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# COMPUTATIONAL COMPLEXITY OF AVALANCHES IN THE KADANOFF TWO-DIMENSIONAL SANDPILE MODEL

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Abstract. In this paper we prove that the *avalanche problem* for the Kadanoff sandpile model (KSPM) is P-complete for two-dimensions. Our proof is based on a reduction from the monotone circuit value problem by building logic gates and wires which work with configurations in KSPM. The proof is also related to the known *prediction problem* for sandpile which is in NC for one-dimensional sandpiles and is P-complete for dimension 3 or greater. The computational complexity of the prediction problem remains open for two-dimensional sandpiles.

## 1. Introduction

Predicting the behavior of discrete dynamical systems is, in general, both the "most wanted" and the hardest task. Moreover, the difficulty is still hard when considering finite phase spaces. Indeed, when the system is not solvable, numerical simulation is the only possibility to compute future states of the system.

In this paper we consider the well-known discrete dynamical system of sandpiles (SPM). Roughly speaking, its dynamics is as follows. Consider the toppling of grains of sand on a (clean) flat surface, one by one. After a while, a sandpile has formed. At this point, the simple addition of even a single grain may cause avalanches of grains to fall down along the sides of the sandpile. Then, the growth process of the sandpile starts again. Remark that this process can be naturally extended to arbitrary dimensions although for $d > 3$, the physical meaning is not clear.

The first complexity results about SPM appeared in [6, 7] where the authors proved the computation universality of SPM. For that, they modelled wires and logic gates with sandpiles configurations. Inspired by these constructions, C. Moore and M. Nilsson considered the *prediction problem* (PRED) for SPM *i.e.* the problem of computing the stable configuration (fixed point) starting from a given initial configuration of the sandpile. C. Moore and M. Nilsson proved that PRED is in $\mathsf{NC}^3$ for dimension 1 and that it is P-complete for $d \geq 3$ leaving $d = 2$ as an open

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problem [12]. (Recall that P-completeness plays for parallel computation a role comparable to NP-completeness for non-deterministic computation. It corresponds to problems which cannot be solved efficiently in parallel (see [9]) or, equivalently, which are *inherently sequential*, unless P =NC). Later, P.B. Miltersen improved the bound for $d = 1$ showing that PRED is in LOGDCFL ($\subseteq$ $\mathsf{AC}^1$) and that it is not in $\mathsf{AC}^{1-\varepsilon}$ for any $\varepsilon > 0$ [11]. Therefore, in any case, one-dimensional sandpiles are capable of (very) elementary computations such as computing the max of $n$ bits.

Both C. Moore and P.B. Miltersen underline that *"having a better upper-bound than P for PRED for two-dimensional sandpiles would be most interesting."*

In this paper, we address a slightly different problem: the avalanche problem (AP). Here, we start with a monotone configuration of the sandpile. We add a grain of sand to the initial pile. This eventually causes an avalanche (a sequence of topples) and we address the question of the complexity of deciding whether a certain given position –initially with no grain of sand– will receive some grains in the future. Like for the (PRED) problem, (AP) can be formulated in higher dimensions. In order to get acquainted with AP, we introduce its one-dimensional version first.

One-dimensional sandpiles can be conveniently represented by a finite sequence of integers $x_1, x_2, \ldots, \ldots, x_n$. The sandpiles are represented as a sequence of *piles* and each $x_i$ represents the number of grains contained in pile $i$. In the classical SPM, a grain falls from pile $i$ to $i+1$ if and only if the height difference $x_i - x_{i+1} \geq 2$. Kadanoff's sandpile model (KSPM) generalises SPM [10, 5] by adding a parameter $p$. The setting is the same except for the local rule: one grain falls to the $p-1$ adjacent piles if the difference between pile $i$ and $i+1$ is greater than $p$.

Assume $x_k = 0$, for a value of $k$ "far away" from the sandpile. The avalanche problem asks whether adding a grain at pile $x_1$ will cause an avalanche such that at some point in the future $x_k \geq 1$, that is to say that an avalanche is triggered and reaches the "flat" surface at the bottom.

This problem can be generalized for two-dimensional sandpiles and is related to the question addressed by C. Moore and P.B. Miltersen.

In this paper we prove that in the two-dimensional case, AP is P-complete. The proof is obtained by reduction from the Circuit Value Problem where the circuit only contains monotone gates — that is, AND's and OR's (see Section 3 for details).

