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From Hermite polynomials to multifractional processes

Renaud Marty

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Abstract

We establish an invariance principle where the limit process is a Hermite-type process. We also prove that this limit process is multifractional. Our main result is a generalization of results from [6] and [11] to a multifractional setting. It also generalizes the main result of [3] to a non-Gaussian framework.

1 Introduction

Hermite processes have attracted a lot of attention for many years because they have nice properties as they generalize fractional Brownian motion [6, 11]. Let \( m \in \mathbb{N}^* \) and \( H \in (1/2, 1) \). The Hermite process \( W_{m,H} \) of order \( m \) and Hurst index \( H \) can be defined for instance in terms of Dobrushin-Wiener-Itô integrals [5] as, for every \( t \in [0, \infty) \),

\[
W_{m,H}(t) = \int_{\mathbb{R}^m} f_{m,H}(x_1, \ldots, x_m, t)d\hat{B}_{x_1} \cdots d\hat{B}_{x_m}
\]  

with

\[
f_{m,H}(x_1, \ldots, x_m, t) = C(m, H) \frac{\exp(it(x_1 + \cdots + x_m)) - 1}{i(x_1 + \cdots + x_m)|x_1 \cdots x_m|^{(2H-2+m)/2m}}
\]

where \( C(m, H) \) is a normalizing constant and \( d\hat{B} \) is the complex random measure corresponding to a standard Brownian motion \( B \). Notice that for \( m = 1 \), the Hermite process \( W_{1,H} \) is the fractional Brownian motion with Hurst index \( H \).

An important property of Hermite processes is the invariance principle [4, 6, 10, 11], which can be stated as follows. Let \( X = \{X_j\}_{j \in \mathbb{N}} \) be a Gaussian stationary sequence of centered random variables with \( \mathbb{E}[X_0^2] = 1 \) and satisfying the property

\[
\mathbb{E}[X_0X_j] \sim cj^{2(H-1)/m} \quad \text{as } j \to \infty
\]

where \( c \) is a positive real number. Notice that (2) is a long range property. We consider a function \( \phi \in L^2(e^{-x^2/2}dx) \) with Hermite rank equal to \( m \), and define the partial sum \( S_{\phi,H}^N(t) \) for every \( N \in \mathbb{N} \) and \( t \in [0, \infty) \) as

\[
S_{\phi,H}^N(t) = \frac{1}{N^m} \sum_{j=1}^{[Nt]} \phi(X_j).
\]

The invariance principle establishes that the finite-dimensional distributions of \( S_{\phi,H}^N \) converge, as \( N \) goes to infinity, to the Hermite process \( W_{m,H} \) with a suitable constant \( C(m, H) \).

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*Institut Elie Cartan de Nancy, Nancy-Université, CNRS, INRIA, B.P. 239, F-54506 Vandoeuvre-lès-Nancy Cedex, France. email : renaud.marty@iecn.u-nancy.fr
As fractional Brownian motion, Hermite processes of index $H$ are $H-$self-similar and $H'$–Hölder-continuous if and only if $H' < H$. As other fractional processes, a drawback of Hermite processes lies in the strong homogeneity of their properties, which are governed by the Hurst index $H$. In order to generalize fractional processes to less homogeneous processes, multifractional processes have been introduced, as for instance the class of multifractional Brownian motions [2, 8]. Multifractional processes have locally, but not globally, the same properties as fractional processes. These properties are governed by a function $h$ that substitute for the constant $H$.

As for fractional Brownian motion and other Hermite processes, some nontrivial multifractional Gaussian processes satisfy invariance principle. Indeed, it is proven in [3] the following result. Let a Gaussian field $\{X_j(H)\}_{j,h\in\mathbb{R}\times(1/2,1)}$ satisfying some long-range assumptions and a continuous function $h$ taking its values in $(1/2,1)$. Then, the finite-dimensional distributions of the process

$$t \to S_h^n(t) = \sum_{j=1}^{[Nt]} \frac{X_j(h(j/N))}{N^{1/2}} \tag{4}$$

converge to those of a centered Gaussian process $S_h$ with covariance given for $t, s \geq 0$ by:

$$\mathbb{E}[S_h(t)S_h(s)] = \int_0^t d\theta \int_0^s d\sigma R(h(\theta), h(\sigma))|\theta - \sigma|^{h(\theta)+h(\sigma)-2} \tag{5}$$

where $R$ is a continuous function and derived from long-range assumptions of the field $\{X_j(H)\}_{j,h\in\mathbb{R}\times(1/2,1)}$. The process $S_h$ is multifractional. If the function $h$ is constant, then the process $S_h$ is a fractional Brownian motion. The result above is then a generalization of classical invariance principle [9].

In this work, we generalize invariance principles presented above. We study the asymptotic behavior of a sequence generalizing both (3) and (4). In particular, this sequence is defined from a Gaussian field $\{X_j(H)\}_{j,h\in\mathbb{R}\times(1/2,1)}$ satisfying long-range properties, a function $\phi \in L^2(e^{-x^2/2}dx)$ with Hermite rank equal to $m$ and a Hurst function $h$. We get as a limit a multifractional process $S_{m,h}$ that depends on the integer $m$ and the function $h$. If the function is a constant $H$, then the limit process is the Hermite process with Hurst index $H$ and Hermite order $m$. If the integer $m$ is equal to 1, then the limit process corresponds to a Gaussian multifractional process of the class obtained in [3].

The paper is organized as follows. In Section 2 we recall some definitions and preliminary results about Hermite polynomials and multiple stochastic integrals, which are used throughout the paper. In Section 3 we establish the main result of the paper. Section 4 is devoted to the proof of the main result.

2 Preliminaries

In this section we give some definitions and recall some results we use throughout this paper.

