Hyperbolic bumps

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Abstract

In many models of working memory, transient stimuli are encoded by feature-selective persistent neural activity. Such stimuli are imagined to induce the formation of a spatially localised bump of persistent activity which coexists with a stable uniform state. As an example, Camperi and Wang [3] have proposed and studied a network model of visuo-spatial working memory in prefontal cortex adapted from the ring model of orientation of Ben-Yishai and colleagues [1]. It is therefore natural to study the emergence of spatially localised bumps for the structure tensor model. This modelization was introduced by Chossat and Faugeras in [4] to describe the representation and the processing of image edges and textures in the hypercolumns of the cortical area V1. The key entity, the structure tensor, intrinsically lives in a non-Euclidean, in effect hyperbolic, space. Its spatio-temporal behaviour is governed by nonlinear integro-differential equations defined on the Poincaré disc model of the two-dimensional hyperbolic space. In this paper, we present an original study, based on non-Euclidean, hyperbolic, analysis, of a spatially localised bump solution.

Keywords: Neural fields; nonlinear integro-differential equations; functional analysis; stability analysis; hypergeometric functions; bumps.

1 Introduction

Chossat and Faugeras in [4] have introduced a new and elegant approach to model the processing of image edges and textures in the hypercolumns of area V1 that is based on a nonlinear representation of the image first order derivatives called the structure tensor. They assumed that this structure tensor was represented by neuronal populations in the hypercolumns of V1 that can be described by equation similar to those proposed by Wilson and Cowan [10].

We recall that the structure tensor is a way of representing the edges and textures of a 2D image $\mathcal{I}(x, y)$ [2, 9]. Moreover, a structure tensor can be seen as a 2×2 symmetric positive matrix.
We assume that a hypercolumn of V1 can represent the structure tensor in the receptive field of its neurons as the average membrane potential values of some of its membrane pouplations. Let $\mathcal{T}$ be a structure tensor. The average potential $V(\mathcal{T}, t)$ of the column has its time evolution that is governed by the following neural mass equation adapted from [4] where we integrate over the set SPD(2), the set of 2×2 symmetric definite-positive matrices:

$$\partial_t V(\mathcal{T}, t) = -\alpha V(\mathcal{T}, t) + \int_{\mathrm{SPD}(2)} W(\mathcal{T}, \mathcal{T}')S(V(\mathcal{T}', t))d\mathcal{T}' + I(\mathcal{T}, t) \quad \forall t > 0 \quad (1)$$

The nonlinearity S is typically taken to be a sigmoidal function which may be expressed as:

$$S(x) = \frac{1}{1 + e^{-\mu(x-\kappa)}}$$

with gain μ and threshold κ. I is an external input.
In this study, we only deal with the reduced case of structure tensors with determinant equal to one. We recall that $\mathrm{SL}(2, \mathbb{R})$ is the set of 2×2 invertible matrices with determinant 1. So in equation (1), the integral is only over $\mathrm{SSPD}(2) = \mathrm{SPD}(2) \cap \mathrm{SL}(2, \mathbb{R})$, the set of special symmetric positive definite matrices. Furthermore, $\mathrm{SSPD}(2) \overset{isom}{=} \mathbb{D}$, where $\mathbb{D}$ is the Poincare Disk, see e.g. [4]. It is well-known [8] that $\mathbb{D}$ (and hence SSPD(2)) is a two-dimensional Riemannian space of constant sectional curvature equal to -1 for the distance noted d_2 defined by

$$d_2(z, z') = \operatorname{arctanh} \frac{|z - z'|}{|1 - \bar{z}z'|}.$$

It is possible to express the volume element $d\mathcal{T}$ in (z_1, z_2) coordinates with $z = z_1 + iz_2$:

$$d\mathcal{T} = \frac{dz_1 dz_2}{(1-|z|^2)^2} \stackrel{def}{=} dm(z)$$

We rewrite (1) in (z) coordinates:

$$\partial_t V(z,t) = -\alpha V(z,t) + \int_{\mathbb{D}} W(z,,z') S(V(z',t)) dm(z') + I(z,t)$$