We stress that our proof for the two-dimensional case needs some further hypothesis/constraints for monotonicity and determinism (see Section 3). If both properties are technical requirements for the proof's sake, monotonicity also has a physical justification. Indeed, if KSPM is used for modelling real physical sandpiles, then the image of a monotone non-increasing configuration has to be monotone non-increasing since gravity is the only force considered here. We have chosen to design the Kadanoff dynamics for $d = 2$ by considering a certain definition of the three-dimensional sandpile which does not correspond to the one of Bak's *et al.* in [1]. This hypothesis is not restrictive. It is just used for constructing the transition rules. Bak's construction was done similarly. Nevertheless, our result depends on the way the three dimensional sandpile is modelled. In our case, we have decided to formalise the sandpile as a monotone decreasing pile in three dimensions where $x_{i,j} \geq \max\{x_{i+1,j}, x_{i,j+1}\}$ (here $x_{i,j}$ denotes the sand grains initial distribution) together with Kadanoff's avalanche dynamics ruled by parameter $p$. The pile $(i, j)$ can give a grain either to every pile $(i+1, j), \ldots, (i+p-1, j)$ or to every pile $(i, j+1), \ldots, (i, j+p-1)$ if the monotonicity is not violated. With such a rule and if we use the height difference for defining the monotonicity, we can define the transition rules of the dynamics for every value of the parameter $p$.

In the case where the value of the parameter $p$ equals 2, we find in our definition of monotonicity something similar with Bak's SPM in two dimensions. Actually, both models are different because the definitions of the three dimensional piles differ. That is the reason why we succeed in proving the $\mathsf{P}$-completeness result which remains an open problem with Bak's definition.

The paper is organized as follows. Section 2 introduces the definitions of the Kadanoff sandpile model in one dimension and presents the avalanche problem. Section 3 generalizes the Kadanoff sandpile model in two dimensions and presents the avalanche problem in two dimension, which is proved $\mathsf{P}$-complete for any value of the Kadanoff parameter $p$. Finally, Section 4 concludes the paper and proposes further research directions.

## 2. Sandpiles and Kadanoff model in one dimension

A sandpile *configuration* is a distribution of sand grains over a lattice (here $\mathbb{Z}$). Each pile of the lattice is associated with an integer which represents its sand content. A finite configuration on $\mathbb{Z}$ can be identified with an ordered sequence of integers ${}^{\omega}x_1, x_2, \ldots, x_n^{\omega}$ in which $x_1$ (resp. $x_n$) is the first (resp. the last) pile such that all the piles on the left of $x_1$ equal $x_1$ (resp. all the piles on the right of $x_n$ equal $x_n$). Given a configuration $x$, $a \in \mathbb{N}$ and $j \in \mathbb{Z}$, we use the notation ${}^{\omega}ax_j$ (resp. $x_ja^{\omega}$) to say that $\forall i \in \mathbb{Z}$, $i < j \Rightarrow x_i = a$ (resp. $\forall i \in \mathbb{Z}$, $i > j \Rightarrow x_i = a$).

Remark that any configuration $x \equiv {}^{\omega}x_1, x_2, \ldots, x_{n-1}, x_n^{\omega}$ can be identified with its *heights differences* sequence

$$ {}^{\omega}0, h_1 = (x_1 - x_2), \ldots, h_{n-1} = (x_{n-1} - x_n), 0^{\omega} \ , $$

$n$ will be referred to as the *length* of the configuration and it is denoted $|x|$. In other words, we associate the initial (infinite) configuration with a finite sequence of integers $h_1, h_2, \ldots, h_{|x|-1}$. This latter representation is more convenient and is widely used in the sequel. A configuration is *finite* if only a finite number of its heights differences sequence has non-zero sand content.

A configuration $x$ is *monotone* if the sequence of its heights differences is monotone *i.e.* $\forall i \in \{1, 2, \ldots, |x| - 1\}$, $h_i \geq 0$. A monotone configuration $x$ is *stable* if the sequence of its heights differences is stable, *i.e.* $\forall i \in \{1, 2, \ldots, |x| - 1\}$, $h_i < p$ *i.e.* if the difference between any two adjacent piles is less than Kadanoff's parameter $p$. Let $\mathrm{SM}(n)$ denote the set of stable monotone configurations of the form ${}^{\omega}x_1, x_2, \ldots, x_{n-1}, x_n^{\omega}$ and of length $n$, for $x_i \in \mathbb{N}$.

Consider a stable monotone configuration ${}^{\omega}x_1, x_2, \ldots, x_n^{\omega}$. Adding one more sand grain, say at pile $i$, may cause that the pile $i$ topples some grains to its adjacent piles. In their turn the adjacent piles receive a new grain and may also topple, and so on. This phenomenon is called an *avalanche* which ends when the system evolves to a new stable configuration.

