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On a representation of weighted distributions

Jarosław Bartoszewicz *
University of Wrocław, Poland

Abstract

The representation of weighted distributions given by Błażej [Błażej, P., Preservation of classes of life distributions under weighting with a general weight function. Statist. Probab. Lett. (2008) 78, 3056-3061] is developed. New relations between weighted distributions and classes of life distributions and stochastic orders are established.

MSC: 60E05, 60E15, 62N05

Keywords: Partial orders; Weighted distributions; Length-biased distributions; Lorenz curve; Gamma distribution; IFR; DFR; IRFR; DRFR.

*Corresponding address: Mathematical Institute, University of Wrocław, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland. E-mail address: jarbar@math.uni.wroc.pl.

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1. Preliminaries

Let X and Y be two random variables, F and G their respective probability distribution functions and f and g their respective density functions, if they exist. Denote by $\overline{F} = 1-F$ the tail (or survival function) of F, by $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $u \in (0,1)$, the quantile (or reversed) function and by $F^{-1}(0)$ and $F^{-1}(1)$ the lower and upper bounds of the support of F, respectively, for G analogously. We identify distribution functions F and G with respective probability distributions and denote their supports by S_F, S_G respectively.

We use *increasing* in place of *nondecreasing* and *decreasing* in place *nonincreasing.*

A distribution F is said to be IFR (or DFR) if $\log \overline{F}$ is concave (or convex) on S_F which is an interval. A distribution F with $S_F = [a,b]$, $-\infty \leq a < b < \infty$, is said to be IRFR (*increasing reversed failure rate*) if $\log F$ is convex on S_F. A distribution F is said to be DRFR (*decreasing reversed failure rate*) if $\log F$ is concave on S_F. It is well known that each DFR distribution is DRFR and each IRFR distribution is IFR.

We deal with some stochastic orders. Recall their definitions and some properties for completeness. Similarly to Shaked and Shanthikumar (2007) we use notation involving random variables. However stochastic orders are relations between probability distributions.

We say that X *is smaller than* Y *in the likelihood ratio order* ($X \leq_{\mathrm{lr}} Y$) if $g(x)/f(x)$ is increasing. We say that X *is smaller than* Y *in the hazard rate order* ($X \leq_{\mathrm{hr}} Y$) if $[\overline{G}(x)]/[\overline{F}(x)]$ is increasing or $r_F(x) \geq r_G(x)$ for every x if F and G are absolutely continuous, where $r_F(x) = f(x)/\overline{F}(x)]$ is *the hazard rate function* of F (for r_G analogously). We say that X *is smaller than* Y *in the reversed hazard rate order* ($X \leq_{\mathrm{rh}} Y$) if $G(x)/F(x)$ is increasing or $\breve{r}_F(x) \leq \breve{r}_G(x)$ for every x if F and G are absolutely continuous, where $\breve{r}_F(x) = f(x)/F(x)$ is *the reversed hazard rate function* of F (for $\breve{r}_G$ analogously). We say that X is *stochastically smaller than* Y ($X \leq_{\mathrm{st}} Y$) if $F(x) \geq G(x)$ for every x. We say that X *is smaller than* Y *in the dispersive order* ($X \leq_{\mathrm{disp}} Y$) if $F^{-1}(b) - F^{-1}(a) \leq G^{-1}(b) - G^{-1}(a)$ whenever $0 < a \leq b < 1$.

It is also well known that

$$\begin{array}{ccc} X \leq_{\mathrm{lr}} Y & \Rightarrow & X \leq_{\mathrm{hr}} Y \\ \Downarrow & & \Downarrow \\ X \leq_{\mathrm{rh}} Y & \Rightarrow & X \leq_{\mathrm{st}} Y \end{array}$$

Let $w : \mathbb{R} \to \mathbb{R}^+$ be a function for which $0 < E[w(X)] < \infty$.

