}_\cdot(\xi_T))$. Then, $\rho_\cdot(\xi_T) \in (\mathcal{D}_{\exp})$ implies $\underline{\rho}_\cdot(\xi_T) \in (\mathcal{D}_{\exp})$.
The comparison with the other solutions is a simple consequence of the fact that $-Y$ is a solution of GBSDE$(q, -\xi_T)$, and therefore bigger than $\rho_\cdot(-\xi_T) = -\underline{\rho}_\cdot(\xi_T)$.
The rest of (iii) is a straightforward consequence of Theorem 3.2. ☐

The question of the existence of solutions of the $(q, \text{or } \underline{q})$-BSDEs is more delicate to tackle and does not admit explicit representation. These difficulties also appear in the Brownian framework when the vector martingale N is defined from a limited number of components of the generating Brownian motion. However, the linearization method motivated below will allow us to overcome these difficulties and to represent solutions as value functionals of some optimization problems.

Variational point of view It is well-known (see for instance, Frittelli [20], Föllmer & Schied [19] or Sircar & Toussaint [41]) that the entropic risk measure $\rho_0(\xi_T)$ admits a variational representation based upon probability measures $\mathbb{Q}$, which are absolutely continuous with respect to $\mathbb{P}$ and have a finite relative entropy $H(\mathbb{Q}/\mathbb{P}) = H^{\text{ent}}(\frac{d\mathbb{Q}}{d\mathbb{P}}) = \mathbb{E}_{\mathbb{P}}[L_{\mathbb{Q}} \ln(L_{\mathbb{Q}})] < \infty$, where $L_{\mathbb{Q}} \equiv \frac{d\mathbb{Q}}{d\mathbb{P}}$. In particular if $\xi_T \in \mathbb{L}^1_{\exp}$, $\rho_0(\xi_T)$ is finite and

$$\rho_0(\xi_T) = \sup_{\mathbb{Q}} \{\mathbb{E}_{\mathbb{Q}}(\xi_T) - H(\mathbb{Q}/\mathbb{P}) |\, H(\mathbb{Q}/\mathbb{P}) < +\infty\}. \tag{22}$$

Moreover, when the random variable ξ_T itself is associated with a finite relative entropy probability measure $\mathbb{Q}^{\xi_T}$ defined by its density $L_T^{\xi_T} = e^{(\xi_T - \rho_0(\xi_T))}$, we can prove by a simple verification that the supremum is attained for $\mathbb{Q}^{\xi_T}$. In the case when $\mathbb{Q}^{\xi_T}$ does not have a finite relative entropy, but ξ_T is *bounded from below*, we can approximate $\rho_0(\xi_T)$ by the increasing sequence $\rho_0(\xi_T \wedge n)$, and prove that the supremum in the optimization problem (22) may be restricted to the family of *equivalent* probability measures.
Comment: Such an assumption appears in different papers dealing with the question of pricing in incomplete markets and entropy (see for instance Delbaen, Grandits, Rheinlander, Samperi, Schweizer & Stricker [10] or Frittelli [20]). Note that our natural assumption that $\rho_0(|\xi_T|) < +\infty$ does not appear in this literature. It implies that the random variable $|\xi_T|$ belongs to $\mathbb{L}^1(\mathbb{Q})$ for any $\mathbb{Q}$ with finite entropy and that $\mathbb{E}_{\mathbb{Q}}\big(|\xi_T|\big)$ is bounded in the family of probability measures $\mathbb{Q}$ such that $H(\mathbb{Q}/\mathbb{P})$ is uniformly bounded".

