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2D electrostatic problems with rounded corners

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Abstract—The second order terms of a multiscale expansion
for dealing with rounded corners in 2D electrostatic problems are
studied. The main ideas and the sequence of problems to consider
are recalled and finite element simulations are presented to assess
the accuracy of the method.

I. INTRODUCTION

High-voltage applications require a precise knowledge of
the electric field in the area where the geometry of the
structure is sharp. On a real device, the geometry is not
“exactly sharp” but present rounded edges or corners. The
accurate description of these rounded shapes, especially when
the geometry involves several corners, can be cumbersome in
a numerical model. In addition most of the time only a rough
(statistical) description of the rounded shape is available due
to the manufacturing tolerance. Dauge et al. have proposed
in [1] a theoretical approach to tackle the “rounded shape”
problems by an accurate asymptotic analysis.

Define the potentials \( v_\epsilon \) in \( \Omega_\epsilon \) and \( v_0 \) in \( \Omega \) (see Fig. 1) by

\[
\begin{align*}
\triangle v_\epsilon &= 0, & \text{on } \partial \Omega_\epsilon \\
v_\epsilon &= 0, & \text{on } \partial \Omega_0 \\
v_\epsilon &= 1, & \text{on } \Gamma_\epsilon \\
\partial_\nu v_\epsilon &= 0, & \text{on } \Gamma^N 
\end{align*}
\]

and

\[
\begin{align*}
\triangle v_0 &= 0, & \text{on } \partial \Omega \\
v_0 &= 0, & \text{on } \partial \Omega^0 \\
v_0 &= 1, & \text{on } \Gamma^1 \\
\partial_\nu v_0 &= 0, & \text{on } \Gamma^N
\end{align*}
\]

where \( \epsilon \) characterizes the “size” of the rounded corner and \( \partial_\nu \) denotes the unitary outward normal on the
boundary of the domain. Throughout the paper \( \omega \) denotes the angle of the sharp corner and \( \alpha = \pi/\omega \). The main idea of [1]
consists in expanding \( v_\epsilon \) into two sums in power of \( \epsilon^n \). Using a smooth radial cut-off function \( \varphi \) defined by

\[
\varphi(\rho) = \begin{cases} 
1, & \text{if } \rho \geq d_1 \\
0, & \text{if } \rho \leq d_0 
\end{cases}, \quad \text{with } d_0 < d_1
\]

\( d_0, d_1 \) being fixed corner distances, the expansion writes for
any integer \( n \geq 1 \) (\( \varphi(\cdot/\epsilon) \) is the function \( t \mapsto \varphi(t/\epsilon) \))

\[
v_\epsilon = \varphi(\cdot/\epsilon) v_0 + \sum_{p=1}^{N} b_p \epsilon^{\alpha_p} v_{p\alpha} + (1 - \varphi) \sum_{p=1}^{N} B_p \epsilon^{\alpha_p} V_{p\alpha} + r_{n\alpha}^r,
\]

where \( b_p \) and \( B_p \) are real parameters and \( r_{n\alpha}^r \) is such that

\[
\exists \tilde{\epsilon} > 0, \, \exists C_n > 0, \, \forall \epsilon < \tilde{\epsilon}, \, \sqrt{\int_{\Omega} \| \nabla r_{n\alpha}^r \|^2 dx} < C_n \epsilon^{(n+1)\alpha},
\]

i.e. the energy norm of the error converges as \( \epsilon^{(n+1)\alpha} \) to 0, which is written \( r_{n\alpha}^r = O_H(\epsilon^{(n+1)\alpha}) \). The functions \( v_{p\alpha} \),
satisfy a boundary value problem in the sharp domain \( \Omega \) (see
Fig 1(b)), whereas the functions \( V_{p\alpha} \) are the so-called profile
terms that satisfy a boundary value problem in the infinite domain
\( \Omega_\infty \) of \( \mathbb{R}^2 \), which is a localization of the rounded corner (see
Fig 1(c)).

The paper [2] was devoted to provide numerically the first order
terms of the theoretical expansion of [1]. The following
remarks have been observed:

- The exact solutions close to the corner, computed for
  several values of the curvature radius \( \epsilon \), are quasi-similar,
  up to a “scaling factor” (related to \( \epsilon \)). It is also noticed
  that the “shape” of the solutions close to the corner (their
  “shape” but not their amplitude) weakly depend on other
  elements of the studied structure, such as the distance to the
  boundaries: if the corner geometry is self-similar, it is
  also said that the dominant term of the solutions close to
  the corner is self-similar.

- The exact solutions far from the corner are weakly influ-
  enced by the change of the curvature radius \( \epsilon \), and they
  converge to the solution on the domain with the sharp
  corner when \( \epsilon \) goes to zero.

The aim of the present paper is to push forward numerically the
expansion (3) on the numerical model studied in [2],
especially when non-symmetric structures are involved.