In order to construct exact bump solutions, we consider the high gain limit $\mu \to \infty$ of the sigmoid function S such that $S(x) = H(x-\kappa)$, where H is the Heaviside function defined by $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ otherwise. Equation (1) is rewritten as:

$$\partial_t V(z,t) = -\alpha V(z,t) + \int_{\mathbb{D}} W(z,z') H(V(z',t)-\kappa) \mathrm{dm}(z') + I(z,t) \tag{2}$$

$$= -\alpha V(z,t) + \int_{\{z' \in \mathbb{D} | V(z',t) \geq \kappa\}} W(z,z') \mathrm{dm}(z') + I(z)$$

We make the assumption that the external input I does not depend upon the variables t and the connectivity function depends only on the hyperbolic distance between two points of $\mathbb{D}$: $W(z,z') = W(d_2(z,z'))$.

2 Stationary pulses

Our aim is to construct a hyperbolic radially symmetric stationary pulse. Let us first consider a general stationary pulse:

$$\alpha V(z) = \int_{\{z' \in \mathbb{D} | V(z') \geq \kappa\}} W(z,z') \mathrm{dm}(z') + I(z)$$

We assume that the set $K = \{z \in \mathbb{D} | V(z) \geq \kappa\} \subset \mathbb{D}$ is compact. We note $M(z,K)$ the integral $\int_K W(z,z') \mathrm{dm}(z')$. The relation $V(z) = \kappa$ holds for all $z \in \partial K$.

2.1 Helgason Fourier transform

Let b be a point on the circle $\partial\mathbb{D}$. For $z \in \mathbb{D}$, we define the "inner product" $< z, b >$ to be the algebraic distance to the origin of the (unique) horocycle based at b through z (see [4]). Note that $< z, b >$ does not depend on the position of z on the horocycle. The Helgason Fourier transform in $\mathbb{D}$ is defined as (see [7]):

$$\widetilde{h}(\lambda, b) = \int_{\mathbb{D}} h(z) e^{(-i\lambda+1)<z,b>} dm(z) \quad \forall (\lambda, b) \in \mathbb{R} \times \partial\mathbb{D}$$

for a function $h : \mathbb{D} \to \mathbb{C}$ such that this integral is well-defined.
We define $\mathbf{W}(z) \stackrel{def}{=} W(d_2(z, O))$.

Lemma 2.1. *The Fourier transform in $\mathbb{D}$, $\widetilde{\mathbf{W}}(\lambda, b)$ of $\mathbf{W}$ does not depend upon the variable $b \in \partial\mathbb{D}$.*

In order to calculate M, we use the Fourier transform. First we rewrite M as a convolution product:

$$M(z,K) = \int_K W(z,z') \mathrm{dm}(z') = \int_{\mathbb{D}} W(z,z') \mathbb{1}_K(z') \mathrm{dm}(z') = \mathbf{W} * \mathbb{1}_K(z)$$

In [7], Helgason proves an inversion formula for the hyperbolic Fourier transform and we apply this result to $\mathbf{W}$. Let $\Phi_\lambda(z) = \int_{\partial\mathbb{D}} e^{(i\lambda+1)<z,b>} db$ then:

$$\mathbf{W}(z) = \frac{1}{4\pi} \int_{\mathbb{R}} \widetilde{\mathbf{W}}(\lambda) \Phi_\lambda(z) \lambda \tanh(\frac{\pi}{2}\lambda) d\lambda \tag{3}$$

Then,

$$M(z,K) = \frac{1}{4\pi} \int_{\mathbb{R}} \widetilde{\mathbf{W}}(\lambda) \Phi_\lambda * \mathbb{1}_K(z) \lambda \tanh(\frac{\pi}{2}\lambda) d\lambda$$

2.2 Study of $M(z,K)$ when $K = B_h(0,\omega)$

We now consider the special case where K is a hyperbolic disk centered at the origin of hyperbolic radius ω, noted $B_h(0,\omega)$.

Lemma 2.2. *For all $\omega > 0$ the following formula holds:*

$$\int_{B_h(0,\omega)} \Phi_\lambda(z) dm(z) = \pi \sinh(\omega)^2 \cosh(\omega)^2 \Phi_\lambda^{(1,1)}(\omega)$$

where $B_h(0,\omega)$ is the hyperbolic ball and:

$$\Phi_\lambda^{(\alpha,\beta)}(\omega) = F(\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda); \alpha+1; -\sinh(\omega)^2),$$

where $\alpha + \beta + 1 = \rho$ and F is the hypergeometric function of first kind.