When the probability measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$, we can use an exponential representation of the likelihood $\frac{d\mathbb{Q}}{d\mathbb{P}} \equiv L_T^{\mathbb{Q}} = \mathcal{E}_T(M^{\mathbb{Q}})$. Thanks to Proposition 2.11, this property is equivalent to the martingale $\mathcal{E}_{.}(M^{\mathbb{Q}})$ belongs to $\mathbb{H}^1(\mathbb{P})$, and it implies that $H(\mathbb{Q}/\mathbb{P}) = \mathbb{E}_{\mathbb{Q}}\big(\frac{1}{2}\langle M\rangle_T\big)$. In particular when $\rho_0(\xi_T) < \infty$ and $(\xi_T)^-$ is bounded, or $\xi_T + \ln|\xi_T| \in \mathbb{L}^1_{\exp}$, we can write

$$\rho_0(\xi_T) = \sup_{M^{\mathbb{Q}}}\{\mathbb{E}_{\mathbb{Q}}(\xi_T) - \frac{1}{2}\langle M^{\mathbb{Q}}\rangle_T |\, \mathbb{Q} = \mathcal{E}_T(M^{\mathbb{Q}}).\mathbb{P},\ \mathbb{E}_{\mathbb{Q}}(\langle M\rangle_T^{\mathbb{Q}}) < +\infty\}. \quad (23)$$

This representation admits a dynamic version, useful in optimisation, if $\rho_0(\xi_T) < \infty$ and $(\xi_T)^-$ bounded,

$$\rho_t(\xi_T) = \operatorname{ess\,sup}_{M^{\mathbb{Q}}}\{\mathbb{E}_{\mathbb{Q}}\big(\xi_T - \frac{1}{2}\langle M^{\mathbb{Q}}\rangle_{t,T}|\mathcal{F}_t|\mathbb{E}_{\mathbb{Q}}(\langle M\rangle_{t,T}^{\mathbb{Q}}) < +\infty\big)\}. \quad (24)$$

In terms of q-BSDEs, this leads to the following dual representation presented in Proposition 4.2, and its approximations based on the solutions of convex BSDEs with linear growth (see for instance El Karoui, Hamadène & Matoussi [16]).
Note that for the sake of generality in the presentation of the results, we adopt the general BSDE framework with $c_. \equiv 0, l_. \equiv 0$, with a continuous orthogonal martingale N and characterize a martingale M by the parameters of its orthogonal decomposition $M_. = Z.N_. + M_.^{\perp} = (Z_., M_.^{\perp})$. When working with an exponential martingale, we write $L_. = \mathcal{E}_.(M) = \mathcal{E}_.(Z.N)\mathcal{E}_.(M^{\perp}) \equiv L_.^Z\, L_.^{\perp}$. The symbol $\mathbb{Q}^{Z,\perp}$ refers to the probability measure with density $L_T^Z\, L_T^{\perp} = L_T^{Z,\perp}$ with respect to $\mathbb{P}$. When the orthogonal martingale part is 0, we simply write $\mathbb{Q}^Z$.

Proposition 4.2. *Let* $q(z) \equiv \frac{1}{2}|z|^2 = \sup_{\nu}\{\nu.z - \frac{1}{2}|\nu|^2\}$ *be the canonical quadratic coefficient, and* $l_\nu(z) = \nu.z - \frac{1}{2}|\nu|^2$ *the family of affine coefficients associated with it.*
(i) *Let* $q_n(z) = \frac{1}{2}|z|^2\mathbf{1}_{\{|z|\leqslant n\}} + (n|z| - \frac{1}{2}n^2)\mathbf{1}_{\{|z|>n\}})$. *The sequence* $q_n(z)$ *is an increasing sequence of positive convex coefficients with both linear growth and quadratic growth, whose the limit is* $q(z)$. *Moreover, the dual representation* $q_n(z) = \sup_\nu\{l_\nu(z)\,|\,|\nu| \leqslant n\}$ *holds true.*
(ii) *Assume that* $|\xi_T| \in \mathbb{L}^1_{\exp}$. *Then, the BSDE* (q_n, ξ_T) *admits a unique solution* $(Y_.^n, Z_.^n, M^{n,p}) \in \mathbb{H}^2(\mathbb{R}^+) \otimes \mathbb{H}^2(\mathbb{R}^n)$ *such that for* $t \leqslant T$,

$$Y_t^n = \operatorname{ess\,sup}_{|\nu_.|\leqslant n}\{\mathbb{E}_{\mathbb{Q}^\nu}\big(\xi_T - \frac{1}{2}\langle \nu_..N\rangle_{t,T}|\mathcal{F}_t\big)\}. \quad (25)$$