For each positive integer $m \in \mathbb{N}$, the $m$th Hermite polynomial $P_m$ of is defined as, for every $x \in \mathbb{R}$,

$$P_m(x) = (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2}.$$  

The family of the Hermite polynomials $\{P_m, m \in \mathbb{N}\}$ is an orthogonal basis of the space $L^2(e^{-x^2/2}dx)$ defined by

$$L^2(e^{-x^2/2}dx) = \left\{ \phi : \mathbb{R} \to \mathbb{C}, \phi \text{ measurable and } \int_{\mathbb{R}} |\phi(x)|^2 e^{-x^2/2}dx < \infty \right\}$$

with the inner product $\langle \cdot, \cdot \rangle$ defined as, for every $\phi_1$ and $\phi_2$ in $L^2(e^{-x^2/2}dx)$,

$$\langle \phi_1, \phi_2 \rangle = \int_{\mathbb{R}} \phi_1(x) \overline{\phi_2(x)} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

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and whose the corresponding norm will be denoted as $\| \cdot \|$. For every function $\phi \in L^2(e^{-x^2/2}dx)$, there exists an integer $m_\phi$ such that $\langle \phi, P_{m_\phi} \rangle \neq 0$ and $\langle \phi, P_m \rangle = 0$ for every $m = 0, \ldots, m_\phi - 1$. The integer $m_\phi$ is called the Hermite index of the function $\phi$. Hence, for every $\phi \in L^2(e^{-x^2/2}dx)$,

$$\phi = \sum_{m=0}^{\infty} \frac{\langle \phi, P_m \rangle}{m!} P_m = \sum_{m=m_\phi}^{\infty} \frac{\langle \phi, P_m \rangle}{m!} P_m$$  \hspace{1cm} (6)

where the convergence of the series holds for the norm $\| \cdot \|$. If $X$ and $Y$ are two Gaussian random variables $\mathcal{N}(0, 1)$, then, for every $j$ and $k$ in $\mathbb{N}^*$,

$$E[P_j(X)P_k(Y)] = \begin{cases} k!E[XY]^j & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$  \hspace{1cm} (7)

As a consequence, for every $\phi \in L^2(e^{-x^2/2}dx)$ and $X \sim \mathcal{N}(0, 1)$,

$$E[|\phi(X)|^2] = \sum_{m=0}^{\infty} \frac{\langle \phi, P_m \rangle^2}{m!} < \infty$$  \hspace{1cm} (8)

Other objects we strongly use in this paper are multiple Wiener-Itô integrals [5, 7]. Many notions of multiple Wiener-Itô integrals with respect to Brownian motion exist and are used to define processes as Hermite processes [6, 11]. Here we have chosen to use the so-called Dobrushin-Wiener-Itô integrals introduced in [5]. Let $d \in \mathbb{N}^*$, $f : \mathbb{R}^d \to \mathbb{C}$ be a square-integrable function, and $B = \{B_t\}_{t \in \mathbb{R}}$ be a standard Brownian motion in $\mathbb{R}$. In this paper, the Dobrushin-Wiener-Itô integral of $f$ is denoted $\int_{\mathbb{R}^d} f \, d\hat{B}^{\otimes d}$ or

$$\int_{\mathbb{R}^d} f(x_1, \ldots, x_d) d\hat{B}_{x_1} \cdots d\hat{B}_{x_d}.$$

It is well-defined if $f$ is even and symmetric, that is, if $f$ satisfies, for every $(x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$f(x_1, \ldots, x_d) = f(-x_1, \ldots, -x_d),$$

and for every permutation $\varsigma$ on $\{1, \ldots, d\}$,

$$f(x_1, \ldots, x_d) = f(x_{\varsigma(1)}, \ldots, x_{\varsigma(d)}).$$

We refer the reader to [5] for the precise definition of $\int_{\mathbb{R}^d} f \, d\hat{B}^{\otimes d}$. Here we only recall some properties that we use in the proof of the main result. The integral $\int_{\mathbb{R}^d} f \, d\hat{B}^{\otimes d}$ is Gaussian if and only if $d = 1$. In any case, it is a centered random variable and we can express its variance as

$$\mathbb{E} \left[ \left( \int_{\mathbb{R}^d} f \, d\hat{B}^{\otimes d} \right)^2 \right] = d! \int_{\mathbb{R}^d} |f(x_1, \ldots, x_d)|^2dx_1 \cdots dx_d.$$

We have a substitution formula for multiple integrals, using the self-similarity of the Brownian motion. For every $a > 0$, we have the equality in distribution

$$\int_{\mathbb{R}^d} f(x_1, \ldots, x_d) d\hat{B}_{x_1} \cdots d\hat{B}_{x_d} \overset{\text{dist}}{=} a^{d/2} \int_{\mathbb{R}^d} f(ax_1, \ldots, ax_d) d\hat{B}_{x_1} \cdots d\hat{B}_{x_d}.$$  \hspace{1cm} (9)

Another formula for change of variables is applied in this paper and is a consequence of Proposition 4.2 of [5]. Let $z : \mathbb{R} \to \mathbb{C}$ be a bounded and measurable function satisfying $z(x) = z(-x)$ and $|z(x)| = 1$ for every $x \in \mathbb{R}$. Then, we have the equality in distribution

$$\int_{\mathbb{R}^d} f(x_1, \ldots, x_d) d\hat{B}_{x_1} \cdots d\hat{B}_{x_d} \overset{\text{dist}}{=} \int_{\mathbb{R}^d} f(x_1, \ldots, x_d) z(x_1) \cdots z(x_d) d\hat{B}_{x_1} \cdots d\hat{B}_{x_d}.$$  \hspace{1cm} (10)

By linearity of the integral and the bounded convergence theorem, we can prove the following convergence lemma.
Lemma 1. Let \( \{f_N\}_N \) be a sequence of even and symmetric functions in \( L^2(\mathbb{R}^d, \mathbb{C}) \). We assume that there exist two even and symmetric functions \( f \) and \( f^* \) in \( L^2(\mathbb{R}^d, \mathbb{C}) \) such that, for a.e. \( x \in \mathbb{R}^d \), \( \lim_{N \to \infty} f_N(x) = f(x) \) and \( \sup_N |f_N(x)| \leq f^*(x) \). Then,

\[
\lim_{N \to \infty} \mathbb{E} \left[ \left| \int_{\mathbb{R}^d} f_N \, d\hat{B}^\otimes d - \int_{\mathbb{R}^d} f \, d\hat{B}^\otimes d \right|^2 \right] = 0.
\]

To conclude this section, we mention a result that relate Hermite polynomials to multiple integrals. Let \( \psi \) be an even and symmetric function in \( L^2(\mathbb{R}, \mathbb{C}) \), \( m \) be a positive integer and \( P_m \) be the \( m \)th Hermite polynomial defined as previously. The random integral \( \int_{\mathbb{R}} \psi(\xi)d\hat{B}_\xi \) defines a centered Gaussian variable. If \( \mathbb{E} \left[ \int_{\mathbb{R}} |\psi(\xi)|^2 d\xi \right] = 1 \) then we have almost surely

\[
P_m \left( \int_{\mathbb{R}} \psi(x) d\hat{B}_x \right) = \int_{\mathbb{R}^m} \psi(x_1) \cdots \psi(x_m) d\hat{B}_{x_1} \cdots d\hat{B}_{x_m}.
\]