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Such a function is called *the weight function*. Then

$$F_w(x) = \frac{1}{E[w(X)]} \int_{-\infty}^{x} w(u)dF(u) = \frac{1}{E[w(X)]} \int_{0}^{F(x)} w(F^{-1}(z))dz \tag{1}$$

is a distribution function, so called *the weighted distribution* of X with the weight function w. If a density f of F exists, then

$$f_w(x) = \frac{w(x)f(x)}{E[w(X)]}$$

is a density of F_w. If $F(0) = 0$ and $w(x) = x$, we call F_w *the length-biased* (or *size-biased*) distribution of X and denote it simply by $\hat{F}$ and $\hat{f}$ for a density). Then if $0 < E(X) < \infty$, we have

$$\hat{F}(x) = \frac{1}{E(X)} \int_{0}^{x} udF(u), \qquad x \geq 0,$$

and

$$\hat{f}(x) = \frac{xf(x)}{E(X)}, \qquad x \geq 0.$$

Denote also by X_w and $\hat{X}$ random variables with respective distributions F_w and $\hat{F}$, *the weighted version* and *the length-biased version* of X, respectively.

We refer to Patil and Rao (1977, 1978) and to Rao (1985) for a survey of statistical applications of weighted distributions, especially to the analysis of data relating to human populations and ecology. Gupta and Keating (1986) obtained relations for reliability measures of the length-biased distribution and some characterization results. Kochar and Gupta (1987) studied properties of weighted distributions in comparison with those of the original distributions for positive random variables and obtained bounds on the moments of weighted distributions. Jain et al. (1989) and Nanda and Jain (1999) studied relations of weighted distributions with classes of life distributions. Navarro et al. (2001) developed characterizations through reliability measures from weighted distributions. Belzunce et al. (2004) studied relations of weighted distributions with stochastic orders and classes of distributions generated by measures of uncertainty. Bartoszewicz and Skolimowska (2006), Misra et al. (2008) and Błażej (2008) studied preservation of stochastic orders and classes of life distribution under weighting.

2. Representations of weighted distributions

We start with a representations of weighted distributions given by Błażej (2008). Let w be a weight function. From (1) we have that $F_w(x) = F^*(F(x))$, where

$$F^*(u) = \frac{1}{E(W)} \int_{0}^{u} w(F^{-1}(z))dz \tag{2}$$

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is an absolutely continuous distribution function on the interval $[0,1]$. Particularly, if w is increasing left-continuous, $F^*(u) = L_W(u)$ and if w is decreasing left-continuous, $F^*(u) = 1 - L_W(1-u)$, where L_W is the Lorenz curve of W, see Bartoszewicz and Skolimowska (2006).

Using the Błażej's result we will derive a simple relation between weighting and monotone transformations.

Theorem 1. *Let $w : \mathbf{R} \to \mathbf{R}^+$ be a weight function of the form $w(x) = \phi(v(x))$, where v is a strictly monotone left-continuous function. Then*

$$X_w =^{\mathrm{st}} v^{-1}(V_\phi)$$

or equivalently

$$v(X_w) =^{\mathrm{st}} V_\phi,$$

where $V = v(X)$ and V_ϕ is the weighted version of V with the weight function ϕ.

Proof. (a) Suppose v is increasing. From (2) it follows, that the distribution F^* has a density

$$f^*(x) = \frac{w(F^{-1}(x))}{E[w(X)]} = \frac{\phi(v(F^{-1}(x)))}{E[w(X)]}.$$

Denote by H the distribution function of V. It is clear that $H(x) = F(v^{-1}(x))$. Let H_ϕ be the weighted distribution of V with the weight function ϕ. Thus from (1) and (2) we have

$$H_\phi(x) = H^*(H(x)) = H^*(F(v^{-1}(x))), \tag{3}$$

where H^* is an absolutely continuous distribution on $[0,1]$ with density

$$h^*(u) = \frac{\phi(H^{-1}(u))}{E[\phi(V)]} = \frac{\phi(v(F^{-1}(u))}{E[w(X)]} = f^*(u), \qquad u \in [0,1],$$

i.e. $F^* = H^*$. Then from (3) we have

$$P\{v^{-1}(V_\phi) \le x\} = P\{V_\phi \le v(x)\} = H_\phi(v(x)) = H^*(F(x)) = F^*(F(x)) = P\{X_w \le x\},$$

i.e. $X_w =^{\mathrm{st}} v^{-1}(V_\phi)$.