The processes $|Y^n_.|$ are dominated by $\rho_.(|\xi_T|)$; the sequence $Y^n_.$ is increasing and converges almost surely to the minimal solution dominated by $\rho_.(|\xi_T|)$ of the BSDE(q,ξ_T), $dY_t = -\frac{\delta}{2}|Z_t|^2 dK_t - Z_t.dN_t - dM_t^\perp$ with terminal condition $Y_T = \xi_T$.

Proof. (i) is directly obtained from standard calculation on quadratic functions and their Fenchel transforms.
(ii) We proceed by verification as to prove that $\mathbb{E}_{\mathbb{Q}^\nu}\big(\xi_T - \frac{1}{2}\langle \nu_..N\rangle_{t,T}|\mathcal{F}_t\big)$ is solution of the BSDE(l_ν,ξ_T). Using a comparison and stability theorem applied to monotone coefficients with linear growth, and more generally the standard results on convex BSDEs with uniformly linear growth (see for instance Theorem 8.7 in El Karoui, Hamadène & Matoussi [16]), we obtain representation of the unique solution of the BSDE(q_n,ξ_T) as given by Equation 4.4. Since we only take the supremum over the set of probability measures $\mathbb{Q}^\nu$, $|\nu_.| \leqslant n$, Y^n is dominated by $\rho_.(\xi_T)$ and bounded from below by $\mathbb{E}(\xi_T) \geqslant -\rho_.(-\xi_T)$. The hypothesis of the stability Theorem 3.6 are satisfied, and the sequence Y^n converges to a solution of the BSDE(q,ξ_T). This solution is minimal, because any other solution dominated in norm by $\rho_.(|\xi_T|)$ is bigger than Y^n. □

4.2 Existence result for BSDEs in the class $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$

We are now interested in quadratic BSDEs satisfying the general structure condition $|\,g(.,t,y,z)| \leqslant Q(t,y,z) \equiv |l_t| + c_t|y| + \frac{1}{2}|z|^2$, $\quad d\mathbb{P}\otimes dt-$a.s., and are looking for solution in $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$ only. As before, the method relies on a regularization of the quadratic coefficient it-self through inf-convolution as to transform it into a coefficient with *both* linear and quadratic growth. This double structure of the transformed coefficient leads to results both in terms of existence and estimation. The previous stability Theorem 3.5 can then be applied to obtain the existence of a solution, after having proved that the approximate solutions are also $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$-semimartingales.

The proof of this fundamental result is based on the following lemma involving classical regularization by inf-convolution techniques introduced by Lepeltier & San Martin [28] in a BSDEs framework. The functions $q_n(z)$ introduced in Proposition 4.2 are an example of regularization by inf-convolution of the canonical function $q(z) = \frac{1}{2}|z|^2$. Let us first observe that the appropriate regularization when dealing with $\underline{q}(z) = -\frac{1}{2}|z|^2$ is a sup-convolution since $\underline{q}(z)$ is concave. To overcome this difficulty, we proceed in two steps, by first assuming that g is bounded from below by some basic function with both a linear and quadratic growth $d_p(t,y,z) = -(l_t + c_t|y| + q_p(z))$ where $q_p(z) = \frac{1}{2}|z|^2\mathbf{1}_{\{|z|\leqslant p\}} + (p|z| - \frac{1}{2}p^2)\mathbf{1}_{\{|z|>p\}})$. Recall that the processes c and l can be still assumed to be bounded by some universal constant $\bar{C}$.