3 Main result

We let \( m \in \mathbb{N}^* \) and define

\[
b_m = 1 - \frac{1}{2m}.
\]

We consider the Gaussian field \( X = \{X_n(H), n \in \mathbb{N}, H \in (b_m, 1)\} \) defined as, for every \( n \geq 0 \) and \( H \in (b_m, 1) \),

\[
X_n(H) = \int_{-\pi}^{\pi} \exp(\imath nx)g(H, x)|x|^{1/2-H} d\hat{B}_x
\]

where \( B \) is a standard Brownian motion and \( g : (b_m, 1) \times (-\pi, \pi) \to \mathbb{C} \) is a measurable function satisfying the following properties.

- For every \( (H, x) \in (b_m, 1) \times [-\pi, \pi] \), \( g(H, x) = \overline{g(H, -x)} \). This property ensures that the field \( X \) is real.

- For every \( H \in (b_m, 1) \),

\[
\int_{-\pi}^{\pi} |g(H, x)|^2 |x|^{1-2H} dx = 1
\]

so that \( \mathbb{E}[X_n(H)^2] = 1 \).

- The function \( g \) is twice continuously differentiable on \( (b_m, 1) \times (-\pi, \pi) \). We then define, for every \( (H, x) \in (b_m, 1) \times (-\pi, \pi) \), \( g_0(H) = g(H, 0) \) and \( g_1(H, x) = \int_{0}^{\pi} (\partial g/\partial \xi)(H, \xi) d\xi \) so that \( g = g_0 + g_1 \) and, for every compact set \( K \) of \( (b_m, 1) \),

\[
\lim_{x \to 0} \sup_{H \in K} \left( |g_1(H, x)| + \left| \frac{\partial g_1}{\partial H}(H, x) \right| \right) = 0.
\]

The assumptions above ensure that the covariance function satisfies the uniform long-range property of [3]. In particular, for every compact set \( K \subset (b_m, 1) \), we have

\[
\lim_{j-k \to \infty} \sup_{(H_1, H_2) \in K^2} \left| (j-k)^{2-H_1-H_2} \mathbb{E}[X_j(H_1)X_k(H_2)] - R(H_1, H_2) \right| = 0
\]

where

\[
R(H_1, H_2) = g_0(H_1)g_0(H_2) \int_{\mathbb{R}} \exp(\imath x)|x|^{1-H_1-H_2} dx
\]

for every \( (H_1, H_2) \in (b_m, 1) \).
We consider a continuously differentiable function \( h : [0, \infty) \to (1/2, 1) \) and a function \( \phi \in L^2(e^{-\xi^2}/dx) \) with Hermite rank equal to \( m \in \mathbb{N}^* \). We let
\[
\tilde{h} := 1 + \frac{h - 1}{m} : [0, \infty) \to (b_m, 1).
\]
We define for every \( t \geq 0 \) and \( N > 0 \)
\[
S_{\phi,h}^N(t) := \sum_{j=1}^{[Nt]} \phi(X_j(h_j^N)) \tag{16}
\]
with
\[
h_j^N := 1 + \frac{h(j/N) - 1}{m} = \tilde{h}(j/N).
\]
Now we can state the main result of this paper.

**Theorem 1.** As \( N \to \infty \), the finite-dimensional distributions of \( S_{\phi,h}^N \) converge to those of \( S_{m,h} \) defined for every \( t \geq 0 \) as
\[
S_{m,h}(t) := \int_{\mathbb{R}^m} f_{m,h}(x_1, \ldots, x_m, t) d\tilde{B}_{x_1} \cdots d\tilde{B}_{x_m} \tag{17}
\]
with, for every \( (x_1, \ldots, x_m, t) \in \mathbb{R}^m \times [0, \infty) \),
\[
f_{m,h}(x_1, \ldots, x_m, t) = \int_0^t \exp \left( i\theta \sum_{l=1}^m x_l \right) \tilde{g}(\theta) \left| x_1 \cdots x_m \right|^{1/2 - \tilde{h}(\theta)} d\theta
\]
where
\[
\tilde{g} = \frac{\langle \phi, P_m \rangle}{m!} (g_0 \circ \tilde{h})^m.
\]
The process \( S_{m,h} \) is continuous (up to a modification) and locally self-similar: for every \( t \geq 0 \),
\[
\lim_{\varepsilon \to 0^+} \text{dist.} \left\{ S_{m,h}(t + \varepsilon u) - S_{m,h}(t) \right\}_{u \geq 0} = \{ T_{m,h,t}(u) \}_{u \geq 0}
\]
where \( \lim_{\varepsilon \to 0^+} \) stands for the limit in distribution in the space of continuous functions endowed with the uniform norm on every compact set and, for every \( u \geq 0 \),
\[
T_{m,h,t}(u) = \tilde{g}(t) \int_{\mathbb{R}^m} \frac{\exp (iu \sum_{l=1}^m x_l) - 1}{i \sum_{l=1}^m x_l} \left| x_1 \cdots x_m \right|^{1/2 - \tilde{h}(t)} d\tilde{B}_{x_1} \cdots d\tilde{B}_{x_m}.
\]

Theorem 1 establishes that sequences of processes defined as (16), in particular from a Hurst function \( h \), converge to multifractional processes with Hurst function \( h \). This has been observed in [3] in the particular case \( \phi \equiv 1 \) where the limit process is \( S_{1,h} \), which is a centered Gaussian process of covariance
\[
(t, s) \mapsto \mathbb{E} [S_{1,h}(t)S_{1,h}(s)] = \int_0^t \int_0^s d\theta d\sigma R(h(\theta), h(\sigma)) \tag{18}
\]
with \( R \) defined by (15). Theorem 1 is then an extension of the main result of [3], which assumes \( \phi \equiv 1 \), to any case where \( \phi \in L^2(e^{-\xi^2}/dx) \).