In this paper, topplings are controlled by the *Kadanoff's parameter* $p \in \mathbb{N}, p \geq 2$ which completely determines the model and its dynamics. In $\mathrm{KSPM}(p)$, $p-1$ grains will fall from pile $i$ if $h_i = (x_i - x_{i+1}) \geq p$ and the new configuration becomes

$$ {}^{\omega}x_1 \cdots (x_{i-1})(x_i - p + 1)(x_{i+1} + 1) \cdots (x_{i+p-2} + 1)(x_{i+p-1} + 1)(x_{i+p}) \cdots x_n 0^{\omega} \ . $$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_1$+1 | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | 0 | 0 | |
| $x_1$-1 | $x_2$+1 | $x_3$+1 | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | 0 | 0 | |
| $x_1$-1 | $x_2$+1 | $x_3$-1 | $x_4$+1 | $x_5$+1 | $x_6$ | $x_7$ | $x_8$ | $x_9$ | 0 | 0 | |
| $x_1$-1 | $x_2$+1 | $x_3$-1 | $x_4$+1 | $x_5$-1 | $x_6$+1 | $x_7$+1 | $x_8$ | $x_9$ | 0 | 0 | |
| $x_1$-1 | $x_2$+1 | $x_3$-1 | $x_4$+1 | $x_5$-1 | $x_6$+1 | $x_7$-1 | $x_8$+1 | $x_9$+1 | 0 | 0 | |
| $x_1$-1 | $x_2$+1 | $x_3$-1 | $x_4$+1 | $x_5$-1 | $x_6$+1 | $x_7$-1 | $x_8$+1 | $x_9$-1 | 1 | 1 | |
| $x_1$-1 | $x_2$-1 | $x_3$ | $x_4$+2 | $x_5$-1 | $x_6$+1 | $x_7$-1 | $x_8$+1 | $x_9$-1 | 1 | 1 | |
| $x_1$-1 | $x_2$-1 | $x_3$ | $x_4$ | $x_5$ | $x_6$+2 | $x_7$-1 | $x_8$+1 | $x_9$-1 | 1 | 1 | |
| $x_1$-1 | $x_2$-1 | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$+2 | $x_9$-1 | 1 | 1 | |
| $x_1$-1 | $x_2$-1 | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | 2 | 1 | |

Figure 1: Avalanches for $p = 3$ with 9 piles. Here, $x_i+1$ (resp. $x_i+2$) indicates that pile $i$ has received some grains once (resp. twice), $x_i - 1$ that pile $i$ has given some grains according to the dynamics; a dark shaded pile indicates the toppling pile, a light shaded pile indicates a pile that could topple in the future. Times goes top-down.

In other words, the pile $i$ distributes one grain to each of its $(p-1)$ right adjacent piles. Equivalently, if we measure the heights differences after applying the dynamics, we get $(h_{i-1}+p-1)(h_i-p)(h_{i+1})(h_{i+2})\cdots(h_{i+p-2})(h_{i+p-1}+1)$, and all remaining heights do not change. In other words, the height difference $h_i$ gives raise to an increase of $(p-1)$ grains of sand to height $h_{i-1}$, a decrease of $p$ grains to height $h_i$ and an increase of one grain to height $h_{i+p-1}$.

We consider the problem of deciding whether some pile on the right of pile $x_n$ (more precisely for $x_k$ for $n < k \leq n+p-1$) will receive some grains according to the Kadanoff's dynamics. Since the initial configuration is stable, it is not difficult to prove that avalanches will reach at most the pile $n+p-1$ (see Fig. 1 for example).

Remark that given a configuration, several piles could topple at the same time. Therefore, at each time step, one might have to decide which pile or piles are allowed to topple. According to the update policy chosen, there might be different images of the same configuration. However, it is known [8] that for any given initial number of sand grains $n$, the orbit graph is a lattice and hence, for our purposes, we may only consider one decision problem to formalize AP:

**Problem AP**

**Instance:** A configuration $x \in \mathrm{SM}(n)$ and $k \in \mathbb{N}$ s.t. $n < k \leq n+p-1$.
**Question:** Does there exist an avalanche such that $x_k \geq 1$?

Let us consider some examples. Let $p = 3$ and consider a stable configuration whose height differences are as follows ${}^{\omega}00\underline{2}2022120000^{\omega}$. We add a single grain at $x_1$ (underlined in the configuration). Then, the next step should probably be ${}^{\omega}02\underline{0}21222120000^{\omega}$. In one step we see that no avalanche can be triggered, hence the answer to $AP$ is negative. As a second example, consider the following sequence of height differences (always with $p = 3$): ${}^{\omega}0\underline{3}122122221201200^{\omega}$. There are several possibilities for avalanches from the left to the right but none of them arrives to the rightmost 0's region. So the answer to the decision problem is still negative. To get an idea of what happens for a positive instance of the problem, consider the initial configuration: ${}^{\omega}0\underline{3}12222100^{\omega}$ with $p = 3$.

The full proof of Theorem 2.1 is a bit technical and will only be sketched here.