Proposition 2.1. *If $K = B_h(0,\omega)$ and $z = \tanh(r)e^{i\theta} \in K$, $M(z,K)$ is given by the following formula:*

$$M(z, B_h(0,\omega)) \stackrel{def}{=} \mathcal{M}(r,\omega) = \frac{1}{4\pi} \int_{\mathbb{R}} \widetilde{\mathbf{W}}(\lambda) \Phi_\lambda(\tanh(r)) \Psi_\lambda(\omega) \lambda \tanh(\frac{\pi}{2}\lambda) d\lambda \tag{4}$$

where

$$\Psi_\lambda(\omega) \stackrel{def}{=} \int_{B_h(0,\omega)} e^{(i\lambda+1)<z',1>} dm(z') \tag{5}$$

We are now in a position to obtain an analytic form for $\mathcal{M}(r,\omega)$. It is given in the following theorem.

Theorem 2.1. *For all $(r,\omega) \in \mathbb{R}^+ \times \mathbb{R}^+$:*

$$\mathcal{M}(r,\omega) = \frac{1}{4} \sinh(\omega)^2 \cosh(\omega)^2 \times \int_{\mathbb{R}} \widetilde{\mathbf{W}}(\lambda) \Phi_\lambda^{(0,0)}(r) \Phi_\lambda^{(1,1)}(\omega) \lambda \tanh(\frac{\pi}{2}\lambda) d\lambda \tag{6}$$

2.3 Discussion

Let us point out that our result can be linked to the work of Folias and Bressloff in [6]. They constructed a two-dimensional pulse for a general, radially symmetric synaptic weight function. They obtain a similar formal representation of the integral of the connectivity function w over the disk $B(O,a)$ centered at the origin O and of radius a. Using their notations,

$$M(a,r) = \int_0^{2\pi}\int_0^a w(|\mathbf{r}-\mathbf{r}'|)r'\,dr'\,d\theta = 2\pi a\int_0^\infty \breve{w}(\rho)J_0(r\rho)J_1(a\rho)\,d\rho$$

where $J_\nu(x)$ is the Bessel function of the first kind and $\breve{w}$ is the real Fourier transform of w. In our case, instead of the Bessel function, we find $\Phi_\lambda^{(\nu,\nu)}(r)$ which is linked to the hypergeometric function of the first kind, as explained in lemma 2.2.

We next adapt the results proved by Folias and Bressloff in [6] to the hyperbolic case.

3 A hyperbolic radially symmetric stationary-pulse

We note $V(r)$ a hyperbolic radially symmetric stationary-pulse solution of (2) where V depends only upon the variable r and is such that:

$$V(r) > \kappa, \quad r \in [0,\omega[,$$

$$V(\omega) = \kappa,$$

$$V(r) < \kappa, \quad r \in]\omega,\infty[,$$

and

$$V(\infty) = 0.$$

Substituting into (2) yields:

$$\alpha V(r) = \mathcal{M}(r,\omega) + I(r) \qquad (7)$$

where $\mathcal{M}(r,\omega)$ is defined in equation (6) and $I(r) = \mathcal{I}e^{-\frac{r^2}{2\sigma^2}}$ is a Gaussian input.
The condition for the existence of a stationary pulse is given by:

$$\alpha\kappa = \mathcal{M}(\omega) + I(\omega) \overset{def}{=} N(\omega) \qquad (8)$$

where

$$\mathcal{M}(\omega) \overset{def}{=} \mathcal{M}(\omega,\omega) = \frac{1}{4}\sinh(\omega)^2\cosh(\omega)^2 \times \int_{\mathbb{R}} \widetilde{\mathbf{W}}(\lambda)\Phi_\lambda^{(0,0)}(\omega)\Phi_\lambda^{(1,1)}(\omega)\lambda\tanh(\frac{\pi}{2}\lambda)d\lambda$$

The function $N(\omega)$ is plotted in figure 1 for a range of the input amplitude $\mathcal{I}$. The horizontal dashed lines indicate different values of $\alpha\kappa$, the points of intersection determine the existence of stationary pulse solutions.