If we assume that \( h \equiv H \in (b_m, 1) \), then Theorem 1 is the main result of [6, 11]. In particular, the limit process \( S_{m,h} \) can be written as \( W_{m,H} \) in (1) with the constant
\[
C(m, H) = \frac{\langle \phi, P_m \rangle}{m!} \left( g_0 \left( \tilde{H} \right) \right)^m = \frac{\langle \phi, P_m \rangle}{m!} \left( \frac{R \left( \tilde{H}, \tilde{H} \right)}{\int_\mathbb{R} e^{\xi^2/2} |\tilde{H}|^{-1 - 2H} d\xi} \right)^{m/2}
\]
where
\[
\tilde{H} := 1 + \frac{H - 1}{m} \in (b_m, 1).
\]
4 Proof of Theorem 1

The proof of Theorem 1 is organized as follows. In Subsection 4.1 we establish a technical lemma we then use throughout the proof of Theorem 1. We prove the convergence of $S_{\phi,h}^N$ in Subsection 4.2 and the regularity properties of $S_{m,h}$ in Subsection 4.3.

4.1 Technical lemma

In the following lemma, we prove for every $T > 0$ the existence of a function $\tilde{f}_T$ that is useful in the sequel of the proof to establish uniform bounds.

**Lemma 2.** For every $T > 0$, there exists a function $\tilde{f}_T \in L^2(\mathbb{R}^m, \mathbb{R})$ so that, for almost every $x \in \mathbb{R}^m$ and for every $t \in [0, T]$ and $H \in [\min \hat{h}, \max \hat{h}]$,

$$\left| \frac{e^{it \sum_{i=1}^m x_i} - 1}{|x_1 \cdots x_m|^{H+1/2} \sum_{i=1}^m x_i} \right| (1 + |\ln |x_1 \cdots x_m||) \leq \tilde{f}_T(x),$$

**Proof.** For every $(x_1, \ldots, x_m) \in \mathbb{R}^m$ we define

$$L(x_1, \ldots, x_m) = (1 + |\ln |x_1 \cdots x_m||)^2$$

and

$$\tilde{f}_T(x_1, \ldots, x_m) = \max_{H \in [\min \hat{h}, \max \hat{h}]} L(x_1, \ldots, x_m)$$

We fix $T > 0$. For every $t \in [0, T]$, we can write

$$\left| \frac{e^{it \sum_{i=1}^m x_i} - 1}{\sum_{i=1}^m x_i} \right|^2 \leq \tilde{f}_T(x_1, \ldots, x_m)^2.$$

It is then enough to prove that, for $H \in [\min \hat{h}, \max \hat{h}]$, the function

$$(x_1, \ldots, x_m) \mapsto \frac{T^2 \sum_{i=1}^m x_i |x_i|^{-1} |x_i|^{-2} L(x_1, \ldots, x_m)}{|x_1 \cdots x_m|^{2H+1}}$$

is integrable. We successively make the substitutions $y_j = x_1 + \cdots + x_j$ for every $j \in \{1, \ldots, m\}$, $z_k = y_{k'}/y_{k'+1}$ for every $k \in \{1, \ldots, m-1\}$ and $z_m = y_m$ to get

$$\int_{\mathbb{R}^m} \frac{T^2 \sum_{i=1}^m x_i |x_i|^{-1} |x_i|^{-2} L(x_1, \ldots, x_m)}{|x_1 \cdots x_m|^{2H+1}} dx_1 \cdots dx_m$$

$$= \int_{\mathbb{R}^m} \frac{T^2 \sum_{i=1}^m x_i |x_i|^{-1} |x_i|^{-2} L(y_1, y_2, \ldots, y_m)}{|y_1|^{2H+1}} dy_1 \cdots dy_m$$

$$= \int_{\mathbb{R}^m} \frac{T^2 \sum_{i=1}^m x_i |x_i|^{-1} |x_i|^{-2} L(z_1, z_2, \ldots, z_m)}{|z_m|^{2m(H+1)+1}} dz_m \int_{\mathbb{R}} \frac{dy_1 \cdots dy_m}{|1 - z_{m-1}|^{2H+1} |z_{m-1}|^{2m-1}(H+1)+1} \times \cdots$$

$$\cdots \times \int_{\mathbb{R}} \frac{dy_1 |z_1|^{2H+1} L(\prod_{k=1}^m z_k, (1-z_1) \prod_{k=2}^m z_k, \ldots, (1-z_{m-1})z_m)}{|1 - z_{m-1}|^{2H+1} |z_{m-1}|^{2m-1}(H+1)+1} \times \cdots$$

The right-hand side above can be bounded by a finite sum of terms of the form

$$\int_{\mathbb{R}^m} \frac{T^2 |z_{m-1}|^{2m(H+1)+1} dz_m}{|z_m|^{2m(H+1)+1}} \int_{\mathbb{R}} \frac{dz_{m-1}}{|1 - z_{m-1}|^{2H+1} |z_{m-1}|^{2m-1}(H+1)+1} \times \cdots$$

$$\cdots \times \int_{\mathbb{R}} \frac{dy_1 |z_1|^{2H+1} L(\prod_{k=1}^m z_k, |z_1|^{2H+1})}{|1 - z_{m-1}|^{2H+1} |z_{m-1}|^{2m-1}(H+1)+1} \times \cdots$$

where $k, j, \mu$ and $\nu$ are integer. The terms of the form (19) are finite since $H \in (1-1/(2m), 1)$ and Bertrand’s test. This concludes the proof. □
4.2 Convergence of $S_{\varphi,h}^N$

We first deal with the study of $S_{P_{m,h}}^N$ defined for every $t \geq 0$ by

$$S_{P_{m,h}}^N(t) = \sum_{j=1}^{\lfloor Nt \rfloor} P_m(X_j(h_j^N)) / N^{h_j/N}.$$  

From now on, we denote $\prod_{i=1}^{m} d\tilde{B}_{x_i}$ by $d\tilde{B}_x^\otimes_m$ when $x = (x_1, \cdots, x_d)$.