**Theorem 2.1.** *AP is in* $\mathsf{NC}^1$ *for* KSPM *in dimension* 1 *and* $p > 1$.

*Sketch of the proof.* The first step is to prove that, in this situation, the Kadanoff's rule can only be applied once at each pile for any initial monotone stable configuration. Using this result one can see that a pile $k$ such that $h_k = 0$ in the initial configuration and $h_k > 1$ in the final one, must have received grains from pile $k - p$. This pile, in its turn, must have received grains from $k - 2p$ and so on until a "firing" pile $i$ with $i \in [\![1, p-1]\!]$. The height difference for all of these piles must be $p - 1$. The existence of this sequence and the values of the height differences can be checked by a parallel iterative algorithm on a PRAM in time $\mathcal{O}(\log n)$. ■

## 3. Sandpiles and Kadanoff model in two dimensions

There are several possibilities to extend the Kadanoff dynamics to two-dimensional sandpiles. We first generalise the definitions introduced in Section 2.

A two-dimensional sandpile *configuration* is a distribution of sand grains over the $\mathbb{N}\times\mathbb{N}$ lattice. Therefore, a configuration on $\mathbb{N}\times\mathbb{N}$ will be identified by a mapping from $\mathbb{N}\times\mathbb{N}$ into $\mathbb{N}$, giving a number of grains of sand to every position in the lattice. Thus, a configuration will be denoted by $x_{i,j}$ as $(i, j) \mapsto \mathbb{N}$. A configuration $x$ is *monotone* if $\forall i, j \in \mathbb{N} \times \mathbb{N}$, $x_{i,j}$ is such that $x_{i,j} \geq 0$ and $x_{i,j} \geq \max\{x_{i+1,j}, x_{i,j+1}\}$. So we get a monotone sandpile, in the same sense as in [2]. Example 3.1 illustrates the case which violates the condition of monotonicity of the Kadanoff dynamics.

**Example 3.1.** Consider the initial configuration given in the bottom left matrix

$$\begin{array}{cccc} 0 & 1 & 0 & 0 \\ \boxed{2} & 3 & 0 & 0 \\ 8 & 4 & 2 & 2 \\ 8 & 4 & 3 & 2 \end{array}$$

$$\uparrow v$$

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 8 & \boxed{6} & 2 & 2 \\ 8 & 4 & 3 & 2 \end{array} \quad \xrightarrow{h} \quad \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 8 & 4 & 3 & \boxed{3} \\ 8 & 4 & 3 & 2 \end{array}$$

Values count for the number of grains of a pile. We see that we cannot apply the Kadanoff's dynamics for a value of parameter $p = 3$ from the square boxed pile. Indeed, the resulting configurations do not remain monotone neither by applying the dynamics horizontally nor vertically (resp. $\uparrow v$ and $\xrightarrow{h}$). A pile which violates the condition has been highlighted by an oval box in the resulting configurations (it might be not unique). ■

Any configuration can be identified by the mapping of its *horizontal heights differences* (resp. vertical): $h_{\rightarrow} : (i, j) \mapsto x_{i,j} - x_{i+1,j}$ (resp. $h_{\uparrow} : (i, j) \mapsto x_{i,j} - x_{i,j+1}$). A configuration is *finite* if only a finite number of its heights differences matrices has non-zero sand content. A monotone configuration $x$ is *horizontally stable* (resp. *vertically*) if $\forall i, j \in \mathbb{N}\times\mathbb{N}$, $h_{\rightarrow}(i, j) = x_{i,j} - x_{i+1,j} < p$ (resp. $\forall i, j \in \mathbb{N}\times\mathbb{N}$, $h_{\uparrow}(i, j) = x_{i,j} - x_{i,j+1} < p$) and is *stable* if both horizontally and vertically stable.

In other words, it is a generalisation of the Kadanoff model in one dimension, which requires the configuration to be stable if the difference between any two adjacent piles is less than the Kadanoff parameter $p$. To this configuration, we apply the Kadanoff dynamics for a given integer $p \geq 1$. This can be done if and only if the new configuration remains monotone. Said differently, prior its application the dynamics requires to test if the local application gives a non-negative configuration.

The Kadanoff dynamics applied to pile $(i, j)$ for a given $p$ consists in giving a grain of sand to any pile in the horizontal or vertical line, *i.e.* $\{(i, j + 1), ....(i, j + p - 1)\}$ or $\{(i + 1, j), ...(i + p - 1, j)\}$. Notice that when considering the dynamics defined over height differences, we work with a different lattice though isomorphic to the initial one. Fig. 2 explains how the dark pile with coordinates $(i, j)$ with a height difference of $p$ gives grains either horizontally (Fig. 2 left) or vertically (Fig. 2 right). Fig. 2 also depicts the relationship between the sandpiles lattice and the heights differences lattice. The local dynamics depicted by Fig. 2 will be called *Chenilles* (horizontal and vertical, respectively).