†Roughly, it means that a single parameter, here \( \epsilon \), and a basic geometry are
sufficient to describe the corners for any value of \( \epsilon \). For a precise definition,
refer to [1].
II. SECOND ORDER EXPANSION

The heuristics for the construction of the two first orders are detailed in [1, subsection 4.1]. The roughest approximation \( v_0 \) of \( v_\epsilon \), far from the corner, that is defined in \( \Omega \) by (1), writes in the neighborhood of the corner

\[
v_0(x) \approx \sum_{p=1}^{\infty} a_p \rho^{p\alpha} \sin(p\alpha \theta) = \sum_{p=1}^{\infty} a_p \rho^{p\alpha}(\rho, \theta),
\]

where \((\rho, \theta)\) are the polar coordinates. As the behaviors of \( v_0 \) and \( v_\epsilon \) are different in the corner, \( v_0 \) should be truncated in the corner. Expansion (5) enforces the coefficient \( B_1 \) of (3) to equal the first singular coefficient \( a_1 \), while the profile term \( V_\alpha \) satisfies, for \( p = 1 \),

\[
\begin{cases}
-\Delta X V_{p\alpha} = [\triangle X; \varphi] s^{p\alpha}, \quad \text{in } \Omega_\infty, \\
v_{p\alpha} |_{\Gamma_\infty} = 0, \\
\lim_{R \to +\infty} V_{p\alpha} = 0.
\end{cases}
\]

For any couple \((\nu, u)\), we remind that \( [\Delta; \nu] u = \Delta(\nu u) - \nu \Delta u \).

In [1], it is especially shown that in the neighborhood of \(+\infty\)

\[
V_\alpha(X) \approx \sum_{p=-1}^{+\infty} A_p s^{-p\alpha}(X).
\]

Then, necessarily, \( b_2 \) and \( B_2 \) respectively equal \( a_1 A_1 \) and \( a_2 \) given by (5) and (7), leading to the second order:

\[
v_\epsilon = \varphi \left( \frac{1}{\epsilon} \right) v_0 + \epsilon \alpha (1 - \varphi) a_1 V_\alpha \left( \frac{1}{\epsilon} \right) \\
+ \epsilon^2 a_1 A_1 v_{2\alpha} + (1 - \varphi) a_2 v_{2\alpha} \left( \frac{1}{\epsilon} \right) + O_H(\epsilon^{3\alpha}),
\]

where \( V_{2\alpha} \) is the profile term that satisfies (6) for \( p = 2 \) and \( v_{2\alpha} \) is the correction far from the corner defined by

\[
\begin{cases}
-\triangle v_{2\alpha} = [\triangle; (1 - \varphi)] s^{-\alpha}, \quad \text{on } \Omega, \\
v_{2\alpha} = 0, \quad \text{on } \Gamma_0, \quad \text{and } u = 0, \quad \text{on } \Gamma^1, \\
\partial_\nu v_{2\alpha} = 0, \quad \text{on } \Gamma^N.
\end{cases}
\]

In [2, page 9], the profile problem (25) satisfied by \( v_\infty \) did not involve the cut-off function \( \varphi \) as in (6). Actually, the profiles \( V_\alpha \) and \( v_\infty \) can be linked through the equality

\[
V_\alpha = v_\infty - \varphi s^{\alpha}.
\]

However, for profiles of order \( p > 2 \), the relationship is more complex and the use of the cut-off function seems to be more convenient [1].

III. NUMERICAL RESULTS

Two L-shape geometries with a rounded corner are considered; their dimensions are specified on Fig. 2. For both cases, \( \alpha \) equals \( 2/3 \). The finite element method has been used for solving (1), (6), and (9) as it was done in [2].

The errors in the energy norm of the first and second order expansions, plotted on Fig. 3, behave respectively as \( \epsilon^{2\alpha} \) and \( \epsilon^{3\alpha} \), independently of the geometry. This is remarkable since \( a_2 = 0 \) in the symmetric configuration (see Fig. 2(a)) and we could think that the first order approximation would have lead to a better approximation than in the non symmetric configuration (see Fig. 2(b)). Nonetheless, this intuition is obviously not correct and the correction \( v_{2\alpha} \) far from the corner plays an equivalent role than the correction close to the corner regarding the energy norm in the whole domain.

The normal electric field along the electrodes in the non symmetric configuration (from the bottom right to the top left of the electrode, see Fig. 2(b)) for two values of \( \epsilon \) are presented in Fig. 4. This normal field has been computed respectively from \( v_\epsilon \), from first order expansion and from second order expansion given by (8). The behavior of the normal field is “closer” for the order 2, in particular the location of the maximum is roughly equivalent to the correct solution. However the maximum electric field is overestimated by both approximations, requiring to go further in the expansion.

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