**Lemma 3.** The process $S_{P_{m,h}}^N$ is equal in distribution to the process $\tilde{S}_{m,h}^N$ defined for every $t \geq 0$ by

$$\tilde{S}_{m,h}^N(t) = \int_{(-N\pi,N\pi)^m} d\tilde{B}_x^\otimes_m \frac{1}{N} \sum_{j=1}^{\lfloor Nt \rfloor} \prod_{i=1}^{m} \exp(i j x_i / N) g(h_j^N, x_i / N) |x_i|^{1/2-h_j^N}.$$  

**Proof.** Using (11) we obtain, almost surely,

$$P_m(X_j(h_j^N)) = \int_{(-\pi,\pi)^m} \prod_{i=1}^{m} \exp(i j x_i) g(h_j^N, x_i) |x_i|^{1/2-h_j^N} d\tilde{B}_x.$$  

We then have

$$S_{P_{m,h}}^N(t) = \sum_{j=1}^{\lfloor Nt \rfloor} \frac{1}{N^{1-m/2}} \int_{(-\pi,\pi)^m} d\tilde{B}_x^\otimes_m \prod_{i=1}^{m} \exp(i j x_i) g(h_j^N, x_i) |x_i|^{1/2-h_j^N}. $$

Making the substitution $x \rightarrow x/N$ and using (9) we get

$$S_{P_{m,h}}^N \equiv t \sum_{j=1}^{\lfloor Nt \rfloor} \frac{1}{N} \int_{(-\pi,\pi)^m} d\tilde{B}_x^\otimes_m \prod_{i=1}^{m} \exp(i j x_i / N) g(h_j^N, x_i / N) |x_i|^{1/2-h_j^N}. $$

This concludes the proof by linearity of the multiple integral. \(\square\)

Now we aim to prove the convergence of $\tilde{S}_{m,h}^N(t)$ in $L^2(\Omega, \mathbb{R})$ for every $t$. To this goal, we introduce the functions

$$f^N : [0, \infty) \times \mathbb{R}^m \rightarrow \mathbb{C}$$

$$(t, x) \mapsto 1_{(-\pi,\pi)^m}(x) \frac{1}{N} \sum_{j=1}^{\lfloor Nt \rfloor} \prod_{i=1}^{m} \exp(i j x_i / N) g(h_j^N, x_i / N) / |x_i|^{h_j^N - 1/2}$$

and we state the following lemma.

**Lemma 4.** For every $t \geq 0$, there exists a function $f_t^* \in L^2(\mathbb{R}^m, \mathbb{R})$ so that, for every $x \in \mathbb{R}^m$ and $N \in \mathbb{N}$,

$$|f^N(t, x)| \leq f_t^*(x).$$

**Proof.** We have

$$f^N(t, x) = 1_{(-\pi,\pi)^m}(x) \frac{i \sum_{i=1}^{m} x_i / N}{1 - e^{-1} \sum_{i=1}^{m} x_i / N} \sum_{j=1}^{\lfloor Nt \rfloor} \frac{e^{ij \sum_{i=1}^{m} x_i / N} - e^{i(j-1) \sum_{i=1}^{m} x_i / N} i \sum_{i=1}^{m} x_i / N}{i \sum_{i=1}^{m} x_i / N} G_j^N(x)$$

where

$$G_j^N(x) = \prod_{i=1}^{m} g(h_j^N, x_i / N) / |x_i|^{h_j^N - 1/2}.$$
We write

\[ f^N(t, x) = f^{N,1}(t, x) - f^{N,2}(t, x) \]

with

\[
\begin{align*}
\frac{f^{N,1}(t, x)}{1 - e^{-\sum_{l=1}^m x_l / N}} & \left| i \sum_{l=1}^m x_l / N \right| \sum_{j=1}^{[Nt]} \frac{1}{\sum_{l=1}^m x_l} \\
& \times \left( G_j^N(x) \left( e^{i(\sum_{l=1}^m x_l / N - 1)} - G_{j-1}^N(x) \left( e^{i(j-1)\sum_{l=1}^m x_l / N} - 1 \right) \right) \right) \\
= & \sum_{j=1}^{[Nt]} \frac{e^{i(j-1)\sum_{l=1}^m x_l / N} - 1}{i \sum_{l=1}^m x_l} \left( G_j^N(x) - G_{j-1}^N(x) \right).
\end{align*}
\]

We first deal with \( f^{N,1} \). Because \( g \) is bounded, there exists \( M_1 > 0 \) such that for every \( x \) and \( N \)

\[
\left| f^{N,1}(t, x) \right| \leq M_1 \left| e^{i[Nt] \sum_{l=1}^m x_l / N} - 1 \right| \left| \sum_{l=1}^m x_l / N \right|.
\]

Then, by Lemma 2, there exists a function \( \tilde{f}_{1,1} \in L^2(\mathbb{R}^m, \mathbb{R}) \) so that for every \( x \) and \( N \),

\[
\left| e^{i[Nt] \sum_{l=1}^m x_l / N} - 1 \right| \left| \sum_{l=1}^m x_l / N \right| \leq \tilde{f}_{1,1}(x),
\]

so that we get

\[
\left| f^{N,1}(t, x) \right| \leq M_1 \tilde{f}_{1,1}(x). \tag{20}
\]

Now we deal with \( f^{N,2} \). By using Taylor formula we obtain, for almost every \( x \),

\[
\left| G_j^N(x) - G_{j-1}^N(x) \right| \leq \frac{\max \left| h \right|}{N} \max_{H \in [\min h, \max h]} \left| h_{[Nt]}^{-1} \sum_{l=1}^m \frac{x_l}{N} \right| \left| \prod_{l=1}^m g \left( H, \frac{x_l}{N} \right) \right| \left| \prod_{l=1}^m g \left( H, \frac{x_l}{N} \right) \right|.
\]

Since \( g \) and \( \frac{\partial g}{\partial H} \) are bounded, there exists a constant \( M_2 > 0 \), which depends only on \( h \) and \( g \), such that for almost every \( x \) and every \( N \)

\[
\left| f^{N,2}(t, x) \right| \leq \frac{M_2}{N} \sum_{j=1}^{[Nt]} \left| e^{i(j-1)\sum_{l=1}^m x_l / N} - 1 \right| \max_{H \in [\min h, \max h]} \left| \prod_{l=1}^m g \left( H, \frac{x_l}{N} \right) \right|.
\]

As for \( f^{N,2} \), by Lemma 2, there exists a function \( \tilde{f}_{1,2} \in L^2(\mathbb{R}^m, \mathbb{R}) \) so that for almost every \( x \) and every \( N \) and \( j \),

\[
\left| e^{i(j-1)\sum_{l=1}^m x_l / N} - 1 \right| \left| \prod_{l=1}^m g \left( H, \frac{x_l}{N} \right) \right| \leq \tilde{f}_{1,2}(x),
\]

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so that we get

$$|f^{N,2}(t, x)| \leq M_2 \tilde{f}_{2, t}(x).$$

(21)

Hence, taking $\tilde{f}_t = M_1 \tilde{f}_{1, t} + M_2 \tilde{f}_{2, t}$ and combining (20) and (21) we conclude the proof.