For a better understanding of the dynamics, recall that in one dimension an avalanche at pile $i$ changes the heights of piles $i-1$, $i$ and $i+p-1$. In two dimensions, there are height changes on the line but also to both sides of it. The dynamics is simpler to depict than to write down formally. An example of the Kadanoff's dynamics applied horizontally (resp. vertically) is given in Fig. 3. More precisely, the Kadanoff's dynamics for a value of parameter $p = 4$ is depicted in Fig. 4. Observe that we do not need to take into account the number of grains of sand in the piles. It suffices to take the graph of the edges adjacent to each pile (depicted by thick lines) and to store the height differences. So, from now on, we will restrict ourselves to the lattice and to the dynamics defined over the height differences. In Fig. 4, we only keep the information required for applying the dynamics in the simplified view.

**Example 3.2** (Obtaining Bak's)**.** In the case $p = 2$ and if we assume the real sandpile is defined as in [2] (*i.e.* $x_{i,j} \geq \max\{x_{i+1,j}, x_{i,j+1}\}$), we get the templates from Fig. 5. ■

### 3.1. P-completeness

Changing from dimension 1 to 2 (or greater), the statement of AP has to be adapted. Consider a finite configuration $x$ which is non-zero for piles $(i, j)$ with $i, j \geq 0$, stable and monotone and let $Q$ be the sum of the height differences. Let us denote by $n$ the maximum index of non-zero height differences along both axes. Then, SM($n$) denotes the set of monotone stable configurations of the form given by a lower-triangular matrix of size $n \times n$ (a matrix where the entries above the main diagonal are zero). To generalise the avalanche problem in two dimensions, we have to find a generic position which is far enough from the initial sandpile but close enough to be attained. To get rough bounds, the following approach was followed. For the upper bound, the worst case occurs when all the grains are arranged on a single pile (with $Q$ as a height difference) which is at an end of one of the axes -at distance $n$ from the origin- and they fall down. For the lower bound, the pile containing the grains is at the origin and the grains fall along the main diagonal. Thus, our decision problem can be restated:

**Problem AP (dimension 2)**

**Instance:** A configuration $x \in \mathrm{SM}(n)$, $(k,\ell) \in \mathbb{N} \times \mathbb{N}$ such that $x_{k,\ell} = 0$ and $\frac{\sqrt{2}}{2}n \leq \|(k,\ell)\| \leq n + Q$ (with $Q$ the sum of the height differences and $\|.\|$ the standard Euclidean norm).

**Question:** Does there exist an avalanche (obtained by using the vertical and horizontal chenilles) such that $x_{k,\ell} \geq 1$?

To prove the P-completeness of AP we proceed by reduction from the monotone circuit value problem MCVP, proved P-complete under many-one $NC^1$ reduction [9, Theorem 6.2.2]. Observe that the original proof [4] uses a logspace reduction but it should be noted that any logspace reduction is also a NC many-one reduction [9, page 54]. MCVP statement is: given the standard encoding of a Boolean circuit (which ensures a topological numbering of the gates) with $n$ inputs $\{\alpha_1, ..., \alpha_n\}$, a designated output $\beta$ and logic gates AND, OR we want to decide the truth of the output value $\beta$ on binary input $\{\alpha_1, ..., \alpha_n\}$ [9, page 122]. Wlog., we also assume that each gate of the circuit has a fan in of two and fan out of at most two and that the gates are laid out in levels with connections only going to adjacent levels. The problem remains P-complete with these restrictions [4]. For the reduction, we have to construct, by using sandpile configurations, wires and turning the signal on the grid (Fig. 7), logic AND gates (Fig. 8 (Right)), logic OR gates (Fig. 9 (Left)), cross-overs (Fig. 8 (Left)) and signal multipliers for starting the process (Fig. 9 (Right)) and eventually doubling the output of a gate. We also need to define a way to deterministically update the network; to do this, we can apply the chenille's templates in any way such that it is spatially periodic, for instance from the left to the right and from the top to the bottom. Our main result is thus:

**Theorem 3.3.** AP *is P-complete for* KSPM *in dimension two and any* $p \geq 2$.

*Proof.* The fact that AP is in P is already known since C. Moore and M. Nilsson paper [12]. Their proof is done by proving that the total number of avalanches required to relax a sandpile is polynomial in the system's size.

For the reduction, one has to take an arbitrary instance of the MCVP variant previously defined and to build an initial configuration of a sandpile for the Kadanoff's dynamics for $p = 2$ (or greater). Thus, we have to design the following gadgets:

- a wire and how to turn the signal (Fig. 7);
- the crossing of information (Fig. 8 (Left));
- a AND gate (Fig. 8 (Right));
- a OR gate (Fig. 9 (Left));
- a signal multiplier (Fig. 9 (Right)).

Simulated gates can be made up like classical gates (up to an additive constant depending upon their size) with a fan in of two parallel wires and a fan out upper-bounded by two. The sandpile circuit is built directly from the standard encoding of the instance of the MCVP variant. This construction can be done by a NC algorithm [4, 9].