Lemma 4.3. *Let $g : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be a continuous function with linear growth in y, and quadratic growth in z, positive, and bounded by below by some d_p function;*

$$-(l_t + c_t|y| + q_p(z)) \equiv d_p(t,y,z) \leqslant g(t,y,z) \leqslant Q(t,y,z) = l_t + c_t|y| + \frac{1}{2}|z|^2. \tag{26}$$

Recall that the processes c and l are bounded by some universal constant $\bar{C}$. The regularizing functions are the convex functions with linear growth $b_n(u,w) = n|u| + n|w|$. The sequences $d_{n,p}(t,y,z) = d_p(t,y,z)\,\square\, b_n(t,y,z)$, $Q_n(t,y,z) = Q\,\square\, b_n(t,y,z)$ and $g_n(t,y,z) = \inf_{u,w}\big(g(t,u,w) + n|y-u| + n|z-w|\big) = g\,\square\, b_n(t,y,z)$ defined respectively as the inf-convolution of the functions d_p, Q and g with the function b_n, have the following properties, for $n \geqslant \sup(\bar{C},p)$:
(i) *$0 \leqslant Q_n(t,y,z) = l_t + c_t|y| + q_n(z) \leqslant l_t + c_t|y| + \frac{1}{2}|z|^2$ and $d_{n,p}(t,y,z) = d_p(t,y,z)$;*
(ii) *$|g_n(t,y,z)| \leqslant l_t + c_t|y| + \sup(q_p(z), q_n(z)) = Q_n(t,y,z) \leqslant l_t + c_t|y| + \frac{1}{2}|z|^2$;*
(iii) *the sequences g_n and Q_n are increasing;*
(iv) *the functions g_n are Lipschitz continuous in (y,z) with Lipschitz constant n;*
(v) *if $(y_n,z_n) \to (y,z)$, then $g_n(t,y_n,z_n) \to g(t,y,z)$.*

The important point is to prove that the solutions to the BSDEs associated with the coefficients g_n, which are Lipschitz with linear growth, are in the class $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$. The argument is a slight extension of the one used in the proof of Proposition 4.2.

Lemma 4.4. *Let $|\eta_T|$ be a $\mathcal{F}_T$-random variable such that $\mathbb{E}[\exp(e^{C_{0,T}}|\eta_T| + \int_0^T e^{C_{t,s}}d\Lambda_s)] < +\infty$. Let g and g_n, Q and Q_n as in Lemma 4.3. The coefficients g_n and Q_n are standard uniformly Lipschitz coefficients. For any $|\xi_T| \leqslant |\eta_T|$, let $(Y^n_., Z^n_., M^{n,\perp}_.)$ and $(U^n_., V^n_., W^{n,\perp}_.)$ be the unique solution of the BSDE(g^n,ξ_T) and BSDE$(Q^n,|\eta_T|)$ in the appropriate space.*
(i) *The sequences $(Y^n_.)$ and $(U^n_.)$ are increasing, and $|Y^n_.| \leqslant U^n_.$, a.s.*
(ii) *For any stopping time $\sigma \leqslant T$*

$$U^n_\sigma \equiv \operatorname*{ess\,sup}_{|\nu_t|\leqslant n}\Big\{\mathbb{E}_{\mathbb{Q}}\big[e^{C_{\sigma,T}}|\eta_T| + \int_\sigma^T e^{C_{\sigma,t}}d\Lambda_t - \frac{1}{2}\int_\sigma^T e^{C_{\sigma,t}}d\langle\nu.N\rangle_t|\mathcal{F}_\sigma\big];\ \frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(\nu * N)_T\Big\}.$$

The random variable U^n_σ is dominated by $\rho_\sigma(e^{C_{\sigma,T}}|\eta_T|+\int_\sigma^T e^{C_{\sigma,t}}d\Lambda_t)$, and both sequences $(Y^n_.)$ and $(U^n_.)$ are $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$-quadratic semimartingales.
(iii) *The sequence $(Y^n_., Z^n_., M^{n,\perp}_.)$ converges uniformly in probability to a minimal solution $(Y_., Z_., M^\perp_.)$ of the BSDE(g,ξ_T).*