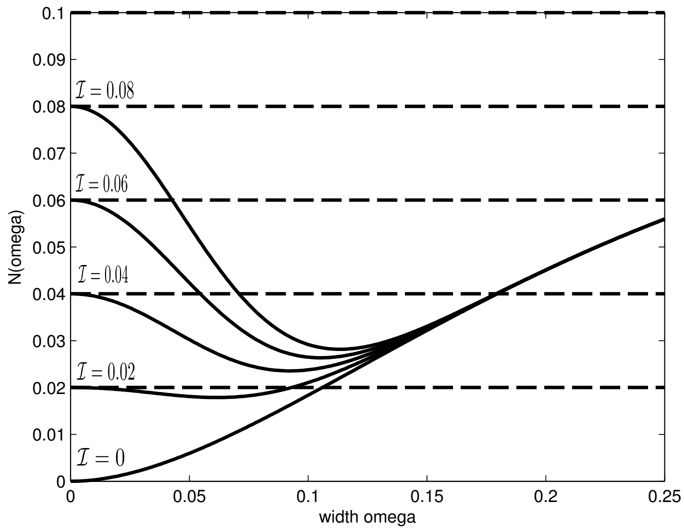

Figure 1: Plot of $N(\omega)$ defined in (8) as a function of the pulse width ω for several values of the input amplitude $\mathcal{I}$ and for a fixed input width $\sigma = 0.05$. The connectivity function has been set to $W(r) = e^{-\frac{r}{b}}$, with $b = 0.2$.

3.1 Monotony of the pulse

We now show that for a general monotonically decreasing weight function W, the function $\mathcal{M}(r,\omega)$ is necessarily a monotonically decreasing function of r. This will ensure that the hyperbolic radially symmetric stationary-pulse solution (7) is also a monotonically decreasing function of r in the case of a Gaussian input.

For $r > \omega$: Differentiating $\mathcal{M}$ with respect to r yields to $\frac{\partial\mathcal{M}}{\partial r}(r,\omega) < 0$ for $r > \omega$, since $W' < 0$ (hypothesis on the monotony of the weight function).

For $r < \omega$: To see that it is also negative for $r < \omega$, we differentiate equation (6) with respect to r and we deduce that:

$$\mathbf{sgn}(\frac{\partial\mathcal{M}}{\partial r}(r,\omega)) = \mathbf{sgn}(\frac{\partial\mathcal{M}}{\partial r}(\omega,r)).$$

Consequently, $\frac{\partial\mathcal{M}}{\partial r}(r,\omega) < 0$ for $r < \omega$.

Hence, we have the following Lemma:

Lemma 3.1. *V is monotonically decreasing in r for any monotonically decreasing synaptic weight function W.*

3.2 Linear stability analysis

We now analyse the evolution of small time-dependent perturbations of the hyperbolic stationary-pulse solution through linear stability analysis.

3.2.1 Spectral analysis of the linearized operator

Equation (2) is linearized about the stationary solution $V(r)$ by introducing the time-depndent perturbation:

$$v(z,t) = V(r) + \phi(z,t)$$

This leads to the linear equation:

$$\partial_t\phi(z,t) = -\alpha\phi(z,t) + \int_{\mathbb{D}} W(z,z')H'(V(r')-\kappa)\phi(z',t)\mathrm{dm}(z').$$

We separate variables by setting $\phi(z,t) = \phi(z)e^{\beta t}$ to obtain the equation:

$$(\beta+\alpha)\phi(z) = \int_{\mathbb{D}} W(z,z')H'(V(r')-\kappa)\phi(z')\mathrm{dm}(z')$$

Introducing the hyperbolic polar coordinates $z = \tanh(r)e^{i\theta}$ and using the result:

$$H'(V(r)-\kappa) = \delta(V(r)-\kappa) = \frac{\delta(r-\omega)}{|V'(\omega)|}$$

we obtain:

$$\begin{aligned}(\beta+\alpha)\phi(z) &= \frac{1}{2}\int_0^\omega\int_0^{2\pi} W(\tanh(r)e^{i\theta}, \tanh(r')e^{i\theta'}) \\ &\quad\times \frac{\delta(r'-\omega)}{|V'(\omega)|}\phi(\tanh(r')e^{i\theta'})\sinh(2r')dr'd\theta' \\ &= \frac{\sinh(2\omega)}{2|V'(\omega)|}\int_0^{2\pi} W(\tanh(r)e^{i\theta}, \tanh(\omega)e^{i\theta'}) \\ &\qquad\qquad \times \phi(\tanh(\omega)e^{i\theta'})d\theta'\end{aligned}$$