The construction of each gadget is shown graphically for $p = 2$ but can be done for greater values. As an example, we give the construction for the AND gate for $p = 3$ in Fig. 6. Generalising for greater values of $p$ is not hard though tedious and would have exceeded the number of pages. For $p = 2$, the horizontal and vertical chenilles are given in Fig. 5. Recall that the decision problem only adds a sand grain to one pile, say $(0,0)$. To construct the entry vector to an arbitrary circuit we have to construct from the starting pile wires to simulate any variable $\alpha_i = 1$. (If $\alpha_i = 0$ nothing is done: we do not construct a wire from the initial pile. Else, there will be a wire to simulate the value 1). Also remark that the signal propagation does not require a global synchronising clock. Actually, this helps for designing the circuit since one signal arriving at a logical gate can "wait" for the second signal to arrive.

Then, by construction, positive instances of the MCVP variant are in 1:1 correspondence with positive instances of AP. ■

**Remark 3.4.** In the case $p = 2$, KSPM corresponds to Bak's model [1] in two dimensions with a sandpile such that $x_{i,j} \geq \max\{x_{i+1,j}, x_{i,j+1}\}$.

## 4. Conclusion and future work

We have proved that the avalanche problem for the KSPM model in two dimensions is $\mathsf{P}$-complete with a sandpile defined as in [2] and for every value of the parameter $p$. Let us also point out that in the case where $p = 2$, this model corresponds to the two-dimensional Bak's model with a pile such that $x_{i,j} \geq 0$ and $x_{i,j} \geq \max\{x_{i+1,j}, x_{i,j+1}\}$. In this context, we also proved that this physical version (with a two dimension sandpile interpretation) is $\mathsf{P}$-complete. It is important to notice that, by directly taking the two-dimensional Bak's tokens game (given a graph such that a vertex has a number of token greater or equal than its degree, it gives one token to each of its neighbors), its computation universality was proved in [7] by designing logical gates in non-planar graphs. Furthermore, by using the previous construction, C. Moore *et al.* proved the $\mathsf{P}$-completeness of this problem for lattices of dimensions $d$ with $d \geq 3$. But the problem remained open for two-dimensional lattices. Furthermore, it was proved in [3] that, in the latter case, it is not possible to build circuits because the information is impossible to cross. The two-dimensional Bak's operator corresponds, in our framework, to the application of the four rotations of the template (see Fig. 10). But this model is not anymore the representation of a two-dimensional sandpile as presented in [2], that is with $x_{i,j} \geq 0$ and $x_{i,j} \geq \max\{x_{i+1,j}, x_{i,j+1}\}$.

To define a reasonable two-dimensional model, consider a monotone sandpile decreasing for $i \geq 0$ and $j \geq 0$. Over this pile we define the extended Kadanoff's model as a local avalanche in the growing direction of the $i-j$ axis such that monotonicity is allowed. Certainly, one may define other local applications of Kadanoff's rule which also match with the physical sense of monotonicity. For instance, by considering the set $(i+1,j), (i+1,j+1), (i,j+1)$ as the piles to be able to receive grains from pile $(i,j)$. In this sense it is interesting to remark that the two-dimensional sandpile defined by Bak (i.e for nearest neighbors, also called the von Neumann neihborhood, a pile gives a token to each of its four neighbors if and only if it has enough tokens) can be seen as the application of the Kadanoff rule for $p = 2$ by applying to a pile, if there are at least four tokens, the horizontal ($\rightarrow$) and the vertical ($\downarrow$) chenille simultaneously (see Fig. 10). Similarily, for an arbitrary $p$, one may simultaneously apply other conbinations of chenilles which, in general, allows us to get $\mathsf{P}$-complete problems. For instance, when there are enough tokens, the applications of the four chenilles (*i.e.* $\leftarrow$,$\rightarrow$,$\uparrow$ and $\downarrow$) give raise to a new family of local templates called *butterflies* (because of their four wings). It is not so difficult to construct wires and circuits for butterflies. Hence, for this model, the decison problem will remain $\mathsf{P}$-complete. One thing to analyze from an algebraic and complexity point of view is to classify every local rule derivated from the chenille application. Further, one may define a more general sandpile dynamics which contains both Bak's and Kadanoff's ones: i.e given an integer $p \geq 2$, we allow the application of every Kadanoff's update for $q \leq p$. That means that an active pile with more than $p$ grains can distribute up to $q$ grains to the adjacent piles. We are studying this dynamics and, as a first result, we observe yet that in one dimension there are several fixed points and also, given a monotone circuit with depth $m$ and with $n$ gates, we may simulate it on a line with this generalized rule for a given $p \geq m + n$.

For the one-dimensional avalanche problem as defined in Section 2, it can be proved that it belongs tho the class $\mathsf{NC}$ for $p = 2$ and that it remains in the same class when the first $p$ piles contain more than one grain (*i.e.* that there is no hole in the pile). We are in the way to prove the same in the general case.