(b) Let now v be decreasing. Then $H(x) = 1 - F(v^{-1}(x))$, $H_\phi(x) = H^*(1 - F(v^{-1}(x)))$ and

$$h^*(u) = \frac{\phi(H^{-1}(u))}{E[\phi(V)]} = \frac{\phi(v(F^{-1}(1-u)))}{E[w(X)]} = f^*(1-u), \qquad u \in [0,1].$$

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Therefore $H^*(u) = 1 - F^*(1-u)$, $u \in [0,1]$, and now we obtain

$$P\{v^{-1}(V_\phi) \le x\} = P\{V_\phi \ge v(x)\} = 1 - H_\phi(v(x) - 0) =$$

$$1 - H^*(1 - F(x)) = 1 - [1 - F^*(1 - [1 - F(x)])] = F^*(F(x)) = P\{X_w \le x\},$$

i.e. $X_w =^{\mathrm{st}} v^{-1}(V_\phi)$. □

Particularly, from Theorem 1 it follows that for the monotone left-continuous function w and the length-biased version of W we have $X_w =^{\mathrm{st}} w^{-1}(\hat{W})$

From the proof of Theorem 1 it follows that the Błażej's representation is *invariant* with respect to monotone transformations of X. Theorem 1 of Błażej (2008) may be reformulated in the following way.

THEOREM 2. *Let $w(x) = \phi(v(x))$ be a weight function, where v is strictly monotone left-continuous. Let H be the distribution function of $v(X)$ and $H_\phi(x) = H^*(H(x))$ be the weighted distribution of V with the weight function ϕ, where H^* is an absolutely continuous distribution on* $[0,1]$.

A. *If v is increasing, then:*

(a) *$X \le_{\mathrm{lr}} X_w$, $(X_w \le_{\mathrm{lr}} X)$ iff H^* is convex (concave) on* $[0,1]$*;*

(b) *$X \le_{\mathrm{hr}} X_w$, $(X_w \le_{\mathrm{hr}} X)$ iff $1 - H^*(1-u)$ is anti-star shaped (star shaped) on* $[0,1]$*;*

(c) *$X \le_{\mathrm{rh}} X_w$, $(X_w \le_{\mathrm{rh}} X)$ iff H^* is star shaped (anti-star shaped) on* $[0,1]$*;*

(b) *$X \le_{\mathrm{st}} X_w$, $(X_w \le_{\mathrm{st}} X)$ iff $H^*(u) \le u$ $(H^*(u) \ge u)$ on* $[0,1]$.

B. *If v is decreasing, then:*

(a) *$X \le_{\mathrm{lr}} X_w$, $(X_w \le_{\mathrm{lr}} X)$ iff $1 - H^*(1-u)$ is convex (concave) on* $[0,1]$*;*

(b) *$X \le_{\mathrm{hr}} X_w$, $(X_w \le_{\mathrm{hr}} X)$ iff H^* is anti-star shaped (star shaped) on* $[0,1]$*;*

(c) *$X \le_{\mathrm{rh}} X_w$, $(X_w \le_{\mathrm{rh}} X)$ iff $1 - H^*(1-u)$ is star shaped (anti-star shaped) on* $[0,1]$*;*

(b) *$X \le_{\mathrm{st}} X_w$, $(X_w \le_{\mathrm{st}} X)$ iff $H^*(u) \ge u$ $(H^*(u) \le u)$ on* $[0,1]$.

3. Applications of the result

We will apply Theorem 1 for proving some properties of stochastic order relations between weighted distributions and classes of life distributions.

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3.1. Dispersive ordering of weighted distributions

Bartoszewicz and Skolimowska (2006) and Misra et al (2008) studied preservation of the dispersive order under weighting. We may apply Theorem 1 to obtain a new result. First we give the following useful lemmas.