With a slight abuse of notation we are led to study the solutions of the integral equation:

$$(\beta+\alpha)\phi(r,\theta) = \frac{\sinh(2\omega)}{2|V'(\omega)|}\int_0^{2\pi}\mathcal{W}(r,\omega;\theta'-\theta)\phi(\omega,\theta')d\theta' \tag{9}$$

where:

$$\mathcal{W}(r,\omega;\varphi) \overset{def}{=} W\circ\tanh^{-1}\left(G^{\varphi}_{r,\omega}\right)$$

with:

$$G^{\varphi}_{r,\omega} \overset{def}{=} \sqrt{\frac{\tanh(r)^2+\tanh(\omega)^2-2\tanh(r)\tanh(\omega)\cos(\varphi)}{1+\tanh(r)^2\tanh(\omega)^2-2\tanh(r)\tanh(\omega)\cos(\varphi)}}$$

3.2.2 Essential spectrum

If the function ϕ satisfies the condition

$$\int_0^{2\pi}\mathcal{W}(r,\omega;\theta')\phi(\omega,\theta-\theta')d\theta' = 0 \quad \forall r,$$

then equation (9) reduces to:

$$\beta+\alpha = 0$$

yielding the eigenvalue:

$$\beta = -\alpha < 0$$

This part of the essential spectrum is negative and does not cause instability.

3.2.3 Discrete spectrum

If we are not in the previous case we have to study the solutions of the integral equation (9).
This equation shows that $\phi(r,\theta)$ is completely determined by its values $\phi(\omega,\theta)$ on the circle of equation $r=\omega$. Hence, we need only to consider $r=\omega$, yielding the integral equation:

$$(\beta+\alpha)\phi(\omega,\theta) = \frac{\sinh(2\omega)}{2|V'(\omega)|}\int_0^{2\pi}\mathcal{W}(\omega,\omega;\theta')\phi(\omega,\theta-\theta')d\theta'$$

The solutions of this equation are exponential functions $e^{\gamma\theta}$, where γ satisfies:

$$(\beta+\alpha) = \frac{\sinh(2\omega)}{2|V'(\omega)|}\int_0^{2\pi}\mathcal{W}(\omega,\omega;\theta')e^{-\gamma\theta'}d\theta'$$

By the requirement that ϕ is 2π-periodic in θ, it follows that $\gamma = in$, where $n\in\mathbb{Z}$. Thus the integral operator with kernel $\mathcal{W}$ has a discrete spectrum given by:

$$(\beta_n+\alpha) = \frac{\sinh(2\omega)}{2|V'(\omega)|}\int_0^{2\pi}\mathcal{W}(\omega,\omega;\theta')e^{-in\theta'}d\theta'$$

Lemma 3.2. *The following properties hold:*

1. $\forall n\in\mathbb{N}$, β_n *is real,*
2. $\forall n\in\mathbb{N}$, $\beta_n\le\beta_0$

If we set:

$$\mathcal{W}_0(\omega) \overset{def}{=} \frac{\sinh(2\omega)}{2}\int_0^{2\pi}\mathcal{W}(\omega,\omega;\theta')d\theta'$$

then we have the reduced stability condition:

$$\frac{\mathcal{W}_0(\omega)}{|V'(\omega)|} < \alpha \tag{10}$$

3.2.4 Rewriting of (10)

From (7) we have:

$$V'(\omega) = \frac{1}{\alpha}(-\mathcal{M}_r(\omega)+I'(\omega))$$

where

$$\mathcal{M}_r(\omega) \overset{def}{=} -\frac{\partial\mathcal{M}}{\partial r}(\omega,\omega)$$