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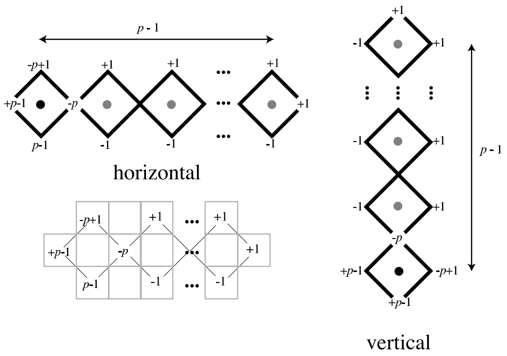


Figure 2: Horizontal resp. vertical (on the left resp. on the right) chenilles in the $\mathbb{N} \times \mathbb{N}$ lattice for arbitrary parameter $p \geq 2$. The pile $(i, j)$ –denoted by a black bullet– gives one grain of sand to the pile $(i + p - 1, j)$ horizontally resp. one grain of sand to the pile $(i, j + p - 1)$ vertically. The figure gives the height differences of this dynamics and the change of the lattice structure between the dynamics on the grains and the corresponding dynamics on the height difference. In the sequel, we will adopt the representation on the left and on the bottom for defining the templates.

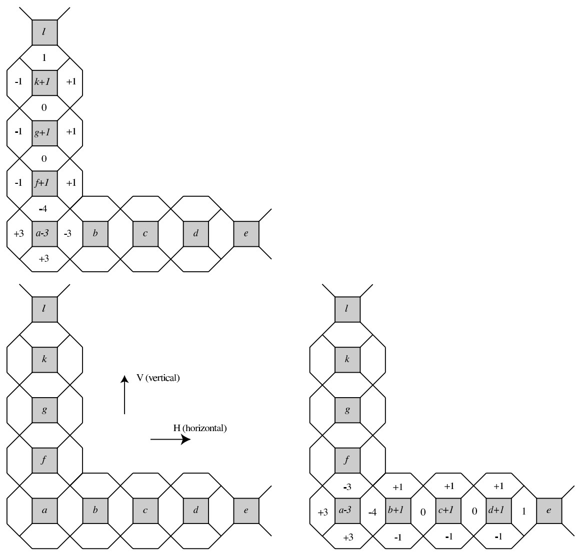


Figure 3: Horizontal and vertical chenilles for $p = 4$. Shaded squares count the number of grains on each pile and the hexagons between the squares the height difference between the corresponding two adjacent piles. The initial configuration is on the bottom-left. The Kadanoff's dynamics is applied from the shaded pile labelled $a$ horizontally or vertically (resp. $\uparrow V$ and $\xrightarrow{H}$) to get the resulting configurations.

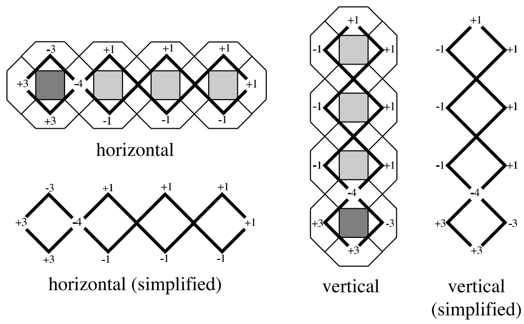


Figure 4: Horizontal and vertical chenilles for $p = 4$. The dynamics is applied to the dark-shaded pile (the leftmost one on the left part of the figure and the lowest one on the right part of the figure). The numbers express the height differences after the application of the dynamics. The two simplified views remove the number of sand grains information and only keeps the height difference information. It corresponds to a change in the lattice structure if the grains are considered or the height difference.

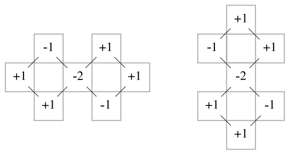

Figure 5: Templates for Bak's dynamics with $p = 2$.

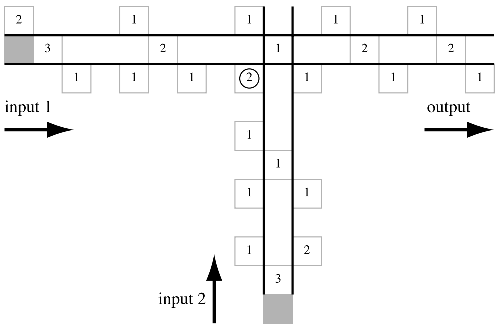


Figure 6: The logic AND gate for two inputs for $p = 3$. The circled "2" is put in order to get enough tokens for the horizontal and vertical inputs.