Proof. The proof relies on classical properties of BSDEs solutions associated with standard coefficients, in a $\mathbb{H}^2$-space. In particular, existence, uniqueness and comparison ho ld true in this case, that implies (i) .
(ii) The dual representation of U^n, solution of a BSDE with convex coefficient Q^n, is a slight generalization of the dual representation of the BSDE$(q_n,|\eta_T|)$ solution in Proposition 4.2, based on Theorem 8.7 in El Karoui, Hamadène & Matoussi [16]. This dual representation implies first that $U^n_\sigma \leqslant \rho_\sigma\big(e^{C^n_{\sigma,T}}|\eta_T|+\int_\sigma^T e^{C^n_{\sigma,t}}d\Lambda^n_t\big)$, and then that we can drop out the index n in the right side of the inequality, and even replace the bounded processes C and Λ by the processes $\overline{\Lambda}$, and $\overline{C}$. Then, by construction, $(Y^n_.)$ and $(U^n_.)$ are $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$-quadratic semimartingales.
(iii) Finally, using the stability Theorem 3.5, we obtain the convergence of this sequence to a solution of the BSDE(g,ξ_T) in the space $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$. □

It remains to overcome the assumption made on the coefficient of a linear quadratic growth lower bound. Given a coefficient g with decomposition $g = g^+ - g^-$, where both functions g^+ and g^- have the same quadratic structure. Let $g_p \equiv g^+ - g^- \square b_p$. Then g_p satisfies the condition (26), and the BSDE(g_p,ξ_T) admits a minimal solution; the sequence of solutions Y^p is decreasing, and belongs to the space $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$. Once again, we use the stability theorem to conclude that the sequence Y^p converges to a solution of the BSDE(g,ξ_T). We summarize the general form of our results in the following theorem.

Theorem 4.5. *Let $|\eta_T|$ be a $\mathcal{F}_T$-random variable such that $\mathbb{E}[\exp(e^{C_{0,T}}\delta|\eta_T|+\int_0^T e^{C_{t,s}}d\Lambda_s)] < +\infty$, and let $g(t,y,z)$ be a quadratic coefficient satisfying the structure condition (2), $|\,g(.,t,y,z)| \leqslant \frac{1}{\delta}|l_t| + c_t|y| + \frac{\delta}{2}|z|^2$.*
For any ξ_T, $|\xi_T| \leqslant |\eta_T|$, there exists at least a solution $(Y,Z,M^\perp)$ in $\mathcal{S}_Q(|\eta_T|,\Lambda,C,\delta)$ of the BSDE(g,ξ_T).

Remark 3. *When both $\Lambda, C \equiv 0$, the theorem becomes: if $|\,g(.,t,y,z)| \leqslant \frac{\delta}{2}|z|^2$, and $\mathbb{E}[\exp|\xi_T|] < +\infty$, their exists at least a solution in the class $\mathcal{D}_{\exp}$.*
In their recent paper [4], Bao, Delbaen & Hu have shown that when the coefficient g is convex, we can only assume that $\xi_T \in L^1_{exp}$ and $g(t,y,z) \leqslant l + \frac{1}{2}|z|^2$, i.e. the quadratic inequality holds only from above.

Comment on the uniqueness of the solution The question of the uniqueness of the solution to a general quadratic BSDE is more challenging. In the standard framework where the terminal condition is bounded, Kobylanski [26] obtains the uniqueness of the solution under some Lipschitz style assumptions. Recently, Tevzadze [42] gives a direct proof of uniqueness still in the bounded case. In the case of an unbounded terminal condition, Briand & Hu [9] work under the additional assumption that the coefficient g is convex with respect to the variable z. This allows them to derive the comparison theorem, which is needed to obtain the uniqueness. Their methodology can be adapted and generalized to our framework without any particular difficulty. In a very recent paper [34], Mocha & Westray have considered general quadratic BSDEs under some stronger assumptions of exponential moment of order $p > 1$ and boundedness of the increasing processes. They obtain some interesting results for the uniqueness of the solution.

Acknowledgements

Both authors would like to thank Mingyu Xu and Anis Matoussi for their helpful comments and discussions at various stages of the paper.