The convergence of $\tilde{S}^N_{m,h}$ can now be established.

**Lemma 5.** For every $t \geq 0$, as $N \to \infty$, $\tilde{S}^N_{m,h}(t)$ converges in $L^2(\Omega, \mathbb{R})$ to $\tilde{S}_{m,h}(t)$ given by

$$\tilde{S}_{m,h}(t) = \int_{\mathbb{R}^m} d\tilde{B}_x \int_0^t \exp \left( i\theta \sum_{l=1}^m x_l \right) g_0(\tilde{h}(\theta))^m |x_1 \cdots x_m|^{1/2-\tilde{h}(\theta)} d\theta.$$ 

**Proof.** Because of Lemmas 1 and 4, it suffices to prove that the function $f^N(t, x)$ converges for almost every $x$ to $f^\infty(t, x)$ defined by

$$f^\infty(t, x) := \int_0^t \exp \left( i\theta \sum_{l=1}^m x_l \right) g_0(\tilde{h}(\theta))^m |x_1 \cdots x_m|^{1/2-\tilde{h}(\theta)} d\theta.$$ 

We let

$$G^N_{j,0}(x) = g_0(h_j^N)^m |x_1 \cdots x_m|^{1/2-h_j^N} \quad \text{and} \quad G^N_{j,1}(x) = G^N_{j}(x) - G^N_{j,0}(x)$$

where $G^N_{j}(x)$ is defined as in the proof of Lemma 4. We also consider the same decomposition $f^N = f^{N,1} - f^{N,2}$ as in the proof of Lemma 4 and we let

$$f^{N,1} = f^{N,1,0} - f^{N,1,1} \quad \text{and} \quad f^{N,2} = f^{N,2,0} - f^{N,2,1}$$

where, for $\kappa \in \{0, 1\}$,

$$f^{N,1,\kappa}(t, x) = \mathbb{I}_{(-N\pi, N\pi)}(x) \frac{i \sum_{l=1}^m x_l / N}{1 - e^{-i \sum_{l=1}^m x_l / N}} G^N_{\left[N\kappa\right],\kappa}(x) \frac{e^{i t \sum_{l=1}^m x_l / N} - 1}{i \sum_{l=1}^m x_l}$$

and

$$f^{N,2,\kappa}(t, x) = \mathbb{I}_{(-N\pi, N\pi)}(x) \frac{i \sum_{l=1}^m x_l / N}{1 - e^{-i \sum_{l=1}^m x_l / N}} \times \sum_{j=1}^{\left[N\kappa\right]} G^N_{j,\kappa}(x) - G^N_{j-1,\kappa}(x).$$

Because $h$ and $g_0$ are continuously differentiable we get, for almost every $x$,

$$\lim_{N \to \infty} f^{N,1,0}(t, x) = g_0(\tilde{h}(t))^m \frac{e^{i t \sum_{l=1}^m x_l} - 1}{i |x_1 \cdots x_m|^{h(t)-1/2} \sum_{l=1}^m x_l}$$

and

$$\lim_{N \to \infty} f^{N,2,0}(t, x) = \int_0^t e^{i \theta \sum_{l=1}^m x_l - 1} \tilde{h}'(\theta) \frac{\partial}{\partial H} \left( \frac{g_0(H)^m}{|x_1 \cdots x_m|^{H-1/2}} \right) \bigg|_{H=\tilde{h}(\theta)} d\theta,$$

so that

$$\lim_{N \to \infty} (f^{N,1,0}(t, x) - f^{N,2,0}(t, x)) = f^\infty(t, x).$$

Now we deal with $f^{N,1}$ and $f^{N,2}$. We remark that we can express $G^N_{j,1}(x)$ as

$$G^N_{j,1}(x) = |x_1 \cdots x_m|^{1/2-h_j^N} \sum_{k=1}^m g_1 \left( h_j^N, \frac{x_k}{N} \right) g_0(h_j^N)^{k-1} \prod_{l=k+1}^m g\left( h_j^N, \frac{x_l}{N} \right).$$
Then, because of Lemma 2 and the boundedness of \(g_0\) and \(g\), there exist a constant \(M_3 > 0\) and a function \(\tilde{f}_{1,3} \in L^2(\mathbb{R}^n, \mathbb{R})\) such that for almost every \(x\),

\[
|f^{N,1,1}(t,x)| \leq M_3 \tilde{f}_{1,3}(x) \sum_{k=1}^{m} \sup_{H \in [\min h, \max h]} |g_1 \left( H, \frac{H_k}{N} \right)|,
\]

so that \(\lim_{N \to \infty} f^{N,1,1}(t,x) = 0\). Similarly, using Lemma 2, there exist a constant \(M_4 > 0\) and a function \(\tilde{f}_{1,4} \in L^2(\mathbb{R}^n, \mathbb{R})\) such that for almost every \(x\),

\[
|f^{N,2,1}(t,x)| \leq M_4 \tilde{f}_{1,4}(x) \sum_{k=1}^{m} \sup_{H \in [\min h, \max h]} \left( |g_1 \left( H, \frac{H_k}{N} \right)| + \left| \frac{\partial g_1}{\partial H} \left( H, \frac{H_k}{N} \right) \right| \right),
\]

so that \(\lim_{N \to \infty} f^{N,2,1}(t,x) = 0\) and then

\[
\lim_{N \to \infty} \left( f^{N,2,1}(t,x) - f^{N,2,1}(t,x) \right) = 0,
\]

which concludes the proof. \(\square\)

The following lemma establishes that the convergence of \(S_{\phi,h}^N\) can be reduced to the one of \(S_{P_{m,h}}^N\) and, as a consequence of Lemma 3, to the one of \(\hat{S}_{m,h}^N\).