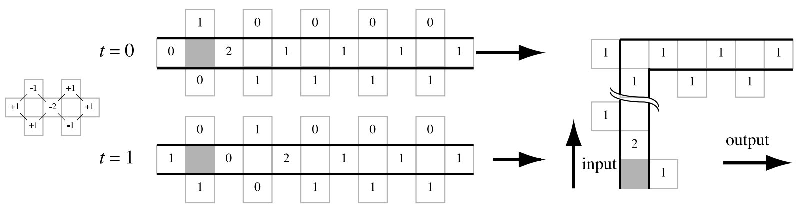


Figure 7: Information propagation in a wire for $p = 2$ at times $t = 0$ and $t = 1$ using the templates for Baks dynamics (recalled on the left of the figure) and wire for turning the information to the right.

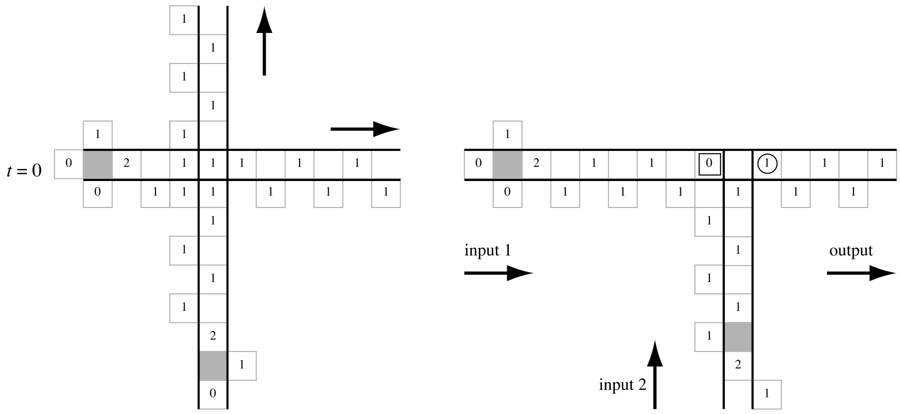


Figure 8: (Left) Crossing over two wires for $p = 2$; arrows show the directions of propagation. (Right) A logic AND gate with two inputs for $p = 2$. The upcoming "2" has to reach the horizontal "2" to change the value of the boxed "0" to "1". Then, the upcoming "2" can apply the vertical chenille template and changes the circled "1" into "2". In other words, the AND is computed by applying 3 horizontal chenilles and 4 vertical ones.

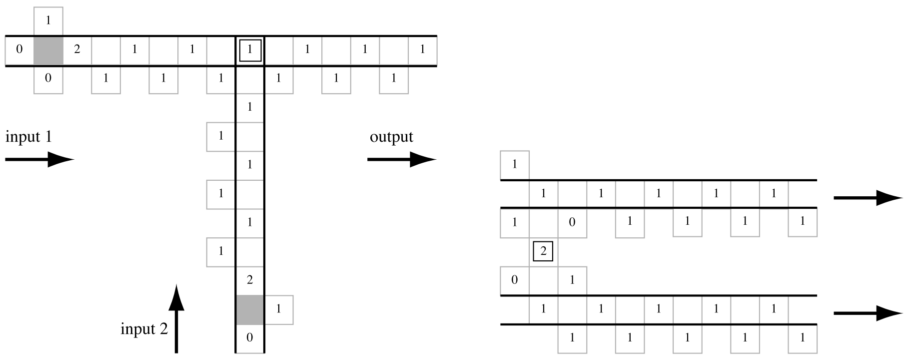


Figure 9: (Left) A logic OR gate with two inputs for $p = 2$. The boxed cell indicates the OR gate point of computation. (Right) A signal multiplier for $p = 2$. The signal starts on boxed pile with value 2 (the input) and applying the first vertical chenille ruled by the Kadanoff's dynamics multiplies the signal on both horizontal wires. Then, we use horizontal chenilles to move both signals according to the arrows.

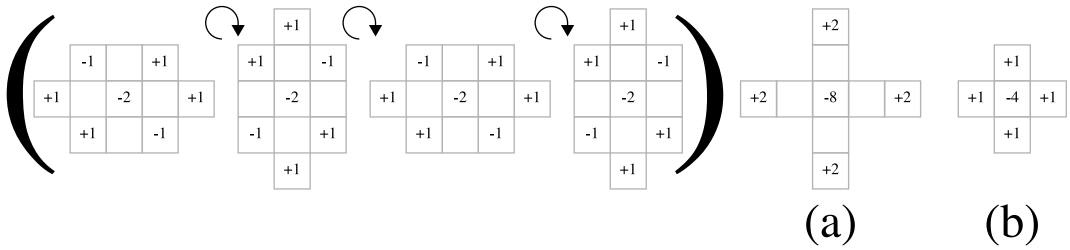


Figure 10: From Bak's to Kadanoff's operators. All the Kadanoff's operators between the brackets have been applied to get the pattern (a). The Bak's pattern (b) is obtained by eliminating the holes in (a) and by dividing the number of tokens by two.