We have previously established that $\mathcal{M}_r(\omega)>0$ and $I'(\omega)$ is negative by definition. Hence, letting $\mathcal{D}(\omega) = |I'(\omega)|$, we have

$$|V'(\omega)| = \frac{1}{\alpha}(\mathcal{M}_r(\omega)+\mathcal{D}(\omega)).$$

By substitution we obtain another form of the reduced stability condition:

$$\mathcal{D}(\omega) > \mathcal{W}_0(\omega) - \mathcal{M}_r(\omega) \tag{11}$$

We also have:

$$\begin{aligned}\mathcal{M}'(\omega) = \frac{d}{d\omega}\mathcal{M}(\omega,\omega) &= \frac{\partial\mathcal{M}}{\partial r}(\omega,\omega) + \frac{\partial\mathcal{M}}{\partial\omega}(\omega,\omega) \\ &= \mathcal{W}_0(\omega) - \mathcal{M}_r(\omega),\end{aligned}$$

and

$$N'(\omega) = \mathcal{M}'(\omega) + I'(\omega) = \mathcal{W}_0(\omega) - \mathcal{M}_r(\omega) - \mathcal{D}(\omega),$$

showing that the stability condition (11) is satisfied when $N'(\omega) < 0$ and is not satisfied when $N'(\omega) > 0$.

4 Numerical simulations

The aim of this section is to numerically solve (2) for different values of the parameters. This implies developing a numerical scheme that approaches the solution of our equation, and proving that this scheme effectively converges to the solution.
Since equation (2) is defined on $\mathbb{D}$, computing the solutions on the whole hyperbolic disk has same the complexity as computing the solutions of usual Euclidean neural field equations defined on $\mathbb{R}^2$.

4.1 Numerical schemes

Let us consider the modified equation of (2):

$$\begin{cases} \partial_t V(z,t) = -\alpha V(z,t) + \int_{B(0,a)} W(z,z') \\ \times H(V(z',t)-\kappa)\mathrm{dm}(z') + I(z) \\ V(z,0) = V_0(z) \end{cases} \quad (12)$$

We express z in (Euclidean) polar coordinates such that $z = re^{i\theta}$, $V(z,t) = V(r,\theta,t)$ and $W(z,z') = W(r,\theta,r',\theta')$. The integral in equation (12) is then:

$$\int_{B(0,a)} W(z,z')S(V(z',t))\mathrm{dm}(z') = \int_0^a \int_0^{2\pi} W(r,\theta,r',\theta') \times H(V(r',\theta',t)-\kappa)\frac{r'dr'd\theta'}{(1-r'^2)^2}$$

We define $\mathcal{R}$ to be the rectangle $\mathcal{R} \stackrel{def}{=} [0,a] \times [0,2\pi]$.
We discretize $\mathcal{R}$ in order to turn (12) into a finite number of equations. For this purpose we introduce $h_1 = \frac{a}{N}$, $N \in \mathbb{N}^* = \mathbb{N}\backslash\{0\}$ and $h_2 = \frac{2\pi}{M}$, $M \in \mathbb{N}^*$,

$$\forall i \in [\![1, N+1]\!] \quad r_i = (i-1)h_1,$$

$$\forall j \in [\![1, M+1]\!] \quad \theta_j = (j-1)h_2,$$

and obtain the $(N+1)(M+1)$ equations:

$$\frac{dV}{dt}(r_i,\theta_j,t) = -\alpha V(r_i,\theta_j,t) + \int_{\mathcal{R}} W(r_i,\theta_j,r',\theta') \times H(V(r',\theta',t)-\kappa)\frac{r'dr'd\theta'}{(1-r'^2)^2} + I(r_i,\theta_j)$$

which define the discretization of (12):

$$\begin{cases} \frac{d\tilde{V}}{dt}(t) = -\alpha\tilde{V}(t) + \mathbf{W}\cdot H(\tilde{V}-\kappa)(t) + \tilde{I} \quad t \in J \\ \tilde{V}(0) = \tilde{V}_0 \end{cases} \quad (13)$$

where $\tilde{V}(t) \in \mathcal{M}_{N+1,M+1}(\mathbb{R})$ [1], $\tilde{V}(t)_{i,j} = V(r_i,\theta_j,t)$. Similar definitions apply to $\tilde{I}$ and $\tilde{V}_0$. Moreover:

$$\mathbf{W}\cdot H(\tilde{V}-\kappa)(t)_{i,j} = \int_{\mathcal{R}} W(r_i,\theta_j,r',\theta') \times H(V(r',\theta',t)-\kappa)\frac{r'dr'd\theta'}{(1-r'^2)^2}$$