**Lemma 6.** For every \(t \geq 0\), we have

\[
\lim_{N \to \infty} \mathbb{E} \left[ \left( S_{\phi,h}^N(t) - \frac{\langle \phi, P_m \rangle}{m!} S_{P_{m,h}}^N(t) \right)^2 \right] = 0.
\]

**Proof.** Since (6) we have

\[
\mathbb{E} \left[ \left( S_{\phi,h}^N(t) - \frac{\langle \phi, P_m \rangle}{m!} S_{P_{m,h}}^N(t) \right)^2 \right] = \mathbb{E} \left[ \left( \sum_{j=1}^{[Nt]} \frac{1}{N^{h(j/N)}} \sum_{n=m+1}^{N^{h(j/N)+k(N)}} \frac{\langle \phi, P_n \rangle}{n!} P_n(\phi_n(h_j^N)) \right)^2 \right] = \sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} \frac{\langle \phi, P_n \rangle^2}{n!} \mathbb{E}[P_n(X_j(h_j^N))P_n(X_k(h_k^N))].
\]

Because of (7) and (13) we get, for every \(n \geq m + 1\),

\[
\mathbb{E}[P_n(X_j(h_j^N))P_n(X_k(h_k^N))] = n!E[X_j(h_j^N)X_k(h_k^N)]^n \leq n! \mathbb{E}[X_j(h_j^N)X_k(h_k^N)]^m,
\]

so that

\[
\mathbb{E} \left[ \left( S_{\phi,h}^N(t) - \frac{\langle \phi, P_m \rangle}{m!} S_{P_{m,h}}^N(t) \right)^2 \right] \leq \left( \sum_{n=m+1}^{[Nt]} \frac{\langle \phi, P_n \rangle^2}{n!} \right) \sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} \frac{\mathbb{E}[X_j(h_j^N)X_k(h_k^N)]}{N^{h(j/N)+h(k/N)}}^m.
\]

Let \(\eta > 0\). Using the representation of the field \(X\), there exists \(N_\eta \in \mathbb{N}^*\) such that, for \(|j - k| > N_\eta\) and \(N \in \mathbb{N}^*\), \(\mathbb{E}[X_j(h_j^N)X_k(h_k^N)] \leq \eta\), so that

\[
\sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} \frac{\mathbb{E}[X_j(h_j^N)X_k(h_k^N)]}{N^{h(j/N)+h(k/N)}}^m \leq \sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} \frac{1_{|j-k| \leq N_\eta}}{N^{h(j/N)+h(k/N)}} + \eta \sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} 1_{|j-k| \geq 1} \frac{\mathbb{E}[X_j(h_j^N)X_k(h_k^N)]}{N^{h(j/N)+h(k/N)}}.
\]

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There exists $C_1(\eta) > 0$ such that

$$
\sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} \frac{1_{|j-k|\leq N_n}}{N^{k(j/N)+k(k/N)}} \leq \frac{C_\eta}{N^{2 \min h - 1}}.
$$

Moreover, because of the assumptions on $X$, there exists a constant $C_2 > 0$, which is independent on $\eta$, such that, for every $j$, $k$ and $N$,

$$
|E[X_j(h_j^N)X_k(h_k^N)]| \leq C_2 |j - k|^{h_j^N + h_k^N - 2}.
$$

We then obtain

$$
\sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} \frac{|E[X_j(h_j^N)X_k(h_k^N)]|^{m+1}}{N^{h(j/N)+k(k/N)}} \leq \frac{C_\eta}{N^{2 \min h - 1}} + \frac{\eta C_2}{N^2} \sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} 1_{|j-k|\geq 1} \frac{|j - k|^{m(h_j^N + h_k^N - 2)}}{N}.
$$

Hence, for every $\eta > 0$

$$
\limsup_{N \to \infty} E \left[ \left( \frac{S_{\phi,h}^N(t)}{m!} - \frac{\langle \phi, P_m \rangle}{m!} S_{\phi,h}^N(t) \right)^2 \right] \leq \eta C_2 \left( \sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n \rangle^2}{n!} \right) \int_0^t \int_0^t |\theta - \sigma|^{h(\theta) + h(\sigma) - 2} d\theta d\sigma.
$$

The constants $\sum_{n=m+1}^{\infty} \langle \phi, P_n \rangle^2 / n!$ and $\int_0^t \int_0^t |\theta - \sigma|^{h(\theta) + h(\sigma) - 2} d\theta d\sigma$ are finite since (8) and $h > 1/2$ respectively. This concludes the proof. \qed

Now we conclude this subsection by the following lemma.

**Lemma 7.** As $N \to \infty$, the finite-dimensional distributions of $S_{\phi,h}^N$ converge to those of $S_{m,h}$, which can be defined for every $t \geq 0$ as:

$$
S_{m,h}(t) := \frac{\langle \phi, P_m \rangle}{m!} \int_{\mathbb{R}^m} f^{\infty}(t, x_1, \ldots, x_m) d\hat{B}_{x_1} \cdots d\hat{B}_{x_m}.
$$

**Proof.** We fix $n \in \mathbb{N}$, $(t_1, \ldots, t_n) \in [0, \infty)^n$ and a Lipschitz bounded function $\Psi : \mathbb{R}^n \to \mathbb{R}$. We define $\phi_m = \langle \phi, P_m \rangle / m!$. We have

$$
|E[\Psi(S_{\phi,h}^N(t_1), \ldots, S_{\phi,h}^N(t_n))] - E[\Psi(S_{m,h}(t_1), \ldots, S_{m,h}(t_n))]| \leq E_1^N + E_2^N
$$

where

$$
E_1^N = |E[\Psi(S_{\phi,h}^N(t_1), \ldots, S_{\phi,h}^N(t_n)) - \Psi(\phi_m S_{P_m,h}^N(t_1), \ldots, \phi_m S_{P_m,h}^N(t_n))]|
$$

and

$$
E_2^N = |E[\Psi(\phi_m S_{P_m,h}^N(t_1), \ldots, \phi_m S_{P_m,h}^N(t_n)) - E[\Psi(S_{m,h}(t_1), \ldots, S_{m,h}(t_n))]|.
$$

Because $\Psi$ is Lipschitz and using Cauchy-Schwartz inequality, there exists $C_1 > 0$ so that, for every $N$,

$$
E_1^N \leq C_1 \sum_{j=1}^{n} \sqrt{E \left[ \left( S_{\phi,h}^N(t_j) - \phi_m S_{P_m,h}^N(t_j) \right)^2 \right]}.
$$

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Thus, as for $x$ almost every $C$ with family where with the proof of the local self-similarity property in the space of continuous functions.