References

[1] Ankirchner, S., Imkeller, P. and Popier, A. (2009). On measure solutions of Backward Stochastic Differential Equations. *Stochastic Processes and their Applications* **119** 2744–2772.

[2] Ankirchner, S., Imkeller, P. and Reis, G. (2007). Classical and variational differentiability of BSDEs with quadratic growth. *Electronic Journal of Probability* **12** 1418–1453.

[3] Azéma, J., Gundy, R.F. and Yor, M. (1980). *Sur l'intégrabilité uniforme des martingales continues.* Séminaire de Probabilités (Strasbourg) XIV 1978/79, LNM 784, Springer-Verlag, 53–61.

[4] Bao,X., Delbaen,F. and Hu, Y. (2010). *Existence and non-uniqueness of solutions for BSDEs.* In: Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen (eds: C. Chiarella, A. Novikov) Springer-Verlag, 123–134.

[5] Barlow, M. and Protter, P. (1989). *On convergence of semimartingales.* Technical Report 89-07, Department of Statistics, Purdue University.

[6] Barrieu, P. and El Karoui, N. (2004). *Optimal Derivatives Design under Dynamic Risk Measures.* In: Mathematics of Finance (eds: G. Yin, Q. Zhang), Contemporary Mathematics - A.M.S. Proceedings, 13–26.

[7] Barrieu, P. and El Karoui, N. (2009). *Pricing, hedging and optimally designing derivatives via minimization of risk measures.* In: Volume on Indifference Pricing (ed: René Carmona) Princeton University Press, 77–146.

[8] Briand,P. and Hu,Y. (2006). BSDE with quadratic growth and unbounded terminal value. *Prob. Theory & Related Fields* **136** 604–618.

[9] Briand,P. and Hu,Y. (2008). Quadratic BSDEs with convex generators and unbounded terminal conditions. *Prob. Theory & Related Fields* **141** 543–567.

[10] Delbaen, F., Grandits, P., Rheinlander, T., Samperi, D., Schweizer, M. and Stricker, C. (2002).: Exponential hedging and entropic penalties. *Mathematical Finance* **12** 99–123.

[11] Dellacherie, C. (1979). *Inégalités de convexité pour les processus croissants et les sousmartingales.* Séminaire de probabilités (Strasbourg) XIII 1977/78, LNM 721, Springer-Verlag 371–377.

[12] Dellacherie, C. and Meyer, P.A. (1975). *Probabilités et Potentiel.* Hermann, Paris.

[13] Delong, L. and Imkeller, P. (2010). *Backward Stochastic Differential Equations with time delayed generators - results and counterexamples.* Working Paper arXiv:1005.4701.

[14] Doléans-Dade, C. and Meyer, P.A. (1979). *Inégalités de normes avec poids.* Séminaire de Probabilités (Strasbourg) XIII 1977/78, LNM 721 Springer-Verlag 313–331.

[15] El Karoui and N., Hamadène, S. (2003). BSDE and risk-sensitive control, zero-sum and non-zero-sum game problems of stochastic functional differential equations. *Stochastic Processes and their Applications* **107** 145–169.

[16] El Karoui, N., Hamadène, S. and Matoussi, A. (2009). *BSDEs and Applications*. In: Volume on Indifference Pricing (ed: René Carmona) Princeton University Press, 267–320.

[17] El Karoui, N. and Huang, S.J. (1997). *A general result of existence and uniqueness of Backward Stochastic Differential Equations*. In: Backward Stochastic Differential Equations (eds: Nicole El Karoui and Laurent Mazliak0, Pitman Research Notes in Mathematics Series 364, Longman 27–36.

[18] El Karoui, N., Peng, S. and Quenez, M.C. (1997). Backward Stochastic Differential Equations in finance. *Mathematical Finance* **7** 1–71.

[19] Föllmer, H. and Schied, A. (2004). *Stochastic Finance: An Introduction in discrete Time (revised edition)*, De Gruyter Studies in Mathematics.