LEMMA 1. (Bartoszewicz, 1985, Bagai and Kochar, 1986) *Let X and Y be two nonnegative random variables.*
(a) *If $X \leq_{\text{hr}} Y$ and F or G is DFR, then $X \leq_{\text{disp}} Y$.*
(b) *If $X \leq_{\text{disp}} Y$ and F or G is IFR, then $X \leq_{\text{hr}} Y$.*
LEMMA 2. (Bartoszewicz and Skolimowska, 2006) *Let F and G be absolutely continuous. If $X \leq_{\text{rh}} Y$ and $\breve{r}_G(x)/\breve{r}_F(x)$ is increasing, then $X_w \leq_{\text{lr}} Y_w$ and hence $X_w \leq_{\text{rh}} Y_w$ and $X_w \leq_{\text{hr}} Y_w$.*
LEMMA 3. *If a random variable X has an IRFR distribution and v is a decreasing convex function on S_F, then the distribution of $v(X)$ is DFR.*

PROOF. We have $H(x) = P\{v(X) \leq x\} = P\{X \geq v^{-1}(x)) = 1 - F(v^{-1}(x))$. Then $\ln[1-H(x)] = \ln F(v^{-1}(x))$ is convex as a composition of two monotone convex functions, i.e. H is DFR. □

LEMMA 4. (Misra et al. 2008) *If F is DFR and the weight function w is increasing and log-convex on S_F, then F_w is DFR.*

THEOREM 3. *Let F and G be absolutely continuous, F be DFR and G be IRFR. Let w be a weight function being of the form $w(x) = \phi(v(x))$, where v is positive decreasing log-convex on $A = S_F \cup S_G$ and ϕ is positive increasing log-convex on the set $v(A)$. If $X \leq_{\text{disp}} Y$, then $X_w \leq_{\text{disp}} Y_w$.*

PROOF. It is well known that if G is IRFR then it is IFR. Since $X \leq_{\text{disp}} Y$ and G is IFR, then by Lemma 1(b) we have $X \leq_{\text{hr}} Y$ and then $v(Y) \leq_{\text{rh}} v(X)$ (see Shaked and Shanthikumar, 2007, Theorem 1.C.8). Denote by F^v and G^v distribution functions of $v(X)$ and $v(Y)$ respectively. Easy calculations show that

$$\frac{\breve{r}_{F^v}(x)}{\breve{r}_{G^v}(x)} = \frac{r_F(v^{-1}(x))}{r_G(v^{-1}(x))}$$

and it is increasing. Therefore from Lemma 2 we have

$$[v(Y)]_\phi \leq_{\text{hr}} [v(X)]_\phi, \tag{4}$$

where $[v(X)]_\phi$ is the weighted version of $[v(X)]$ with the weight function ϕ, for $[v(Y)]_\phi$ similarly. Since v is log-convex decreasing, it is also convex decreasing, and then by

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Lemma 3 it follows that $v(Y)$ has a DFR distribution and next from Lemma 4, $[v(Y)]_\phi$ has also a DFR distribution. Therefore from Lemma 1(a) and (4) we obtain

$$[v(Y)]_\phi \leq_{\rm disp} [v(X)]_\phi.$$

Since (4) implies $[v(Y)]_\phi \leq_{\rm st} [v(X)]_\phi$, and v^{-1} is decreasing convex, then from Theorem 3.B.10(b) of Shaked and Shanthikumar (2007) we obtain

$$v^{-1}([v(X)]_\phi) \leq_{\rm disp} v^{-1}([v(Y)]_\phi),$$

which is equivalent to $\quad X_w \leq_{\rm disp} Y_w \quad$ by Theorem 1. □

3.2. A property of the gamma distribution

It is easy to prove the following facts: if the distribution F of X is IFR (DFR) and v is an increasing concave (convex) function on S_F, then $v(X)$ has an IFR (DFR) distribution. Therefore it follows: if a random variable Z has the gamma distribution with shape parameter $p \geq 1$ and $\alpha \in (0,1]$, then the distribution of Z^α is IFR and if Z has the gamma distribution with with shape parameter $p \in (0,1]$ and $\alpha \in [1,\infty)$, then the distribution of Z^α is DFR.