By an integration by part, we have

$$E_2^N = \left| E \left[ \phi_m \check{S}_{m,h}(t_1), \ldots, \phi_m \check{S}_{m,h}(t_n) \right] - \Psi(S_{m,h}(t_1), \ldots, S_{m,h}(t_n)) \right|.\]

Thus, as for $E_1^N$, because $\Psi$ is Lipschitz and using Cauchy-Schwartz inequality, there exists $C_2 > 0$ so that, for every $N$,

$$E_2^N \leq C_2 \sum_{j=1}^n \sqrt{E \left[ \left( \phi_m \check{S}_{m,h}(t_j) - S_{m,h}(t_j) \right)^2 \right]}.$$

As a consequence, from Lemma 5,

$$\lim_{N \to \infty} E_2^N = 0.\]

We conclude the proof by combining (23), (24) and (25).

\[\square\]

4.3 Continuity and local self-similarity of $S_{m,h}$

We first prove the local self-similarity in the sense of the finite-dimensional distributions. Then we prove the continuity of $S_{m,h}$. Finally, we establish a tightness property for the family $\{ \varepsilon^{-h(t)}(S_{\phi,h}(t + \varepsilon u) - S_{\phi,h}(t)) \}_{u \geq 0}$ using Kolmogorov lemma [1] to conclude with the proof of the local self-similarity property in the space of continuous functions.

By making the substitution $\theta \to \varepsilon \theta + t$, we get

$$S_{m,h}(t + \varepsilon u) - S_{m,h}(t) = \int \exp \left( it \sum_{l=1}^m x_l \right) \hat{f}_1(t, u, x, \varepsilon) d\hat{B}_{x_1} \cdots d\hat{B}_{x_m},$$

where

$$\hat{f}_1(t, u, x, \varepsilon) = \varepsilon^{-h(t)} \int_0^u \exp \left( i \theta \sum_{l=1}^m x_l \right) \tilde{g}(\varepsilon \theta + t) |x_1 \cdots x_m|^{1/2 - \hat{h}(\varepsilon \theta + t)} d\theta.$$

Since (10) and (9) we have

$$\left\{ S_{m,h}(t + \varepsilon u) - S_{m,h}(t) \right\}_{u \geq 0} \sim \text{dist} \left\{ \int \hat{f}_2(t, u, x, \varepsilon) d\hat{B}_{x_1} \cdots d\hat{B}_{x_m} \right\}_{u \geq 0}$$

with

$$\hat{f}_2(t, u, x, \varepsilon) = \int_0^u \varepsilon^{m(\hat{h}(\varepsilon \theta + t) - \hat{h}(t))} \exp \left( i \theta \sum_{l=1}^m x_l \right) \tilde{g}(\varepsilon \theta + t) |x_1 \cdots x_m|^{1/2 - \hat{h}(\varepsilon \theta + t)} d\theta.$$

For almost every $x \in \mathbb{R}^m$, every $u$ and $t$,

$$\lim_{\varepsilon \to 0} \hat{f}_2(t, u, x, \varepsilon) = \tilde{g}(t) \frac{\exp (iu \sum_{l=1}^m x_l) - 1}{i(\sum_{l=1}^m x_l)|x_1 \cdots x_m|^{\hat{h}(t) - 1/2}}.$$

By an integration by part, we have

$$\hat{f}_2(t, u, x, \varepsilon) = \varepsilon^{m(\hat{h}(\varepsilon u + t) - \hat{h}(t))} \tilde{g}(\varepsilon u + t) \frac{\exp (iu \sum_{l=1}^m x_l) - 1}{i(\sum_{l=1}^m x_l)|x_1 \cdots x_m|^{\hat{h}(\varepsilon u + t) - 1/2}}$$

$$+ \varepsilon \int_0^u \varepsilon^{m(\hat{h}(\varepsilon \theta + t) - \hat{h}(t))} \frac{\exp (i \theta \sum_{l=1}^m x_l) - 1}{i(\sum_{l=1}^m x_l)|x_1 \cdots x_m|^{\hat{h}(\varepsilon \theta + t) - 1/2}}$$

$$\times \left\{ \tilde{g}(\varepsilon \theta + t) + \hat{h}'(\varepsilon \theta + t) \tilde{g}(\varepsilon \theta + t)(\ln (\varepsilon^m) - \ln |x_1 \cdots x_m|) \right\} d\theta.$$
We fix $T > 0$ and $U > 0$. As a consequence of the identity above and because of the
regularity of $\hat{h}$ and $\hat{g}$, there exists a constant $M_{T,U} > 0$ such that, for every $u \in [0,U]$, $t \in [0,T]$, $\varepsilon \in (0,1]$ and $x \in \mathbb{R}^m$,
\[
|f_2(t, u, x, \varepsilon)| \leq M_{T,U} \tilde{f}_{T+U,2}(x)
\]  
(26)
where $\tilde{f}_{T+U,2}(x)$ is defined in Lemma 2. Since $\tilde{f}_{T+U,2}$ is square integrable and because of
Lemma 1, this proves the local self-similarity of $S_{m,h}$ in the sense of the finite-dimensional
distributions.

To prove the continuity of the $S_{m,h}$ we use Kolmogorov lemma. By making the same
calculations as above we have, for every $t > s > 0$,
\[
\mathbb{E} \left[ (S_{\phi,h}(t) - S_{\phi,h}(s))^2 \right] = m!(t-s)^{2h(s)} \int_{\mathbb{R}^m} \left( \tilde{f}_2(s, 1, x, t-s) \right)^2 dx_1 \cdots dx_m
\]
If $t - s < 1$, because of (26) we then have
\[
\mathbb{E} \left[ (S_{\phi,h}(t) - S_{\phi,h}(s))^2 \right] \leq m!M_{1,1}(t-s)^{2h(s)} \int_{\mathbb{R}^m} \tilde{f}_{2,2}(x)^2 dx_1 \cdots dx_m,
\]
which concludes the proof of the continuity of $S_{\phi,h}$.

Finally, in a similar way as just previously there exists a constant $C > 0$ such that, for
every $u$ and $v$ satisfying $|u - v| < 1$,
\[
\mathbb{E} \left[ \left( \frac{S_{\phi,h}(t + \varepsilon u) - S_{\phi,h}(t + \varepsilon v)}{\varepsilon^{h(t)}} \right)^2 \right] \leq C|u - v|^{2h(t)},
\]
This prove the tightness of the family $\left\{ \{ \varepsilon^{-h(t)}(S_{\phi,h}(t + \varepsilon u) - S_{\phi,h}(t)) \}_{u \geq 0} \right\}_{\varepsilon > 0}$ thanks to
Kolmogorov lemma [1], and then the local self-similarity property of $S_{\phi,h}$.

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