[20] Frittelli, M. (2000). The minimal Entropy Martingale Measure and the Valuation Problem in Incomplete Markets. *Mathematical Finance* **10** 39–52.

[21] Grandits, P. and Rheinlander, T. (2002). On the minimal entropy martingale measure. *Annals of Probability* **30** 1003–1038.

[22] Harremoës, P. (2008). Some New Maximal Inequalities. *Statistics and Probability Letters* **78** 2776–2780.

[23] Hu, Y., Imkeller, P. and Muller, M. (2005). Utility maximization in incomplete markets. *Annals of Applied Probability* **15** 1691–1712.

[24] Hu, Y. and Schweizer, M. (2009). *Some new results for an infinite-horizon stochastic control problem.* WNCCR FINRISK working paper No. 587, National Centre in Competence in Research Financial Valuation and Risk Management, $<$ www.nccr-finrisk.uzh.ch/media/pdf/wp/$WP587D1$.pdf $>$.

[25] Kazamaki, N. (1994). *Continuous Exponential Martingales and BMO*, Lecture Notes 1579, Springer-Verlag.

[26] Kobylanski, M. (2000).: Backward Stochastic Differential Equations and Partial Differential Equations with Quadratic Growth. *Annals of Probability* **28** 558–602.

[27] Lenglart, E., Lépingle, D. and Pratelli, M. (1980). *Présentation unifiée de certaines inégalités de la théorie de martingales*. Séminaire de Probabilités (Strasbourg) XIV 1978/79, LNM 784, Springer Verlag 26–48.

[28] Lepeltier, J.P. and San Martin, J. (1997). Backward Stochastic Differential Equations with Continuous Coefficient. *Statistics and Probability Letters* **32** 425–430.

[29] Lepeltier, J.P. and San Martin, J. (1998). Existence of BSDE with Superlinear Quadratic Coefficient. *Stochastics and Stochastic Report* **63** 227–240.

[30] Mania, M., Santacroce, M. and Tevzadze, R. (2003). A Semimartingale BSDE Related to the Minimal Entropy Martingale Measure. *Finance and Stochastics* **7** 385–402.

[31] Mania, M. and Santacroce M. (2010). Exponential utility maximization under partial information. *Finance and Stochastics* **14** 419–448.

[32] Mania, M. and Schweizer, M. (2005). Dynamic exponential utility indifference valuation. *Annals of Applied Probability* **15** 2113–2143.

[33] Meyer, P.A. (1976). *Un cours sur les intégrales stochastiques*. Séminaire de Probabilité X, LNM 511, Springer-Verlag, 245–400.

[34] Mocha, M. and Westray, N. (2011). *Quadratic semimartingale BSDEs under and exponential moments condition*, Working Paper arXiv:1101.2582v1.

[35] Morlais, M.A. (2009). Quadratic BSDEs driven by a continuous martingale and application to Utility Maximization. *Finance and Stochastics* **13** 121–150.

[36] Morlais, M.A. (2011). A new existence result for BSDEs with jumps and application to the utility maximization problem. To appear in *Stochastic Processes and their Applications*.

[37] Neveu, J. (1972). *Martingales à Temps Discret*. Masson, Paris.

[38] Pardoux, E. and Peng, S. (1990). Adapted Solution of a Backward Stochastic Differential Equation. *Systems and Control Letters* **14** 55–61.

[39] Protter, P. (2005). *Stochastic Integration and Differential Equations, Second Edition*. Springer Verlag.

[40] Rouge, R. and El Karoui, N. (2000). Pricing via utility maximization and entropy. *Mathematical Finance* **10** 259–276.

[41] Sircar, R. and Toussaint, A. (2009). *A Framework for Dynamic Hedging under Convex Risk Measures*. In: Proceedings of 2008 Ascona Seminar on Stochastic Analysis, Random Fields and Applications (eds. R. Dalang, M. Dozzi, F. Russo), Birkhauser.

[42] Tevzadze, R. (2008). Solvability of Backward Stochastic Differential Equations with Quadratic Growth. *Stochastic processes and their Applications* **118** 503–515.