We will apply Theorem 1 for proving distribution results for powers of gamma random variables. The following lemma will be useful.

Lemma 5. (Bartoszewicz and Skolimowska, 2006) *Let F be absolutely continuous and w be a monotone left continuous function weight function.*
(a) *If $w(x)$ is increasing and $w(x)r_F(x)$ is decreasing, then F_w is DFR.*
(b) *If $w(x)$ is decreasing and $w(x)r_F(x)$ is increasing, then F_w is IFR.*
(c) *If $w(x)$ is increasing and $w(x)\breve{r}_F(x)$ is decreasing, then F_w is DRFR.*
(d) *If $w(x)r_F(x)$ is decreasing, then F_w is DRFR.*

Theorem 4. *Let Z be a random variable with the gamma distribution with shape parameter $p > 0$ and scale parameter 1.*
(a) *If $0 < \alpha \leq 1$ and $0 \leq \alpha(p-1) \leq 1-\alpha$, then the distribution of $Z^{1/\alpha}$ is DFR.*
(b) *If $1 \leq \alpha < \infty$ and $1 \leq \alpha p \leq \alpha$, then the distribution of $Z^{1/\alpha}$ is IFR.*
(c) *If $\alpha(p-1) < 0 < \alpha$, then the distribution of $Z^{1/\alpha}$ is DRFR.*
(d) *If $\alpha(p-1) < 0 < \alpha$, then the distribution of $Z^{-1/\alpha}$ is DRFR.*

Proof. Let X have the exponential distribution with mean 1. Let $v(x) = x^{1/\alpha}$, $\alpha > 0$. It is well known that $V = v(X)$ has the Weibull distribution $H(x) = 1 - e^{-x^\alpha}$, $x > 0$, with

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the failure rate function $r_H(x) = \alpha x^{\alpha-1}$, which is decreasing for $0 < \alpha \le 1$ and increasing for $\alpha \ge 1$.

Let $\phi(v) = v^{\alpha(p-1)}$. Thus $w(x) = \phi(v(x)) = x^{p-1}$ and it is easily seen that $X_w =^{\mathrm{st}} Z$. If $0 < \alpha \le 1$ and $0 \le \alpha(p-1) \le 1-\alpha$, then $\phi(x)$ is increasing and $\phi(x)r_H(x)$ is decreasing and hence from Lemma 5(a) (applied to H and ϕ), V_ϕ has a DFR distribution. Applying Theorem 1 we obtain Theorem 4(a).

The proofs of theorems (b) and (c) are similar. In the case (b), $\phi(x)$ is decreasing but $\phi(x)r_H(x)$ is increasing and the result follows from Lemma 5(b) and Theorem 1. In the case (c), $\phi(x)r_H(x)$ is decreasing and the result follows from Lemma 5(d) and Theorem 1.

Let now $v(x) = x^{-1/\alpha}$, $\alpha > 0$. Then $V = v(X)$ has the type II extreme value distribution (see Pal et al., 2006) with density

$$h(x) = \alpha x^{-(\alpha+1)} \exp(-x^{-\alpha}) \qquad x > 0,$$

and the reversed failure rate $\breve{r}_G(x) = \alpha x^{-(\alpha+1)}$, which is decreasing for $\alpha > 0$. Let now $\phi(v) = v^{\alpha(1-p)}$ and then $w(x) = x^{p-1}$, which implies that $X_w =^{\mathrm{st}} Z$ also. Otherwise $\phi(x)\breve{r}_G(x) = \alpha x^{-(\alpha p+1)}$ is decreasing and Theorem 3(d) follows from Lemma 5(c) and Theorem 1. $\square$

Remark. If we take, for example, $\alpha = 1/2$ and $p = 3/2$, we have from Theorem 4(a) that an increasing convex transformation of an IFR random variable may have a DFR distribution. Similarly, if we take $\alpha = 2$ and $p = 1/2$, we obtain from Theorem 4(b) that a concave transformation of a DFR random variable may be IFR distributed. Thus the reversed statements to those at the beginning of this subsection are not true.

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