It remains to discretize the integral term. For this as in [5], we use the rectangular rule for the quadrature so that for all $(r,\theta) \in \mathcal{R}$ we have:

$$\int_0^a \int_0^{2\pi} W(r,\theta,r',\theta')H(V(r',\theta',t)-\kappa)\frac{r'dr'd\theta'}{(1-r'^2)^2} \cong h_1h_2 \sum_{k=1}^{N+1}\sum_{l=1}^{M+1} W(r,\theta,r_k,\theta_l)H(V(r_k,\theta_l,t)-\kappa)\frac{r_k}{(1-r_k^2)^2}$$

We end up with the following numerical scheme, where $\mathcal{V}_{i,j}(t)$ (resp. $\mathcal{I}_{i,j}$) is an approximation of $\tilde{V}_{i,j}(t)$ (resp. $\tilde{I}_{i,j}$), $\forall(i,j) \in [\![1,N+1]\!] \times [\![1,M+1]\!]$:

$$\frac{d\mathcal{V}_{i,j}}{dt}(t) = -\alpha\mathcal{V}_{i,j}(t) + h_1h_2 \sum_{k=1}^{N+1}\sum_{l=1}^{M+1} \widetilde{W}_{k,l}^{i,j} H(\mathcal{V}_{k,l}-\kappa)(t) + \mathcal{I}_{i,j}$$

with $\widetilde{W}_{k,l}^{i,j} \stackrel{def}{=} W(r_i,\theta_j,r_k,\theta_l)\frac{r_k}{(1-r_k^2)^2}$.

4.2 Numerical simulation

We give a numerical solution of (2) in the case where the connectivity function is an exponential function, $W(x) = e^{-\frac{|x|}{b}}$, with b a positive parameter. Only excitation is present in this case.We set $\alpha = 1$, $\kappa = 0.04$, $\omega = 0.18$. We fix the input to be of the form:

$$I(z) = \mathcal{I}e^{-\dfrac{d_2(z,0)^2}{\sigma^2}}$$

with $\mathcal{I} = 0.04$ and $\sigma = 0.05$. Then the condition of existence of a stationary pulse (8) is satisfied, see figure 1. We plot a bump solution according to (8) in figure 2.

5 Conclusion

We have presented a study of spatially localised bumps in the context of the structure tensor model in the high-gain limit of the sigmoid function. This study was based on non-Euclidean functional analysis and Helgason Fourier transform allowed us to write explicit formula for our bumps. However, it is true that networks with Heaviside nonlinearities are not very realistic from the neurobiological perspective and lead to difficult mathematical considerations. But, taking the high-gain limit is instructive since it allows the explicit construction of stationary solutions which is

[1] $\mathcal{M}_{n,p}(\mathbb{R})$ is the space of the matrices of size $n \times p$ with real coefficients.

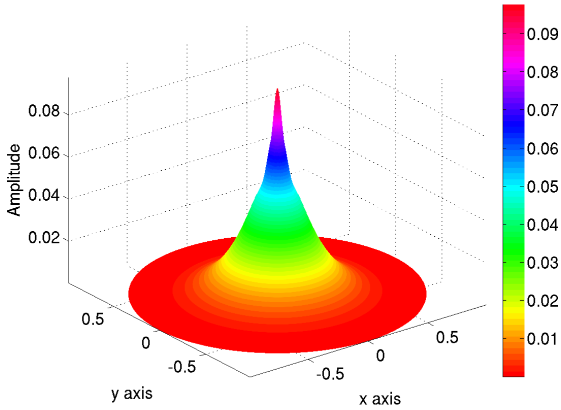

Figure 2: Plot of a bump solution of equation (7) for the values $\alpha = 1$, $\kappa = 0.04$, $\omega = 0.18$ and for $b = 0.2$ for the width of the connectivity, see text.

impossible with sigmoidal nonlinearities. We also developped a linear stability analysis adapted from [6] of what we called hyperbolic radially symmetric stationary-pulse. Finally, we illustrated our theoretical results with numerical simulations based on rigorously defined numerical schemes.

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